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Local existence for the semilinear Schrödinger equations in one space dimension

By

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1. Introduction

In this paper we study the initial value problem for the semilinear Schrödinger equations in one space dimension:

$$(1.1) \qquad u_t - iu_{xx} = F(u, u_x) \quad \text{in} \quad (0, \infty) \times \mathbf{R} ,$$

$$(1.2) \qquad u(0, x) = u_0(x) \quad \text{in} \quad \mathbf{R} ,$$

where $u(t, x)$ is complex-valued, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, and $u_{xx} = \partial^2 u/\partial x^2$. We assume that the nonlinear term $F: \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is

$$(1.3) \qquad F(u, q) \in C^\infty(\mathbf{R}^2 \times \mathbf{R}^2; \mathbf{C}) \ , \quad |F(u, q)| \leq C(|u|^2 + |q|^2) \quad \text{near} \quad (u, q) = 0 \ .$$

We regard the second variable q as u_x. Let $\partial/\partial u = 1/2\,(\partial/\partial v - i\partial/\partial w)$, $\partial/\partial \bar{u} = 1/2\,(\partial/\partial v + i\partial/\partial w)$, $\partial/\partial q = 1/2\,(\partial/\partial \xi - i\partial/\partial \eta)$, and $\partial/\partial \bar{q} = 1/2\,(\partial/\partial \xi + i\partial/\partial \eta)$ where $u = v + iw$, $q = \xi + i\eta$ and $v, w, \xi, \eta \in \mathbf{R}$.

The purpose of this paper is to show the local existence of solutions to (1.1) – (1.2). When we try to get a classical energy estimate, Im $\partial F/\partial q\,(u, u_x)$ which is imaginary part of coefficient of u_x gives the loss of derivatives, and then we cannot derive the estimate. Our idea to resolve this difficulity comes from the theory of linear Schrödinger type equations. More precisely, let us consider the following linear problem:

$$(1.4) \qquad u_t - iu_{xx} + b(x)u_x + c(x)u = f(t, x) \quad \text{in} \quad \mathbf{R} \times \mathbf{R} ,$$

$$(1.5) \qquad u(0, x) = u_0(x) \quad \text{in} \quad \mathbf{R} ,$$

where $b(x)$, $c(x) \in \mathscr{B}^\infty(\mathbf{R})$, $u_0(x) \in L^2(\mathbf{R})$, and $f(t, x) \in L^1_{\text{loc}}(\mathbf{R}; L^2(\mathbf{R}))$. According to Takeuchi [10] (see also Mizohata [6]), a necessary and sufficient condition for L^2–wellposedness to (1.4) – (1.5) is

$$(1.6) \qquad \sup_{x \in \mathbf{R}} \left| \int_0^x \mathrm{Im}\, b(y)\, dy \right| < +\infty \ .$$

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In fact, if we assume (1.6), then the following transformation

$$(1.7)\qquad u(x)\ \mapsto\ v(x)=u(x)\exp\left(\frac{i}{2}\int_0^x b(y)dy\right)$$

is automorphic in $L^2(\mathbf{R})$ and (1.4) − (1.5) become

$$(1.8)\qquad v_t-iv_{xx}+\tilde{c}(x)v=\tilde{f}(t,x)\qquad \text{in}\quad \mathbf{R}\times\mathbf{R}\ ,$$

$$(1.9)\qquad v(0,x)=v_0(x)\qquad \text{in}\quad \mathbf{R}\ ,$$

where $\tilde{c}(x)=-b_x(x)/2+ib^2(x)/4+c(x)\in\mathscr{B}^\infty(\mathbf{R})$, $v_0(x)=u_0(x)\exp\left(\frac{i}{2}\int_0^x b(y)\,dy\right)\in L^2(\mathbf{R})$, and $\tilde{f}(t,x)=f(t,x)\exp\left(\frac{i}{2}\int_0^x b(y)\,dy\right)\in L^1_{\mathrm{loc}}(\mathbf{R};L^2(\mathbf{R}))$. Since the first order term is eliminated in (1.8), we can easily obtain the energy inequality in $L^2(\mathbf{R})$. Thus, this shows that (1.6) is a sufficient condition for L^2−wellposedness.

Studies on semilinear equations have been mainly concerned with the case of $\mathrm{Im}\partial F/\partial q\equiv 0$ (Klainerman-Ponce [4], Shatah [8], Cohn [1].). The reason is that it is difficult even to show the local existence because of the loss of derivatives. But Hayashi [2] succeeded in treating some cubic polynominals which include the type of $\mathrm{Im}\partial F/\partial q\not\equiv 0$, in some classes of analytic functions.

We prove the local existence of solutions to (1.1) - (1.2) with general nonlinear terms, by using a modified method of linear theory. The main results are following.

Theorem 1. *Let m be an integer ≥ 3. Then there exist a constant $\alpha_0>0$ and a time $T>0$ depending only on α_0 such that for any $u_0\in X^m$ with $\|u_0\|_{X^m}\leq\alpha_0$, an initial value problem (1.1) − (1.2) possesses a unique solution $u\in L^\infty(0,T;X^m)$, where $X^m=H^m(\mathbf{R})\cap\Sigma$ and $\Sigma=\{u\in L^2(\mathbf{R});\ xu\in L^2(\mathbf{R})\}$ with $\|u\|_\Sigma=\|\langle x\rangle u\|_{L^2}$.*

We introduce the following quantity

$$(1.10)\qquad \left(\sum_{k=0}^m\int|\partial_x^k u|^2\exp\left(\int_{-\infty}^x \mathrm{Im}\frac{\partial F}{\partial q}(u,u_x)(y)dy\right)dx\right)^{1/2},$$

which corresponds to the transformation (1.7). If $\int_{-\infty}^x \mathrm{Im}\frac{\partial F}{\partial q}(u,u_x)(y)\,dy$ is bounded, then (1.10) is equivalent to H^m norm. Namely this condition is the same as (1.6). To ensure this, we introduced the function space Σ (see Lemma 4).

Recently Soyeur [9] has succeeded in solving the next type of the semilinear equation:

$$\partial_t u - i\Delta u = -\frac{2i\bar{u}}{1+|u|^2}\sum_{j=1}^{N}(\partial_{x_j}u)^2 \quad \text{in} \quad (0, \infty)\times\mathbf{R}^N . \tag{1.11}$$

(Actually he solved the Ishimori equation. But this is essentially the same to (1.11) from our point of view.) The equation (1.11) is in the stereographic representation of Heisenberg ferrowmagnetic model equation (HFM). He introduced

$$\left|\sum_{k=0}^{m}\int\frac{|\partial_x^k u|^2}{(1+|u|^2)^2}dx\right|^{1/2} , \tag{1.12}$$

as a quantity corresponding to conservation laws of HFM. Takeuchi's method is available to the case of general space dimensions:

$$\left(\partial_t - i\Delta + \sum_{j=1}^{N} b_j(x)\,\partial_{x_j} + c(x)\right)u = f(t, x) \quad \text{in} \quad \mathbf{R}\times\mathbf{R}^N ,$$

if

$$\partial_{x_j}\mathrm{Im}\,b_i(x) - \partial_{x_i}\mathrm{Im}\,b_j(x) = 0 \quad \text{for} \quad i, j = 1, \ldots, N .$$

And for (1.11) we have

$$\exp\left(\int_{-\infty}^{x_j}\mathrm{Im}\frac{\partial F}{\partial q_j}(u, \partial_x u)\,(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_N)\,dy_j\right)$$
$$= \exp\left(-2\int_{-\infty}^{x_j}\frac{(|u|^2)_{x_j}}{1+|u|^2}dy_j\right) = \frac{1}{(1+|u|^2)^2} , \quad j=1, \ldots, N .$$

Thus, the quantity (1.12) is useful to solve the equation (1.11).

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2. Some Lemmata

In this section we prepare some lemmata to prove Theorem 1 by using viscosity method.

By Taylor's formula near $(u, q) = 0$ and the assumption (1.3), we have the following decomposition of the nonlinear term $F(u, q)$:

$$F(u, q) = Q(u, q) + C(u, q) \quad \text{near} \quad (u, q) = 0 ,$$

where

$$Q(u,q)=Q(z)=\frac{1}{2}\sum_{j,k=1}^{4}\left(\frac{\partial^2 F}{\partial z_j \partial z_k}(0)\right)z_j z_k\ ,$$

$$C(u,q)=C(z)=\frac{1}{2}\sum_{j,k,l=1}^{4}\left(\int_0^1 (1-s)^2 \frac{\partial^3 F}{\partial z_j \partial z_k \partial z_l}(sz)\,ds\right)z_j z_k z_l\ ,$$

near $z=(u,\bar{u},q,\bar{q})=0$. Since we consider only small solutions, we can treat $C(u,q)$ as if it were a second order polynominal. Thus we will prove Theorem 1 when $F(u,q)$ is a second order polynominal and we will give the correction for the case of general nonlinear term in remarks. Now the properties of F are following.

Lemma 2. *Let m be an integer ≥ 2. For $u, v\in H^m(\mathbf{R})$ $(\subset W^{1,\infty}(\mathbf{R}))$, the following bounds hold:*

$$\|F(u,u_x)\|_{H^{m-1}}\leq C\|u\|_{W^{1,\infty}}\|u\|_{H^m}\ , \tag{2.1}$$

$$\|F(u,u_x)-F(v,v_x)\|_{H^{m-1}}\leq CR\|u-v\|_{H^m}\ , \tag{2.2}$$

where $R=\max\{\|u\|_{H^m}, \|v\|_{H^m}\}$.

Proof. Let $w, z=u, \bar{u}, u_x, \bar{u}_x$ and $k=0, 1, \ldots, m-1$. It is sufficient to treat the product of w and z, namely

$$\|\partial_x^k F(u,u_x)\|_{L^2}\sim \|\partial_x^k(wz)\|_{L^2}\ .$$

Leibniz' formula implies that

$$\leq\sum_{j=0}^{k}\binom{k}{j}\|\partial_x^{k-j}w\,\partial_x^j z\|_{L^2}\ .$$

By Hölder inequalities, we have

$$\leq\sum_{j=0}^{k}\binom{k}{j}\|\partial_x^{k-j}w\|_{L^{2k/(k-j)}}\|\partial_x^j z\|_{L^{2k/j}}\ . \tag{2.3}$$

Gagliardo-Nierenberg inequality yields

$$\|\partial_x^{k-j}w\|_{L^{2k/(k-j)}}\leq C\|\partial_x^k w\|_{L^2}^{(k-j)/k}\|w\|_{L^\infty}^{1-(k-j)/k},\ \|\partial_x^j z\|_{L^{2k/j}}\leq C\|\partial_x^k z\|_{L^2}^{j/k}\|w\|_{L^\infty}^{1-j/k}\ . \tag{2.4}$$

Substituting (2.4) to (2.3), we obtain (2.1). Similarly we can derive (2.2).

Remark 1. If $F(u,q)$ is general, then the properties of $F(u,q)$ are following:

$$\|F(u,u_x)\|_{H^{m-1}}\leq C\|u\|_{W^{1,\infty}}\|u\|_{H^m}\ , \tag{2.5}$$

$$\|F(u,u_x)-F(v,v_x)\|_{H^{m-1}}\leq CR\|u-v\|_{H^m}\ , \tag{2.6}$$

for any $u, v\in H^m(\mathbf{R})$ $(m\geq 2)$ with $\|u\|_{W^{m-1,\infty}}, \|v\|_{W^{m-1,\infty}}\leq 1$. The proof of (2.5), (2.6) is similar to Moser's lemma [7] (see also Klainerman [3]).

We use the viscosity method with $\varepsilon\in(0,1]$ in some sense:

$$u_t^\varepsilon-\varepsilon u_{xx}^\varepsilon-iu_{xx}^\varepsilon=F(u^\varepsilon,u_x^\varepsilon)\quad\text{in}\quad(0,\infty)\times\mathbf{R}\ ,\tag{2.7}$$

$$u^\varepsilon(0,x)=u_0(x)\quad\text{in}\quad\mathbf{R}\ .\tag{2.8}$$

We can easily solve (2.7)-(2.8): namely

Lemma 3. *Let m be an integer ≥ 2. For any $u_0\in H^m(\mathbf{R})$, there exixts a time $T_\varepsilon=T(\varepsilon,\|u_0\|_{H^m})$ such that an initial value problem (2.7)-(2.8) possesses a unique solution $u^\varepsilon\in C([0,T_\varepsilon);H^m(\mathbf{R}))$. If the maximal existence time T_ε is finite, then*

$$\limsup_{t\uparrow T_\varepsilon}\|u^\varepsilon(t)\|_{H^m}=+\infty\ .\tag{2.9}$$

Proof. The idea of proof is due to contraction type arguments in $L^\infty(0,T;H^m(\mathbf{R}))$. Let $S^\varepsilon(t)$ be a semigroup generated by the linear equation:

$$U_t-(\varepsilon+i)U_{xx}=0\ .\tag{2.10}$$

Let $\mathcal{T}$ be a nonlinear map defined by

$$\mathcal{T}u(t)=S^\varepsilon(t)u_0+\int_0^t S^\varepsilon(t-\tau)F(u,u_x)(\tau)d\tau\ .\tag{2.11}$$

We have only to show that the map $\mathcal{T}$ has a unique fixed point in $L^\infty(0,T;H^m(\mathbf{R}))$ provided T is small enough, because it is easy to check that the fixed point u^ε is in $C([0,T];H^m(\mathbf{R}))$ and (2.9) holds. Let $Y_T=L^\infty(0,T;H^m(\mathbf{R}))$ and let $B_R(Y_T)=\{u\in Y_T;\ \|u\|_{Y_t}\leq R\}$

Assume $u\in Y_T$, then we have

$$\|\mathcal{T}u(t)\|_{H^m}\leq\|S^\varepsilon(t)u_0\|_{H^m}+\int_0^t\|S^\varepsilon(t-\tau)F(u,u_x)(\tau)\|_{H^m}d\tau\ ,$$

with Plancherel's formula

$$\begin{aligned}&=\|\langle\xi\rangle^m e^{-\varepsilon\xi^2 t}\hat{u}_0\|_{L^2}+\int_0^t\|\langle\xi\rangle^m e^{-\varepsilon\xi^2(t-\tau)}\widehat{F(u,u_x)}(\tau)\|_{L^2}d\tau\\&\leq\|u_0\|_{H^m}+\int_0^t\sup_{\xi\in\mathbf{R}}\left(\langle\xi\rangle e^{-\varepsilon\xi^2(t-\tau)}\right)\|\langle\xi\rangle^{m-1}\widehat{F(u,u_x)}(\tau)\|_{L^2}d\tau\\&\leq\|u_0\|_{H^m}+\int_0^t\sup_{\xi\in\mathbf{R}}\left(\langle\xi\rangle e^{-\varepsilon\xi^2(t-\tau)}\right)\|F(u,u_x)(\tau)\|_{H^{m-1}}d\tau\ ,\end{aligned}\tag{2.12}$$

where $\langle\xi\rangle=(1+\xi^2)^{1/2}$. Note that

$$\sup_{\xi\in\mathbf{R}}\langle\xi\rangle e^{-\varepsilon\xi^2(t-\tau)}\leq C_\varepsilon\left(1+\frac{1}{\sqrt{t-\tau}}\right)\ ,\qquad\text{for}\quad t-\tau>0\ ,\tag{2.13}$$

and substituting (2.1) (in Lemma 2) and (2.13) to (2.12), we have

$$\|\mathcal{T}u(t)\|_{H^m} \leq \|u_0\|_{H^m} + C_\varepsilon \int_0^t \left(1 + \frac{1}{\sqrt{t-\tau}}\right) \|u(\tau)\|_{H^m}^2 d\tau \ .$$

If $u \in B_R(Y_T)$, we obtain

$$\|\mathcal{T}u\|_{Y_T} \leq \|u_0\|_{H^m} + C_\varepsilon (T + T^{1/2}) R^2 \ .$$

Here we remark that C_ε is independent of T. If $u, v \in B_R(Y_T)$, then similar evaluations imply that

$$\|\mathcal{T}u - \mathcal{T}v\|_{Y_T} \leq C_\varepsilon (T + T^{1/2}) R \|u - v\|_{Y_T} \ .$$

Let $R > \|u_0\|_{H^m}$ and let T be sufficiently small, then there exists a contraction factor δ with $0 < \delta < 1$ such that

$$\|\mathcal{T}u\|_{Y_T}, \|\mathcal{T}v\|_{Y_T} \leq R \ , \quad \|\mathcal{T}u - \mathcal{T}v\|_{Y_T} \leq \delta \|u - v\|_{Y_T}$$

for any $u, v \in B_R(Y_T)$. Thus, the map $\mathcal{T}$ is contraction in $B_R(Y_T)$, and therefore we have a unique fixed point $u_\varepsilon \in L^\infty(0, T; H^m(\mathbf{R}))$.

As we mentioned before, we need bounds for weights in (1.10) to ensure the condition (1.6). Now we define the following two weights:

$$\theta_0(t, x; u) = \exp\left(\int_{-\infty}^x \int_0^1 \mathrm{Im} \frac{\partial F}{\partial q}(\sigma u, \sigma u_x)(t, y) d\sigma dy\right) \ , \tag{2.14}$$

$$\theta_1(t, x; u) = \exp\left(\int_{-\infty}^x \mathrm{Im} \frac{\partial F}{\partial q}(u, u_x)(t, y) dy\right) \ . \tag{2.15}$$

Concerning the bounds for θ_0 and θ_1, we obtain the following

Lemma 4. *Assume that u is a solution to (2.7)-(2.8) with $u \in L^\infty(0, T; X^m)$, $(m \geq 3)$. Then there exists a constant C_1 which is independent of $\varepsilon \in (0, 1]$, such that we bound*

$$\exp(-C_1 \|u(t)\|_{X^m}) \leq \theta_i(t, x; u) \leq \exp(C_1 \|u(t)\|_{X^m}) \ , \tag{2.16}$$

$$|\partial_t(\theta_i(t, x; u))| \leq C_1 \theta_i(t, x; u) (\|u(t)\|_{H^3} + \|u(t)\|_{H^3}^2) \ , \tag{2.17}$$

for $(t, x) \in (0, T) \times R$ and $i = 0, 1$.

Proof. It is enough to show the case $i=1$ because the proof of the other case is similar to that of $i=1$. Since F is a second order polynomimal of $u, \bar{u}, q, \bar{q}$, there exist constants $\alpha, \beta \in \mathbf{C}$ such that

$$\mathrm{Im} \frac{\partial F}{\partial q}(u, u_x) = (\alpha u + \overline{\alpha u}) + (\beta u_x + \bar{\beta} \overline{u_x}) \ .$$

Integrating over $(-\infty, x)$, we have

$$\int_{-\infty}^x \mathrm{Im} \frac{\partial F}{\partial q}(u, u_x) dy = \int_{-\infty}^x (\alpha u + \overline{\alpha u})(t, y) dy + (\beta u + \bar{\beta} \bar{u})(t, x) \ .$$

Then we get

$$\sup_{x\in\mathbf{R}}\left|\int_{-\infty}^{x}\mathrm{Im}\frac{\partial F}{\partial q}(u, u_x)\,dy\right|\leq C(\|u\|_{L^1}+\|u\|_{L^\infty}) .$$

Note that

$$\|u\|_{L^1}=\int|u|dx=\int\langle x\rangle^{-1}\langle x\rangle|u|dx ,$$

with Schwarz inequality

$$\leq\left(\int\langle x\rangle^{-2}dx\right)^{1/2}\left(\int\langle x\rangle^{2}|u|^2dx\right)^{1/2}\leq C\|u\|_{\Sigma} , \tag{2.18}$$

where $\langle x\rangle=(1+x^2)^{1/2}$. By (2.18) and Sobolev's embeddings, we obtain

$$\sup_{x\in\mathbf{R}}\left|\int_{-\infty}^{x}\mathrm{Im}\frac{\partial F}{\partial q}(u, u_x)\,dy\right|\leq C_1\|u\|_{H^1\cap\Sigma} .$$

This shows (2.16). Concerning (2.17), we differentiate $\theta_1(t, x; u)$ with respect to t, then

$$\begin{aligned}\partial_t(\theta_1(t, x; u))&=\partial_t\left(\exp\left(\int_{-\infty}^{x}\mathrm{Im}\frac{\partial F}{\partial q}(u, u_x)\,dy\right)\right)\\&=\theta_1(t, x; u)\,\partial_t\left(\int_{-\infty}^{x}\mathrm{Im}\frac{\partial F}{\partial q}(u, u_x)\,dy\right)\\&=\theta_1(t, x; u)\left(\int_{-\infty}^{x}(\alpha u_t+\overline{\alpha u_t})\,dy+(\beta u_t+\bar{\beta}\overline{u_t})\right) .\end{aligned} \tag{2.19}$$

Since u is a solution to (2.7) - (2.8), (2.19) becomes

$$\begin{aligned}\partial_t(\theta_1(t, x; u))&=2\theta_1(t, x; u)\,\mathrm{Re}\Big\{\alpha\int_{-\infty}^{x}\{(\varepsilon+i)u_{xx}+F(u, u_x)\}dy+\\&\qquad\qquad\beta(\varepsilon+i)u_{xx}+\beta F(u, u_x)\Big\}\\&=2\theta_1(t, x; u)\,\mathrm{Re}\Big\{\alpha(\varepsilon+i)u_x+\alpha\int_{-\infty}^{x}F(u, u_x)\,dy+\\&\qquad\qquad\beta(\varepsilon+i)u_{xx}+\beta F(u, u_x)\Big\} .\end{aligned}$$

Here we note that $\varepsilon\in(0, 1]$, there exists a constant C_1 which is independent of $\varepsilon\in(0, 1]$ such that

$$|\partial_t(\theta_1(t, x; u))|\leq C_1\theta_1(t, x; u)\,(\|u\|_{W^{1,\infty}}+\|u\|_{H^1}^2+\|u\|_{W^{2,\infty}}+\|u\|_{W^{1,\infty}}^2) .$$

This completes the proof of Lemma 4.

Remark 2. When the nonlinear term $F(u, q)$ is general, them Im $\partial F/\partial q$ are following:

$$\mathrm{Im}\frac{\partial F}{\partial q}(u, q)=\mathrm{Im}\frac{\partial Q}{\partial q}(u, q)+\mathrm{Im}\frac{\partial C}{\partial q}(u, q) ,$$

$$\operatorname{Im}\frac{\partial C}{\partial q}(u, q)=\operatorname{Im}\frac{\partial C}{\partial q}(z)=\frac{1}{2}\operatorname{Im}\frac{\partial}{\partial q}\left\{\sum_{j,k,l=1}^{4}\left(\int_0^1(1-s)^2\frac{\partial^3 F}{\partial z_j\partial z_k\partial z_l}(sz)\,ds\right)z_jz_kz_l\right\}$$

$$=\frac{3}{2}\operatorname{Im}\sum_{j,k=1}^{4}\left(\int_0^1(1-s)^2\frac{\partial^3 F}{\partial z_j\partial z_k\partial q}(sz)\,ds\right)z_jz_k$$

$$+\frac{1}{2}\operatorname{Im}\sum_{j,k,l=1}^{4}\left(\int_0^1 s(1-s)^2\frac{\partial^4 F}{\partial z_j\partial z_k\partial z_l\partial q}(sz)\,ds\right)z_jz_kz_l\ .$$

If $u\in L^\infty(0, T; X^m)$ $(m\geq 3)$ is a solution to (2.7) - (2.8) with $\sup_{t\in[0,t]}\|u(t)\|_{W^{2,\infty}}\leq 1$ then similarly we obtain

$$\text{(2.20)}\qquad \exp(-C_1\{\|u(t)\|_{X^m}+\|u(t)\|_{X^m}{}^2\})\leq\theta_i(t, x; u)\leq$$
$$\leq\exp(C_1\{\|u(t)\|_{X^m}+\|u(t)\|_{X^m}{}^2\})\ ,$$

$$\text{(2.21)}\qquad |\partial_t(\theta_i(t, x; u))|\leq C_1\theta_i(t, x; u)\,(\|u(t)\|_{H^3}+\|u(t)\|_{H^3}^2)\ ,$$

for $(t, x)\in(0, T)\times\mathbf{R}$ and $i=0, 1$.

3. Energy estimates

In this section we derive the energy estimates for (2.7) - (2.8) to get the uniform bound for the solutions. We use the quantities such as (1.10) to resolve the loss of derivatives. Now let us introduce the following notations:

$$\text{(3.1)}\qquad [u(t)]_m=\left\{\int\langle x\rangle^2\theta_0(t, x; u)|u(t)|^2dx+\sum_{k=1}^{m}\int\theta_1(t, x; u)|\partial_x^k u(t)|^2dx\right\}\ .$$

In order to estimates for $[u^\varepsilon(t)]_m$, $\varepsilon\in(0, 1]$, the following lemma holds.

Lemma 5. *Let m be an integer $m\geq 3$, $u_0\in X^m$, and $u\in L^\infty(0, T; X^m)$ be a solution to* (2.7) - (2.8)*. Then we have*

$$\text{(3.2)}\qquad [u(t)]_m\leq[u_0]_m+C_2\int_0^t\sup_{x\in\mathbf{R}}\{\theta_0(\tau, x; u)+\theta_1(\tau, x; u)\}^{1/2}\times$$
$$\{1+\|u(\tau)\|_{H^m}+\|u(\tau)\|_{H^m}^2\}\,\|u(\tau)\|_{X^m}d\tau\ ,$$

for $t\in[0, T]$, where C_2 is independent of $\varepsilon\in(0, 1]$.

Proof. We show the ineqality (3.2) in two steps.

Step 1. First we consider the first term in the right hand side of (3.2). We use the mean value theorem to the nonlinear term of the equation (2.7) with u in place of u^ε, then (2.7) becomes

$$\text{(3.3)}\qquad u_t-\varepsilon u_{xx}-iu_{xx}=u\int_0^1\frac{\partial F}{\partial u}(\sigma u, \sigma u_x)\,d\sigma+\overline{u}\int_0^1\frac{\partial F}{\partial \overline{u}}(\sigma u, \sigma u_x)\,d\sigma$$

$$+u_x\int_0^1\frac{\partial F}{\partial q}(\sigma u,\ \sigma u_x)\,d\sigma+\bar{u}\int_0^1\frac{\partial F}{\partial q}(\sigma u,\ \sigma u_x)\,d\sigma\ .$$

If we multiply $2\bar{u}\theta_0(t,\ x;\ u)\langle x\rangle^2$ to (3.3) and take the real part, then we have

$$\begin{aligned}(3.4)\qquad &(|u|^2)_t\theta_0\langle x\rangle^2-\varepsilon(u_{xx}\bar{u}+\bar{u}_{xx}u)\theta_0\langle x\rangle^2-i(u_{xx}\bar{u}-\bar{u}_{xx}u)\theta_0\langle x\rangle^2\\ &=2|u|^2\theta_0\langle x\rangle^2\mathrm{Re}\int_0^1\frac{\partial F}{\partial u}d\sigma+2\mathrm{Re}\Big(\bar{u}^2\theta_0\langle x\rangle^2\int_0^1\frac{\partial F}{\partial u}d\sigma\Big)\\ &\quad+i(u_x\bar{u}-\bar{u}_xu)\theta_0\langle x\rangle^2\int_0^1\mathrm{Im}\frac{\partial F}{\partial q}d\sigma+(|u|^2)_x\theta_0\langle x\rangle^2\int_0^1\mathrm{Re}\frac{\partial F}{\partial q}d\sigma\\ &\quad+\mathrm{Re}\Big((\bar{u}^2)_x\theta_0\langle x\rangle^2\int_0^1\frac{\partial F}{\partial q}d\sigma\Big)\ ,\end{aligned}$$

here we used the identity $\alpha\beta+\bar{\alpha}\bar{\beta}=(\alpha+\bar{\alpha})(\beta+\bar{\beta})/2+(\alpha-\bar{\alpha})(\beta-\bar{\beta})/2$, for $\alpha,\ \beta\in\mathrm{C}$ to the third term of the right hand side. We remark that $\theta_0\langle x\rangle^2\int_0^1\mathrm{Im}\frac{\partial F}{\partial q}d\sigma=\{\theta_0\langle x\rangle^2\}_x-2\theta_0x$, then (3.4) becomes

$$\begin{aligned}(3.5)\qquad &\{|u|^2\theta_0\langle x\rangle^2\}_t-|u|^2\partial_t\theta_0\langle x\rangle^2\\ &-\varepsilon(u_{xx}\bar{u}+\bar{u}_{xx}u)\theta_0\langle x\rangle^2-i(u_{xx}\bar{u}-\bar{u}_{xx}u)\theta_0\langle x\rangle^2\\ &=2|u|^2\theta_0\langle x\rangle^2\mathrm{Re}\int_0^1\frac{\partial F}{\partial u}d\sigma+2\mathrm{Re}\Big(\bar{u}^2\theta_0\langle x\rangle^2\int_0^1\frac{\partial F}{\partial u}d\sigma\Big)\\ &\quad+i(u_x\bar{u}-\bar{u}_xu)\,[\{\theta_0\langle x\rangle^2\}_x-2\theta_0x]+(|u|^2)_x\theta_0\langle x\rangle^2\int_0^1\mathrm{Re}\frac{\partial F}{\partial q}d\sigma\\ &\quad+\mathrm{Re}\Big((\bar{u}^2)_x\theta_0\langle x\rangle^2\int_0^1\frac{\partial F}{\partial q}d\sigma\Big)\ .\end{aligned}$$

Integrating over $\mathbf{R}$ and integrating by parts, the left hand side of (3.5) becomes

$$\begin{aligned}(3.6)\qquad \mathrm{L.H.S.}&=\frac{d}{dt}\int|u|^2\theta_0\langle x\rangle^2dx-\int|u|^2\partial_t\theta_0\langle x\rangle^2dx\\ &\quad+2\varepsilon\int|u_x|^2\theta_0\langle x\rangle^2dx-\varepsilon\int|u|^2\{\theta_0\langle x\rangle^2\}_{xx}dx\\ &\quad+i\int(u_x\bar{u}-\bar{u}_xu)\{\theta_0\langle x\rangle^2\}_xdx\ .\end{aligned}$$

Since the third term of (3.6) is non-negative, we can neglect it. And we here note that

$$\begin{aligned}(3.7)\qquad |\{\theta_0(t,\ x;\ u)\langle x\rangle^2\}_{xx}|&=\Big|\Big\{\theta_0\langle x\rangle^2\int_0^1\mathrm{Im}\frac{\partial F}{\partial q}d\sigma+2\theta_0x\Big\}_x\Big|\\ &=\theta_0(t,\ x;\ u)\Big|\langle x\rangle^2\Big(\int_0^1\mathrm{Im}\frac{\partial F}{\partial q}d\sigma\Big)^2+4x\int_0^1\mathrm{Im}\frac{\partial F}{\partial q}d\sigma\\ &\qquad+\langle x\rangle^2\Big(\int_0^1\mathrm{Im}\frac{\partial F}{\partial q}d\sigma\Big)_x+2\Big|\\ &\le C\theta_0(t,\ x;\ u)\langle x\rangle^2(1+\|u\|_{W^{2,\infty}}+\|u\|_{W^{1,\infty}}^2)\ .\end{aligned}$$

Substituting (2.17) (in Lemma 4) and (3.7) to (3.6) we obtain

$$\text{(3.8)} \qquad \text{L.H.S.} \geq \frac{d}{dt}\int |u|^2\theta_0\langle x\rangle^2 dx - C_3(1+\|u\|_{H^3}+\|u\|_{H^3}^2)\int |u|^2\theta_0\langle x\rangle^2 dx$$
$$-2\int \operatorname{Im}(u_x\bar{u})\,\{\theta_0\langle x\rangle^2\}_x dx\ ,$$

where $C_3>0$ is independent of $\varepsilon\in(0,1]$. On the other hand, after some computations the right hand side becomes

$$\text{(3.9)} \qquad \text{R.H.S.} \leq C\|u\|_{W^{1,\infty}}\int |u|^2\theta_0\langle x\rangle^2 dx$$
$$-2\int \operatorname{Im}(u_x\bar{u})\,\{\theta_0\langle x\rangle^2\}_x dx + 4\int \operatorname{Im}(u_x\bar{u})\,\theta_0 x dx$$
$$-\int |u|^2\Big(\theta_0\langle x\rangle^2\int_0^1 \operatorname{Re}\frac{\partial F}{\partial q}d\sigma\Big)_x dx$$
$$-\operatorname{Re}\int \bar{u}^2\Big(\theta_0\langle x\rangle^2\int_0^1 \frac{\partial F}{\partial q}d\sigma\Big)_x dx\ .$$

For the third term of (3.9), by Schwarz inequality we obtain

$$\text{(3.10)} \qquad \left|\int \operatorname{Im}(u_x\bar{u})\,\theta_0 x dx\right|$$
$$\leq C\sup_{x\in\mathbf{R}}\theta_0^{1/2}(t,x;u)\Big(\int |u_x|^2 dx\Big)^{1/2}\Big(\int |u|^2\theta_0 x^2 dx\Big)^{1/2}$$
$$\leq C\sup_{x\in\mathbf{R}}\theta_0^{1/2}(t,x;u)\,\|u\|_{H^1}\Big(\int |u|^2\theta_0 x^2 dx\Big)^{1/2}\ ,$$

and similarly to (3.7) we have

$$\text{(3.11)} \qquad \left|\Big(\theta_0(t,x;u)\langle x\rangle^2\int_0^1 \operatorname{Re}\frac{\partial F}{\partial q}d\sigma\Big)_x\right|,\quad \left|\Big(\theta_0(t,x;u)\langle x\rangle^2\int_0^1 \frac{\partial F}{\partial q}d\sigma\Big)_x\right|$$
$$\leq C\theta_0(t,x;u)\langle x\rangle^2(\|u\|_{W^{2,\infty}}+\|u\|_{W^{1,\infty}}^2)\ .$$

Substituting (3.10) and (3.11) to (3.9), we have

$$\text{(3.12)} \qquad \text{R.H.S.} \leq C(\|u\|_{H^3}+\|u\|_{H^3}^2)\int |u|^2\theta_0\langle x\rangle^2 dx$$
$$+C\sup_{x\in\mathbf{R}}\theta_0^{1/2}(t,x;u)\,\|u\|_{H^3}\Big(\int |u|^2\theta_0 x^2 dx\Big)^{1/2}$$
$$-2\int \operatorname{Im}(u_x\bar{u})\,\{\theta_0\langle x\rangle^2\}_x dx$$

Since the last term of (3.8) is equal to that of (3.12), they are canceled. Thus we obtain

$$\frac{d}{dt}\Big(\int |u|^2\theta_0\langle x\rangle^2 dx\Big)^{1/2} \leq C_4(1+\|u\|_{H^3}+\|u\|_{H^3}^2)\sup_{x\in\mathbf{R}}\theta_0^{1/2}(t,x;u)\,\|u\|_{X^m}\ ,$$

where C_4 is independent of $\varepsilon\in(0,1]$.

Step 2. Next we derive a higher order estimate. Differentiate the equation (2.7) with respect to x up to order k, $(k=1, \dots, m)$

$$(\partial_x^k u)_t - \varepsilon (\partial_x^k u)_{xx} - i(\partial_x^k u)_{xx} = \partial_x^{k-1}\left(\frac{\partial F}{\partial u}u_x + \frac{\partial F}{\partial \bar{u}}\bar{u}_x + \frac{\partial F}{\partial q}u_{xx} + \frac{\partial F}{\partial \bar{q}}\bar{u}_{xx}\right) ,$$

Leibniz' formula yields

$$\begin{aligned} &= \sum_{j=0}^{k-1}\binom{k-1}{j}\left(\partial_x^{k-1-j}\frac{\partial F}{\partial u}\partial_x^{j+1}u + \partial_x^{k-1-j}\frac{\partial F}{\partial \bar{u}}\partial_x^{j+1}\bar{u}\right) \\ &+ \sum_{j=0}^{k-2}\binom{k-1}{j}\left(\partial_x^{k-1-j}\frac{\partial F}{\partial q}\partial_x^{j+2}u + \partial_x^{k-1-j}\frac{\partial F}{\partial \bar{q}}\partial_x^{j+2}\bar{u}\right) \\ &+ \frac{\partial F}{\partial q}\partial_x^{k+1}u + \frac{\partial F}{\partial \bar{q}}\partial_x^{k+1}\bar{u} \ . \end{aligned} \tag{3.13}$$

Similarly, we multiply $2\theta_1\partial_x^k\bar{u}$, integrate over $\mathbf{R}$, and take the real part. Then the left hand side becomes

$$\begin{aligned} \text{L.H.S.} = &\frac{d}{dt}\int |\partial_x^k u|^2\theta_1 dx - \int |\partial_x^k u|^2 \partial_t \theta_1 dx \\ &+ 2\varepsilon \int |\partial_x^{k+1}u|^2\theta_1 dx - \varepsilon\int |\partial_x^k u|^2 \partial_x^2\theta_1 dx \\ &+ i\int (\partial_x^{k+1}u\partial_x^k\bar{u} - \partial_x^{k+1}\bar{u}\partial_x^k u)\,\partial_x\theta_1 dx \ . \end{aligned}$$

Here we note that

$$|\partial_x^2\theta_1(t, x; u)| \leq C\theta_1(t, x; u)\,(\|u\|_{W^{2,\infty}} + \|u\|_{W^{1,\infty}}^2) \ .$$

After the same computations as in (3.8), there exists a constant $C_5 > 0$ which is independent of $\varepsilon \in (0, 1]$, such that

$$\begin{aligned} \text{L.H.S.} \geq &\frac{d}{dt}\int |\partial_x^k u|^2\theta_1 dx - C_5(\|u\|_{H^3} + \|u\|_{H^3}^2)\int |\partial_x^k u|^2\theta_1 dx \\ &- 2\int \operatorname{Im}(\partial_x^{k+1}u\partial_x^k\bar{u})\,\partial_x\theta_1 dx \ . \end{aligned} \tag{3.14}$$

by the way, the right hand side of (3.13) becomes

$$\begin{aligned} \text{R.H.S.} = &2\operatorname{Re}\sum_{j=0}^{k-1}\int\left(\partial_x^{k-1-j}\frac{\partial F}{\partial u}\partial_x^{j+1}u\partial_x^k\bar{u} + \partial_x^{k-1-j}\frac{\partial F}{\partial \bar{u}}\partial_x^{j+1}\bar{u}\partial_x^k\bar{u}\right)\theta_1 dx \\ &+ 2\operatorname{Re}\sum_{j=0}^{k-2}\int\left(\partial_x^{k-1-j}\frac{\partial F}{\partial q}\partial_x^{j+2}u\partial_x^k\bar{u} + \partial_x^{k-1-j}\frac{\partial F}{\partial \bar{q}}\partial_x^{j+2}\bar{u}\partial_x^k\bar{u}\right)\theta_1 dx \\ &- 2\int \operatorname{Im}\frac{\partial F}{\partial q}\operatorname{Im}(\partial_x^{k+1}u\partial_x^k\bar{u})\,\theta_1 dx \\ &+ \int \operatorname{Re}\frac{\partial F}{\partial q}(|\partial_x^k u|^2)_x\theta_1 dx + \operatorname{Re}\int\frac{\partial F}{\partial \bar{q}}\{(\partial_x^k\bar{u})^2\}_x\theta_1 dx \ . \end{aligned} \tag{3.15}$$

Now, we estimate the first and second terms in (3.15). Let $l_1, l_2=1, 2$, $v, w=u, \bar{u}$. It is enough to treat only the form $\partial_x^{k-2+l_1-j}v\partial_x^{j+l_2}w\partial_x^k\bar{u}$. If $j=0$, then the terms of this type can be evaluated by

$$C\|\partial_x^{l_2}w\|_{L^\infty}\sum_{j=1}^{k}\int|\partial_x^j u|^2\theta_1 dx\leq C\|u\|_{H^3}\sum_{j=1}^{k}\int|\partial_x^j u|^2\theta_1 dx\ ,$$

because $1\leq k-2+l_1\leq k$ and $l_2\leq 2$. On the other hand, if $j\geq 1$, this type of term can be estimated by

$$C\|\partial_x^{k-2+l_1-j}w\|_{L^\infty}\sum_{j=1}^{k}\int|\partial_x^j u|^2\theta_1 dx\leq C\|u\|_{H^k}\sum_{j=1}^{k}\int|\partial_x^j u|^2\theta_1 dx\ ,$$

because $k-2+l_1-j\leq k-1$ and $1\leq l_2+j\leq k$. Thus we have

$$\begin{aligned}(3.16)\qquad \text{R.H.S.}\leq & C\|u\|_{H^m}\sum_{j=1}^{k}\int|\partial_x^j u|^2\theta_1 dx\\ & -2\int \mathrm{Im}\frac{\partial F}{\partial q}\mathrm{Im}\,(\partial_x^{k+1}u\partial_x^k\bar{u})\,\theta_1 dx\\ & -\int|\partial_x^k u|^2\Big(\mathrm{Re}\frac{\partial F}{\partial q}\theta_1\Big)_x dx-\mathrm{Re}\int(\partial_x^k\bar{u})^2\Big(\frac{\partial F}{\partial \bar{q}}\theta_1\Big)_x dx\ .\end{aligned}$$

Similar caluculations to (3.7) give

$$(3.17)\qquad \left|\Big(\mathrm{Re}\frac{\partial F}{\partial q}\theta_1\Big)_x\right|,\ \left|\Big(\frac{\partial F}{\partial \bar{q}}\theta_1\Big)_x\right|\leq C\theta_1(t, x;\, u)\,(\|u\|_{W^{2,\infty}}+\|u\|_{W^{1,\infty}}^2)\ .$$

By using (3.17) to (3.16), we have

$$\begin{aligned}(3.18)\qquad \text{R.H.S.}\leq & C\,(\|u\|_{H^m}+\|u\|_{H^m}^2)\sum_{j=1}^{k}\int|\partial_x^j u|^2\theta_1 dx\\ & -2\int \mathrm{Im}\frac{\partial F}{\partial q}\mathrm{Im}\,(\partial_x^{k+1}u\partial_x^k\bar{u})\,\theta_1 dx\ .\end{aligned}$$

Combining (3.14) and (3.18), and summing up on k, we get

$$\frac{d}{dt}\Big(\sum_{k=1}^{m}\int|\partial_x^k u|^2\theta_1 dx\Big)^{1/2}\leq C_6\,(\|u\|_{H^m}+\|u\|_{H^m}^2)\,\Big(\sum_{k=1}^{m}\int|\partial_x^k u|^2\theta_1 dx\Big)^{1/2}\ ,$$

where C_6 is independ of $\varepsilon\in(0, 1]$.

Remark 3. When the nonlinear term $F(u, q)$ is general, if we consider the solution $u\in C([0, T];\, X^m)$ to (2.7) - (2.8) with $\sup_{t\in[0,T]}\|u\|_{W^{m-1,\infty}}\leq 1$ then we can obtain the energy inequality such as (3.2).

4. End of proofs

In this last section we complete the proof of Theorem 1. Concerning the existence, we derive the uniform bound for $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ by using the energy estimates in Lemma 5. And therefore we can construct a solution to (1.1) - (1.2) provided $\varepsilon\downarrow 0$.

Proof of Existence. Since we consider small solutions, there exists a constant $C_7>0$ which is independent of $\varepsilon\in(0,1]$, such that (3.2) becomes

$$(4.1)\qquad [u^\varepsilon(t)]_m\leq [u_0]_m+C_7\int_0^t e^{C_1\|u^\varepsilon(\tau)\|_{X^m}}\|u(\tau)\|_{X^m}d\tau\ .$$

Here we used (2.16). Let $\alpha_0=\min(1/2,\ 1/10C_1)$ and let $\alpha=\|u_0\|_{X^m}\leq\alpha_0$. Now we define

$$(4.2)\qquad T_\varepsilon^*=\sup\{T>0 \mid \|u^\varepsilon(t)\|_{X^m}<2\alpha\ ,\quad 0\leq t<T\}\ ,$$

and it is clear that $T_\varepsilon^*>0$. When $t\in[0,T_\varepsilon^*]$, (4.1) is valid because $\|u^\varepsilon(t)\|_{X^m}\leq 1$. Using (2.16) to the left hand side of (4.1), we have

$$\|u^\varepsilon(t)\|_{X^m}\leq e^{3C_1\alpha}\alpha+C_7e^{4C_1\alpha}\int_0^t\|u^\varepsilon(\tau)\|_{X^m}d\tau\ ,\qquad \text{for}\quad t\in[0,T_\varepsilon^*]\ .$$

Gronwall's lemma yields

$$(4.3)\qquad \|u^\varepsilon(t)\|_{X^m}\leq e^{3C_1\alpha}\alpha\exp(C_7e^{4C_1\alpha}t)\ .$$

If we put $t=T_\varepsilon^*$, (4.3) becomes

$$2\alpha\leq e^{3C_1\alpha}\alpha\exp(C_7e^{4C_1\alpha}T_\varepsilon^*)\ ,$$

which implies

$$T_\varepsilon^*\geq\frac{\log 2-3C_1\alpha}{C_7e^{4C_1\alpha}}\geq\frac{\log 2-3C_1\alpha_0}{C_7e^{4C_1\alpha_0}}\equiv T>0\ .$$

Since $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;X^m)$ and $\{u_t^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^{m-2}(\mathbf{R}))$, $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $C^{0,1}(0,T;H^{m-2}(\mathbf{R}))$, and then $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $C^{0,\delta/2}([0,T];H^{m-\delta}(\mathbf{R}))$ for any $\delta>0$ with the interpolation. We remark that $L^\infty(0,T;X^m)$ is the dual space of $L^1(0,T;X^m)$ which is a separable Banach space. Thus, by Banach-Alaoglu theorem, there exist a subsequence and $u\in L^\infty(0,T;X^m)$ such that

$$u^\varepsilon\xrightarrow{w*}u\qquad\text{in}\quad L^\infty(0,T;X^m)\qquad(\text{as }\varepsilon\downarrow 0)\ .$$

By Rellich's theorem and Ascoli-Arzelà, theorem we obtain

$$u^\varepsilon\longrightarrow u\qquad\text{in}\quad C([0,T];H^{m-\delta'}_{\mathrm{loc}}(\mathbf{R}))\qquad(\text{as }\varepsilon\downarrow 0)\ ,\quad(\delta'>0)\ .$$

(c.f. Aubin's compactness theorem. see Lions [5] Théorème 5.1. in CHA-

PITRE 1.) It is easy to check that u is a solution to (1.1)-(1.2), and thus the proof of the existence is completed.

Finally we prove the uniqueness of solution by the same technique used in the energy estimates.

Proof of Uniqueness. Let $u, v \in L^\infty(0, T; X^m)$ be solutions such that $u(0, x) = v(0, x)$, let $w = u - v$, and let $R = \max\{\|u\|_{L^\infty(0,T;X^m)}, \|u\|_{L^\infty(0,T;X^m)}\}$. We here define the weight:

$$\eta(t, x; u, v) = \exp\Big(\int_{-\infty}^{x}\int_0^1 \mathrm{Im}\frac{\partial F}{\partial q}(\sigma u + (1-\sigma)v, \sigma u_x + (1-\sigma)v_x)(y)\,d\sigma dy\Big) , \tag{4.4}$$

and it is clear that the similar estimates in Lemma 4 for $\eta(t, x; u, v)$ hold:

$$e^{-CR} \leq \eta(t, x; u, v) \leq e^{CR} , \tag{4.5}$$

$$|\partial_t\{\eta(t, x; u, v)\}| \leq C(R+R^2)\eta(t, x; u, v) , \tag{4.6}$$

for $(t, x) \in [0, T] \times \mathbf{R}$. Taking the difference between two equations, we have

$$w_t - iw_{xx} = F(u, u_x) - F(v, v_x)$$

Mean value theorem yields

$$\begin{aligned} &= w\int_0^1 \frac{\partial F}{\partial u}(\sigma u + (1-\sigma)v, \sigma u_x + (1-\sigma)v_x)\,d\sigma \\ &+ \bar{w}\int_0^1 \frac{\partial F}{\partial \bar{u}}(\sigma u + (1-\sigma)v, \sigma u_x + (1-\sigma)v_x)\,d\sigma \\ &+ w_x\int_0^1 \frac{\partial F}{\partial q}(\sigma u + (1-\sigma)v, \sigma u_x + (1-\sigma)v_x)\,d\sigma \\ &+ \bar{w}_x\int_0^1 \frac{\partial F}{\partial \bar{q}}(\sigma u + (1-\sigma)v, \sigma u_x + (1-\sigma)v_x)\,d\sigma . \end{aligned} \tag{4.7}$$

Multiplying $2\bar{w}\eta$ to (4.7) and after similar calculations, we obtain

$$\begin{aligned} &\frac{d}{dt}\int |w|^2\eta dx - \int |w|^2\partial_x\eta dx + i\int (w_x\bar{w} - \bar{w}_x w)\,\partial_x\eta dx \\ &= 2\mathrm{Re}\int\Big(|w|^2\int_0^1 \frac{\partial F}{\partial u}d\sigma + \bar{w}^2\int_0^1 \frac{\partial F}{\partial \bar{u}}d\sigma\Big)\eta dx \\ &+ i\int (w_x\bar{w} - \bar{w}_x w)\,\partial_x\eta dx \\ &+ \int (|w|^2)_x\eta\int_0^1 \mathrm{Re}\frac{\partial F}{\partial q}d\sigma dx + \mathrm{Re}\int (\bar{w}^2)_x\eta\int_0^1 \frac{\partial F}{\partial \bar{q}}d\sigma dx . \end{aligned}$$

Similarly it follows that

$$\frac{d}{dt}\int |w|^2\eta dx \leq C(R+R^2)\int |w|^2\eta dx .$$

Thus, Gronwall's lemma yields

$$\int |w|^2 \eta dx = 0 \ , \quad t \in [0, T] \ . \tag{4.8}$$

Note (4.5), (4.8) shows the uniqueness of solution.

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