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Almost transversality theorem in the classical dynamical system 1

By

Kiyoshi ASANO

1. Problem and main result

The purpose of this paper is to show that the Lagrange plane moving along the Hamilton flow is almost transversal to the base space, if the Hamiltonian satisfies a non-degeneracy condition.

Let Ω be a domain in R^n, $\tilde{\Omega}=\Omega\times R^n$ be the phase space on Ω (i.e., the cotangent bundle $T^*(\Omega)$) and $H(t, x, \xi)$ be a smooth function defined on $R\times\tilde{\Omega}$. We assume

$$[\mathrm{H.1}]\qquad \partial_x^\alpha\partial_\xi^\beta H(t, x, \xi)\in C^1(R\times\tilde{\Omega})\quad \text{for}\quad |\alpha|+|\beta|\leq 2\,,$$

We consider the Hamilton flow defined by H, i.e. the characteristic curve defined by the differential equation

$$(1.1)\qquad \frac{dX}{dt}=\frac{\partial H}{\partial\xi}(t, x, \Xi)\qquad (1.1)_s\qquad X|_{t=s}=x\in\Omega\,,$$

$$\frac{d\Xi}{dt}=-\frac{\partial H}{\partial x}(t, X, \Xi)\,,\qquad \Xi|_{t=s}=\xi\in R^n\,.$$

The solution $(X(t), \Xi(t))$ of the intial value problem $(1.1)-(1.1)_s$ exists uniquely in a maximal time interval $I_0=I_0(s, x, \xi)$, which is described as

$$(1.2)\qquad X(t)=X(t, x, \xi)=X(t, s, x, \xi)=X\,,$$

$$\Xi(t)=\Xi(t, x, \xi)=\Xi(t, s, x, \xi)=\Xi\,,$$

or

$$(1.2)'\qquad (X(t), \Xi(t))=S(t, s)(x, \xi)\,.$$

The mapping $S(t, s)$ is a local diffeomorphism in $\tilde{\Omega}$ and satisfies

$$(1.3)\qquad S(t, s)S(s, r)=S(t, r)\quad \text{(transitive law)}\,,$$

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$S(s, s)=I=\text{identity}$.

Taking a cotangent vector $z=(y, \eta)\in R_x{}^n\times R_\xi{}^n=T^*_{(x,\xi)}(\tilde{\Omega})$, we have

(1.4) $$\frac{d}{dt}X^1(t)z=H_{\xi x}(t, X, \Xi)X^1(t)z+H_{\xi\xi}(t, x, \Xi)\Xi^1(t)z\,,$$

$$\frac{d}{dt}\Xi^1(t)z=-H_{xx}(t, X, \Xi)X^1(t)z-H_{x\xi}(t, x, \Xi)\Xi^1(t)z\,,$$

$(1.4)_s$ $$(X^1(t)z, \Xi^1(t)z)|_{t=s}=z\,.$$

Here $(X^1(t), \Xi^1(t))$ is a bundle map from $T^*_{(x,\xi)}(\tilde{\Omega})$ into $T^*_{(X,\Xi)}(\tilde{\Omega})$ induced from a differentiable map $S(t, s)$, and

(1.5) $$H_{\xi x}=(\partial^2 H/\partial\xi_i\partial x_j)\,, \qquad H_{x\xi}=(\partial^2 H/\partial x_i\partial\xi_j)\,,$$

$$H_{\xi\xi}=(\partial^2 H/\partial\xi_i\partial\xi_j)\,, \qquad H_{xx}=(\partial^2 H/\partial x_i\partial x_j)\in C^1(R\times\tilde{\Omega})\,.$$

If we take a standard (orthogonal) coordinate system $(x, \xi)=(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ in $\Omega\times R^n=\tilde{\Omega}$, then we can express the linear mappings $X^1(t)$ and $\Xi^1(t)$ as

(1.6) $$X^1(t)z=\sum_{i=1}^n y_i\frac{\partial X}{\partial x_i}(t)+\sum_{j=1}^n \eta_j\frac{\partial X}{\partial\xi_j}(t)$$

$$\Xi^1(t)z=\sum_{i=1}^n y_i\frac{\partial\Xi}{\partial x_i}(t)+\sum_{j=1}^n \eta_j\frac{\partial\Xi}{\partial\xi_j}(t)\,, \quad z={}^t(y, \eta)\in R^{2n}\,.$$

In what follows $X^1(t)$ and $\Xi^1(t)$ are considered as the mappings from $T^*_{(x\xi)}(\tilde{\Omega})$ to R^n. Sometimes we use a simpler expression

$(1.5)'$ $$Z^1(t, x, \xi)={}^t(X^1(t)z, \Xi^1(t)z)\,,$$

$(1.4)'$ $$\frac{d}{dt}Z^1=H^2(t, x, \xi)Z^1\,, \quad H^2=\begin{pmatrix} H_{\xi x} & H_{\xi\xi} \\ -H_{xx} & -H_{x\xi}\end{pmatrix}.$$

As is well known, the differential equation (1.4) is called the variational equation associated with (1.1), and has a unique solution $Z^1(t, x, \xi)z$ in I_0 for each $z={}^t(y, \eta)\in R^{2n}$. Linear independence of $\{z_j\}\subset R^{2n}=T^*_{(x,\xi)}(\tilde{\Omega})$ implies the linear independence of $\{Z^1(t, x, \xi)z_j\}$. In this paper, however, we are concerned with the rank of $\{X^1(t, x, \xi)z_j\}$.

We introduce a symplectic form (skew-symmetric non-degenerate inner product) $[\ ,\ ]$ of R^{2n}. Denote by $\langle\ ,\ \rangle$ the usual inner product in R^n, and put

(1.7) $$[w, z]=\langle v, \zeta\rangle-\langle y, \eta\rangle \quad \text{for} \quad w={}^t(v, \eta)\,, \quad z={}^t(y, \zeta)\in R^{2n}\,.$$

For convenience we define two projections p and q from R^{2n} to R^n by

(1.8) $pz=y, \quad qz=\zeta \quad \text{for} \quad z={}^t(y, \zeta)$.

With these notations we write

(1.9) $pZ^1(t)=X^1(t), \quad qZ^1(t)=\Xi^1(t)$.

The form [,] is invariant along the Hamilton flow, i.e.

$$(1.10) \qquad \frac{d}{dt}[Z^1(t)w, Z^1(t)z]=\frac{d}{dt}\{\langle X^1(t)v, \Xi^1(t)\zeta\rangle - \langle X^1(t)y, \Xi^1(t)\eta\rangle\}=0\,.$$

We call a linear subspace L of R^{2n} a *null plane*, if $[w, z]=0$ for each w, $z\in L$. An n-dimensional null plane of R^{2n} is called a *Lagrange plane*. (1.10) means that $Z^1(t, x, \xi)$ maps a null plane L of $T^*_{(x,\xi)}(\tilde{\Omega})$ onto a null plane $Z^1(t, x, \xi)L$ of $T^*_{(X,\Xi)}(\tilde{\Omega})$. The following Lemma is useful in later discussions.

Lemma 1.1. *Let L be a Lagrange plane of R^{2n}, and* $\dim pL=k$, $0\le k\le n$. *Then, there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of R^n such that $\{z_i={}^t(e_i, \mu_i e_i); 1\le i\le k\}\cup\{z_j={}^t(0, e_j); k+1\le j\le n\}$ is a basis of L and $\{e_1, \cdots, e_k\}$ is a basis of pL.*

Proof. Let $\{w_i={}^t(v_i, \eta_i); 1\le i\le n\}$ be a basis of L. We can assume that $\{v_1, \cdots, v_k\}$ is an orthonormal set, $\langle v_i, v_j\rangle=\delta_{ij}$, $1\le i, j\le k$, and $v_{k+1}=\cdots=v_n=0$. The symplectic relation:

$$[w_i, w_j]=\langle v_i, \eta_j\rangle-\langle v_j, \eta_i\rangle=0$$

implies that η_j, $k+1\le j\le n$, is orthogonal to $\{v_1, \cdots, v_k\}$, that is, $\eta_j\perp pL$ for $k+1\le j\le n$. Applying a suitable transformation to $\{w_i; 1\le i\le n\}$, we can assume that $\{\eta_j; k+1\le j\le n\}$ is an orthonormal set and $\{\eta_i; 1\le i\le k\}$ is orthogonal to η_j, $k+1\le j\le n$.

Taking an orthogonal matrix Q, we make a new basis $\{z_i={}^t(y_i, \zeta_i); 1\le i\le k\}\cup\{z_j={}^t(0, \eta_j); k+1\le j\le n\}$ of L by the transformation

$$(1.11) \qquad (y_1, \cdots, y_k)=(v_1, \cdots, v_k)Q\,,$$
$$(\zeta_1, \cdots, \zeta_k)=(\eta_1, \cdots, \eta_k)Q\,.$$

Then we have a matrix-equalities

$$(1.12) \qquad (\langle y_i, \zeta_j\rangle)={}^tQ(\langle v_i, \eta_j\rangle)Q={}^tQ((\langle\langle v_j, \eta_i\rangle)Q=((\langle y_j, \zeta_i\rangle)\,,$$

$$(1.13) \qquad (\langle y_i, y_j\rangle)={}^tQ(\langle v_i, v_j\rangle)Q={}^tQ(\delta_{ij})Q=E_k\,.$$

A suitable orthogonal matrix Q diagonalizes the symmetric matrix $(\langle v_i, \eta_j\rangle)$, i.e.

(1.14) $\langle y_i, \zeta_j\rangle=\mu_i\delta_{ij}\,,\qquad 1\leq i, j\leq k\,.$

(1.13) means that $\{y_1, \cdots, y_k\}$ is an orthonormal set, and (1.11) means that $\{\zeta_1, \cdots, \zeta_k\}$ is orthogonal to $\{\eta_{k+1}, \cdots, \eta_n\}\subset(pL)^{\perp}$. Hence $\{\zeta_1, \cdots, \zeta_k\}\subset pL$ which is spnned by $\{y_1, \cdots, y_k\}$. This fact and (1.14) prove

(1.15) $\zeta_i=\mu_i y_i\,,\qquad 1\leq i\leq k\,.$ Q.E.D.

Remark. We call the basis $\{z_1, \cdots, z_n\}$ the standard basis of L.

Take a complementary pair $\{L, M\}$ of Lagrange planes such that

(1.16) $L\oplus M=R^{2n}$ (orthogonal direct sum).

Then we re-coordinate $\tilde{\Omega}\Omega\times R^n$ by

(1.17) $\Omega\times R^n\ni{}^t(x, \xi)=w+\eta\equiv{}^t(w, \eta)\in L\oplus M\,.$

We define a family of the moving Lagrange planes by

(1.18) $M(t, w, \eta)=Z^1(t, w, \eta)M\equiv Z^1(t, x, \xi)M\,.$

$M(t, w, \eta)$ may be defined only locally. We put

(1.19) $m(t, w, \eta)\equiv\dim X^1(t, w, \eta)=\dim pZ^1(t, w, \eta)M\,.$

Then we have the following theorem describing the bifurcation of singularities where $X^1(t, w, \eta)$ is not transversal to Ω.

Theorem 1. *Assume that* $H(t, x, \xi)$ *satisfies* $[H, 1]$ *and*

$[H.2]$ $H_{\xi\xi}(t, x, \xi)>0$ *(positive definite) for* $(t, x, \xi)\in R\times\tilde{\Omega}\,.$

Take a pair of Lagrange planes $\{L, M\}$ *satisfying* (1.16), *let* M *be the initial Lagrange plane at* $(s, x, \xi)=(s, w, \eta)\in\tilde{\Omega}$, *and define by* (1.18) *the Lagrange planes* $M(t, x, \xi)\equiv M(t, w, \eta)$ *moving along the Hamilton flow* $(X(t, x, \xi), \Xi(t, x, \xi))$ *starting from* (s, x, ξ).

Assume

$[D.1]$ $m(t_0, w_0, \eta_0)=m<n\,,\quad (w_0, \eta_0)\equiv(x_0, \xi_0)\in\tilde{\Omega},\ t_0\in I_0(s, x_0, \xi_0)\,.$

If $(X_0, \Xi_0)=(X(t_0, x_0, \xi_0), \Xi(t_0, x_0, \xi_0))$ *is not a stasionary point of the equation* (1.1) *near* $t=t_0$, *then: there exist neghibourhoods* U_0 *of* (x_0, ξ_0) *and* $V_0=[t_0-\delta, t_0+\delta]\times U_0$ *of* (t_0, x_0, ξ_0), *and a family of functions* $\{\psi_j(x, \xi)=\psi_j(w, \eta);\ m+1\leq j\leq n\}\subset\mathrm{Lip}(U_0)$ *such that*

$[D.2]$ (1) $\psi_j(x_0, \xi_0)=0,\ m+1\leq j\leq n,$

(2) $(t, x, \xi)=(t, w, \eta)\in V_0$ *satisfies* $m(t, x, \xi)<n$ *if and only if* $t=t_0+\psi_j(x, \xi)$ *for some* $j,\ m+1\leq j\leq n.$

Here Lip(U_0) *is the set of Lipschitz continuous functions defined on* U_0.

2. Proof of Theorem 1

We note that the stationary solution $(X, \Xi)\equiv(x, \xi)$ may appear only for the discrete set of $\xi\in T_x{}^*(\Omega)$, if $x\in\Omega$ is fixed. Under the conditions stated in Theorem 1, we have the following Lemma 2.1 which is a trivial version of Lemma 1.1.

Lemma 2.1. *Assume* $m(t_0, x_0, \xi_0)=\dim X^1(t_0, x_0, \xi_0)=m\leq n$. *Then: there exist a basis* $\{\eta_1{}^0, \cdots, \eta_n{}^0\}$ *of* M *and an orthogonal basis* $\{e_1{}^0, \cdots, e_n{}^0\}$ *of* $R^n=T^*_{X_0}(\Omega)$, $X_0=X(t_0, x_0, \xi_0)$, *such that*

(2.1) (1) $e_i{}^0=X^1(t_0, x_0, \xi_0)\eta_i{}^0=pZ^1(t_0, x_0, \xi_0)\eta_i{}^0$,

$\Xi^1(t_0, x_0, \xi_0)\eta_i{}^0=qZ^1(t_0, x_0, \xi_0)\eta_i{}^0=\mu_i{}^0e_i{}^0$, $1\leq i\leq m$,

$\{e_1{}^0, \cdots, e_m{}^0\}$ *is a basis of* $X^1(t_0, x_0, \xi_0)M$,

(2) $e_j{}^0=\Xi^1(t_0, x_0, \xi_0)\eta_j{}^0$,

$X^1(t_0, x_0, \xi_0)\eta_j{}^0=0$, $m+1\leq j\leq n$.

In what follows we assume $L=T_x{}^*(\Omega)$ and $M=T_\xi{}^*(R^n)$ for simplicity, and identify the point $(y, \eta)\in\tilde{\Omega}$ with the element of $T^*_{(t,\xi)}(\tilde{\Omega})$.

Suggested by Lemma 2.1, we define two linear mappings $K(t, x, \xi)$ and $J(t, x, \xi)$ acting in R^n. (1) If $m(t, x, \xi)=n$, we put

$$K(t, x, \xi)=\Xi^1(t, x, \xi)X^1(t, x, \xi)^{-1}\,, \quad \text{i.e.} \tag{2.2}$$

$$K(t, x, \xi)X^1(t, x, \xi)\eta=\Xi^1(t, x, \xi)\eta\,, \quad \eta\in M\,.$$

The symplectic relation

$$\langle X^1\eta, \Xi^1\zeta\rangle=\langle X^1\zeta, \Xi^1\eta\rangle\,, \quad \eta, \zeta\in M=R^n\,,$$

shows that $K(t, x, \xi)$ is a symmetric operator in R^n.

(2) If $m(t_0, x_0, \xi_0)<n$, we put with a suitabl real number λ

$$A(t_0, x_0, \xi_0, \lambda)=\lambda X^1(t_0, x_0, \xi_0)-\Xi^1(t_0, x_0, \xi_0)\,, \tag{2.3}$$

$$J(t_0, x_0, \xi_0, \lambda)=X^1(t_0, x_0, \xi_0)A(t_0, x_0, \xi_0, \lambda)^{-1}\,.$$

More presicely, we take bases $\{\eta_1{}^0, \cdots, \eta_n{}^0\}$ of M and $\{e_1{}^0, \cdots, e_n{}^0\}$ of R^n specified in Lemma 2.1, and observe

$$A(t_0, x_0, \xi_0, \lambda)\eta_i{}^0=\begin{cases}(\lambda-\mu_i)e_i{}^0\,, & 1\leq i\leq m\,,\\ -e_i{}^0\,, & m<i\leq n\,.\end{cases} \tag{2.4}$$

If we choose $\lambda\in D(\sigma)=\{\lambda\in R;\ \sigma\leq|\lambda|\leq\bar{\sigma},\ \max|\mu_i^0|<\sigma\leq\bar{\sigma}/4\}$, the continuity of $A(t, x, \xi, \lambda)$ implies that there exists a neighbourhood V_0 of (t_0, x_0, ξ_0) such that there holds

$$A(t, x, \xi, \lambda)M=R^n \quad \text{for} \quad (t, x, \xi)\in V_0, \quad \lambda\in D(\sigma). \tag{2.5}$$

Hence the mapping $J(t, x, \xi, \lambda)$ is well defined for $(t, x, \xi, \lambda)\in V_0\times D(2\sigma)$ i.e.

$$J(t, x, \xi, \lambda)\{\lambda X^1(t, x, \xi)-\Xi^1(t, x, \xi)\}\eta=\Xi^1(t, x, \xi)\eta, \quad \eta\in M. \tag{2.3$'$}$$

If $m(t, x, \xi)=n$, the definition (2.3) (or (2.3)$'$) reduces to

$$J(t, x, \xi, \lambda)=\{\lambda-K(t, x, \xi)\}^{-1}. \tag{2.6}$$

The linear mapping $J(t, x, \xi, \lambda)$ is also symmetric.

We note

$$m(t, x, \xi)=m \Longleftrightarrow \operatorname{rank} J(t, x, \xi, \lambda)=m. \tag{2.7}$$

For abbreviation we write

$$z=(x, \xi), \quad z_0=(x_0, \xi_0), \quad \tilde{z}=(t, x, \xi), \quad \tilde{z}_0=(t_0, x_0, \xi_0),$$

$$A=A(\tilde{z}, \lambda), \quad A_0=A(\tilde{z}_0, \lambda), \quad X^1=X^1(\tilde{z}) \quad X_0^1=X^1(\tilde{z}_0) \quad \text{etc}.$$

Since A and A_0 are invertible on M, we have

$$\begin{aligned} A^{-1}-A_0^{-1}&=A_0^{-1}BA^{-1}=A^{-1}BA_0^{-1}, \\ B=B(t, x, \xi, \lambda)&=A_0-A \\ &=-\lambda\{X^1(t, z)-X^1(t_0, z_0)\}+\{\Xi^1(t, z)-\Xi^1(t_0, z_0)\}. \end{aligned} \tag{2.8}$$

If we choose a sufficiently small neighbourhood V_0, the operator norm $\|A_0^{-1}B\|$ is also small, e.g. $\|A_0^{-1}B\|\leq 1/2$ for $(\tilde{z}, \lambda)\in V_0\times D(2\sigma)$. Hence we obtain

$$(1-A_0^{-1}B)^{-1}=1+A_0^{-1}B+(A_0^{-1}B)^2+\cdots, \tag{2.9}$$

$$\begin{aligned} A^{-1}&=(1-A_0^{-1}B)^{-1}A_0=A_0^{-1}+A_0^{-1}BA_0^{-1}+A_0^{-1}BCBA_0^{-1}, \\ C&=(1-A_0^{-1}B)^{-1}A_0^{-1}. \end{aligned} \tag{2.10}$$

From (1.4) it follows

$$\begin{aligned} X^1-X_0^1&=(t-t_0)\{H_{\xi x}^0X_0^1-H_{\xi\xi}^0\Xi_0^1\}+Y(t, z), \\ Y(\tilde{z}_0)&=\frac{\partial Y}{\partial t}(\tilde{z}_0)=0, \quad H_{\xi x}^0=H_{\xi x}(\tilde{z}_0), \quad H_{\xi\xi}^0=H_{\xi\xi}(\tilde{z}_0) \quad \text{etc}. \end{aligned} \tag{2.11}$$

$$\Xi^1-\Xi_0^1=-(t-t_0)\{H_{xx}^0X_0^1-H_{x\xi}^0\Xi_0^1\}+\theta(t, z), \tag{2.12}$$

$$\theta(\tilde{z}_0)=\frac{\partial\theta}{\partial t}(\tilde{z}_0)=0\,.$$

We also note

$$J_0e_i{}^0=X_0{}^1A_0{}^{-1}e_i{}^0=\begin{cases}(\lambda-\mu_i)^{-1}e_i{}^0\,, & 1\leq i\leq m\,,\\ 0\,, & m<i\leq n\,,\end{cases}\tag{2.13}$$

$$L_0e_i{}^0\equiv\Xi_0{}^1A_0{}^{-1}e_i{}^0=\begin{cases}\mu_i{}^0(\lambda-\mu_i{}^0)^{-1}e_i{}^0\,, & 1\leq i\leq m\,,\\ e_i{}^0\,, & m<i\leq n\,.\end{cases}\tag{2.14}$$

Let P_0 and Q_0 be the orthogonal projections from R^n to the subspaces spanned by $\{e_1{}^0,\cdots,e_m{}^0\}$ and $\{e_{m+1}^0,\cdots,e_n{}^0\}$, respectively. Then we have the trivial equalities:

$$L_0Q_0=-Q_0\,,\quad P_0-\lambda J_0=-L_0P_0\,.\tag{2.15}$$

Combining (2.8)–(2.15) with (2.3)′ and putting $\tau=t-t_0$, we obtain

$$\begin{aligned}J&=X^1A^{-1}\\&=J_0+\tau\{-L_0P_0H_{\xi x}^0J_0-J_0H_{x\xi}^0L_0P_0-J_0H_{xx}^0L_0P_0-L_0P_0H_{\xi\xi}^0L_0P_0\}\\&\quad+\tau\{Q_0H_{\xi x}^0J_0+Q_0H_{\xi\xi}^0L_0P_0\}+\tau\{J_0H_{x\xi}^0Q_0+L_0P_0H_{\xi\xi}^0Q_0\}\\&\quad-\tau Q_0H_{\xi\xi}^0Q_0+\mathcal{J}\,,\end{aligned}\tag{2.16}$$

$$\mathcal{J}(\tilde{z}_0)=\frac{\partial\mathcal{J}}{\partial t}(\tilde{z}_0)=0\,.$$

Introducing a new linear mapping G_0, we rewrite (2.16) as

(2.16)′ $\quad J=J_0+\tau G_0-\tau D_0-\mathcal{J}$, where

(2.17) $\quad G_0$ is independent of $\tilde{z}$ and $Q_0G_0Q_0=0$,

$$D_0=Q_0H_{\xi\xi}^0Q_0=Q_0D_0=D_0Q_0>0 \text{ on } Q_0R^n\,.$$

Note that the linear mappings J_0, G_0, D_0 and $\mathcal{J}$ are symmetric in R^n.

Temporalily we consider these mappings acting in C^n. Define the orthogonal projections $P(\tilde{z})$ and $Q(\tilde{z})$ by

$$Q(\tilde{z})=\frac{1}{2\pi i}\int_\Gamma(\mu-J(\tilde{z},\lambda))^{-1}d\mu\,,\quad (Q(\tilde{z}_0)=Q_0)\,,\tag{2.18}$$

$$P(\tilde{z})=1-Q(\tilde{z})\,,\quad (P(\tilde{z}_0)=P_0)\,,$$

where Γ is a circle $\{\mu\in C;\ |\mu|=\varepsilon>0\}$ containing 0 and excluding $\{(\lambda-\mu_i{}^0)^{-1};\ 1\leq i\leq m\}$. If we take a sufficiently small neighbouhood V_0, $Q(\tilde{z})$ and $P(\tilde{z})$

are in $C^1(V_0)$ and orthogonal projections in R^n. Moreover we have

(2.19) $J(\tilde{z}, \lambda)Q(\tilde{z})=Q(\tilde{z})J(\tilde{z}, \lambda)$, $J(\tilde{z}, \lambda)P(\tilde{z})=P(\tilde{z})J(\tilde{z}, \lambda)$,

(2) $\operatorname{rank} J(\tilde{z}, \lambda)=\operatorname{rank} J(\tilde{z}, \lambda)Q(\tilde{z})+\operatorname{rank} J(\tilde{z}, \lambda)P(\tilde{z})$,

$\operatorname{rank} J(\tilde{z}, \lambda)P(\tilde{z})=m$ for $\tilde{z}\in V_0, \lambda\in D(\sigma)$.

(2.20) (1) $J(\tilde{z}, \lambda)Q(\tilde{z})=-\tau D(\tilde{z})+E(\tilde{z})$,

(2) $D(\tilde{z})=Q(\tilde{z})D_0Q(\tilde{z})>0$ on $Q(\tilde{z})R^n$,

(3) $D(\tilde{z}), E(\tilde{z})=Q(\tilde{z})\mathcal{J}(\tilde{z})Q(\tilde{z})\in C^1(V_0)$, $E(\tilde{z}_0)=\frac{\partial E}{\partial t}(\tilde{z}_0)=0$,

(2.21) (1) $\operatorname{rank} J(\tilde{z}, \lambda)Q(\tilde{z})=\operatorname{rank}((-\tau+F(\tilde{z}))|_{Q(\tilde{z})R^n})$, $\tilde{z}\in V_0$,

(2) $F(\tilde{z})=D(\tilde{z})^{-1/2}E(\tilde{z})D(\tilde{z})^{-1/2}\in C^1(V_0)$,

$F(\tilde{z}_0)=\frac{\partial F}{\partial t}(\tilde{z}_0)=0$.

Since $F(\tilde{z})$ is symmetric in $Q(\tilde{z})R^n$, we apply the min-max principle to estimate the variance of the eigenvalues $\{\tau_j(\tilde{z})=\tau_j(\tau, z);\ m+1\leq j\leq n\}$ of $F(\tilde{z})$ in $Q(\tilde{z})R^n$. Then we have

(2.22) (1) $\tau_j(\tilde{z})\in \operatorname{Lip}(V_0)$, $\tau_j(\tilde{z}_0)=0$, $m+1\leq j\leq n$,

(2) $|\tau_j(\tau, z)-\tau_j(\tau', z)|\leq|\tau-\tau'|/2$ for $\tau, \tau'\in[-\delta, \delta]$, $z\in U_0$,

if we choose $\delta>0$ and a neighbourhood U_0 of z_0 sufficiently small.

By virtue of (2.22) (2), we can apply the classical implicit function theorem in order to solve he equations

(2.23) $\tau=\tau_j(\tau, z)$, $m+1\leq j\leq n$.

Taking smaller $\delta>0$ and U_0 (if neccessary), we can prove that the equation (2.23) hs a unique solution: $\tau=\psi_j(z)$, for each $z\in U_0$, such that $\psi_j(z)\in \operatorname{Lip}(U_0)$, $\psi_j(z_0)=0$, $m+1\leq j\leq n$. Conversely, if $(\tau, z)\in[-\delta, \delta]\times U_0$ and $\tau=\tau_j(\tau, z)$ for some j, $m<j\leq n$, then it follows $\tau=\psi_j(z)$. This completes the proof.

Institute of Mathematics
Yoshida College
Kyoto University

References

[Arnold, V.I.] Mahthematical methods of classical Mechanics, Springer New York, Heiderberg, Berlin, 1980.

[Milnor, J.] Morse Theory, Princeton University Press, Princeton, 1963.