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The approximation of holomorphic function on Riemann surfaces

By

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Introduction

In the present paper we shall treat the approximation problem of the holomorphic function on a Riemann surface. According to Gunning and Narashimhan [1], on an arbitary open Riemann surface, there exists a locally univalent holomorphic function. The Riemann surface generated by such a function seems to be an unbranched covering surface over the complex plane. It is not easy to study such a function on the general open Riemann surface and so we shall consider a special open Riemann surface i.e. a compact Riemann surface punctured by a point. In this case a locally univalent holomorphic function has in general an essential singularity at the puncture. In the following we shall prove the approximation theorem of a holomorphic function on the punctured surface by meromorphic functions defined on the compact surface. The proof is performed, with some modifications, by the same way as Behnke and Stein did in [2]. In [2] the approximation problem has been treated exclusively at the open Riemann surface, on the contrary here the problem is concerned with the compact Riemann surface.

1. Cauchy kernel

We shall consider a compact Riemann surface R and denote its genus by g. Let $\{A_i, B_i\}_{i=1,\cdots,g}$ be a canonical homology basis and dw_i $(i=1, \cdots, g)$ be the first kind of normal differentials such that $\int_{A_j} dw_i = \delta_{ij}$ and denote the third kind of normal differential by $d\Pi_{p,q}$. $d\Pi_{p,q}$ has a simple pole with residue $1(-1)$ at $p(q)$, respectively and holomorphic eleswhere and all A-periods vanish. The B-periods are given by

$$(1) \qquad \int_{B_j} d\Pi_{p,q} = 2\pi i \int_p^q dw_j \quad (j=1, \cdots, g).$$

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Now let $\Pi_{p,q}^{s,t}=\Pi_{p,q}(s)-\Pi_{p,q}(t)=\int_t^s d\Pi_{p,q}$, then the law of interchang of argument and parameter holds (Osgood [3]):

(2) $\Pi_{p,q}^{s,t}=\Pi_{s,t}^{p,q}$.

For fixed t and q we consider the additive function $\Pi_{p,q}^{s,t}$ of two variables $(s, p)\in R\times R$, then $\partial_s\Pi_{p,q}^{s,t}=d\Pi_{p,q}(s)$ (∂_s ; differential operator with respect to s). It is seen from (2) that $d\Pi_{p,q}(s)$ is a multi-valued function of p and has a simple pole at s and

$$(3)\qquad \begin{aligned} &\int_{A_j}\partial_p(d\Pi_{p,q}(s))=\int_{A_j}\partial_p(\partial_s\Pi_{s,t}^{p,q})=\partial_s\int_{A_j}\partial_p(\Pi_{s,t}^{p,q})=0\\ &\int_{B_j}\partial_p(d\Pi_{p,q}(s))=\partial_s\int_{B_j}\partial_p(\Pi_{s,t}^{p,q})=\partial_s\int_{B_j}d\Pi_{s,t}(p)\\ &=\partial_s\left(2\pi i\int_s^t dw_j\right)=-2\pi i dw_j(s)\quad \text{(by (1))}. \end{aligned}$$

Let U_0 be a parameter neighborhood of q ($q\to z=0$) and set $dw_j=w_j'(z)dz$, then there exist g distinct points $P_1, \cdots, P_g$ ($P_i\to z_i$) in $U_0-\{q\}$ such that

$$(4)\qquad \det|w_j'(z_i)|_{i,j=1,\cdots,g}\neq 0.$$

We remark that the set $\{(P_1, \cdots, P_g);\ \det|w_j'(z_i)|_{i,j=1,\cdots,g}=0\}$ is closed and nowhere dense in R^g (Farkas and Kra [4].)

Suppose $h(p)$ is a meromorphic function on R and has pole at q only and $dh\neq 0$ at P_i ($i=1, \cdots, g$), then it is seen from (4) that the equations

$$(5)\qquad dw_j(s)=\sum_{i=1}^{g}\frac{dw_j(P_i)}{dh(P_i)}dc_i(s)\qquad (j=1, \cdots, g)$$

have 1st kind of differentials $dc_i(s)$ as solution. Let us set

$$(6)\qquad dw(s, p)=d\Pi_{p,q}(s)-\sum_{i=1}^{g}\frac{d\Pi_{p,q}(P_i)}{dh(P_i)}dc_i(s),$$

then it follows from (3) and (5) that

$$\int_{B_j}\partial_p(dw(s, p))=-2\pi i dw_j(s)-\sum_{i=1}^{g}\{-2\pi i\frac{dw_j(P_i)}{dh(P_i)}dc_i(s)\}=0$$

and so $dw(s, p)$ has the following properties;

(i) For variable s $dw(s, p)$ is the 3rd kind of differential and has a simple pole with residue 1 (-1) at $p(q)$, respectively.

(ii) For variable p $dw(s, p)$ is the single valued meromorphic function and has a simple pole at $g+1$ distinct points $s, P_1, \cdots, P_g$, respectively and holomorphic elsewhere.

$dw(s, p)$ is called the elementary differential in [2] (also see: Behnke und Sommer [5]).

2. Lemmas

At first we shall state two lemmas.

Lemma 1. *Suppose* G_0 *and* G_1 *are domains* $(G_0 \Subset G_1 \subset R)$ *and* $H(p)$ *is meromorphic in* G_1 *and has a pole at* $p_0 \in G_1 - \bar{G}_0$ *only and* $\varphi(p)$ *is meromorphic in* G_1 *and is holomorphic on* $\bar{G}_0 \cup \{p_0\}$ *and*

$$|\varphi(p_0)| > \sup_{p \in G_0} |\varphi(p)|.$$

Then $H(p)$ *is expressed by* $\lim_{k\to\infty} H_1(p) \sum_{\mu=0}^{k} a_\mu(\varphi(p))^\mu$ $(a_\mu:$ *constants), where the convergence is locally uniform in* G_0 *and* $H_1(p)$ *is meromorphic in* G_1 *and is holomorphic at* p_0 *and has the poles at the same points as the poles of* $\varphi(p)$.

Proof. Let k be the order of pole at p_0 of $H(p)$ and set

$$H(p) = \frac{H(p)(\varphi(p_0) - \varphi(p))^k}{(\varphi(p_0) - \varphi(p))^k} = \frac{H_1(p)}{(\varphi(p_0) - \varphi(p))^k},$$

then $H_1(p) = H(p)(\varphi(p_0) - \varphi(p))^k$ is meromorphic in G_1 and is holomorphic at p_0 and has the same poles as that of $\varphi(p)$.
Since

$$|\varphi(p)/\varphi(p_0)| < 1 \quad (p \in G_0),$$

we have

$$\frac{1}{(\varphi(p_0) - \varphi(p))^k} = \frac{1}{\varphi(p_0)^k (1 - \frac{\varphi(p)}{\varphi(p_0)})^k}$$

$$= \sum_{\mu=0}^{\infty} a_\mu(\varphi(p))^\mu \quad (a_\mu: \text{constants}),$$

where the convergence is locally uniform in G_0. *q. e. d.*

Set $G = R - \bar{U}_0$ and let K be an arbitrary compact subset of G. Suppose G_0 and D are two domains such that $K \subset G_0 \Subset D \Subset G$ and $R - D$ has precisely one connected component and the boundary $\gamma = \partial D$ is a smooth curve and $dh \neq 0$ on γ. Then by the residue theorem we have for $f(p) \in H(G)$ and $p \in K$

$$f(p) = \frac{1}{2\pi i} \int_\gamma f(s) dw(s,p)$$

$$= \frac{1}{2\pi i} \int_\gamma f(s) \frac{dw(s,p)}{dh(s)} dh(s).$$

Lemma 2. *The holomorphic function* $f(p)$ *can be approximated uniformly* *on K by the Riemenn sum as follows ; for any* $\epsilon>0$ *there exist a finite number of points* $S_\nu \in \gamma$ $(\nu=1, \cdots, n)$ *such that*

$$|f(p)-\frac{1}{2\pi i}\sum_{\nu=1}^{n} f(S_\nu)\frac{dw(S_\nu,p)}{dh(S_\nu)}(h(S_{\nu+1})-h(S_\nu))|<\epsilon \quad (S_{n+1}=S_1).$$

The lemma is proved by the similar way to that of Saks and Zygmund [6] did in the complex plane and so we shall omit its proof. It is seen from (6) that as the function of variable p each term of the Riemann sum has a simple pole at $g+1$ distinct points S_ν, $P_1, \cdots, P_g$, respectively. Here we shall perform the translation of pole S_ν (Polvershiebung), while g points $P_1, \cdots, P_g$ are fixed in $U_0 -\{q\}$.

3. Theorems

Thceorem 1. *Suppose* $f(p)$ *is the holomorphic function in* $G=R-\bar{U}_0$, *then* $f(p)$ *can be approximated locally uniformly in G by the meromorphic functions defined on R as follows ;*
For any $\epsilon>0$ *and any compact* $K\subset G$ *there exists a meromorphic function* $g(p)$ *such that*

$$|f(p)-g(p)|<\epsilon \quad (p\in K),$$

where $g(p)$ has poles at most at properly choiced distinct $g+1$ points Q, $P_1, \cdots, P_g$ in $U_0-\{q\}$ only.

Proof. Let G_0 and D be two domains, $K\subset G_0 \Subset D \Subset G$, as before. Following to the lemma 2 there exist a finite number of points $S_\nu \epsilon \gamma (\nu=1,\cdots,n)$ such that for any $\epsilon>0$

$$(7) \qquad |f(p)-\frac{1}{2\pi i}\sum_{\nu=1}^{n} f(S_\nu)\frac{dw(S_\nu, p)}{dh(S_\nu)}(h(S_{\nu+1})-h(S_\nu))|<\epsilon/2 \quad (p\in K).$$

We choose a smooth simple curve γ_ν joinning S_ν to Q such that γ_ν is contained in $R-D$ and $dh\neq 0$ on γ_ν and $P_i \notin \gamma_\nu$ $(i=1, \cdots, n)$, where Q is a point in $U_0-\{q, P_1, \cdots, P_g\}$. Then we can take a finite number of points $S_\nu=S_{\nu 1}, S_{\nu 2}, \cdots, S_{\nu m+1} =Q$ on γ_ν such that

$$(8) \qquad \left|\frac{dw(S_{\nu\mu+1}, S_{\nu\mu})}{dh(S_{\nu\mu+1})}\right| > \sup_{p\in \bar{G}_0}\left|\frac{dw(S_{\nu\mu+1}, p)}{dh(S_{\nu\mu+1})}\right| \qquad (\nu=1, \cdots, n, \mu=1, \cdots, m).$$

Because $\frac{dw(s, p)}{dh(s)}$ has the pole at $s=p$ and also it is a continous function of two variables $(s, p)\in \gamma_\nu \times \bar{G}_0$, thus for any $s\in\gamma_\nu$ there exists a neighborhood U_s of s such that

$$\left|\frac{dw(s^*, s')}{dh(s^*)}\right| > \sup_{p\in \bar{G}_0}\left|\frac{dw(s^*, p)}{dh(s^*)}\right| \qquad (s^*, s'\in U_s\cap\gamma_\nu).$$

Then γ_ν can be covered by such a finite number of neighborhoods and so we can obtain the desired sequence of points $S_{\nu\mu}$ $(\mu=1, \cdots, m)$ on each γ_ν. Let G_1 be a domain such that G_1 contains $S_{\nu 2}$ and $G_0 \subset D \subset G_1 \subset G$ and set

$$H(p)=\frac{1}{2\pi i}f(S_\nu)\frac{dw(S_\nu, p)}{dh(S_\nu)}(h(S_{\nu+1})-h(S_\nu))$$

$$\varphi(p)=\frac{dw(S_{\nu 2}, p)}{dh(S_{\nu 2})}$$

then $H(p)$ is meromorphic in R and in G_1 it has a pole at $S_\nu \in G_1-\bar{G}_0$ only and $\varphi(p)$ is meromorphic in R and is holomorphic over $\bar{G}_0\cup\{S_\nu\}$ and by (8) we have

$$|\varphi(S_\nu)|=\left|\frac{dw(S_{\nu 2}, S_\nu)}{dh(S_{\nu 2})}\right| > \sup_{p\in \bar{G}_0}\left|\frac{dw(S_{\nu 2}, p)}{dh(S_{\nu 2})}\right| = \sup_{p\in \bar{G}_0}|\varphi(p)|.$$

Thus we have by the lemma 1 for a sufficiently large k

$$(9) \qquad \left|\frac{1}{2\pi i}f(S_\nu)\frac{dw(S_\nu, p)}{dh(S_\nu)}(h(S_{\nu+1})-h(S_\nu))-g_{\nu 1}(p)\right| < \frac{\epsilon}{2mn} \qquad (p\in K),$$

where $g_{\nu 1}(p)=H_{\nu 1}(p)\sum_{\mu=0}^{k}a_\mu\left(\frac{dw(S_{\nu 2}, p)}{dh(S_{\nu 2})}\right)^\mu$ (a_μ: constants) and $H_{\nu 1}(p)$ is meromorphic on R and has the pole at the same points as that of $\frac{dw(S_{\nu 2}, p)}{dh(S_{\nu 2})}$. Thus, by the same way as before, we get mn functions $g_{\nu\mu}(p)$ $(\nu=1, \cdots, n, \mu=1, \cdots, m)$ such that

$$(10) \qquad |g_{\nu\mu+1}(p)-g_{\nu\mu}(p)| < \frac{\epsilon}{2mn} \qquad (p\in K, \mu=2, \cdots, m)$$

and $g_{\nu\mu}(p)=H_{\nu\mu}(p)P_{\nu\mu}[\frac{dw(S_{\nu\mu+1}, p)}{dh(S_{\nu\mu+1})}]$, where $P_{\nu\mu}[x]$ is a polynomial of x and $H_{\nu\mu}(p)$ is meromorphic on R and has the pole at the same points as that of $\frac{dw(S_{\nu\mu+1}, p)}{dh(S_{\nu\mu+1})}$.

Finally we shall set

$$g(p)=\sum_{\nu=1}^{n}g_{\nu m}(p)=\sum_{\nu=1}^{n}H_{\nu m}(p)P[\frac{dw(Q, p)}{dh(Q)}],$$

then $g(p)$ is meromorphic in R and has poles at $g+1$ points $Q, P_1, \cdots, P_g$ only and by (7), (9) and (10) we can get

$$|f(p)-g(p)| < \epsilon \quad (p\in K) \qquad q. e. d.$$

Now let $M_0(R)$ be a family of function such that it is meromorphic on R and has pole at q only. The next step is to diminish the number of poles of approximating function constructed in the theorem 1.

Remarking that g distinct points $P_1, \cdots, P_g$ and a point Q may be taken in an arbitrary neighborhood of q, we get the following

Theorem 2. *In the theorem* 1 *it is possible to take the function belonging to* $M_0(R)$ *as the approximating function.*

Proof. Suppose $\varphi(p)$ is a function belonging to $M_0(R)$ and $U(U\subset U_0)$ is a neighborhood of q such that for any $p'\in U-\{q\}$

$$|\varphi(p')|>\sup_{p\in G}|\varphi(p)|.$$

In the process of construction of $dw(s, p)$ we shall choose g points $P_1, \cdots, P_g$ which belong to $U-\{q\}$ and moreover let Q be in $U-\{q\}$, then a approximating function $g(p)$ has poles, at most, at Q and $P_i(i=1, \cdots, g)$ only. Let k and l_i be the order of pole at Q and P_i, respectively and set

$$H(p)=g(p)(\varphi(Q)-\varphi(p))^k\prod_{i=1}^{g}(\varphi(P_i)-\varphi(p))^{l_i}.$$

Then $H(p)$ is holomorphic at Q and $P_i(i=1, \cdots, g)$ and has pole at q only. For any $p\in G$ we have

$$g(p)=\frac{H(p)}{(\varphi(Q)-\varphi(p))^k\prod_{i=1}^{g}(\varphi(P_i)-\varphi(p))^{l_i}}$$

$$=\frac{H(p)}{\varphi(Q)^k\prod_{i=1}^{g}\varphi(P_i)^{l_i}}\cdot\frac{1}{\left(1-\frac{\varphi(p)}{\varphi(Q)}\right)^k\prod_{i=1}^{g}\left(1-\frac{\varphi(p)}{\varphi(P_i)}\right)^{l_i}}$$

$$=H(p)\sum_{u=0}^{\infty}a_u(\varphi(p))^u \qquad (a_u: \text{constants}),$$

where the convergence is locally uniform in G. q.e.d.

4. Special case and application

Suppose $H(G)$ and $H(R-\{q\})$ are families of holomophic functions in G and $R-\{q\}$, respectively. Then considering the restriction to G we have

$$M_0(R)\subset H(R-\{q\})\subset H(G).$$

The theorem 2 states

$$\overline{M_0(R)}=H(G),$$

where the topology is that of locally uniform convergence in G. Thus we get as the corollary

$$\overline{H(R-\{q\})}=H(G).$$

This is the theorem of Behnke and Stein in the special case.

Suppose K is a compact set in $R-\{q\}$ and is Runge type and $H(K)$ is a family of function $f(p)$ which is continous on K and holomorphic in the interior of K, then we get $M_0(R)\subset H(R-\{q\})\subset H(K)$ on K.
Acccording to the Merigelyan's theorem

$$\overline{H(R-\{q\})}=H(K),$$

where the topology of uniform convergence on K is used. On the other hand

$$\overline{M_0(R)}=H(R-\{q\}),$$

hence we obtain

$$\overline{M_0(R)}=H(K).$$

Thus it is seen that the approximating function in the Merigelyan's theorem may be chosen from $M_0(R)$ in this special case as well.

We shall consider some applications of the theorem 2.
In the case of the genus $g=0$, $\overline{M_0(R)}=H(G)$ shows the Runge's approximation theorem by polynomial. In the case of $g=1$, any point on R is not special, because the 1st kind of differelntial has no zero point on R. We consider the universal covering surface $\hat{\boldsymbol{C}}-\{\infty\}$ of R and denote a fundamental domain by the parallelogram

$$\boldsymbol{P}=\{\lambda\omega_1+\mu\omega_2\,;\ \lambda,\ \mu\in[0,\ 1],\ \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right)>0\}.$$

Suppose G is a simply connected subdomain in $\boldsymbol{P}$ and the lattices lie in the exterior of G and $f(z)$ is a doubly periodic holomorphic function in G, then it follows from the theorem 2 that $f(z)$ can be approximated locally uniformly in G by the doubly periodic meromorphic function $g(z)$ which has poles at the lattice only. It is easy to see that such a elliptic function $g(z)$ can be expressed uniquely as follows ;

$$g(z)=P_1[\wp(z)]+\wp'(z)P_2[\wp(z)],$$

where $\wp(z)$ is the Weierstrass $\wp(z)$ with the fundamental periods ω_i $(i=1,\ 2)$ and $P_i[x]$ $(i=1,\ 2)$ are polynomials of x. Thus $f(z)$ can be approximated locally uniformly in G by $P_1[\wp(z)]+\wp'(z)P_2[\wp(z)]$. Inversely, suppose G_0 is a simply connected bounded domain in $\boldsymbol{C}$ and is contained in a parallelogram $\boldsymbol{P}_0$. then $f(z)$ $(f(z)\in H(G_0))$ can be approximated locally uniformly in G_0 by $P_1[\wp_0(z)]+\wp_0'(z)P_2[\wp_0(z)]$, where $\wp_0(z)$ is the Weierstrass $\wp$-function which has

P_0 as the fundamental period parallelogram.

Now we shall return to the locally univalent holomorphic function $F(p)$ stated in the introduction ([1], Kusunoki and Sainouchi [7], Ripoll [8]). Here we shall consider $F(p)$ on the punctured surface $R-\{q\}$, then q is the isolated singular point of $F(p)$. It is clear that q is not removable. Since $\deg(dF)=2g-2$, if $F(p)$ has the pole at q, then $g=0$. Thus if $g\geqq 1$, then q is the essential singular point, hence according to the Picard theorem $F(p)$ assumes all finite values, with the exception of one at most, an infinite number of times. Let $\{G_n\}_{n=0,1,\cdots}$ be a canonical exhaution of $R-\{q\}$ such that $G_{n+1}\subset G_n$ $(n=1, 2, \cdots)$, $\bigcup_{n=0}^{\infty} G_n=R-\{q\}$ and $R-\bar{G}_n$ is connected and has q as an interior point. Then by the theorem 2 we can get the sequence of meromorphic functions $\{g_n(p)\}$ as follows; Let $\epsilon=\sum_{n=1}^{\infty}\epsilon_n$ $(\epsilon_n>0)$, then for any ϵ_n

$$|F(p)-g_n(p)|<\epsilon_n,\ g_n(p)\in M_0(R) \qquad (p\in\bar{G}_{n-1}).$$

Therefore it is seen that $F(p)$ is approximated by the sequence $\{g_n(p)\}$ locally uniformly in $R-\{q\}$:

$$F(p)=\lim_{n\to\infty} g_n(p).$$

Since $dF\neq 0$ in $R-\{q\}$, it may be assumed that $dg_n\neq 0$ on $\bar{G}_{n-1}$ and so the zeros of dg_n lie in the exterior of $\bar{G}_{n-1}$. Therefore, geometrically speaking, the covering surface R_F generated by $F(p)$ can be approximated by the covering surface R_{g_n} generated by $g_n(p)$ as follows; if n increases, then the number of sheets of R_{g_n} increases and so the number of branch points or its branching order increases as well, nevertheless the projection of image of $\{p\in R: dg_n(p)=0\}$ mapped by $g_n(p)$ approches to the point at infinity.

At last we state the general form of theorem 2. The proof can be obtained, with slight modification, by the same way as before.

Theorem 3. *Suppose G is a subdomain of a compact Riemann surface R and $R-\bar{G}$ has n connected components $U_1, \cdots, U_n$ and each component has an interior point, then $f(p)$ $(f(p)\in H(G))$ can be approximated locally uniformly in G by the meromorphic functions on R, where each meromorphic function has poles at most at n points $Q_i(Q_i\in U_i,\ i=1, \cdots, n)$ only.*

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References

[1] R. C. Gunning and R. Narashimhan, Immersion of open Riemann surfaces, Math. Ann., **174** (1968), 103-108.

[2] H. Behnke und K. Stein, Entwicklungen analytischen Funktionen auf Riemannschen Flächen, Math. Ann., **120**(1948), 430-461.

[3] W. F. Osgood, Lehrbuch der Funktionentheorie, II, Chelsea, 1965.
[4] H. M. Farkas and I. Kra, Riemann surfaces, Springer, 1980.
[5] H. Behnke und F. Sommer, Theorie der analytischen Funktionen einer komplexen Veränderlichen, Springer, 1955.
[6] S. Saks and A. Zygmund, Analytic Functions, Elsevier, 1952.
[7] Y. Kusunoki and Y. Sainouchi, Holomorphic differetials on open Riemann surfaces, J. Math. Kyoto Univ., **11**(1971), 181-194.
[8] P. C. Ripoll, A remark on meromorphic differentials in open Riemann surfaces, L'Enseignement Math., **35**(1989), 55-60.