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Global weak solutions for the equation of isothermal gas around a star

By

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1. Introduction

In this paper we study the spherically symmetric motion of isothermal gas under the gravitational force around a solid star with radius 1 and mass M. This motion is governed by the compressible Euler equation. The dynamics of an isothermal gas around the star in R^3 is written by the following system of equations.

$$
(1.1)\qquad
\begin{aligned}
&\rho_t+(\rho u)_r+\frac{2}{r}\rho u=0,\\
&\rho(u_t+uu_r)+p_r=-\frac{\rho M}{r^2},\\
&p=a^2\rho,
\end{aligned}
$$

on $t\geq 0$ and $1\leq r<\infty$. Here, ρ and p are the density and the pressure respectively, u is the velocity normal to the surface of the star, a is a given constant and $-M/r^2$ means the gravitational force.

We investigate solutions of (1.1) which satisfy the boundary condition

$$
(1.2)\qquad u(t, 1)=0 .
$$

Let us adopt a new function $\tilde{\rho}$. Put $\tilde{\rho}=r^2\rho$. Then (1.1) becomes

$$
(1.3)\qquad
\begin{aligned}
&\tilde{\rho}_t+(\tilde{\rho}u)_r=0,\\
&u_t+uu_r+a^2\frac{\tilde{\rho}r}{\tilde{\rho}}=\frac{2a^2}{r}-\frac{M}{r^2}.
\end{aligned}
$$

Next we introduce the Lagrangian mass coordinate

$$
(1.4)\qquad \tau=t\ ,\quad \xi=\int_1^r\tilde{\rho}(t, s)ds .
$$

Then, from (1.3) we get

$$
(1.5)\qquad
\begin{aligned}
&\tilde{\rho}_\tau+\tilde{\rho}^2u_\xi=0,\\
&u_\tau+a^2\tilde{\rho}_\xi=\frac{2a^2}{r}-\frac{M}{r^2}.
\end{aligned}
$$

Put $v=1/\tilde{\rho}$. Then (1.5) becomes, after changing τ to t and ξ to x,

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$$v_t - u_x = 0 \ ,$$

(1.6) $$u_t + \left(\frac{a^2}{v}\right)_x = \frac{K}{r} - \frac{M}{r^2} \ ,$$

where r is now defined by $r = 1 + \int_0^x v(t, \zeta)\, d\zeta$ and $K = 2a^2$. Let us consider the initial boundary value problem for (1.6) in $t \geq 0$, $x \geq 0$ with the following initial and boundary conditions.

(1.7) $$u(0, x) = u_0(x),\ v(0, x) = v_0(x), \quad \textit{for}\ x > 0 \ ,$$

(1.8) $$u(t, 0) = 0, \quad \textit{for}\ t > 0 \ .$$

$u(t, x)$ and $v(t, x)$ are called weak solutions of initial boundary value problem (1.6), (1.7) and (1.8) if $u, v \in L^\infty((0, T) \times (0, \infty))$ and if they satisfy the integral identities

$$\int_0^T \int_0^\infty v\phi_t - u\phi_x \, dxdt + \int_0^\infty v_0(x)\, \phi(0, x)\, dx = 0 \ ,$$

(1.9) $$\int_0^T \int_0^\infty u\psi_t + \left(\frac{a^2}{v}\right)\psi_x + \int_0^\infty u_0(x)\, \psi(0, x)\, dx$$

$$= -\int_0^T \int_0^\infty \left(\frac{K}{1 + \int_0^x v(t, \zeta)\, d\zeta} - \frac{M}{(1 + \int_0^x v(t, \zeta)\, d\zeta)^2} \right) \cdot \psi \, dxdt \ ,$$

for any test function $\phi \in C_0^\infty([0, T) \times [0, \infty))$ and $\psi \in C_0^\infty([0, T) \times (0, \infty))$ and for any $T > 0$.

Here is our main result.

Theorem 1.1. *Suppose that $v_0(x)$ and $u_0(x)$ are of bounded variation, and that $v_0(x)$ satisfy $\delta_0 < v_0(x) < M_0$ for some positive constants δ_0 and M_0. Then* (1.6), (1.7) *and* (1.8) *admit a global weak solution* (*i.e. T does not depend on the initial data.*) *which satisfies*

$$\|u\|_\infty < \infty,\ 0 < \delta'_0 < v(t, x) < M'_0 \ \textit{a.e. for some}\ \delta'_0 \ \textit{and}\ M'_0 \ .$$

In section 2, we shall prove our main result by using a modified Glimm's scheme. After constructing approximate solutions for (1.6), we shall get uniform estimates of the variations of them and then show that there exist subsequences of approximate solutions which converge to weak solutions by using Glimm's theory (see [4]).

In general, the pressure p satisfies $p = a^2\rho^\gamma$ $(\gamma \geq 1)$ for an isentropic gas. In this paper we restrict ourselves to the case

(1.10) $$\gamma = 1 \ .$$

If $\gamma \geq 1$, (1.6) becomes

$$v_t - u_x = 0 \ ,$$

(1.11) $$u_t + \left(\frac{a^2}{v^\gamma}\right)_x \cdot \frac{1}{r^{2(\gamma-1)}} = \frac{2a^2\gamma v^{1-\gamma}}{r^3 \cdot r^{2(\gamma-2)}} - \frac{M}{r^2} \ .$$

Because (1.11) has variable coefficients if $\gamma \neq 1$, our method can't be applied. It is the reason why we impose the condition (1.10). In the case $\gamma \neq 1$, Makino and Takeno have proved the existence of temporally local weak solution for (1.1) in [11]. For viscous barotropic gas, Okada and Makino [14] have proved the existence of global solutions. In this paper we obtain a global weak solutions first for the nonviscous gas under the gravitational force.

There are many works related to the compressible Euler equation. For one dimensional case, Nishida [13] established the existence of a global weak solution, for the first time, for the case $\gamma = 1$ by using the Glimm's method. DiPerna [3] extended the result to the case of $\gamma = 1 + 2/(2m+1)$ ($m \geq 2$ integers) using the theory of compensated compactness. Ding et al [1], [2] removed this restriction and established the existence of global weak solutions for $1 < \gamma \leq 5/3$.

On the other hand, little is known for the case $n \geq 2$ (n: space dimension). No global solutions have been known to exist, but only local classical solutions (see [5], [6], [8] and [9]). In [10], Makino, Mizohata and Ukai have presented global weak solutions first for this case without external force.

The proof of Theorem 1.1 is similar to that of Theorem 4.2 in [10]. Suppose that solutions are spherically symmetry and $\gamma = 1$ outside a unit ball in R^n. If there is no external force, we obtain

$$
(1.12) \qquad
\begin{aligned}
& v_t - u_x = 0 , \\
& u_t + \left(\frac{a^2}{v}\right)_x = \frac{a^2(n-1)}{1+\int_0^x v(t, \zeta) d\zeta} ,
\end{aligned}
$$

from the compressible Euler equation in the same way.

In [10] we have got uniform a priori estimates of approximate solutions by using the fact that the right-hand side of (1.12) is positive and monotone decreasing in x. However, in this case the right-hand side of (1.8)

$$
(1.13) \qquad \frac{K}{1+\int_0^x v(t, \zeta) d\zeta} - \frac{M}{(1+\int_0^x v(t, \zeta) d\zeta)^2}
$$

is not necessarily positive or monotone decreasing. So we have to calculate a priori estimates of approximate solutions more carefully.

2. The global weak solution for the Lagrangian equation

Let us construct approximate solutions $u^l(t, x)$ and $v^l(t, x)$ by using a modified Glimm's scheme. For $l, h > 0$, define

$$
(2.1) \qquad
\begin{aligned}
& Y = \{(n, m); n = 1, 2, 3, \cdots, m = 1, 3, 5, \cdots\} , \\
& \mathrm{A} = \prod_{(m,n) \in Y} [\{nh\} \times ((m-1)l, (m+1)l)] ,
\end{aligned}
$$

where l/h will be determined later. Choose a point $\{a_{nm}\} \in A$ randomly, and write $a_{nm}=(nh, c_{nm})$. For $n=0$, we put $c_{0m}=ml$. Mesh lengths l and h are chosen so that $l/h>a/(\inf v^l)$ for any given $T>0$. Suppose that u^l and v^l are defined for $0\leq t<nh$. We are going to define u^l and v^l for $nh\leq t<(n+1)h$. For $ml\leq x<(m+2)l$, m: *odd*, we define

$$\begin{aligned} u^l(t, x) &= u_0^l(t, x)+U^l(t, x)\cdot(t-nh)\ , \\ v^l(t, x) &= v_0^l(t, x)\ , \end{aligned} \tag{2.2}$$

where u_0^l and v_0^l are the solutions of

$$\begin{aligned} & v_t-u_x=0\ , \\ & u_t+\left(\frac{a^2}{v}\right)_x=0\ , \end{aligned} \tag{2.3}$$

with initial data $(t=nh)$

$$\begin{aligned} u_0^l(nh, x) &= \begin{cases} u^l(nh-0, c_{nm})\ , & x<(m+1)l\ , \\ u^l(nh-0, c_{nm+2})\ , & x>(m+1)l\ , \end{cases} \\ v_0^l(nh, x) &= \begin{cases} v^l(nh-0, c_{nm})\ , & x<(m+1)l\ , \\ v^l(nh-0, c_{nm+2})\ , & x>(m+1)l\ , \end{cases} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} U^l(t, x) &= \frac{K}{1+\sum_{j=1}^{\frac{m+1}{2}} v^l(nh-0, c_{n2j-1})\cdot 2l} \\ &\quad -\frac{M}{\left(1+\sum_{j=1}^{\frac{m+1}{2}} v^l(nh-0, c_{n2j-1})\cdot 2l\right)^2}\ . \end{aligned} \tag{2.5}$$

For $0\leq x<l$, we define u^l and v^l as (2.2) where u_0^l and v_0^l are the solutions of (2.3) with initial $(t=nh)$ boundary data

$$u_0^l(nh, x)=u^l(nh-0, c_{n1}),\ v_0^l(nh, x)=v^l(nh-0, c_{n1}),\ x>0\ , \tag{2.6}$$

$$u(t, 0)=0\ , \quad t>nh\ , \tag{2.7}$$

and

$$U^l(t, x)=K-M\ . \tag{2.8}$$

System (2.3) is hyperbolic, provided $v>0$, with the characteristic roots and Riemann invariants given by

$$\begin{aligned} \lambda &= -\frac{a}{v}\ , & r&=u+a\log v\ , \\ \mu &= \frac{a}{v}\ , & s&=u-a\log v\ . \end{aligned} \tag{2.9}$$

Concerning shock waves and Riemann invariants, the following three lemmas are well known.

Lemma 2.1. *The 1-shock wave curve S_1 and 2-shock wave curve S_2, starting from (r_0, s_0), can be expressed in the form*

$$(2.10)\qquad \begin{aligned} &S_1\colon s-s_0=f(r-r_0) \text{ for } r\leq r_0\ ,\\ &S_2\colon r-r_0=f(s-s_0) \text{ for } s\leq s_0\ ,\end{aligned}$$

where

$$0\leq f'(x)<1, f''(x)\leq 0, \lim_{x\to-\infty} f'(x)=1\ .$$

The 1-rarefaction wave curve R_1 and 2-rarefaction wave curve R_2, starting from (r_0, s_0), can also be expressed in the form

$$(2.11)\qquad \begin{aligned} &R_1\colon s-s_0=0 \text{ for } r\geq r_0\ ,\\ &R_2\colon r-r_0=0 \text{ for } s\geq s_0\ .\end{aligned}$$

Let us consider (2.3) with following initial data

$$(2.12)\qquad u_0(x)=\begin{cases} u_l\ ,\\ u_r\ ,\end{cases} \quad v_0(x)=\begin{cases} v^l\ , & x<0\ ,\\ v_r\ , & x>0\ .\end{cases}$$

Lemma 2.2 *Let u and v be the solutions of* (2.3) *and* (2.12). *Then,*

$$(2.13)\qquad \begin{aligned} &r(t, x)\equiv r(u(t, x), v(t, x))\geq r_0\equiv\min\ (r(u_r, v_r), r(u_l, v_l))\ ,\\ &s(t, x)\equiv s(u(t, x), v(t, x))\leq s_0\equiv\max\ (s(u_r, v_r), s(u_l, v_l))\ .\end{aligned}$$

Next consider (2.3) in $t\geq 0$, $x\geq 0$ with following initial and boundary conditions

$$(2.14)\qquad u(0, x)=u_0^+\ ,\quad v(0, x)=v_0^+\ ,\quad \text{for } x>0\ ,$$

$$(2.15)\qquad u(t, 0)=0\ ,\quad \text{for } t>0\ .$$

Lemma 2.3. *Let u and v be the soloutions of* (2.3), (2.14) *and* (2.15). *Then,*

$$(2.16)\qquad \begin{cases} r(t, x)\equiv r(u(t, x), s(t, x))\geq r(u_0^+, v_0^+)\ ,\\ s(t, x)\equiv s(u(t, x), s(t, x))\leq\max\ (-r(u_0^+, v_0^+), s(u_0^+, v_0^+))\ .\end{cases}$$

The above three lemmas were proved in [13]. Using these lemmas and the fact that $U^l(t, x)\leq K+M$, we can get the following lemma.

Lemma 2.4. *Put $r_0=\min\ r(u_0(x), v_0(x))$ and $s_0=\max\ s(u_0(x), v_0(x))$.・Then, for $0<t<T$,*

$$(2.17)\qquad \begin{cases} r^l(t, x)\equiv r(u^l(t, x), s^l(t, x))\geq\min\ (r_0, r_0+(K-M)T)\ ,\\ s^l(t, x)\equiv s(u^l(t, x), s^l(t, x))\\ \leq\max\ (-r_0, -r_0-(K-M)T, s_0)+(K+M)T\ .\end{cases}$$

Proof. By using Lemma 2.2 and Lemma 2.3, we get

$$(2.18)\qquad \begin{cases} r^l(t, x) \geq \min\ (r_0, r_0 + (K-M)h)\ , \\ s^l(t, x) \leq \max\ (-r_0, s_0) + (K+M)h\ , \end{cases}$$

for $0 \leq t < h$. Thus we obtain, for $h \leq t < 2h$

$$r^l(t, x) \geq \min\ (r_0, r_0 + 2(K-M)h)\ ,$$

$$\begin{aligned} s^l(t, x) &\leq \max\ (-\min\ (r_0, r_0 + (k-M)h), \max\ (-r_0, s_0) + \\ &\quad + (k+M)h) + (K+M)h \\ &\leq \max\ (-\min\ (r_0, r_0 + (k-M)h) + (K+M)h, \max\ (-r_0, s_0) + \\ &\quad + (k+M)h) + (K+M)h \\ &\leq \max\ (-r_0, -r_0 - (K-M)h, s_0) + 2(K+M)h\ . \end{aligned}$$

Continuing similar calculations successively until $T = Nh$, we can obtain (2.17).

Let us consider Riemann problem (2.3) and (2.12). Denote by Δr (resp Δs) the absolute value of the variation of the Riemann invariant r (resp s) in the first (resp second) shock wave. We denote $P(u_l, v_l, u_r, v_r) = \Delta_r + \Delta_s$. Then the following lemma is known. For the proof, see [10].

Lemma 2.5.

$$(2.19)\qquad P(u_1, v_1, u_3, v_3) \leq P(u_1, v_1, u_2, v_2) + P(u_2, v_2, u_3, v_3)\ ,$$

where u_1, u_2 and u_3 are arbitrary constants and v_1, v_2 and v_3 are arbitrary positive constants.

Denote by $i_0^{n\pm}$ the straight line segments joining the points $\left(0, \left(n \pm \frac{1}{2}\right)h\right)$ and a_{1n}. Let $F(i_0^{n\pm})$ be the absolute value of the variation of the Riemann invariants for all shocks on $i_0^{n\pm}$. Then we also have the following Lemma.

Lemma 2.6.

$$(2.20)\qquad F(i_0^{n+}) \leq F(i_0^{n-}) + 2|M-K|h\ .$$

The proof of this lemma 2.6 is very complicated. We shall prove it in the Appendix.

We denote

$$Z_1 = \{l-0, l+0, 3l-0, \cdots, (2m-1)l-0, (2m-1)l+0, \cdots\}\ ,$$

$$Z_2 = \{2l, 4l, 6l \cdots 2ml, \cdots\}\ .$$

Let $Z_{(n)} = Z_1 \cup Z_2 \cup \{c_{nm}\}$ and line up the elements $z_{n,i}$ of $Z_{(n)}$ so that $z_{n,i} \leq z_{n,i+1}$. (We regard $(2m-1)l-0 < (2m-1)l+0$ for m: integer.)

Let

$$F(nh-0, u^l, v^l) = \frac{1}{2}F(i_0^{n-})$$
$$+\sum_{z_{n,i}\in Z(n)} P(u^l(nh-0, z_{n,i}), v^l(nh-0, z_{n,i}), u^l(nh-0, z_{n,i+1}),$$
$$v^l(nh-0, z_{n,i+1})),$$
$$F(nh+0, u^l, v^l) = \frac{1}{2}F(i_0^{n+}) + \sum_{m:odd} P(u^l(a_{nm}), v^l(a_{nm}), u^l(a_{nm+2}),$$
$$u^l(a_{mm+2})) .$$

Using Lemma 2.5 and Lemma 2.6, we get

$$\text{(2.21)}\qquad F((n+1)h+0, u^l, v^l) \le F((n+1)h-0, u^l, v^l) + |M-K|h .$$

We also get

$$F((n+1)h-0, u^l, v^l) = F((n+1)h-0, u_0^l, v_0^l)$$
$$+\sum_{m:odd} P(u^l((n+1)h-0, ml-0), v^l((n+1)h-0, ml-0) ,$$
$$u^l((n+1)h-0, ml+0), v^l((n+1)h-0, ml+0)) .$$

Lemma 2.7.

$$P(u^l((n+1)h-0, ml-0), v^l((n+1)h-0, ml-0) ,$$
$$\text{(2.22)}\qquad u^l((n+1)h-0, ml+0), v^l((n+1)h-0, ml+0))$$
$$\le 2h\,\{\max\ (U^l(nh, (m-1)l) - U^l(nh, (m+1)l), 0)\}, \ m\text{: odd} .$$

Proof. From the definition,

$$u^l((n+1)h-0, ml-0) = u_0^l(nh, ml) + U^l(nh, (m-1)l)\cdot h ,$$
$$u^l((n+1)h-0, ml+0) = u_0^l(nh, ml) + U^l(nh, (m+1)l)\cdot h ,$$
$$v^l((n+1)h-0, ml-0) = v^l((n+1)h-0, ml+0) = v_0^l(nh, ml) .$$

Therefore we get

$$r^l((n+1)h-0, ml-0) - r^l((n+1)h-0, ml+0)$$
$$\text{(2.23)}\qquad = s^l((n+1)h-0, ml-0) - s^l((n+1)h-0, ml+0)$$
$$= h\times\{U^l(nh, (m-1)l) - U^l(nh, (m+1)l)\} .$$

If $U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \ge 0$, there are two shock waves S_1 and S_2 and then the following inequality holds.

$$\text{(2.24)}\qquad \Delta r,\ \Delta s \le h\,\{U^l(nh, (m-1)l) - U^l(nh, (m+1)l)\} \le \Delta r + \Delta s .$$

From (2.24), we get (2.22). If $U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \le 0$, there are two rarefaction waves R_1 and R_2 and then $\Delta r = \Delta s = 0$. So we get (2.22).

Note that the negative variation of $U^l(t, x)$ is smaller than $(M+K)h$. Using (2.21) and Lemma 2.7, we get

$$F((n+1)h+0, u^l, v^l) \leq F(nh+0, u^l, v^l) + (K+M)h + |M-K|h \ . \tag{2.25}$$

Thus we obtain the following estimate.

$$F(nh+0, u^l, v^l) \leq F(+0, u^l, v^l) + (M+K)T + |M-K|T \ . \tag{2.26}$$

Denote by $G(\tau)$ the absolute value of the sum of negative variation of r^l and s^l for $t=\tau$.
Then for $nh \leq \tau < (n+1)h$, we get

$$\begin{aligned} G(\tau) &\leq G(nh) + 2h \sum_{m:odd} \{\max(0, U^l(nh, (m-1)l) - U^l(nh, (m+1)l))\} \\ &\leq G(nh) + (M+K)h \ . \end{aligned} \tag{2.27}$$

Lemma 2.8.

$$G(nh) \leq 2F(nh+0, u^l, v^l) \ . \tag{2.28}$$

Proof. Denote by δs (resp δr) the absolute value of the Riemann invariant s (resp r) in the first (resp second) shock wave. By Lemma 2.1, $\Delta r + \delta s < 2\Delta r$ on the first shock and $\delta r + \Delta s < 2\Delta s$ on the second shock. So from (2.23), (2.24) and above arguments, we get (2.28).

From (2.26), (2.27) and (2.28), for any τ $(nh \leq \tau < (n+1)h)$,

$$\begin{aligned} G(\tau) &\leq G(nh) + (M+K)h \leq 2F(nh+0, u^l, v^l) + (M+K)h \leq \\ & 2F(+0, u^l, v^l) + 3(M+K)T + 2|M-K|T \equiv M_1 \ . \end{aligned} \tag{2.29}$$

Now we can get uniform a priori estimates of r^l and s^l by using Lemma 2.4 and (2.29). From these estimates, we can obtain uniform estimates of the total variation of u^l, v^l and $\inf v^l$. Denote by $T.V.\ u$ the total variation of u.

Theorem 2.9. *For any $T>0$, the variation of u^l and v^l is bounded uniformly for h and $\{a_{mn}\}$. Their upper bound and lower bound, especially the positive lower bound of v^l, are also uniformly bounded.*

Proof. Denote by $T.V^+.\ u$ (resp $T.V^-.\ u$) the absolute value of the positive (resp negative) variation of u. Put $f^l \equiv 2u^l = r^l + s^l$. Then $0 \leq f^l(t, 0) \leq Kh$. Without loss of generality, we assume that $u_0(x)$ and $v_0(x)$ are constant outside a bounded interval.
Let

$$f^l(t, \infty) = r^l(t, \infty) + s^l(t, \infty) \equiv M_2 \ . \tag{2.30}$$

Then from the definition,

$$f^l(t, 0) + T.V^+ f^l - T.V^- f^l = f^l(t, \infty) \ .$$

Since $T.V^{-}f^{l}(t,\ \cdot\ )\leq G(t)$ for any t, (2.29) yields

$$T.V^{+}f^{l}=f^{l}(t,\infty)+T.V^{-}f^{l}-f^{l}(t,0)\leq M_1+M_2\ .$$

Thus we have

$$T.V.f^{l}=T.V.2u^{l}\leq 2M_1+M_2\ . \tag{2.31}$$

From (2.31), we get

$$|f^{l}|\leq Kh+2M_1+M_2\leq KT+2M_1+M_2\equiv 2M_3\ .$$

Therefore we get

$$|u_l|\leq M_3\ . \tag{2.32}$$

Using Lemma 2.4, we obtain

$$2a\log v^{l}=r^{l}-s^{l}\geq r_0-2(K+M)T-\max(-r_0,-r_0-(K-M)T,s_0)\ .$$

Thus we get

$$v^{l}\geq\exp\frac{r_0-2(K+M)T-\max(-r_0,\ -r_0-(K-M)T,s_0)}{2a}\equiv\frac{1}{M_5}\ . \tag{2.33}$$

From the definition,

$$r^{l}(t,0)+T.V^{+}.r^{l}-T.V^{-}.r^{l}=r^{l}(t,\infty)\ .$$

Using Lemma 2.4 and (2.27),

$$T.V^{+}.r^{l}=-r^{l}(t,0)+T.V^{-}.r^{l}+r^{l}(t,\infty)\leq -r_0+(K+M)T+M_1+r(t,\infty)\ . \tag{2.34}$$

In view of (2.32) and (2.34), there exists a positive constant M_6 such that

$$v^{l}\leq M_6 \tag{2.35}$$

By using standard argument of Glimm's theory, we then can prove Theorem 1.2. This part of the proof almost same to that of Theorem 4.2 in [10]. So we omit this part of the proof in this paper.

3. Remark

When we prove Theorem 1.1, the following problem is found. Does Theorem 1.1 also mean the existence of global solutions for (1.1)? Let us describe it more correctly. We can prove that Euler equations (1.1) and Lagrangian equations (1.6) are equivalent for classical solutions by using the chain rule. However, if solutions are weak solutions, we can not use it. So it is not clear that (1.1) and (1.6) are equivalent for weak solutions. Many mathematicians seem to admit it without proof. Wagner [15] has proved it for the first time. He has showed the equivalence of the Euler and Lagrangian equation of the compressible Euler equation in one space dimension for the Cauchy problem. In [12], Mizohata has proved it for the spherically symmetric solutions. We now state our result for (1.1) in [12]. Consider (1.1) and (1.2) with the initial condition

$$\rho(0,r)=\rho_0(r)\ ,\quad u(0,r)=u_0(r)\ , \tag{3.1}$$

We call u and $\rho\,(\in L^\infty((0,T)\times(1,\infty)))$ weak solutions of (1.1), (1.2) and (3.1) if they satisfy

$$\begin{aligned}
&\int_0^T\int_1^\infty (\rho\phi_t+\rho u\phi_r)\,drdt+\int_1^\infty \rho_0(r)\,\phi(0,r)\,dr\\
&=\int_0^T\int_1^\infty \frac{2}{r}\rho u\phi\ drdt\ ,\\
&\int_0^T\int_1^\infty (\rho u\psi_t+(\rho u^2+p)\,\psi_r)\,drdt+\int_1^\infty \rho_0(r)\,u_0(r)\,\psi(0,r)\,dr\\
&=-\int_0^T\int_1^\infty\left(\frac{2}{r}\rho u^2-\frac{\rho M}{r^2}\right)\psi\ drdt\ ,
\end{aligned} \tag{3.2}$$

for any test function $\phi\in C_0^\infty([0,T)\times[1,\infty))$ and $\psi\in C_0^\infty([0,T)\times(1,\infty))$ and for any $T>0$.

Here is our result for the existence of the global weak solution for (1.1).

Theorem 3.1 *Suppose that $\rho_0(r)$ and $u_0(r)$ are bounded variation, and that $\rho_0(r)$ satisfy $\delta_0/r^2<\rho_0(r)<M_0/r^2$ for some positive constants δ_0 and M_0. Then* (1.1), (1.2) *and* (3.1) *admit a global weak solution.* (*i.e. T does not depend on the initial data.*)

For the detail, see [12].

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Appendix. Proof of Lemma 2.6

To prove Lemma 2.6, we must check the following 12 cases:

[1] $c_{1n}<l$,

(1) S_2 crosses i_0^{n-},

(2) R_2 crosses i_0^{n-},

(3) no wave cross i_0^{n-}.

[2] $c_{1n}l$,

(1) S_2 and S_1 cross i_0^{n-},

(2) R_2 and S_1 cross i_0^{n-},

(3) S_2 and R_1 cross i_0^{n-},

(4) R_2 and R_1 cross i_0^{n-},

(5) S_1 crosses i_0^{n-},

(6) R_1 crosses i_0^{n-},

(7) S_2 crosses i_0^{n-},

(8) S_2 crosses i_0^{n-},

(9) no wave cross i_0^{n-}.

For simplicity, we restrict ourselves to the typical case (1) in [2]. We can check the other cases similarly.

Put $r_+^{n-1}=r^l(a_{1n-1})$, $s_+^{n-1}=s^l(a_{1n-1})$, $r_-^{n-1}=-s_-^{n-1}$ $=r^l((n-1)h+0, 0)$, and $\delta_{n-1}=U^l(a_{1n-1})$.

Put $r_+^{n-1'}=r^l((n-1)h+0, 2l)$ and $s_+^{n-1'}=s^l((n-1)h+0, 2l)$.

Put $A=(r_-^{n-1}, s_-^{n-1})$, $B=(r_+^{n-1}, s_+^{n-1})$ and $B'=(r_+^{n-1'}, s_+^{n-1'})$.

Put $C=(r_+^{n-1'}+\delta_{n-1}h, s_+^{n-1'}+\delta_{n-1}h)$.

In this case S_2 and S_1 cross i_0^{n-}. (See Fig. 1)

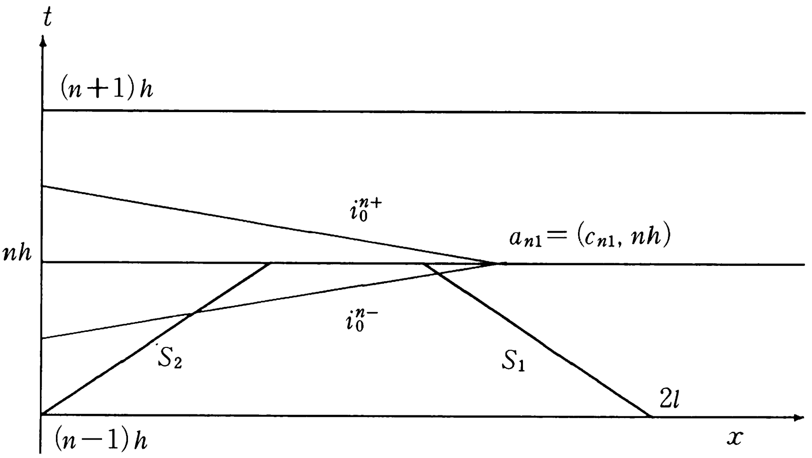

Figure. 1

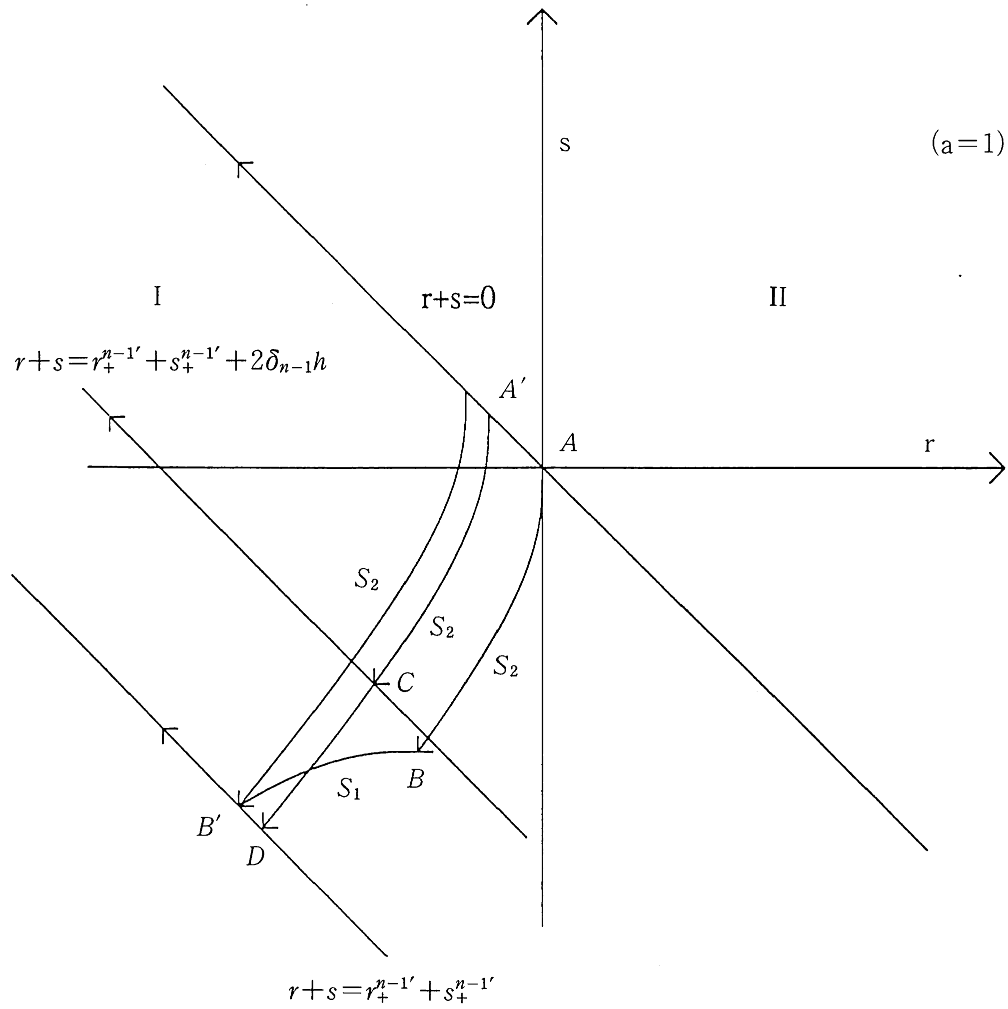

Figure. 2

1) $\delta_{n-1}\geq 0$.

i) $C\in I$.

In this case S_2 cross i_0^{n+}. From figure. 2,

$$F(i_0^{n+})=V(A'C)\leq V(A'D)=V(AD)=V(AB')=F(i_0^{n-}) \ .$$

i) $C\in II$.

In this case R_2 cross i_0^{n+}. Then we obtain

$$F({}_0^{n-})\geq F(i_0^{n+})=0 \ .$$

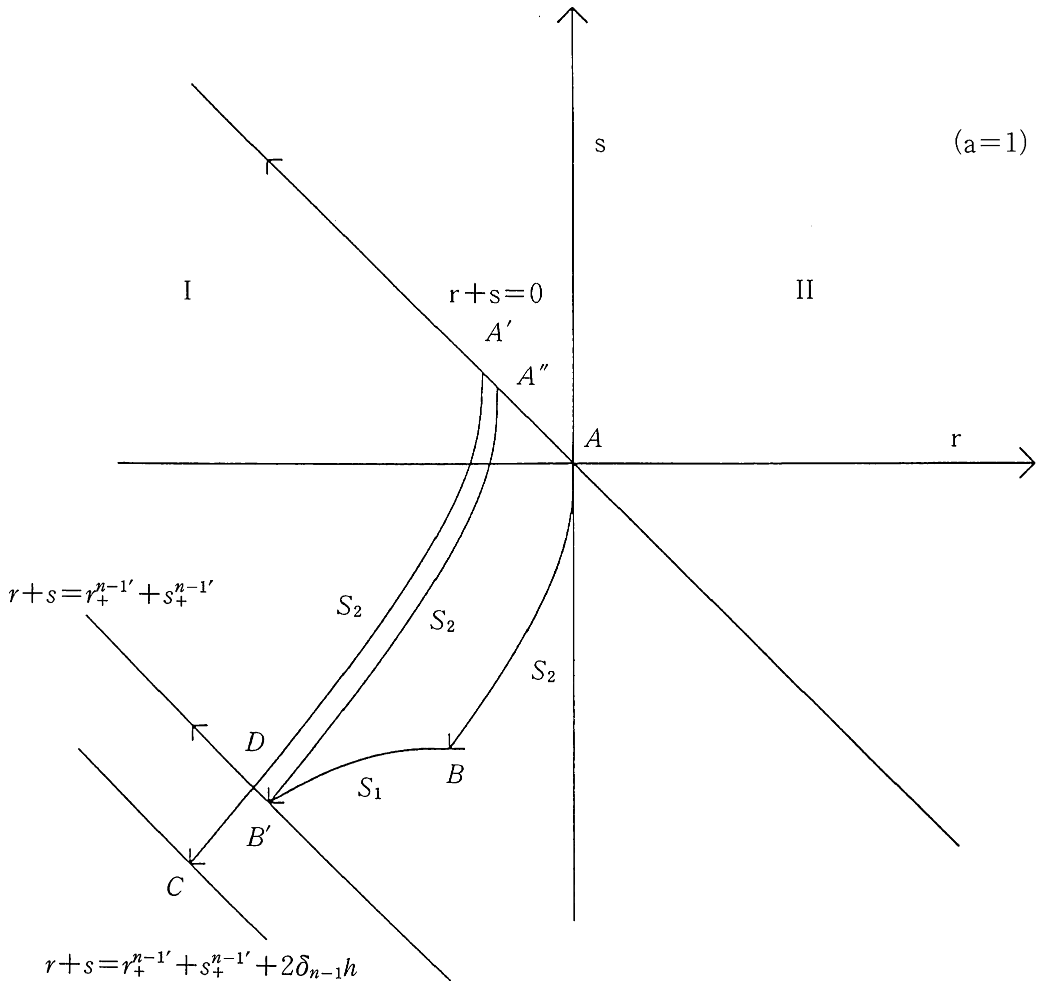

Figure. 3

2) $\delta_{n-1}<0$

In this case S_2 cross i_0^{n+}. Note that if $\delta_{n-1}<0$, $K-M\leq\delta_{n-1}<0$. From figure.3,

$$F(i_0^{n+})=V(A'C)=V(A'D)+V(DC)=V(A''B')+2|\delta_{n-1}|h$$
$$=V(AB')+2|\delta_{n-1}|h\leq F(i_0^{n-})+2|K-M|h \ .$$

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