J. Math. Kyoto Univ. (JMKYAZ)
35-1 (1995) 1-33

Smooth projective varieties with the ample vector bundle $\overset{2}{\wedge} T_X$ in any characteristic

By

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In the present paper we determine the structure of smooth projective varieties with the ample vector bundle $\overset{2}{\wedge} T_X$. If X is a projective space or smooth hyperquadric, $\overset{2}{\wedge} T_X$ is an ample vector bundle. We consider the converse and obtain the following:

Main Theorem. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field whose characteristic is arbitrary. Assume that $\overset{2}{\wedge} T_X$ is ample. Then we have the following:*

1) *if $n \geq 5$, then X is isomorphic to a projective space or a hyperquadric.* (*see Theorem 6.12 and Theorem 7.11*)

2) *if the characteristic of the base field is zero and* $n \geq 3$, *then the same conclusion as in* 1) *holds.* (*see Corollary 4.5 and Theorem 5.6*).

Mori [Mo2] proved that a smooth projective variety with the ample tangent bundle is a projective space in any characteristic. Siu-Yau [S-Y] independently proved Frankel conjecture that an n-dimensional compact Kaehler manifold of positive bisectional curvature is biholomorphic to the projective space. Here we must notice that the positivity of bisectional curvature implies the ampleness of the tangent bundle over the complex number field.

An interesting problem to consider next is to determine the structure of variety with semi-ample tangent bundle. In differential geometry Mok [Mok] showed that if X is a compact complex manifold carrying a kaehler metric with non-negative bisectional curvature, then the universal covering is a product of $\boldsymbol{C}^k$, projective space and Hermitian symmetric manifold of rank ≥ 2. Here we must have in mind that the non-negative bisectional curvature implies the semi-ampleness of the tangent bundle. In this meaning it seems to us that our Main theorem is of significanse as the next step for the study of manifold with semi-ample tangent bundle.

Concerned with the subject stated above we have an attempt to determine the structure of Fano varieties by means of the quantity of rational curves of

Communicated by Prof. M. Maruyama, April 30, 1993
Revised June 9, 1994

the minimal degree. For a Fano variety X, length (X) is defined to be min $\{(-K_X \cdot C) \mid C$ is a rational curve in $X\}$. Then the length of $\boldsymbol{P}^n$ and n-dimensional hyperquadric are $n+1$ and n respectively. In case of $n=3$ Wiśniewski proved the converse over the field of complex numbers in [$W1$].

Now we state the proof of Main theorem. One of the key of the proof is to show that the family $\{\ell_y\}_{y \in Y}$ of rational curves of the minimal degree has the following property: there is a point x in X which is at worst an ordinaly singular point of each curve ℓ_y through the point x as stated below:

(5. I) Let X be a smooth projective variety. Assume that $\overset{2}{\wedge} T_X$ is ample and length $X = \dim X + 1$. Then $S\mathscr{C}$ is a proper set in X. (See $\mathscr{C}$ and $S\mathscr{C}$ for §2 and §5 respectively.)

Thus in characteristic zero we get the desired conclusion by virtue of Kobayashi-Ochiai's theorem. But effecient theorems in characteristic zero (Kodaira's vanishing theorem, Lefschetz's Theorem, Sard's Theorem, Kobayashi-Ochiai's Theorem and so on) do not hold in positive characteristic. Therefore there are several problems we must solve as stated in §6 and §7. For example, for lack of Lefschetz's Theorem, it is very troublesome to deal with the hyperquadric case as in §6 and in the absense of Sard's theorem an unusual case is treated as in (#) of 7. 2.

Recently we learned that the first author and Y. Miyaoka [CM] showed the following conjecture in characteristic zero: An $n\,(\geq 2)$-dimensional Fano variety X of the length n or $n+1$ is isomorphic to a hyperquadric or a projective space respectively. But our theorem holds for any characteristic and hyperquadric case is discussed in entirely different way.

This paper consists of the following sections:

§ 1 . Preliminaries.
§ 2 . The property of the singular curve ℓ_y.
§ 3 . Fano varieties X with $v^*T_X \simeq \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus b} \oplus \mathscr{O}^{\oplus c}$,
and the morphism $g: Z \longrightarrow \boldsymbol{P}(\Omega_X^1)$.
§ 4 . Hyperquadrics (in characteristic zero).
§ 5 . Projective spaces (in characteristic zero) .
§ 6 . Hyperquadrics (in positive characteristic) .
§ 7 . Projective spaces (in positive characteristic) .

In §1 we study the basic property of rational curves ℓ_y of minimal degree in X. First we construct the parameter space Y of such rational curves ℓ_y in X and its (modified) universal space Z which is $\boldsymbol{P}^1$-bundle $Z \longrightarrow Y$. Next we investigate the property of the singular curve in $\{\ell_y\}_{y \in Y}$ in §2. For the purpose we define two subsets $\mathscr{N}$ and $\mathscr{C}$ of Y which consists of nodal curves and cuspidal curves (see (2.1)) respectively. To deal with hyperquadric case in characteristic zero, we get in §3 that Z is naturally contained in $\boldsymbol{P}(\Omega_X)$ as a divisor by virtue of Theorem due to Fulton-Hansen. Moreover we show that the locus consisting of these rational curves in question through a point in X becomes a divisor and particularly it is a cone over an $(n-2)$-dimensional smooth hypersurface in $\boldsymbol{P}^{n-1}$. Therefore we get Theorem 4.5. To prove Main Theorem in characteristic zero, we estimate the dimension of $S\mathscr{C}$. Then the facts 5.2.1 ~ 5.2.6 are available not only in characteristic zero but in positive characteristic. In §6 and §7 we deal with the positive characteristic cases, though the lack of Kobayasi-Ochiai's Theorem and Lefschetz's Theorem in positive characteristic causes complicated arguments. Moreover Wiśniewski's Theorem *A* about Picard group of Fano varieties in [*W*2] is important for our proof.

Conventions and Notations. We work over the algebraically closed field of any characteristic in general. But in §4 and 5, it is supposed that the characteristic of the base field is zero. We use the customary terminology of algebraic geometry. $\mathcal{O}$ (a) denotes the line bundle $\mathcal{O}_{p^1}(1)^{\otimes a}$ on $\boldsymbol{P}^1$. For a vector bundle E on a scheme S, E^V denotes the dual vector bundle of E.

§ 1. Preliminaries

Throughout this paper let X be an n-dimensional smooth Fano variety.
(1.1) Let length (X) be min $\{(C, -K_X)|C$ is a rational curve in $X\}$ and C_0 a rational curve with $(C_0, -K_X)$ = length $(X) = m$. Take the normalization $\varphi: P^1 \longrightarrow C_0$. Then we let H be an irreducible component of the Hilbert scheme Hom (P^1, X) containing the morphism φ where $\dim H \geq \chi(P^1, \varphi^* T_X) = m + \dim X$ by virtue of Proposition 3 in [Mo2].
(1.1.1) Throughout this paper it is supposed that the above H and H_P, H_x defined hereafter are normal varieties.

Let G be Aut $\boldsymbol{P}^1$. Since the natural action of G on $\mathrm{Hom}_k(\boldsymbol{P}^1, X)$ induces the action σ of G on the connected component containing H and, consequently, on H:

$$\sigma: G\times H \longrightarrow H,\ \sigma(g, v)x = v(g^{-1}x),\ g\in G,\ v\in V,\ x\in \boldsymbol{P}^1,$$

G also acts on $H\times \boldsymbol{P}^1$ as follows:

$$\tau: G\times H\times \boldsymbol{P}^1 \longrightarrow H\times \boldsymbol{P}^1,\ \tau(g, v, x) = (\sigma(g, v), gx).$$

Let $\mathrm{Chow}^d X$ be the Chow variety parametrizing 1-dimensional effective cycles C of X with $(C. -K_X) = d$. Then we have a morphism $\alpha: H \longrightarrow \mathrm{Chow}^m X$ with $(v(\boldsymbol{P}^1) \cdot -K_X) = m$ for $v\in H$.

The following proposition can be proved in the same way as Lemma 9 in [Mo2].

Proposition 1.2. 1) *σ is a free action.*
2) *(Y, Γ) is the geometric quotient of H by G in the sense of [Mu] where $Y \longrightarrow \overline{\alpha(H)}$ is the normalization of the closure $\overline{\alpha(H)}$ of $\alpha(H)$ $(\subset \mathrm{Chow}^m X)$ in the field $k(H)^G$ of the G-invariant rational function on H.*

Thus H is a principal fiber bundle over a normal projective variety Y with the group G. Moreover dim $Y \geq \chi(\boldsymbol{P}^1, \varphi^* T_X) - 3 = m + \dim X - 3$.

The following argument can be found before the claim 8. 2. in [Mo2].

(1. 2. 1) Under the above notations, we have a G-invariant morphism:

$$F: H \times \boldsymbol{P}^1 \longrightarrow Y \times X, F(v, x) = (\Gamma(v), v(x)), v \in H, x \in \boldsymbol{P}^1.$$

Let $Z = \mathrm{Spec}_{Y \times X}[(F_* \mathcal{O}_{H \times \boldsymbol{P}^1})^G]$. Then Z is the geometric quotient $H \times \boldsymbol{P}^1/G$ and is a $\boldsymbol{P}^1$-bundle $q: Z \longrightarrow Y$ in the étale topology. Moreover let $p: Z \longrightarrow X$ be a natural projection.

Hereafter we use the morphisms p, q very often.

(1. 1. P) In 1. 1, we fixed the rational curve C_0 on X and studied a family of rational curves on X to which C_0 belongs.

Next we fix a point P at which the curve C_0 is smooth. This condition is effectively used when the geometric quotient of H_P by G_0 stated below is constructed, as shown in Lemma 9 of [Mo2]. We let $\iota: o \longrightarrow P(\in X)$ be a map with a point o in $\boldsymbol{P}^1$ and take an irreducible component H_P of the Hilbert scheme Hom $(\boldsymbol{P}^1, X: \iota)$ containing the morphism φ where Hom $(\boldsymbol{P}^1, X: \iota)$ is closed subscheme $\{v \in \mathrm{Hom}(\boldsymbol{P}^1, X) \mid v(o) = P\}$ of Hom $(\boldsymbol{P}^1, X)$. By Proposition 3 in [Mo2] we can show that

(1. 1. 1. P) H_P is a closed subscheme of H and dim $H_P \geq \dim \chi(\boldsymbol{P}^1, \varphi^* T_X \otimes \mathcal{O}(-1))$.

Let $G_0 = \{v \in \mathrm{Aut}\, \boldsymbol{P}^1 \mid v(o) = o\}$. In the same way as in 1. 2, we get an action $\sigma_P: G_0 \times H_P \longrightarrow H_P$ induced by the action σ.

Proposition 1. 2. P. *Let us maintain the notations of 1. 1. P. Then,*

1) *σ_P is a free action, and*

2) *$(Y(P), \Gamma_P)$ is the geometric quotient of H_P by G_0 in the sense of [Mu] where $\Gamma_P: Y(P) \longrightarrow \overline{\alpha(H_P)}$ is the normalization of the closure $\overline{\alpha(H_P)}$ of $\alpha(H_P)$ ($\subset$ Chow$^m X$) in the field $k(H_P)^{G_0}$ of the G_0-invariant rational functions on H_P. Thus H_P is a principal fiber bundle over a normal projective variety $Y(P)$ with group G_0. Moreover dim $Y(P) \geq \dim \chi(\boldsymbol{P}^1, \varphi^* T_X \otimes \mathcal{O}(-1)) - 2 = m - 2$.*

(1. 2. 1. P) In the next place we consider a G_0-invariant morphism $F_P: H_P \times \boldsymbol{P}^1 \longrightarrow Y(P) \times X$. Then we get the geometric quotient $Z(P)$ and canonical projections $p_P: Z(P) \longrightarrow X$ and $q_P: Z(P) \longrightarrow Y(P)$ in the same manner as in 1. 2. 1.

We state several properties about H and H and H_P.

Proposition 1. 3. *Under the above notations we have the following properties:*

1) *Let φ be as in 1.1. Assume that $\varphi^* T_X$ is generated by global sections. Then X is swept out by rational curves of H.*

2) *For every point x in X, dim $qp^{-1}(x) \geq m - 2$. For each irreducible component*

D of $q^{-1}qp^{-1}(x)$, a canonical morphism $D-p^{-1}(x)\longrightarrow X$ induced by the morphism p is quasi-finite.

3) *If $H^1(\boldsymbol{P}^1, v^*T_X)=0$ for every v in H, H is smooth and therefore Y in Proposition 1.2 is smooth.*

4) *Assume that $H^1(\boldsymbol{P}^1, v^*T_X\otimes\mathcal{O}(-1))=0$ for every point v in H_P. Then H_P is smooth.*

Proof. 1), the former part of 2), 3) and 4) are trivial. For the proof of 2) assume that we can choose a point A in $p(D)-\{x\}$ and an irreducible projective curve C in D so that $p(C)=A$ and C is not contained in $D-p^{-1}(x)$. Then for every point c in a projective curve $q(C)$ each rational curve $pq^{-1}(c)$ passes through two points x and A. It is shown by Theorem 4 in [Mo2] that such a family of rational curves has an element which is a sum of b rational curves with $b\geq 2$, which contradicts the assumption that each rational curve $pq^{-1}(c)$ is of the minimal degree with respect to the ample line bundle $-K_X$.

Corollary 1. 3. 1. *Let P be a point in X and y a point in $qp^{-1}(x)$. Assume that the curve $pq^{-1}(y)$ is smooth at the point P. Then there is a canonical morphism $j: Y(P)\longrightarrow Y$ which is finite and of degree 1.*

Proof. Note that $Y(P)$ is defined by 1. 1. P. The action $\sigma: G\times H\longrightarrow H$ in 1.1 induces the one $\sigma_P: G_0\times H_P\longrightarrow H_P$ canonically. By Proposition 1. 2. a natural morphism $H_P\longrightarrow Y$ is a G_0-invariant morphism. Thus we get a canonical morphism $Y(P)\longrightarrow Y$. Moreover by Proposition 1. 2 and Proposition 1. 2P it suffices to show that the morphism $Y(P)(K)\longrightarrow Y(K)$ between sets of k-rational points is generically injective. But it is trivial. q. e. d.

To show that every Fano manifold is algebraically simply connected we show

Proposition 1. 4. *Let Z and U be smooth projective varieties and $f: U\longrightarrow Z$ an étale finite morphism. Assume that $\chi(U, \mathcal{O}_U)=1$. Then, f is an isomorphism.*

Proof. The assumption says that $f^*T_Z=T_U$. Thus, Hirzebruch Atiyah-Singer Riemann-Roch theorem implies that $\deg f\times\chi(Z, \mathcal{O}_Z)=\chi(U, \mathcal{O}_U)=1$. Hence f is an isomorphism. q. e. d.

Corollary 1. 4. 1. *Any smooth projective Fano variety Z defined over the complex number field is algebraically simply connected.*

Proof. Let $f: U\longrightarrow Z$ be a finite étale morphism from an algebraic scheme U to Z. Then we see that U is a smooth projective variety. Since $f^*K_Z=K_U$, U is a Fano variety. By virtue of Kodaira's vanishing Theorem, we get $H^i(Z, \mathcal{O}_Z)=0$ for $1\leq i\leq\dim Z$, hence $\chi(Z, \mathcal{O}_Z)=1$. Thus, Proposition 1. 4 asserts that f is an isomorphism. q. e. d.

Now under an additional assumption let us study the property of the morphisms p, q which is important in §.3.

Proposition 1.5. *Let us assume that for every point v in H v^*T_X is generated by global sections. Then the morphism $p: Z \longrightarrow X$ is smooth and factors as $Z \xrightarrow{p'} \bar{X} \xrightarrow{j} X$ where p' is a smooth morphism to a smooth variety $\bar{X}$ and all the fibers are irreducible and where j is finite and étale. Finally assume additionally that the characteristic of the base field is zero. Then the morphism j is an isomorphism.*

Proof. By Proposition 1. 2 and 1) of Proposition 1. 3, it suffices to show that the canonical morphism $s: \boldsymbol{P}^1 \times H \longrightarrow X$ is smooth, namely the induced homomorphism $s_*: T_{\boldsymbol{P}^1 \times H} \longrightarrow s^*T_X$ is surjective. Since v^*T_X is generated by global sections for every point v in H, the canonical isomorphism between $H^0(\boldsymbol{P}^1, v^*T_X)$ and the Zariski tangent space $T_{H,v}$ provides us with the surjectivity s_* on $\boldsymbol{P}^1 \times \{v\}$, which yields the desired fact. Hence since $p: Z \longrightarrow X$ is smooth, take the Stein factorisation $j: \mathrm{Spec}_X\, p_* \mathcal{O}_Z \longrightarrow X$ of p and set $\mathrm{Spec}_X\, p_* \mathcal{O}_Z$ as $\bar{X}$. Then we see easily that $\bar{X}$ is smooth and j is ètale and finite. In characteristic zero since X is a Fano variety, the morphism p' is an isomorphism one by Corollary 1.4.1. q. e. d.

Next when M is an n-dimensional smooth projective variety, $\det \overset{2}{\wedge} T_M = -(n-1)K_M$. Thus if $\overset{2}{\wedge} T_M$ is ample, M is a Fano variety. Thus we have

Proposition 1. 6. *Let X be a smooth Fano variety with the ample vector bundle $\overset{2}{\wedge} T_X$. Then length $(X) = n$ or $n+1$. Moreover let C be a rational curve on X with $(-K_X \cdot C) =$ length (X) and $v: \boldsymbol{P}^1 \longrightarrow C$ the normalization of C. Assume that $n \geq 3$. Then v^*T_X is one (#) of the following.*
(This v is said to be #-type when # is one of α, β, γ and σ as stated below.)

If $\deg v^*T_X = n+1$,

α-type) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$

β-type) $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus n-3}.\oplus \mathcal{O}.$

γ-type) $\mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2}.\oplus \mathcal{O}..$

As exceptional cases

$\mathcal{O}(2)^{\oplus 3} \oplus \mathcal{O}(-1)$ (*only in case* $n=4$),

$\mathcal{O}(3)^{\oplus 2} \oplus \mathcal{O}(-2)$ *or* $\mathcal{O}(3) \oplus \mathcal{O}(2) \oplus \mathcal{O}(-1)$ *only in case* $n=3$).

If $\deg v^*T_X = n$,

δ-type) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}.$

As an exceptional case,

$\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(-1)$ (*only in case* $n=3$)

Proof. Letting $v^*T_X = \bigoplus \mathcal{O}(a_i)$ with $a_i \geq a_2 \geq \cdots \geq a_n$, we have $a_1 \geq 2$. Noting $v^* \overset{2}{\wedge} T_X = \sum_{i<j} \mathcal{O}(a_i + a_j)$ and it is ample, we see $a_i + a_j$ is positive. By virtue of Theorem 4 in [Mo**2**], deg $v^*T_X \leq n+1$. Thus we get the desired fact.

q. e. d.

Corollary 1.7. *Let the assumption and notations be as above and as in 1.6. Then for each point y in Y,* $p^*T_{X|q^{-1}(y)}$ *is one (#) of the types as in Proposition 1.6. (Hereafter the point y is said to be #-type).*

§2. The property of singular curves ℓ_y

Throughout in this section we let X be a Fano variety and we maintain notations C_0, H, H_P, Z, Y, p, q and m (=length X) in §.1 and set $pq^{-1}(y)$ as ℓ_y.

In this section we study how many curves in the set $\{\ell_y | y \in Y\}$ of rational curves of minimal degree on X are singular and what the type of the singularity is.

First let us begin with the definition of singular curves which we treat here.

(2. 1) A nodal (or, cuspidal) curve means the rational curve dominated by a plane curve C of degree 3 with only one node (or cusp) point P via a birational morphism v. Moreover the point $v(P)$ of the curve $v(C)$ is said to be nodal (or, cuspidal) point respectively.

Let $\mathcal{N}$ be the set $\{y \in Y | \ \ell_y$ is a nodal curve$\}$ and $\mathcal{C}$ the set $\{y \in Y | \ \ell_y$ is a cuspidal curve$\}$. Moreover let $\mathcal{NC}$ be $\mathcal{N} \cap \mathcal{C}$.

Now a point y in Y is said to be $\bar{\alpha}$-type if $p^*T_{X|l_y}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^b \oplus \mathcal{O}^c$ with $b \geq 1$ and $c \geq 0$.

Proposition 2.1.1. 1) *The set $\mathcal{N} \cup \mathcal{C}$ is a closed subset in Y and $\mathcal{C}$ is closed.*

2) *Assume that $\overset{2}{\wedge} T_X$ is ample, dim $X \geq 4$ and deg $v^*T_X = n+1$ (see Proposition 1.6). Then $\mathcal{C}$ is equal to the image of the set $\{v \in H | \ v$ is γ-type$\}$ via the morphism $\Gamma : H \longrightarrow Y$.*

Proof. 1) is trivial. For 2) we state an easy

Fact: Let $w : \boldsymbol{P}^1 \longrightarrow M$ be a non-constant morphiam to a smooth variety M and o a point of $\boldsymbol{P}^1$. Then the following two conditions are equivalent to each other:

1) the homomorphism $w_* : T_{P_1} \longrightarrow w_*T_M$ induced by the morphism w is injective as a vector bundle.
2) $w(\boldsymbol{P}^1)$ is not a cuspidal curve.

Thus noting that the vector bundle v^*T_X of γ-type has no line bundle $\mathcal{O}(2)$ as a direct summand from Proposition 1.6, we complete the proof. q. e. d.

Next we show

Theorem 2. *Let the notations be as above.*

(2. A) *Set* $\{x \in X | \text{there is a point } y \text{ in } Y \text{ so that } \ell_y \text{ is smooth at the point } x\}$ *as* X_0. *Assume that for a general point* v *in* H, v^*T_X *is generated by global sections and the characteristic of base field is zero. Then there is an open subset* X_1 *in* X_0 *so that for each point* x *in* X_1, *there is a point* y *of* α*-type in* $qp^{-1}(x)$. *Moreover for* x *in* X_1 *the set* $\{y \in pq^{-1}(x) | y \text{ is } \alpha\text{-type}\}$ is open in $qp^{-1}(x)$.

(2. B) *For every point* x *in* X, *the set* $\{y \in Y | x \text{ is a nodal point in } \ell_y\}$ *is at most a finite set.*

(2. C) *Assume that* $\dim \mathcal{N} \geq n$. *Then* $\mathcal{C}$ *is not empty and intersects with the closure* $\overline{\mathcal{N}}$ *of* $\mathcal{N}$ *in* Y.

(2. C') *Suppose that* $\mathcal{C}$ *is empty. For each point* x *in* X, *the set of nodal curves in* Y *passing throuth* x *is at most finite set. Moreover* $\dim \mathcal{N} \leq n-1$. *Namely, there is a open subset* V *in* X *such that for every* y *in* Y, ℓ_y *is smooth in* V.

For the above properties, we need several propositions.

(2. 2) Let E be a direct sum of line bundles $L_1 \oplus L_2$ on a projective curve C. Set $\boldsymbol{P}(E)$ as S and the section $\boldsymbol{P}(L_i)$ as C_i. Now let φ be a morphism from S to a variety so that a fiber of a canonical projection $\pi: S \longrightarrow C$ is mapped to a curve via φ. The we have

Lemma 2.3. *Under the above condition 2.2, let* C_3 *be a section of* π *and* M *a quotient line bundle of* E *which yields the section* C_3. *Assume that* $\varphi(C_3)$ *is a point and* $\dim \varphi(S) = 2$. *Then the morphism* φ *is obtained by a linear system of the line bundle* $(\mathcal{O}_{\boldsymbol{P}(E)}(1) \otimes \pi^* M^{-1})^{\otimes a}$ *with some positive integer* a. *Moreover one of two line bundles* $L_1 \otimes M^{-1}$, $L_2 \otimes M^{-1}$ *is ample and the other is trivial. Namely the curve* C_i *such that* $L_i = M$ *is mapped to a point via* φ *and the other to a curve.*

Proof. Let $W := \mathcal{O}_{P(E)}(\mathrm{a}) \otimes \pi^* N$ be a line bundle which gives the morphism φ where N is a line bundle on C. First since a fiber of π goes to a curve via φ, a is positive. Moreover since $W_{|C_3} = \mathcal{O}_C$, we have $N = M^{-a}$. Hence we infer that $W = (\mathcal{O}_{\boldsymbol{P}(E)}(1) \otimes \pi^* M^{-1})^{\otimes a} = \mathcal{O}_{\boldsymbol{P}(E \otimes M^{-1})}(1)^{\otimes a}$. On the other hand W is semi-ample, so is $W_{|C_i}$. As $W_{|C_i}$ is $(L_i \otimes M^{-1})^{\otimes a}$, $L_i \otimes M^{-1}$ is semi-ample, which says that $\deg L_i \geq \deg M$. Moreover $\dim \varphi(S) = 2$ implies that the self-intersection of $(\mathcal{O}_{\boldsymbol{P}(E)}(1) \otimes \pi^*(-M)^{\otimes a} (= a^2 \sum_i \deg(L_i \otimes M^{-1}))$ is positive. If both of $L_i \otimes M^{-1}$ are ample, so is W. On the other hand since φ is not finite, we have a contradition. Hence we see that the one of $L_i \otimes M^{-1}$ is ample and the other is not ample. Moreover we have an exact sequence:

$L_1 \otimes M^{-1} \oplus L_2 \otimes M^{-1} \longrightarrow \mathcal{O} \longrightarrow 0$, which yields $L_1 \simeq M$ or $L_2 \simeq M$, because either of $L_i \otimes M^{-1}$ has no non-zero section. Thus the last part is trivial.

q. e. d.

Corollary 2.4. *Let the condition and assumption be as in Lemma 2.3. Assume $C_1 \cap C_3 = \phi$. Then $\varphi(C_1)$ is a curve and $C_2 = C_3$.*

Proof. The assumption that $C_1 \cap C_3$ is empty says that $M \simeq L_2$ and E is isomorphic to $L_1 \oplus M$. Since $\varphi(C_3)$ is a point, $L_1 \otimes L_2^{-1}$ is ample by lemma 2. 3 and therefore $\varphi(C_1)$ is a curve. Since C_2 and C_3 are linearly equivalent to $W \otimes \pi^* L_1$, the intersection $C_2 \cdot C_3$ is the degree of $L_2 \otimes L_1^{-1}$, which is negative. Thus we get $C_2 = C_3$. q. e. d.

Corollary 2.5. *Let the condition be as in 2.2. Assume that $\dim \varphi(S) = 2$. Then if φ is not a finite morphism, one of $\varphi(C_1)$ and $\varphi(C_2)$ is a point and the other a curve. In other words, if $\varphi(C_i)$ is a curve for $i = 1, 2$, then φ is a finite morphism.*

Proof. By the assumption there is a point t in $\varphi(S)$ such that $\varphi^{-1}(t)$ contains a curve D with $\pi D = C$. By base change via a morphism $D \longrightarrow C$ we get the same set-up as in Lemma 2. 3, which yields this Corollary. q. e. d.

Here we have a

Proposition 2.6. *Let M be a variety, $\pi : S \longrightarrow C$ a $\boldsymbol{P}^1$-bundle over an irreducible projective curve C and $f: S \longrightarrow M$ a morphism with $\dim f(S) = 2$. We assume that*

1) *For each point c in C, $\pi^{-1}(c)$ is transformed to a curve.*
2) *f is not finite.*

Then we have the following assertions:

1) *The set $\{s \in f(S) \mid \dim f^{-1}(s) \geq 1\}$ consists of only one point A.*
2) *One dimensional part of $f^{-1}(A)$ intersects a general fiber $\pi^{-1}(c)$ at one point.*
3) *If the characteristic of the base field is zero, then one dimensional part of $f^{-1}(A)$ consists of only one rational section of π. (Here a rational section D of π means that $\pi_{|D} : D \longrightarrow C$ is a birational morphism.)*

Proof. By the assumption, we have a point A in $f(S)$ so that $f^{-1}(A)$ contains an irreducible component D which is of one-dimension. Now assume that D intersects with a general fiber of π at more than one point. Let $\bar{D}$ be the normalization of D. Then a canonical morphism $j: \bar{D} \longrightarrow C$ induces a $\boldsymbol{P}^1$-bundle $\bar{\pi}: \bar{D} \times_C S (= \bar{S}) \longrightarrow \bar{D}$ and a section D_2 of $\bar{\pi}$. Letting $h: \bar{S} \longrightarrow S$ the morphism induced by the morphism j, $h^{-1}(D_2)$ has another irreducible curve $D_3 (\neq D_2)$ and the image of D_2 and D_3 by hf: $\bar{S} \longrightarrow M$ is the same point A. Now taking a generic hyperplane section of $f(S)$ not passing through the point A, we have another curve D_1 in $\bar{S}$ which intersects with neither D_2 nor D_3. Therefore after several base change we obtain the same set-up as in Corollary 2. 4 by setting D_i as C_i. Thus we have a contradiction, which yields 2). The rest is trivial. q. e. d.

The above results provide us with the following proposition which is used in §.3.

Proposition 2.7. *Let $\pi: T \longrightarrow V$ be a $\boldsymbol{P}^1$-bundle over a smooth projective variety V and $\varphi: T \longrightarrow U$ a morphism. Assume that*
1) *every fiber of π goes to a curve via φ,*
2) *there is an irreducible divisor D of T which collapses to a point A in U via φ, and*
3) *the restriction of the morphism φ to $T-D$ is quasi-finite. Finally suppose that the characteristic of the base field is zero. Then D is a section of π. Moreover there is a rank-2 vector bundle E on V and its subline bundle M enjoying the following exact sequence on V:*

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

where $T \simeq \boldsymbol{P}(E)$ and $\boldsymbol{P}(\mathcal{O})$ corresponds to the section D. Here M is an ample line bundle and E splits to $\mathcal{O} \oplus M$.

Proof. The assumption 1) implies that the morphism $\pi_{|D}$ is finite. By 2) in Proposition 2. 6 and Zariski Main Theorem we infer that D is a section of π. Thus the section D gives a rank-2 vector bundle E on V and the quotient line bundle M with an exact sequence on V:

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

where $P(\mathcal{O})$ determines the section D canonically.

By the proof in Lemma 2. 3, we infer that φ is obtained by high power of $\mathcal{O}_{\boldsymbol{P}(E)}(1)$. Thus E corresponds to an element σ in $H^1(V, M)$. Now take an irreducible divisor G of T which does not intersect with D and if G is singular, make the desingularization $f: \overline{G} \longrightarrow G$ of G. Then the fiber product $P(E) \times_V \overline{G}$ has another section $\overline{G}$ which does not intersect with the section induces by G. Thus f^*E splits to $\mathcal{O} \oplus f^*M$. This says that there is a canonical homomorphism $f^*: H^1(V, M) \longrightarrow H^1(\overline{G}, f^*M)$ with $f^*\sigma=0$. By Proposition 4. 17 [F], we have $\sigma=0$ (in characteristic zero). Since $\mathcal{O}_{\boldsymbol{P}(E)}(1)_{|\boldsymbol{P}(M)} \simeq M$, the remainder is trivial. Thus we get the proof. q. e. d.

Now we begin the proof of Theorem.

Proof of (2. A). Let $Y_x = qp^{-1}(x)$. The assumption yields an open subset H_0 in H such that for each point v in H_0 v^*T_x is generated by global sections. Thus let Y_1 be the image of H_0 via the geomertic quotient $\gamma: H \longrightarrow Y$. Then we can take an open set X_1 in X_0 so that for each point x in X_1 $Y_x \cap Y_1$ is not empty. Therefore $\dim Y_x = \chi(\boldsymbol{P}^1, v^*T_X \otimes \mathcal{O}(-1)) - \dim \mathrm{Aut}\, G_0 = m-2$ for each point x in X_1 (See 1. 2. P for G_0). Suppose that there are a point x

in X_1 and an open set U_x in Y_x so that for every point Y in U_x y is not $\overline{\alpha}$-type, namely, when $p^*T_{X|l_y}=\bigoplus_{i=1}^{n}\mathcal{O}(a_i)$ with $a_1\geq a_2\geq\cdots\geq a_n\geq 0$, # $\{i|a_i\geq 1\}$ is less than $m-1$. Fix a point $\bar{o}$ of $\boldsymbol{P}^1$ with $o\neq\bar{o}$. First G_0 (1. 1. P) acts on $\boldsymbol{P}^1-\{o\}$ transitively. Let s be the canonical morphism $F_x\cdot p_x:\boldsymbol{P}^1\times H_x\longrightarrow X$ in 2.1. P. Since G_0 acts on $\boldsymbol{P}^1\times H_x$ canonically, we see that $s(\boldsymbol{P}^1\times H_x)-\{x\}$ coincides $s(\{\bar{o}\}\times H_x)-\{x\}$. Thus to study the rank of the homomorphism ds : $T_{\boldsymbol{P}^1\times H_x}\longrightarrow s^*T_x$ on $(\bar{o}, v)$ for a point v in $H_x\cap H_0$ we have only to check the one of $\overline{\mathrm{d}s}: T_{\{o\}\times H_x}\longrightarrow s^*S_x$ on $(\bar{o}, v)$ where $\bar{s}:\{\bar{o}\}\times H_x\longrightarrow X$ is the composite morphism of a closed embedding $i:\{\bar{o}\}\times H_x\longrightarrow\boldsymbol{P}^1\times H_x$ and the morphism s. Noting that the Zariski tangent space $T_{H_x,v}$, of H_x at the point v in H_x is isomorphic to $H^0(\boldsymbol{P}^1, v^*T_x\otimes\mathcal{O}(-1))$, we infer that the rank of ds at $(\bar{o}, v)$ is $\leq m-2$, which implies that the image of the canonical morphism $s:\boldsymbol{P}^1\times H_x\longrightarrow X$ is of $\leq(m-2)$-dimension by Sard's Theorem. Therefore $\dim(Z\times_Y Y_x)=\dim s(\boldsymbol{P}^1\times H_x)$ by 2) of Proposition 1. 3. On the other hand $\dim(Z\times_Y Y_x)=\dim Y_x+1=m-1$ which yields a contradiction.

Proof of (2. B). This is clear by 2) in Proposition 2. 6.

Proof of (2. C). We have only to show

Sublemma 2.8. Let $\mathscr{C}$, $\mathscr{N}$ be as in the first part of this section. Assume that there exists a point P in X and a curve C in $\mathscr{N}$ so that for each point y in C, ℓ_y passes through the point P. Then $\mathscr{C}$ intersects with the closure $\overline{C}$ of C in Y.

Proof. We suppose the contrary. Take the normalization $g:\widetilde{C}\longrightarrow\overline{C}$ of $\overline{C}$ and consider a smooth ruled surface $\widetilde{C}\times_{\overline{C}}q^{-1}(\overline{C})$ $(=R)$. Then by the assumption, we see that R contains sections C_1, C_2 with $C_1\cap C_2=\phi$ satisfying the following property: letting $\bar{p}:R\longrightarrow X$ and $\bar{q}:R\longrightarrow\widetilde{C}$ be canonical morphisms induced by the morphism $R\longrightarrow q^{-1}(\overline{C})$, for every point c in $\widetilde{C}$, $\bar{p}(C_1\cap\bar{q}^{-1}(c))$ coincides with $\bar{p}(C_2\cap\bar{q}^{-1}(c))$ and it is a nodal point of $\ell_{g(c)}$. Since $\dim\bar{p}(R)=2$, R has a curve C_3 so that $\bar{q}(C_3)=\widetilde{C}$ and $\bar{p}(C_3)=P$. Remark that the ruled surface R is isomorphic to $\boldsymbol{P}(L_1\oplus L_2)$ with two line bundles L_1 L_2 on $\widetilde{C}$ so that each line bundle L_i corresponds to the section C_i. Since $\bar{p}$ is not finite, $\varphi(C_1)$ or $\varphi(C_2)$ collapses to a point Q by Corollary 2. 5. Consequently both of the points go to the point Q, which yields a contradition to Corollary 2. 5. q. e. d.

Proof of (2. C'). It is obvious by 2. C and sublemma 2. 8. q. e. d.

§3. Fano varieties X with $v^*T_x\simeq\mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus b}\oplus\mathcal{O}^{\oplus c}$ and the morphism $g:Z\longrightarrow P(\Omega_X^1)$

We maintain notations H, Y, Z, H_P, $Y(p)$ defined in §1.

(3. 1) Assume that for every element v in H,

v^*T_x is $\mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus b}\oplus\mathcal{O}^{\oplus c}$, namely the set $\mathscr{C}$ in § 2 is empty and $(v(\boldsymbol{P}^1)\cdot -K_X)=\text{length } X\geq 3$. (Note that the assumption 1. 1. 1 and 1. 1. 1. P hold automatically by 3) , 4) in Proposition 1. 3.)

Remark 3.1.1. The assumption says that for a point x in X, $\dim qp^{-1}(x)\geq 1$ and therefore there is a point y in $qp^{-1}(x)$ so that ℓ_y is smooth at the point x by B in Theorem 2.

Under the assumption we show the induced morphism $g: Z\longrightarrow \boldsymbol{P}(\Omega_X^1)$ is a closed embedding and next we study the basic property of X obtained in case of $b=n-2$ and $c=1$.

The $\boldsymbol{P}^1$-bundle $q: Z\longrightarrow Y$ yields an exact sequence

$$0\longrightarrow T_{Z/Y}\xrightarrow{\ i\ } T_Z\longrightarrow q^*T_Y\longrightarrow 0.$$

On the other hand by Proposition 1.5 and Proposition 1.6 the morphism $p: Z\longrightarrow X$ gives a surjective homomorphism $p_*: T_Z\longrightarrow p^*T_X$. Thus we consider the composite homomorphism ip$_*$

(3. 2) $T_{Z/Y}\longrightarrow p^*T_X$.

Since the above situation 3.1 means that for any point v in H,

(3.3) the morphism $v: \boldsymbol{P}^1\longrightarrow X$ is unramified, the homomorphism f in 3.2 is injective as a vector bundle on Z, which yields a morphism $g: Z\longrightarrow \boldsymbol{P}(\Omega_X^1)$ satisfying the following diagram:

(3. 4)

$$\begin{array}{ccc} Z & \xrightarrow{\ g\ } & P(\Omega_X^1) \\ {\scriptstyle q}\downarrow & \searrow^{p} & \\ Y & & X \end{array}$$

where η is a tautological line bundle of T_X and $g^*\eta\simeq T_{Z/Y}^{v}$.

Now we consider the case when the morphism g is a closed embedding.

First we recall notations.

(3. 5) For a point x in X, let $Y_x=qp^{-1}(x)$ and $Z_x=q^{-1}qp^{-1}(x)$. Moreover let $L_y=q^{-1}(y)$ and $\ell_y=p(L_y)$.

(3. 6) Now let us study the property of the morphism g on $p^{-1}(x)$, written by g_x. First by Remark 3.1.1, $Y(x)$ is defined. The morphism $j: Y(x)\longrightarrow Y$ in Corollary 1.3.1 has a property that $j(Y(x))\subset qp^{-1}(x)$. Since $p^{-1}(x)$ is

smooth and irreducible (char$k=0$) by Proposition 1. 5 and $\dim Y(x)=\dim q(p^{-1}(x))=m-2$, the morphism $j: Y(x) \longrightarrow qp^{-1}(x)$ induces the natural one $Y(x) \longrightarrow p^{-1}(x)$, which is finite birational and therefore an isomorphism.

Thus we study the morphism $g_x: p^{-1}(x)\ (=Y(x)) \longrightarrow \boldsymbol{P}(\Omega^1_{X,x})$. Let H_x be as in §1. By the canonical morphism $H_x \times \boldsymbol{P}^1 \longrightarrow X$, we can define a morphism

$$\phi: H_x \longrightarrow \boldsymbol{V}(\Omega^1_{X,x}),\ \phi(v)=\mathrm{d}v_{*,o}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

where t is a local parameter of $\boldsymbol{P}^1$ at the fixed point o.

Now hereafter we assume that

(3. 7) $n\geq 4,\ b>c.$

From now on let us show that the morphism g_x is unramified.

First by 3. 7, $v: \boldsymbol{P}^1 \longrightarrow v(\boldsymbol{P}^1)$ is umramified. Thus we see that the image $\phi(H_x)$ is contained in $\boldsymbol{V}(\Omega^1_{X,x})-\{0\}$, which induces the morphism $H_x \longrightarrow \boldsymbol{P}(\Omega^1_{X,x}) \simeq \boldsymbol{P}^{n-1}$. Since this morphism is G_x-invariant, we have the induced morphism $Y(x) \longrightarrow \boldsymbol{P}^{n-1}$, which is just the morphism g_x itself as shown above.

Now by the assumption 3. 7, $v^*T_X \otimes \mathcal{O}(-2)$ is isomorphic to $\mathcal{O}\oplus\mathcal{O}(-1)^{\oplus b} \oplus \mathcal{O}(-2)^{\oplus c}$ and therefore, $\dim H^0(\boldsymbol{P}^1, v^*T_X \otimes \mathcal{O}(-2))=1$. Note that $H^0(P^1, v^*T_X\otimes\mathcal{O}(-2))$ is the Zariski tangent space $T_{\phi^{-1}\phi(v),v}$ of $\phi^{-1}\phi(v)$ at v (see 8. 1 in [Mo2]). Thus $\dim_v\phi^{-1}\phi(v)\leq 1$. On the other hand the algebraic group G_x acts on H_x and $\dim H^0(\boldsymbol{P}^1, T_{P^1}\otimes\mathcal{O}(-2))=1$, and therefore $\dim_v\phi^{-1}\phi(v)=1$. Since $\dim T_{\phi^{-1}(v),v}=\dim_v\phi^{-1}\phi(v)$, we infer that $\phi^{-1}\phi(v)$ is smooth and therefore every fiber of ϕ is smooth. Thus we see that g_x is umramified.

Thus we have the following:

Proposition 3.7.1. *Under the notation in 3.3, assume the condition 3.7 for any x in X. Then g is of maximal rank on every point v in Z. Moreover, for each point x in X, g_x is a closed embedding.*

Proof. The former is shown. The latter is due to the following Theorem by W. Fulton anf J. Hansen.

Theorem (*Proposition* 2 [F-H]). *Let V be a projective variety of dimension n, $h: V \longrightarrow \boldsymbol{P}^m$ an unramified morphism with $m<2n$.* Then h is a closed embedding. q.e.d.

The above Proposition immediately yields

Corollary 3.8. *Let the notation and condition be as in 3.3. Assume the* *condition 3.7. Then g is a closed embedding.*

Now to study the structure of Z_x, we prepare a few notations.

(3. 9) Let $\sigma_x : X_x \longrightarrow X$ be the blow up of X with the point x as the center. For a subvariety W in X, $\sigma_x^{-1}[W]$ denotes the proper transform of W by σ_x.

Now by 2. C' and 3. 1, we can take

(3. 9. 1) a point A in $V(\subset X)$, namely ℓ_y is smooth at the point A for any y in $qp^{-1}(A)$. Therefore the canonical morphism $p^{-1}(A) \longrightarrow qp^{-1}(A)$ is an isomorphism. Thus $p^{-1}(A)$ and Z_A are smooth and therefore $p^{-1}(A) \times_Y Z$ is canonically isomorphic to Z_A.

Let us cosider a morphism $p_A : Z_A \longrightarrow X$ induced by $p : Z \longrightarrow X$. Noting that $p_A^{-1}(A)$ is a Cartier divisor in Z_A, by the universality of blowing-up we get

(3. 10) a morphism $m : Z_A \longrightarrow X_A$ with $m\sigma_A = p_A$ and $m(p_A^{-1}(A)) = \sigma_A^{-1}[D_A]$ where $D_A = p_A(Z_A)$.

Now let us study the behavior of the morphism m on $p_A^{-1}(A)$.

Take a point y in Y_A. Let $\overline{\ell}_y$ be the proper transform of ℓ_y by σ_A and $h : \boldsymbol{P}^1 \longrightarrow \overline{\ell}_y$ the normalization of ℓ_y.

(3. 11) First we remark that for each point y in Y_A

1) $\sigma_A^{-1}[\ell_y]$ intersects with $\sigma_A^{-1}(A)$ transversally,

2) Since $p^*T_{X|L_y}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$, $m^*T_{A|L_y}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus b} \oplus \mathcal{O}(-1)^{\oplus c}$.

To show this, it is sufficient to use the following result in Appendix B. 6. 10. in [H]

(#) Let $X \subset Y$ and $Y \subset Z$ be regular imbeddings. Let $\overline{Z}$ be the blowing-up of Z at X, $\overline{Y}$ the blowing-up of Y at X and E the exceptional divisor of X via the morphism $f : \overline{Z} \longrightarrow Z$. Then $N_{\overline{Y}/\overline{Z}} = f^*N_{Y/Z} \otimes \mathcal{O}_{\overline{Z}}(-E)$.

3) $\sigma_A^{-1}(A) \cap \sigma_A^{-1}[D_A]$ is a smooth subvariety in $\sigma_A^{-1}(A)$ $(= \boldsymbol{P}(\Omega^1_{X,A}) \simeq \boldsymbol{P}^{n-1})$ and it is canonically isomorphic to $p^{-1}(A)$ from Proposition 3. 7. 1 and the above 1). Moreover $\sigma_A^{-1}[D_A]$ is smooth around the subvariety $\sigma_A^{-1}(A) \cap \sigma_A^{-1}[D_A]$.

We study the morphism $Z_A \longrightarrow m(Z_A)$. By (2) of 3. 11, m is of maximal rank at each point z in Z_A. Precisely speaking, the homomorphism $m_* : T_{Z_A} \longrightarrow m^*T_{X_A}$ is injective as a vector bundle. Moreover letting $\overline{m}$ the morphism obtained by restricting m to $p^{-1}(A)$, we see that $\overline{m}$ induces an isomorphism from $p^{-1}(A)$ to $\sigma_A^{-1}(A) \cap \sigma_A^{-1}[D_A]$. Thus the morphism $m : Z_A \longrightarrow m(Z_A)$ is an isomorphism around $p^{-1}(A)$.

Summarizing the above argument in 3. 10 and 3. 11, we get

Proposition 3.12. *Let A be a point in 3.9.1 and m in 3.10. Then two* *morphisms* $Z_A \longrightarrow m(Z_A)$ *and* $Z_A \longrightarrow p_A(Z_A)$ *are birational morphisms. More precisely, there is an open neighborhood* U $(\supset p^{-1}(A))$ *in* Z_A *so that* $m: U \longrightarrow m(U)$ *is an isomorphism and* p_A *is an immersion on* $U - p^{-1}(\mathrm{A})$. *Moreover* $Z_A - p^{-1}(A) \longrightarrow p(Z_A) - \{A\}$ *is finite.*

Proof. We have only to show the last part. But it is obvious by 2) in Propositoin 1. 3. q. e. d.

Now recalling that the set $\mathscr{C}$ of our Fano variety X in question is empty and combining 2. C' and Proposition 2. 7, we get

Corollary 3.13. *Let* A *be a point in 3.9.1. Then* Y_A *is a smooth subvariety in* Y, Z_A *a* $\boldsymbol{P}^1$*-bundle over* Y_A *and* $p^{-1}(A)$ *is a section in* Z_A. *Moreover assume that the characteristic of the base field is zero. Then there is an ample line bundle* M *on* Y_A *so that* $Z_A \simeq \boldsymbol{P}(\mathcal{O} \oplus M)$, *the restricted morphism of* p *to* Z_A *is given by the tautological line bundle of* $\mathcal{O}_A \oplus M$ *and* $\boldsymbol{P}(\mathcal{O}_{Y_A})$ *is* $p^{-1}(A)$.

Finally we assume that $b = n - 2$ and $c = 1$.

Then we show that

(3. 14) There is a point $\bar{A}$ in V (see (2. C') and (3. 9. 1)) so that $p(Z_{\bar{A}})$ is a normal Cartier divisor with at most one isolated singularity $\bar{A}$. Then a natural map $P_{\bar{A}}: Z_{\bar{A}} - p^{-1}(\bar{A}) \longrightarrow p(Z_{\bar{A}}) - \bar{A}$ is an isomorphism.

For a variety T, Sing T denotes the singular locus of T.

Noting 3. 13, assume that

(#) for every point A in V, $p(Z_A)$ is non-normal, equivalently, $\mathrm{codim}_{p(Z_A)}$ Sing $p(Z_A) = 1$ because $p(Z_A)$ is a Cartier divisor in X. More precisely Sing $p(Z_A) - \{A\}$ is a Weil divisor in $p(Z_A) - \{A\}$ since a normal point in $p(Z_A) - \{A\}$ is smooth one there by 3. 9. 1.

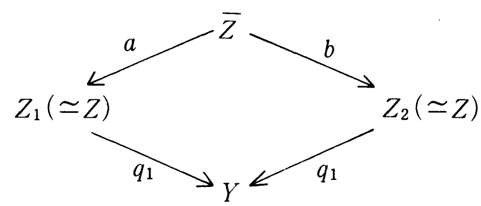
Thus we see for every point x in X, $\mathrm{codim}_{p(Z_x)}$ Sing $p(Z_x) = 1$.

Let $S(Z_x)$ be the closure of a set $p^{-1}(\mathrm{Sing}\, p(Z_x)) - p^{-1}(x)$ in Z_x.

Then (#) yields the property:

(3. 15) 1) For each point x in X, $S(Z_x)$ is of codimension 1 in Z_x and $S(Z_x) \cap p^{-1}(x)$ is at most finite set by (2. B) of Theorem 2. 2) For each point A in V, $S(Z_A)$ is a Cartier divisor in Z_A and $S(Z_A) \cap p^{-1}(A)$ is empty.

Now let $\bar{Z}$ be the fiber product $Z \times_Y Z$ of Z and Z over Y and Δ the diagonal of $\bar{Z}$. Then there is a canonical morphism $h: \bar{Z} \longrightarrow X \times X$ by $(z, z') \longrightarrow (p(z), p(z'))$.



(3. 16) Then we have the following property:
for each point x in X,
1) $\bar{Z}$ is a disjoint union of $p^{-1}(x)\times_Y Z_2$ where x runs over X as a set,
2) Let us set $p^{-1}(x)\times_Y Z$ as $\bar{Z}_x$ and let $\bar{p}_x:\bar{Z}_x\longrightarrow Z_x$ be a canonical morphism. Then $\bar{p}_x$ is a finite and birational morphism by (2. B) in Theorem 2. In particular if x is in V, then $\bar{p}_x$ is an isomorphism.

Let S be a closed set $h^{-1}(\mathrm{Sing}h(\bar{Z}))$ in $\bar{Z}$. Noting that $\mathrm{Sing}\,h(\bar{Z})=\bigcup_{x\in X}\{x\}\times \mathrm{Sing}p(Z_x)$, we see that S is contained in $\bigcup_{x\in X}\bar{p}_x^{-1}(S(Z_x))\cup\Delta$ and is of 2n-2 dimension by 3. 15. Take an irreducible componenet $J(\neq\Delta)$ in S which is a Cartier divisor in $\bar{Z}$. For a general point A in V, $J\cap\bar{Z}_A$ is contained in a disjoint union of $\bar{p}_x^{-1}(S(Z_x))\cup(\{x\}\times p^{-1}(x))$ and does not contain $\Delta\cap\bar{Z}_A$ by 2) of 3. 15 and 2) of 3. 16. On the other hand $J\cap\bar{Z}_A$ is a Cartier divisor in $\bar{Z}_A$. Hence $J\cap\Delta\cap\bar{Z}_A$ is empty. Moreover we can easily see that a Cartier divisor $J\cap\bar{Z}_A$ is connected in $\bar{Z}_A$ and therefore every fiber of a canonical morphism ap : $J\longrightarrow X$ is connected where a : $\bar{Z}\longrightarrow Z$ be the first projection. On the other hand since $J\cap\bar{Z}_x$ is a Cartier divisor in $\bar{Z}_x$ for each point x in X, it is contained in $S(Z_x)$ and disjoint to $\{x\}\times p^{-1}(x)$ by 1) of 3. 15 and 2) of 3. 16. Hence we have

Proposition 3.17. *$\Delta\cap J$ is empty.*

Now since the diagonal Δ is a section of b, we have an exact sequence on Z_2:

(3. 17. 1) $0\longrightarrow\mathcal{O}\longrightarrow E\longrightarrow L\longrightarrow 0$

where E is a rank-2 vector bundle and L a line bundle on Z_2. Here $\bar{Z}$ and Δ are canonically isomorphic to $\boldsymbol{P}(E)$ and $\boldsymbol{P}(L)$ respectively. Consider the fiber product $J\times_{Z_2}\bar{Z}$ $(\simeq J\times_{Z_2}\boldsymbol{P}(E))$. Then Δ and J yield two disjoint sections with respect to the $\boldsymbol{P}^1$-bundle b in the above fiber product. Hence letting $\varphi: J\longrightarrow Z_2$ a canonical projection, we see that the pull-back of the exact sequence 3. 17. 1 via φ splits to $\varphi^*E\simeq\mathcal{O}\oplus\varphi^*L$. Restricting the exact sequence (3. 17. 1) to $M_y:=(q_2)^{-1}(y)$ $(\simeq\boldsymbol{P}^1)$, we get an exact sequence: $0\longrightarrow\mathcal{O}_{\boldsymbol{P}^1}\longrightarrow E_{|M_y}\longrightarrow L_{|P^1}\longrightarrow 0$. On the other hand since $(bq_2)^{-1}(y)\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1\simeq\boldsymbol{P}(E_{|M_y})$, $E_{|M_y}$ is isomorphic to $\mathcal{O}(a)\oplus\mathcal{O}(a)$. Noting that $\varphi^*E\simeq\mathcal{O}\oplus\varphi^*L$, we get $a=0$. Taking the direct image R^0q_{2*} of the exact sequence 3. 16. 1 we obtain an exact sequence by the base change theorem:

(3. 17. 2) $0\longrightarrow\mathcal{O}_Y\longrightarrow F\longrightarrow N\longrightarrow 0$

where F is a rank-2 vector bundle on Y with $q_2^*F\simeq E$ and N a line bundle on $Y$with $q_2^*N\simeq L$. Thus we infer that $\boldsymbol{P}(F)\simeq Z$. On the other hand $\boldsymbol{P}(N)$ yields a unique section $p^{-1}(A)$ in Z_A for each point A in X and therefore $P(N)=\bigcup_{A\in X}p^{-1}(A)$ by Corollary 3. 13. which contradicts to the fact that

$Z = \bigcup_{A \in X} p^{-1}(A)$.

Hence we proved 3. 17. q. e. d.

§4. Hyperquadrics (in characteristic zero)

In this section using the results in §3, we study a smooth projective Fano variety X satisfying the following condition: length $(X) = \dim X = n \geq 2$ and for any rational curve C of the minimal degree on X, $v^* T_X$ is isomorphic to $\mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{P^1}$ where $v : \boldsymbol{P}^1 \longrightarrow C$ is the normalization of C.

First we study the structure of $p(Z_{\bar{A}})$ in 3. 14, written by D. Note that D is a normal irreducible divisor which is a cone with at most one isolated singularity.

By virtue of Theorem A in [W2] note that

(4. 1) when $n \geq 3$, Pic $X \simeq \boldsymbol{Z} L$ with the ample line bundle L in X.

Taking account of the fact that $Z_{\bar{A}} \simeq \boldsymbol{P}(\mathcal{O}_{Y_{\bar{A}}} \oplus M)$ by 3. 13 and D is an ample divisor in X, we have the following:

Proposition 4.2. 1) *Pic*$D \simeq \boldsymbol{Z}\mathcal{O}_D(S)$ *where S is the image of the section $\boldsymbol{P}(M)$ via p.*
2) *The closed embedding $i : D \longrightarrow X$ yields a canonical isomorphism Pic $X \simeq$ PicD if $n \geq 4$.*

Proof. 2) is obtained by Lefschetz's Theorem. As a reference see §1 in [Fuj]. q. e. d.

The intersection number of a fiber of $q : Z_{\bar{A}} \longrightarrow Y_{\bar{A}}$ and the section S in $Z_{\bar{A}}$ is one. Moreover the canonical morphism $p : Z_{\bar{A}} \longrightarrow D (\subset X)$ is birational.

Thus recalling the assumption 3. 7 first, we can show that in case of $n \geq 4$

(4. 3) $-K_X = nL$.

In fact let $-K_X = aL$ by (4. 1). Thus we infer that $n = (\ell_y \cdot -K_X)_X = (\ell_y \cdot aL)_X = (\ell_y \cdot L_D)_D = a(\ell_y, S) = a$ by Proposition 4. 2. Hence by virtue of Theorem due to Kobayashi and Ochiai we see that when dim $X \geq 4$, X is a hyperquadric.

In case of $n = 2$, X is a Del Pezzo surface. Moreover the assumption implies that the surface has no exceptional rational curve of the first kind. Thus we infer that X is a smooth quadric surface.

Finally the case of $n = 3$ is shown by Theorem A in [W2] and Corollary 2. 6 in [W1]. Thus we get

Theorem 4.4. *Let X be an n-dimensional Fano manifold with length $(X) = n$. Assume that for any rational curve C of the minimal degree on X, $v^* T_X$ is isomorphic to $\mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{P^1}$ where $v : \boldsymbol{P}^1 \longrightarrow C$ is the nor-*

malization of C. Then if $n \geq 2$, X is a quadric hypersurface.

Consequentely combining Proposition 1. 6 and Theorem 4. 4 we obtain

Corollary 4.5. *Let X be a smooth projective variety. Assume that* $\overset{2}{\wedge} T_X$ *is ample and* length $X = \dim X \geq 3$. *Then X is a hyperquadric.*

§5. Projective spaces (in characteristic zero)

In this section let X be a Fano variety with length $X = \dim X + 1$ in characteristic zero. In 5. I ~ 5. III, we assume 1. 1. 1 and 1. 1. 1. P. Note, in case of Main Theorem, that the two assumptions automatically follows from the condition.

For a subscheme W of $\mathscr{C}$, let SW be the set $\{x \in X | x$ is a cuspidal point of ℓ_y for a point y in $W\}$. $S\mathscr{C}$ is a closed subset in X.

We prove the following three facts:

(5. I) Assume $\overset{2}{\wedge} T_X$ is ample and $\dim X (=n) \geq 4$. Then $\dim \mathscr{C} \leq n-1$ and therefore $S\mathscr{C}$ is a proper subset of X.

(5. II) If $S\mathscr{C}$ is a proper subset of X, there is an open set U in X so that for each point x in U $p_x : Z_x \longrightarrow X$ is birational. (Here the morphism p_x is the one induced by p which is shown to be generically finite surjective by 2) of Proposition 1. 3)

(5. III) If there is an open set U in X so that for each point x in U $p_x : Z_x \longrightarrow X$ is birational, then $X \simeq \boldsymbol{P}^n$.

To show 5. I. we make a preparation.

By Proposition 2. 1. 1 note that $\mathscr{C} = \{y \in Y | y$ is γ-type$\}$ and hence each cuspidal curve $pq^{-1}(y)$ has only one cuspidal point. Let H_γ be $\{v \in H | v^* T_X$ is γ-type$\}$. H_γ is a closed subscheme of H.

(5. 0) When $n = 4$, let $H_\varepsilon = \{v \in H | v^* T_X \simeq \mathscr{O}(2)^{\oplus 3} \oplus \mathscr{O}(-1)\}$. Then H_ε is closed by semi-continuity of coherent sheaf and $\mathscr{C} \cap \Gamma(H_\varepsilon)$ is empty.

(5. 1) Let R be a plane cubic curve with one cusp singularity P and take a general point y in $\mathscr{C}$. Since ℓ_y has a cuspidal point, there is a canonical birational morphism $\varphi : R \longrightarrow \ell_y$. Thus we can find the following irreducible component H_R of Hom (R, X) containing the morphism φ. Fixing a birational morphism $\mu : \boldsymbol{P}^1 \longrightarrow R$, we have a canonical morphism $\phi : H_R \longrightarrow H (\subset$ Hom $(\boldsymbol{P}^1, X))$ with $\phi(H_R) \subset H_\gamma$ and $\dim \alpha(\phi(H_R)) = \dim \mathscr{C}$ under the notation $\alpha : H \longrightarrow \mathrm{Chow}_X^{n+1}$ in 1. 2 canonically. Note that $\alpha(\phi(H_R))$ is closed in Chow_X^{n+1} by virtue of the latter part in the proof of Lemma 9 ii) in [Mo2]. Take the normalization $g : \mathscr{C}_R \longrightarrow \alpha(\phi(H_R))$ of the closed subvariety $\alpha(\phi(H_R))$. Then we have an irreducible component $\mathscr{C}(R)$ of $\mathscr{C} (\subset Y)$ such that $h(\mathscr{C}(R)) = \alpha(\phi(H_R))$ with the normalization $h : Y \longrightarrow \alpha(H)$ in 1. 2.

Now we show 5. I.

Assuming that

(5. 1) $\dim \mathscr{C} \geq n$,

one has $\dim H_R \geq n+2$ by the fact that Aut(R) is of 2-dimension.

By virtue of Proposition 2 in [Mo2], we have inequalities: $h^0(R, w^*T_X) \geq \dim H_R \geq \chi(R, w^*T_X)$ for each point w in H_R.

Thus we conclude that

(5. 2. 1) for each point w in H_R, $h^0(R, w^*T_X) = n+2$ and $h^1(R, w^*T_X) = 1$. Thus H_R is smooth and of $n+2$ dumension.

In fact, setting w^*T_X as E, we have an isomorphism: $H^1(R, E) \simeq H^0(R, E^v)^v$ since the canonical sheaf ω_R of the curve R is $\mathscr{O}_R$. Remarking that $\chi(R, w^*T_X) = n+1+n\chi(R, \mathscr{O}_R) = n+1$, we have $h^0(R, E^v) \geq 1$ by the assumption. Letting $\mu: \boldsymbol{P}^1 \longrightarrow R$ the normalisation, we see that $\mu^*E^v \simeq \mathscr{O}(-3) \oplus \mathscr{O}(-1)^{\oplus n-2} \oplus \mathscr{O}$ and therefore $h^0(R, E^v) \leq 1$. Thus $h^0(R, w^v) = 1$, $h^0(R, E) = n+2$ and $h^0(R, w^*T_X) = \dim H_R$ as desired.

Now we claim that:

ϕ is a closed embedding.

In fact, the morphism $\phi: H_R \longrightarrow H$ induces the homomorphism of the tangent spaces $d\phi_{[w]}: T_{H_R,[w]} \longrightarrow T_{H,[\mu w]}$ for each point w in H_R. Then it corresponds canonically to the homomorphism: $H^0(R, w^*T_X)) \longrightarrow H^0(\boldsymbol{P}^1, (\mu w)^*T_X))$. Then it is obviously injective. Moreover we see easily that for a morphism v of γ-type in $\mathscr{C}$ there is a unique morphism $w: R \longrightarrow X$ such that $\mu w = v$ and therefore that ϕ is a closed embedding as desired.

Now let o be a point in $\boldsymbol{P}^1$ with $\mu(o) = P$, $G = \mathrm{Aut}\boldsymbol{P}^1$ and $G_R = \mathrm{Aut}(R)$. Then note that G_R is canonically isomorphic to $G_0 (= \{\sigma \in G | \sigma(o) = o\})$ which is a closed subgroup of G. In Proposition 1. 2. 1 we have the free action $\sigma: G \times H \longrightarrow H$ and we see that H_R is stable under the action G_R. Moreover by the natural closed embedding: $G_R \times H \longrightarrow G \times H$, the action σ induces a canonical action $G_R \times H_R \longrightarrow H_R$, which is a free action. In the same way as in 1. 1 (essencially in the way of the proof of Lemma 9 [Mo2]) we can construct the geometric quotient of H_R by G_R which coincides with $\mathscr{C}_R$. Moreover we have a geometric quotient $Z_{\mathscr{C}_R}$ of $R \times H_R$ by G_R and a canonical morphism $\mathscr{C}_R \longrightarrow \mathscr{C}(R)$ to some component $\mathscr{C}(R)$ of $\mathscr{C}$ which is finite and birational. Therefore combining 5. 2. 1, we see

(5. 2. 2) $\mathscr{C}_R$ is a smooth projective variety and therefore so is the fiber product $Z \times_Y \mathscr{C}_R$. Two canonical morphisms $Z \times_Y \mathscr{C}_R \longrightarrow Z \times_Y \mathscr{C}(R)$ and $Z \times_Y \mathscr{C}_R \longrightarrow Z_{\mathscr{C}_R Z}$ are the normalizations.

Let $\overline{p}: Z \times_Y \mathscr{C}_R \longrightarrow X$ and $\overline{q}: Z \times_Y \mathscr{C}_R \longrightarrow \mathscr{C}_R$ be canonical projections.

Now let us cosider the above morphism $\overline{p}$.

Recall that P is a unique cuspidal point of the curve R, take a point w in H_R and fix it hereafter. Note that $H^1(R, \mathscr{O}_R) \simeq k$.

It is easy to see that a non-zero section of E^v (5. 2. 1) gives rise to a trivial line bundle of E^v on R. Set the quotient vector bundle on R as F^v. We have an exact sequence on R:

$$0 \longrightarrow \mathcal{O}_R \longrightarrow E^v \longrightarrow F^v \longrightarrow 0.$$

Since $\mu^* F \simeq \mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2}$, F is ample. Thus we infer that $H^1(R, F) = H^0(R, F^v) = 0$ and we obtain

(5. 2. 3.) E splits to $\mathcal{O} \oplus F$.

Let $V = \{s \in H^0(R, F) | s(P) = 0\}$. Then we have

(5. 2. 4) $\dim V = 2$.

In fact we can find two sections s_1, s_2 in $H^0(R, F)$ which are linearly independent over k with $s_1(P) = s_2(P) = 0$ since rank $F = n-1$ and $h^0(F) = n+1$. Assume that there is another section s of $H^0(R, F)$ where $s(P) = 0$ and s, s_1, s_2 are linearly independent over k. Since $\mu^* F \simeq \mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2}$, the induced three sections $\bar{s}$, $\bar{s}_1$, $\bar{s}_2$ in $H^0(\boldsymbol{P}^1, \mu^* F)$ are also linearly independent over k and can be considered as sections of $\mathcal{H}om(\mathcal{O}(2), \mathcal{O}(3))$ because the mulplicity of the curve R at P is 2. Since $\dim H^0(\boldsymbol{P}^1, \mathcal{O}(1)) = 2$, the above argument yields a contradiction. Thus we have an $(n-1)$-dimensional vector subspace W in $H^0(R, F)$ with $V \cap W = \{0\}$. Then the above argument says that

(5. 2. 5) The sections of vector space $W(\subset H^0(R, w^* T_X))$ generates the vector space $F \otimes k(P)$ $(\subset w^* T_X \otimes k(P))$ at the singular point P of R. Therefore $H^0(R, w^* T_X)$ generates $w^* T_X \otimes k(P)$ at the point P.

Recall that $\mathscr{C}_R$ is smooth and set $Z \times_Y \mathscr{C}_R$ as $\bar{Z}_R$. Since each point y in $\mathscr{C}_R$ induces only one cuspidal point of the cuspidal curve $\bar{p}\,\bar{q}^{-1}(y)$, a $\boldsymbol{P}^1$-bundle $\bar{q} : \bar{Z}_R \longrightarrow \mathscr{C}_R$ has a section S induced by these cuspidal points. Now consider the homomorphism $\bar{p} : T_{\bar{Z}_R} \longrightarrow \bar{p}^* T_X$ induced by the canonical morphism $\bar{p} : \bar{Z}_R \longrightarrow X$. By 5. 2. 5 we see that the morphism $\bar{Z}_R \longrightarrow X$ is of maximal rank around the section S. On the other hand the morphism $\bar{q}$ yields an exact sequence:

$$0 \longrightarrow T_{\bar{q}} \xrightarrow{j} T_{\bar{Z}_R} \longrightarrow \bar{q}^* T_{\mathscr{C}_R} \longrightarrow 0$$

where $T_{\bar{q}}$ is the relative tangent line bundle of $\bar{q}$. Since the composite homomorphism $j\bar{p}$ $T_{\bar{q}} \longrightarrow \bar{p}^* T_X$ is zero on the section S, there is an induced surjective homomorphism on S: $T_{\mathscr{C}_R} \longrightarrow \bar{p}^* T_X$. Therefore we have a property.

(5. 2. 6) The induces morphism $p_S : S \longrightarrow X$ restricted $\bar{p}$ to S is finite and surjective.

Let $\bar{y}$ be a point in $\mathscr{C}$ and P the only one cuspidal point of $\ell_{\bar{y}}$. Then to show the above statement (5. 2. 6), we prove that $\{y \in \mathscr{C} | P$ is the cuspidal point of $\ell_y\}$ is finite set. Consequently it is sufficient to show the following:

Claim: The closed subscheme $\{v \in H | v(o) = P,\ dv_{*,o}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = 0\}$ $(= B)$ is

smooth and of 2-dimensional. (Here t is a local parameter of $\boldsymbol{P}^1$ at the point o).
In fact, we see that the Zariski tangent space $T_{B,v}$ is isomorphic to $H^0(\boldsymbol{P}^1, v^*T_X\otimes\mathcal{O}(-2))$ which is $\mathcal{O}(1)\oplus\mathcal{O}(-1)^{\oplus n-2}\oplus\mathcal{O}(-2)$. Moreover noting that B has a canonical action via the 2-dimensional automorphism G_0 induced by Aut (R) , we get the desired fact. At the same time we see that

(5. 2. 7) the induced morphism $p_S : S \longrightarrow X$ restricted p to S is étale.

(5. 2. 7. 1) S is a section of $\boldsymbol{P}^1$-bundle $\bar{q} : \bar{Z}_R \longrightarrow \mathscr{C}_R$ over the smooth projective variety $\mathscr{C}_R$, $\bar{Z}_R$ is described as $P(J)$ where J is a rank 2 vector bundle over $\mathscr{C}_R$ satisfying the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathscr{C}_R} \longrightarrow J \longrightarrow L \longrightarrow 0$$

where the quotient line bundle L of J on $\mathscr{C}_R$ yields the section S.

Now in characteristic zero, we infer by Corollary 1. 4. 1. that P_S is an isomorphism namely $\mathscr{C}_R$ is isomorphic to X. By virtue of [W2] it is known that Pic$X\simeq \boldsymbol{Z}L_0$ with the ample line bundle L_0. Since $-K_X$ is ample, it follows that $H^1(X, M) = 0$ for any line bundle M on X by Kodaira vanishing theorem. Thus we have: (5. 2. 8) G splits to $\mathcal{O}_{\mathscr{C}_R}\oplus L$.

Let S_0 be the other section induced by the trivial line bundle $\mathcal{O}$ of E.
Note that for a general point x in X, $p^{-1}(x)$ is a smooth curve in $\bar{Z}_R$ which is a rational section over $\bar{q}p^{-1}(x)$ in the meaning in Proposition 2. 6. Thus from Corollary 2. 5 we see that $\bar{q}p^{-1}(x)$ is contained in S_0. Hence the morphism $p : \bar{Z}_R \longrightarrow X$ collapses only the section S_0, which implies that dim $p(\bar{Z}_R) = n+1$. Thus we get a contradiction.

Hence we proved 5. I .

In the next place we prove 5. II .

We assume the contrary.

(5. 3) There is a point x in $X-S\mathscr{C}$ so that p_x is of degree $d>1$. (Note that this is an open condition.) In other words, there is a point y in Y_x and a projective curve E in Y_x where for a general point $\bar{y}$ in E $\ell_{\bar{y}}$ and ℓ_y intersect at a point which is not x.

Then the curve E is the image of some component of $p^{-1}(\ell_y)$ via q and each $\ell_{\bar{y}}$ $(y \neq \bar{y})$ passes through the point x and $\ell_y \cap \ell_{\bar{y}} - \{x\}$ is not empty. Therefore we have more precise situation:

(5.4) there are a point x in $X-S\mathscr{C}$, a curve ℓ_y on X and an irreducible complete curve $C_1(\neq L_y)$ satisfying the following:

1) x is a smooth point of ℓ_y,
2) C_1 is an irreducible component of the closure of $p_x^{-1}(\ell_y - x)$, and
3) for each point c in C_1, $\ell_{q(c)}$ is smooth at the point x (see 2. B).

Thus we consider a ruled surface $q^{-1}(q(C_1))$ $(=S)$ over the projective curve $q(C_1)$. Letting φ: $\bar{C} \longrightarrow q(C_1)$ the normalization, set $\bar{C} \times_{C_1} S$ as the ruled surface $\bar{S} \longrightarrow \bar{C}$. Let $\bar{p}$: $\bar{S} \longrightarrow X$ be the canonical morphism induced by the morphism p and H an ample line bundle on X and f a fiber of $\bar{S} \longrightarrow \bar{C}$. Let C_0 be the minimal section in $\bar{S}$ induced by $p^{-1}(x) \cap q^{-1}(q(C))$ and $e = (C_0 \cdot C_0)$. Then $\bar{p}^*H$ is numerically equivalent to $a(C_0\text{-ef})$ and C_1 to $\alpha C_0 + \beta f$ with integers a, α and β. We get $a > 0$. Note that $\bar{p}(C_1) = \bar{p}(f)$ and $\deg \bar{p}_{|f} = 1$. Then we see that $a\beta = (\bar{p}^*H, C_1) = \deg \bar{p}_{|C_1}(H \cdot \bar{p}(C_1))_X = \deg \bar{p}_{|C_1}(H \cdot \bar{p}(f))_X$. Moreover we have $a = (\bar{p}^*H, f) = \deg \bar{p}_{|f}(H \cdot \bar{p}(f))_X = (H \cdot \bar{p}(f))_X$. Thus we get $\deg \bar{p}_{|C_1} = \beta$. On the other hand $(C_1 \cdot C_0) = \beta + \alpha e$ with $\alpha e \neq 0$.

Thus we will induce a contradiction by Proposition 5. 5 shown below.

We make a preparation for Proposition 5. 5.

Let π: $S \longrightarrow E$ be a geometrical ruled surface over a smooth projective curve E. Let C_0 and C_1 be sections of π. Let us consider a morphism p from S to a smooth variety X with the following properties:

1) $\dim p(S) = 2$.
2) C_0 collapses to a point v via p.
3) the curve $p(C_1)$ is smooth at v.

Now let t be a point on $C_0 \cap C_1$ and $F = \pi^{-1}\pi(t)$.

Proposition 5.5. *Assume that a curves $p(F)$ is smooth at v and the morphism $F \longrightarrow P(F)$ is birational. Letting $I(C_0, C_1; t)$ the intersection of curves C_0 and C_1 at the point t and e_t the ramification index of the morphism $p_{|C_1}$: $C_1 \longrightarrow p(C_1)$ at the point t. Then $I(C_0, C_1; t) = e_t$.*

Proof. Take a local coordinate x, y at the point $C_0 \cap F$ where C_0 is an x-axis and F a y-axis and moreover take a coordinate $z_1, \ldots, z_n$ at the point v where $p(F)$ is a z_1-axis. Thus we can describe the morphism p from a neighbourhood of v in S to X as $(\ldots, z_i, \ldots) = (\ldots, f_i(x, y), \ldots)$ so that $f_i(x, y)$ is a holomorphic function near a neighborfood at $C_0 \cap F$ and $f_i(0, 0) = 0$ for any i. Now since the section C_0 collapses to the point v via the morphism p, $f_i(x, 0)$ is zero and therefore for any i $f_i(x, y)$ can be written as $y^{m_i} g_i(x, y)$ where $m_1 \geq 1$ and $g_i(0, y) \neq 0$. Noting that the morphism $F \longrightarrow p(F)$ is birational and $p(F)$ is a z_1-axis we get $m_1 = 1$ and $g_1(0, 0) \neq 0$. Letting $C_1 = \{y = x^m\}$ locally, we can describe the morphism p restricted to the section C_1 as the mapping $(x^m g_1(x, x^m), x^{mm_2} g_2(x, x^m), \ldots, x^{mm_i} g_i(x, x^m), \ldots)$. Noting $g_1(0, 0) \neq 0$, we get the desired fact. q. e. d.

Thus we get 5. II.

Finally we show 5. III. Take a general point x in U. Then since the characteristic of the ground field is zero and length $X = n+1$, the morphism $Z_x \longrightarrow X$ is separable and the induced homomorphism $T_{q^{-1}(Y_{x,\mathrm{reg}})} \longrightarrow p^*T_X$ is generically surjective. Hence for a general point y in Y_x we see $p^*T_{X|L_y}$ is

$\mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus n-1}$ by 2. A. Note that $p^{-1}(x)\longrightarrow Y_x$ is a finite birational morphism and $p: Z_x-p^{-1}(x)\longrightarrow X-\{x\}$ is also a finite birational morphism by the assumption of 5. Ⅲ. Take the normalization $\bar{S}\longrightarrow p^{-1}(x)$ of $p^{-1}(x)$. Letting $j: Z_x\times_{Y_x}\bar{S}(=\bar{Z}_x)\longrightarrow Z_x$ the canonical morphism by the base change $\bar{S}\longrightarrow Y_x$, we see that j is a finite birational morphism. Hence we infer that the composite morphism jp is a birational morphism. Thus the section S induced by $\bar{S}$ in $\bar{Z}_x$ gives a rank-2 vector bundle E and its subline bundle M on $p^{-1}(x)$ with the following exact sequence:

$$(\#)\quad 0\longrightarrow M\longrightarrow E\longrightarrow \mathcal{O}\longrightarrow 0$$

where $\boldsymbol{P}(E)=\bar{Z}_x$ and $\boldsymbol{P}(\mathcal{O})=S$ canonically. Letting $h:\tilde{S}\longrightarrow S$ be a desingularization of S, we see that $h_*\mathcal{O}_{\tilde{S}}=\mathcal{O}_S$ because h is a birational morphism and S is normal. Moreover the canonical homomorphism $h^1(S,M)\longrightarrow h^1(\tilde{S},h^*M)$ is injective and h^*E splits to $\mathcal{O}\oplus h^*M$ by the argument in Proposition 2. 7. Thus we see that E splits to $\mathcal{O}\oplus M$. At the same time since $jp:\bar{Z}_x-S\longrightarrow Z-\{x\}$ is quasi-finite, M is ample. Then we get a birational morphism $\varphi:\boldsymbol{P}(E)\longrightarrow\overline{X}$ from $\boldsymbol{P}(E)$ to the normal cone $\overline{X}$ where via $\varphi:\boldsymbol{P}(E)-\boldsymbol{P}(\mathcal{O})\longrightarrow X-\{x\}$ is an isomorphism and $P(\mathcal{O})$ goes to the point x. Thus we have a canonical morphism $\sigma:\overline{X}\longrightarrow X$ which is a finite and birational morphism. Since X is smooth, σ is an isomorphism. Setting a section $\boldsymbol{P}(M)$ as D, we infer that $\varphi: D\longrightarrow\varphi(D)\ (\subset X)$ is a birational morphism. Hence we see that $(l\cdot p(D))=1$. Since Pic $X\simeq\boldsymbol{Z}$ by [W2] and $(-K_X\cdot\ell)=n+1$, $-K_X$ is $(n+1)p(D)$. Thus we are done by Theorem due to Kobayashi and Ochiai [KO].

Hence we complete the proof of 5.Ⅲ.

(5. 6) Thus we show 2) of Main Theorem.
By Proposition 1. 6 we see that length (X) is n or $n+1$. In the former case X is a smooth hyperquadric from Corollary 4. 5. In the other case if $n\geq 4$, we infer that X is a projective space by 5. Ⅰ ~ 5. Ⅲ. Moreover if $n=3$, it is proved by Corollary 2.6 in [W1] that the same conclusion holds.

§ 6. Hyperquadric (in positive characteristic)

(6.1) In this section and the last section we prove Main Theorem in positive characteristic. All the results of §1 and §2 except the ones stated below hold in positive characteristic:

Corollary 1. 4. 1, the latter part of Proposition 1. 5, 2. *A*, 3) of 2. 6, 2. 7.

What we must check is the first two facts. Thus we consider Proposition 1. 5 first.

Let $\tau:\boldsymbol{P}^1\times H\longrightarrow Z$ be a canonical projection. Since the natural morphism $q^{-1}(y)\longrightarrow pq^{-1}(y)$ is birational, so is a canonical morphism $q^{-1}(y)\longrightarrow p'q^{-1}(y)$. Thus $\tau p':\boldsymbol{P}^1\times H\longrightarrow\overline{X}$ yields a canonical morphism $\varepsilon: H\longrightarrow$

Hom $(\boldsymbol{P}^1, \overline{X})$. Consequently we have a component $\overline{H}$ of Hom $(\boldsymbol{P}^1, \overline{X})$ which contains $\varepsilon(H)$. Since the morphism j is étale and therefore $j^*T_X \simeq T_{\overline{X}}$ and moreover v^*T_X is generated by global sections for each v in H, there is a canonical isomorphism: $v^*T_X \simeq \varepsilon(v)^*T_{\overline{X}}$. Thus $\overline{H}$ is smooth and $\dim \overline{H} = h^0(v^*T_{\overline{X}}) = \dim H$. Moreover the induced isomorphism $H^0(\boldsymbol{P}^1, v^*T_X) \simeq H^0(\boldsymbol{P}^1, \varepsilon(v)^*T_{\overline{X}})$ corresponds to the homomorphism $d\varepsilon_{*,v}: T_{H,v} \longrightarrow T_{\overline{H},\varepsilon(v)}$ induced by a canonical morphism $\varepsilon: H \longrightarrow \overline{H}$. Hence we infer that $\overline{H}$ contains $\varepsilon(H)$ as an open set. Moreover a composite morphism $\tau j: \boldsymbol{P}^1 \times \overline{H} \longrightarrow X$ yields a morphism $\overline{H} \longrightarrow$ Hom $(\boldsymbol{P}^1, X)$. Consequently we have a natural morphism $\overline{\varepsilon}: \overline{H} \longrightarrow H$ so that $\varepsilon\overline{\varepsilon}: \overline{H} \longrightarrow \overline{H}$ is an identity. Hence we infer that the morphism $\varepsilon: H \longrightarrow \overline{H}$ is an isomorphism and that Y and Z are the geometric quotients of $\overline{H} \times \boldsymbol{P}^1$ by G respectively. Therefore we observe $\overline{X}$ instead of X. Moreover we show a fact corresponding to Corollary 1. 4. 1.

Proposition 6.2. *Let $\overline{X}$ be as above. If $\overline{f}$ is a projective space or a smooth hyperquadric, the étale finite morphism $j: \overline{X} \longrightarrow X$ is an isomorphism.*

Proof. By Proposition 1. 4, we get the desired fact. q. e. d.

Therefore we have only to show that $\overline{X}$ is a projective space of a smooth hyperquadric. Then without the fear of confusion we use the same notation X.

Hereafter in this section it is supposed that

$n \geq 5.$

Now we check the facts in §3 in positive characteristic. Corollary 3. 13 is the only one to consider. Then a section $p^{-1}(A)$ of 3. 9. 1 is a hypersurface in $\boldsymbol{P}^{n-1} (n \geq 5)$. Letting $S = p^{-1}(A)$, we see from Corollary 3.2 of §4 in [H] that

(6. 3) Pic$S \simeq \boldsymbol{Z}\mathcal{O}_S(1)$ and hence $H^i(S, M) = 0$ for every line bundle M on S and $1 \leq i \leq n-3$ where $\mathcal{O}_S(1) = \mathcal{O}_{\boldsymbol{P}^{n-1}}(1)_{|S}$.

Thus we get

(6. 3. 1) Corollary 3. 13 (=the splitness) holds,

Therefore results in §3 hold in positive characteristic.

Next in the remainder part of this section we show that Fano variety X with the ample vector bundle $\overset{2}{\wedge} T_X$ and length $X = n (\geq 5)$ is a smooth hyperquadric.

Take a general point x in V in 3. 14, and set the normal divisor $p(Z_x)$ as D. Different from the case in characteristic zero we show, in positive characteristic, that D is a divisor in $\boldsymbol{P}^n$ and next that X is a smooth Cartier divisor in the weighted projective space . Thus we can get the desired fact easily.

For the purpose we make a preparation.

Note that Proposition 4. 2 1) is characteristic free.

(6. 4) Pic $D \simeq \mathbf{Z}\mathcal{O}_D(S)$.

We set $\mathcal{O}_D(S)$ as L_D. Then we have

Proposition 6.5. Pic $X \simeq \mathbf{Z}L$ *with the ample line bundle* L.

Proof. Note that Wiśniewski's Theorem A in [W2] holds in positive characteristic. In fact to construct the closed subscheme F in Fano variety X induced by the extremal ray R_1, we need not use the contraction map which Wisniewski adopted in his proof in [W2]. Moreover we can check easily that any curve C in F belongs to the vector space generated by R_1. The rest of the proof of Wisniewski's Theorem holds in positive characteristic. Also see the statement of the last part in [W3]. q. e. d.

(6. 6) Moreover we show that

a canonical homomorphism Pic $X \longrightarrow$ Pic D induced by the closed embedding $i: D \longrightarrow X$ is an isomorphism if $n \geq 5$.

For the proof we use the following

Theorem (SGA2 originally or Theorem 3. 1 of Chapter IV in [H]. Let A be a complete non-singular variety and let B be a closed subscheme. Assume that

i) Leff (A, B),

ii) B meets every effective divisor on A, and

iii) $H^i(B, I^n/I^{n+1}) = 0$ for $i=1, 2$ and all $n \geq 1$ where I is the sheaf of ideals of B.

Then the natural map Pic $A \longrightarrow$ Pic B is an isomorphism.

Since D is an ample divisor in X by Proposition 6. 5, Leff (X, Y) follows from Proposition 1. 3, Theorem 1. 5 and its proof in §4 in [H]. As for iii) it suffices to show $H^i(D, rN_{D/X}) = 0$ for any positive integer r and $i=1, 2$.

For the purpose we show

Lemma 6.7. *Let* $L_S = L_{D|S}$. *Assume dim* $D = n-1 \geq 4$. *Then for every integer* r, *we have*

1) $H^1(S, rL_S) = H^2(S, rL_S) = 0$.

2) $H^i(D, rL_D) = 0$ *for* $i=1, 2$.

Proof. 1) is trivial from 6. 3.

Next we have the following exact sequence on D:

$$0 \longrightarrow \mathcal{O}_D(-S) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Tensoring rL_D we get

$$0 \longrightarrow (r-1)L_D \longrightarrow rL_D \longrightarrow rL_S \longrightarrow 0$$

From 1) we obtain a surjective homomorphim : $H^1(D, (r-1)\ LD) \longrightarrow H^1(D, rL_D)$ and an injection : $H^2(D, (r-1)L_D) \longrightarrow H^2(D, rL_D)$. Let the canonical sheaf $\omega_D = wL_D$ with an integer w by Proposition 6. 5. Thus by virtue of Serre's duality we get $H^1(D, rL_D) \simeq H^{n-2}(D, (w-r)\ L_D)$ and therefore we see that $H^{n-2}(D, (w-r)\ L_D) = 0$ for a sufficiently large integer $-r$ by Serre vanishing theorem, which yields the desired fact in case of $i=1$. The remainder case is obtained in the same way. Hence we complete the proof. q. e. d.

Thus we get

Corollary 6.8. *Assume that $n \geq 5$. A canonical homomorphism* $PicX \longrightarrow \text{Pic}\ D$ *induced by the closed embedding $i: D \longrightarrow X$ is an isomorphism. Thus $L_{|D} = L_D (= \mathcal{O}_D(S))$.*

Now take a point $\bar{A}$ in V in 3. 14 and set a normal Cartier divisor $p(Z_{\bar{A}})$ as D where $Z_{\bar{A}} \simeq \boldsymbol{P}(\mathcal{O} \oplus M)$ in 3. 13. Then we have shown that the induced morphism $p_A : Z_{\bar{A}} \longrightarrow D$ is a blow-down of $\boldsymbol{P}(\underline{\mathcal{O}})$ and D is a cone over the smooth projective variety $(\simeq S)$ with the vertex $\bar{A}$. Moreover we see from 6. 3 that the normal bundle $N_{S/D}$ is isomorphic to $\mathcal{O}_S(b)$ with $b>0$. Then we have

Proposition 6.9. b$=1$, *namely* $N_{S/D} \simeq \mathcal{O}_S(1)$.

Proof. We study the cone singularity $(D, \bar{A})$. Letting (R, M) the local ring $\mathcal{O}_{D, \bar{A}}$, $N = N_{S/D}$ and $T = \bigoplus_{t \geq 1} H^0(S, tN)$, we see that R is the localisation of T at T_+ where $T_+ = \bigoplus_{t \geq 1} H^0(S, tN)$. Thus since D is a Cartier divisor in the smooth variety X, we have $\dim H^0(S, N) \leq \dim M/M^2 \leq n$. Consequently we get $b=1$ from the following exact sequence and computation:

$$h^0(S, \mathcal{O}_S(b)) = h^0(\boldsymbol{P}^{n-1}, \mathcal{O}_{\boldsymbol{P}^{n-1}}(b)) - h^0(\boldsymbol{P}^{n-1}, \mathcal{O}_{\boldsymbol{P}^{n-1}}(b-d))$$

obtaind by the sequence:

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^{n-1}}(b-d) \longrightarrow \mathcal{O}_{\boldsymbol{P}^{n-1}}(b) \longrightarrow \mathcal{O}_S(b) \longrightarrow 0,$$

with $d = \deg S$, and

$$h^0(\mathcal{O}_{\boldsymbol{P}^m}(b)) - h^0(\mathcal{O}_{\boldsymbol{P}^m}(c)) = {}_{m+b}C_b - {}_{m+c}C_c$$

$$= \begin{cases} {}_{m+c}C_c\left(\dfrac{(m+b)\cdots(m+c+1)}{b\cdots(c+1)} - 1\right) \geq {}_{m+2}C_2 - 1 \geq m+2, & \text{when } b>c\geq 2. \\ {}_{m+b}C_b - (m+1) \geq {}_{m+2}C_2 - (m+1) = {}_{m+1}C_2 \geq m+2, & \text{when } b>c=1. \\ {}_{m+b}C_b - 1 \geq {}_{m+2}C_2 - 1 \geq m+2, & \text{when } b>1,\ c\leq 0. \end{cases}$$

q. e. d.

We have come to the final stage.

First we show

Proposition 6.10. *Let the notations be as in §. 4. Assume $n \geq 5$. Then $K_X = -nL$.*

Proof. The intersection number of the fiber of q and $p^{-1}(\bar{A})$ in $Z_{\bar{A}}$ is one. Noting that $p_{\bar{A}}$ is birational, and that $-K_X = aL$ from Proposition 6. 3, we have $n = (\ell_y \cdot -K_X) = (\ell_y \cdot aL) = a(\ell_y \cdot L_D)_D = a(\ell_y \cdot S)_D = $ a from Corollary 6. 8. q. e. d.

Finally we show that X is a quadric hypersurface.

Noting that $p^{-1}(\bar{A})$ ($\simeq$ S) is a smooth hypersurface of degree d in $\boldsymbol{P}(\Omega^1{}_{X,\bar{A}})$, we let f be a defining equation of S where $S \simeq \mathrm{Proj}\ k[x_0, \ldots, x_{n-1}]/(f)$ in P^{n-1} and the weight of $x_i = 1$ for every i. Moreover recalling that $L_S = \mathcal{O}_S(1) = N_{S/D}$ from 6. 3 and Proposition 6.9, we have, by virutue of Theorem 3.6 in [Mo 1]

Proposition 6.10. *D is a hypersurface in $\boldsymbol{P}^n$ which is isomorphic to Proj $k[x_0, \ldots, x_n]/(\bar{f})$ in $\boldsymbol{P}^n$ where the weight of $x_n = 1$, $\bar{f}$ is a homogeneous polynomial $(= x_n^d + a_{n-1}x_n^{d-1} +, \ldots, a_1x_n + \mathrm{f})$ of degree d, a_i a homogeneous polynomial of degree $d-\mathrm{i}$ in $k[x_0, \ldots, x_{n-1}]$ and $\bar{f}(x_0, \ldots, x_{n-1}, 0) = f$.*

Therefore we see that the above S is an intersection of D and a hyperplane in $\boldsymbol{P}^{n+1}$. Let $\mathcal{O}_X(D) = cL$. Then using Theorem 3. 6 in [Mol] again, we see that X is isomorphic to $\mathrm{Proj}k[x_0, \ldots, x_{n+1}]/(F)$ $(= X(F))$ in the weighted projective space $\boldsymbol{Q}(1, \ldots, 1, c)$ where F is a weighted homogeneous polynomial $(= x_{n+1}^e + b_{e-1}x_{n+1}^{e-1} + \cdots + b_1x_{n+1} + \bar{f})$ in $k[x_0, \ldots, x_{n+1}]$ of degree d $(= ce)$, b_i a homogeneous polynomial of degree $d - ic$ in $k[x_0, \ldots, x_n]$ and $F(x_0, \ldots, x_n, 0] = \bar{f}$. On the other hand we know

(6. 11) $K_X = (d - (n+1+c))L$ by virtue of Proposition 3. 3 in [Mol]

Hence combining 6. 9, 6. 11 and $d = ce$, we have $c = 1$ and $e = d = 2$. Thus we can prove that

Theorem 6.12. *Let X be a smooth projective variety. Assume that $\overset{2}{\wedge} T_X$ is ample and length$(X) = \dim \mathrm{X} \geq 5$. Then X is a hyperquadric.*

§ 7. Projective spaces (in positive characteristic)

In this section it is assumed that $n = \dim X \geq 5$.

Here we prove that if a smooth projective variety X is of length $n+1$ so that the second exterior power of T_X is ample, then X is isomorphic to $\boldsymbol{P}^n$ in positive characteristic by the same manner way as in §5. But several phenomena peculiar to positive characteristic happen. The particularly complicated one is about the separability of a canonical morphism $Z_X \longrightarrow X$. For the purpose we must show that there exists a curve ℓ_y of α-type as stated in 7. 2.

Noting that facts (5. 2. 1) ~ (5. 2. 7) for 5. I are characteristic free, we first obtain

Proposition 7.1. dim $\mathscr{C} \leq n=1$ *and* $S\mathscr{C}$ *is a proper subset in* X.

Proof. Assume $\dim \mathscr{C} \geq n$. Then as stated in 5. 2. 7 there exists the smooth variety S in $\bar{Z}_R$ induced by cuspidal points which is an ètale cover over the given variety X with the ample vector bundle $\overset{2}{\wedge} T_X$. Then we have a claim:

$$\text{Pic } S(\simeq \text{Pic } \mathscr{C}_R) \simeq \boldsymbol{Z}$$

In fact since $p_S: S \longrightarrow X$ in 5. 2. 6 is ètale, $\overset{2}{\wedge} T_X$ is also an ample vector bundle and therefore S is a Fano variety of length $n+1$. Thus we get the desired fact by [W2]. By $S \simeq \mathscr{C}_R$, we use the notation S rather than $\mathscr{C}_R$. Here recall the exact sequence in 5. 2. 7. 1:

$$0 \longrightarrow \mathscr{O}_S \longrightarrow J \longrightarrow L \longrightarrow 0$$

where L is a line bundle on S. Now letting (g_{ij}) be the transition matrix of the vector bundle J, we denote the vector bundle induced by the Frobenius morphism of S by $J^{(P)}$ whose transition matrix is $(g_{ij}{}^p)$. Moreover repeating the procedure by m-times Frobenuis maps of S, we get $J^{(P^m)}$. Since the canonical surjective morphism $P(J^{(P^m)}) \longrightarrow X$ has 1-dimensional fiber, L is not a trivial line bundle, namely L is positive or negative by virtue of Pic $S \simeq \boldsymbol{Z}$. Then since $H^1(S, L^{\otimes(-a)}) = H^{n-1}(S, K_S \otimes L^{\otimes a})$ by Serre's duality, $H^1(S, L^{\otimes(-a)})$ is 0 for a large number a. Thus we infer that $J^{(P^m)}$ splits into $\mathscr{O} \oplus L^{(P^m)} (=J')$, which implies that there is a birational but not finite morphism f from $\boldsymbol{P}(J')$ to a cone T which collapes either section $\boldsymbol{P}(\mathscr{O})$ or $\boldsymbol{P}(L^{(P^m)})$ to a vertex. Let $a: \boldsymbol{P}(J') \longrightarrow \boldsymbol{P}(J)$ be a canonical S-morphism. Thus we have three non-finite and non-constant morphisms: the $\boldsymbol{P}^1$-bundle: $\boldsymbol{P}(J') \longrightarrow S$, a morphism $a\bar{p}: \boldsymbol{P}(J') \longrightarrow X$ and a birational morphism $f: \boldsymbol{P}(J') \longrightarrow T$ and see that three line bundles on $\boldsymbol{P}(J')$ corresponding to the above morphisms are different from each other. On the other hand since Pic $\boldsymbol{P}(J') \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$, the pseudo-ample cone has two boundarys each of which corresponds to a line bundle which is neither ample nor trivial. Thus we get a contradiction.

q. e. d.

From Propositin 7. 1 the argument of 5. II says in positive characteristic

Remark 7.1.1. For a general point x in $X - S\mathscr{C}$ a canonical morphism $p_x: Z_x \longrightarrow X$ is bijective on $Z_x - p^{-1}(x)$.

To complete the proof of (5. II) In positive characteristic and to develope the argument for (5. III), we need to prove that a canonical morphism $p_x: Z_x \longrightarrow X$ is separable.

For the purpose we have the following claim:

(7. 2) Assume that $\overset{2}{\wedge} T_X$ is ample and length $(X)=n+1$. Moreover assume $n\geq 5$. Then X has a curve ℓ_y of type α. (The proof continues till 7. 5.)
In fact, if otherwise, we can assume by Proposition 1. 6 that (#) for every point y in Y, ℓ_y is of β or γ-type.

Noting that dim $\mathscr{C}\leq n-1$ from 7. 1, in order to obtain a contradiction, we divide into two cases:

(7. 2. 1) There are points y, y' in Y so that ℓ_y is β-type and $\ell_{y'}$ is γ-type.
(7. 2. 2) For every point y in Y ℓ_y is β-type.

Hereafter we prove that neither 7. 1. 2 nor 7. 2. 2 happen.

First we consider first case.

For every point y in Y, $P^*T_{X|,q^{-1}(y)}$ is a direct sum of a trivial line bundle and ample vector bundle. Hence considering the canonical homomorphism: $q^*q_*p^*\Omega^1{}_X \longrightarrow p^*\Omega^1{}_X$, we infer that $q_*p^*\Omega^1{}_X$ is a line bundle on Y and the homomorphism is injective as a vector bundle by virtue of the base change theorem. Let D be the cokernel of the homomorphism. Then D is a vector bundle of rank-$(n-1)$ on Z and for each point y in Y, $D_{|q^{-1}(y)}$ is $\mathcal{O}(3)\oplus\mathcal{O}(1)^{\oplus n-2}$ or $\mathcal{O}(2)^{\oplus 2}\oplus\mathcal{O}(1)^{\oplus n-3}$. Moreover the latter vector bundle is more general than the former. Noting that $\mathscr{C}$ is a closed subscheme of Y where for each point y in $\mathscr{C}$ $D_{|q^{-1}(y)}\simeq\mathcal{O}(3)\oplus\mathcal{O}(1)^{\oplus n-2}(=E)$ we see that the codimension of $\mathscr{C}$ in Y is not bigger than $\dim H^1(M, T_M)$ where $M=\boldsymbol{P}(E)$. For the proof see, for example, proposition 2. 3 in [S]. Thus noting that dim $H^1(M, T_M)\leq\dim H^1(M, E\otimes E^\vee)$ by virtue of Leray spectral sequence, we see that dim $H^1(M, E\otimes E^\vee)=n-2$, namely $\operatorname{codim}_Y\mathscr{C}\leq n-2$. On the other hand it is already shown that $\operatorname{codim}_Y\mathscr{C}\geq n-1$ from 7. 1. This is a contradiction. Thus we conclude that the case 7. 2. 1 does not occur.

Next we treat with the case 7. 2. 2 in 7. 3～7. 5. Since Y has no cuspidal curve, there is a point x where ($) each curve ℓ_y through the piont x is smooth from 2. C'.

Thus we fix the point x. Let us consider a morhism $g: p^{-1}(x)\longrightarrow \boldsymbol{P}(\Omega^1{}_{X,x})\simeq\boldsymbol{P}^{n-1}$ as in 3. 6. We can first check that under the case 7. 2. 2,

(7. 3) g is a finite surjective morphism. Moreover it is purely inseparable.

In fact, since dim $p^{-1}(x)=n-1$, for the former part it is sufficient to show

(7. 4) Claim: Let W be a closed curve in $p^{-1}(x)$. Assume that $g(W)$ is a point. Then there is a point $\bar{z}$ in W so that $\ell_{q(\bar{z})}$ is not smooth at the point x. (See ($))

In fact assume that for each point z in W $\ell_{q(\bar{z})}$ is smooth at the point x in X. Then we see that for points z and z' in W the curve $\ell_{q(z)}$ tangents to the other curve $\ell_{q(z')}$ at the point x. Then we can take a general hyperplane section D in X through the point x so that D intersects transversally with all $\ell_{q(z)}$

$(z \in W)$ at x. This implies that the intersection $pq^{-1}q\,(W) \cap D$ of a surface $pq^{-1}q\,(W)$ and the ample divisor D has a component which consists of one point x. This is a contradiction.

Next we show the latter part. For the purpose we have only to prove that g is generically bijective. First fix a general point x in $X - S\mathscr{C}$. Let $S := p_x^{-1}(x)$ and choose an open set U in Y_x so that $S \cap q^{-1}(U)$ is a Cartier divisor in $q^{-1}(U)$. Take the blowing-up $\sigma_x : X_x \longrightarrow X$ along the point x and let E_x the exceptional divisor in X_x via σ_x. Then by the universality of the blowing-up, we have a canonical morphism $m : q^{-1}(U) \longrightarrow X_x$ with $\sigma_X\,\mathrm{m} = p_x$. Then we see that m is injective from Remark 7. 1. 1. Note that $m_{|S \cap q^{-1}(U)}$ is equal to $g_{|S \cap q^{-1}(U)}$. Hence m is bijective.

Thus under the assumption 7. 2. 2. we get 7. 3.

Remark 7.4.1. The latter argument of 7. 4 says that

Let x be a point in $X - S\mathscr{C}$. If a canonical morphism $g_x : p^{-1}(x) \longrightarrow \boldsymbol{P}(\Omega_{X,x})$ is surjective, then g_x is generically one to one without the conditions of the types α, β, γ of $\ell_y\,(y \in Y_x)$ (from Remark 7. 1. 1).

Moreover we continue the argument to show

(7. 5) the fact 7. 3 yields a cotradiction.

By 7. 3 there is a Frobenius morphism $F : \boldsymbol{P}^{n-1} \longrightarrow \boldsymbol{P}^{n-1}$ and a purely inseparable morphism $\bar{g} : \boldsymbol{P}^{n-1} \longrightarrow p^{-1}(x) = A$ with $F = \bar{g}g$. Since for each point y in Y

$p^*T_{X|q^{-1}(y)} \simeq \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus n-3} \oplus \mathcal{O}$, p^*T_X has the following three exact sequences:

(7. 5. 1) $0 \longrightarrow T_{Z/Y} \xrightarrow{\ i\ } p^*T_X \longrightarrow$ the coker of $i\,(=M) \longrightarrow 0$,

obtained by the unramified morphism $q^{-1}(y) \longrightarrow \ell_y$ as in 3. 2.

(7. 5. 2) $0 \longrightarrow G \xrightarrow{\ j\ } p^*T_X \longrightarrow$ the coker of $j\,(=N) \longrightarrow 0$,

where G denotes a rank 2-bundle $q^*q_*(p^*T_X \otimes T^{\vee}_{Z/Y}) \otimes T_{Z/Y}$ on Z by virtue of the base change theorem of Grothendieck and

(7. 5. 3) $0 \longrightarrow$ the kernel of $k\,(=H) \longrightarrow p^*T_X \xrightarrow{\ k\ } L \longrightarrow 0$

finally in the same manner as in 7. 5. 2 where L is the dual bundle of the line bundle $q^*q_*p^*\Omega^1{}_X$.

Then we see that G is a subbundle of H and $T_{Z/Y}$ a canonical line subbundle of G. Thus restricting each exact sequence 7. 5. 1 ~ 7. 5. 3 to the fiber A and pulling back them to $\boldsymbol{P}^{n-1}$ via the morphism $\bar{g}$, we have

$$0 \longrightarrow \bar{T} \longrightarrow \mathcal{O}_{\boldsymbol{P}^{n-1}}{}^{\oplus n} \longrightarrow \bar{M} \longrightarrow 0.$$
$$0 \longrightarrow \bar{G} \longrightarrow \mathcal{O}_{\boldsymbol{P}^{n-1}}{}^{\oplus n} \longrightarrow \bar{N} \longrightarrow 0.$$
$$0 \longrightarrow \bar{H} \longrightarrow \mathcal{O}_{\boldsymbol{P}^{n-1}}{}^{\oplus n} \longrightarrow \bar{L} \longrightarrow 0.$$

Hence we have three vector bundles $\bar{T}$, $\bar{G}$ and $\bar{H}$ on $\boldsymbol{P}^{n-1}$ with $\bar{T} \subset \bar{G} \subset \bar{H}$.

The Chern polynomial of $\overline{H}$ is described as $\sum_{i=0}^{n-1}(qt)^{i}$ with a variable t and some natural integer q. On the other hand since $\overline{T}$ is a line subbundle of a rank-2 vector bundle $\overline{G}$ on $\boldsymbol{P}^{n-1}$, $\overline{G}$ splits to a sum of two line bundles. Therefore the polynomial becomes zero at two non-zero integers with the same sign, but it is impossible.

Thus we complete the proof of 7. 2.

Therefore by 7. 2 choosing a general point x in $X-S\mathscr{C}$, we have an open set U in Y_x so that for each point y ℓ_y is α-type.

Thus we get from Remark 7. 1. 1,

(7. 6) $p_x: Z_x \longrightarrow X$ is separable for a general point x in $X-S\mathscr{C}$ and therefore birational.

In fact since there is a morphism: $H_x \times p^1 \xrightarrow{F_x} Z_x \xrightarrow{p_x} X$ as stated in 1. 2. 1. P, it suffices to show that the induced morphism $H_x \times \boldsymbol{P}^1 \longrightarrow X$ is separable and therefore $H_x \times (\boldsymbol{P}^1 - \{o\}) \longrightarrow X$ is generically smooth. But since for a general point v in H_x, v^*T_X is isomorphic to $\mathscr{O}(2)\oplus\mathscr{O}(1)^{\oplus(n-1)}$, from Proposition 3 in [Mo2] it is trivial as shown in (8. 2) of [Mo2].

Moreover in the same way as in (3. 6) and (3. 7) we infer from Remark 7. 4. 1 that

(7. 7) for a general point x in $X-S\mathscr{C}$ the morphism g_x is birational.

Let $A=p^{-1}(x)$ and let us recall that A is smooth from Proposition 1. 5 and Proposition 1. 6 and therefore that A is canonically isomorphic to $Y(x)$ $(=H_x/G_0)$ from Proposition 1. 3 and 3. 6.

As the finial stage we prepare a claim to show that g is a finite morphism. First let $\mathring{A}=\{z\in A|g^{-1}g(z)$ is a finite set$\}$. Then we remark that

(7. 8) The morphism $g: A \longrightarrow \boldsymbol{P}^{n-1}$ is an open immersion on $\mathring{A}$ by virtue of Zariski Main Theorem and $\mathring{A}$ is equal to the subset $\{z\in A|g$ is isomorphism around the point $z\}$

Then we have

Proposition 7.9. $\mathring{A}=\{z\in A|_{q(z)}$ is α-type$\}$.

For the proof we have only to show that

(7. 9. 1) Let z be a point in A. Then

1) If z is in $\mathring{A}$, then $\ell_{q(z)}$ is α-type,

2) If z is not in $\mathring{A}$, then $\ell_{q(z)}$ is not α-type.

First recall the notations. Let $\phi: H_x \longrightarrow \boldsymbol{V}(\Omega_{X,x})$ be a canonical morphism (3. 6) and $\Gamma_x: H_x \longrightarrow A$ the geometric quotient by G_0 (Proposition 1. 2. P). Take a point z in A. Let v be a point of H_x with $\Gamma_x(v)=z$ and H_v a componet of $\phi^{-1}(\phi(v))$ containing the point v. Moreover let pr be the canonical projection $\boldsymbol{V}(\Omega_{X,x})-\{0\}$. Let z be in $\mathring{A}$. Since g is an isomorphism at the point z, we infer that the composite morphism $\Gamma_x g: H_x \longrightarrow \boldsymbol{P}(\Omega_{X,x})$ is smooth at the point v, $\Gamma_x g=\phi pr$ and therefore ϕ is smooth at the point v. Thus we see that

H_v is smooth at the point v and therefore the Zariski tangent space ZT_v of H_v at the point v is isomorphic to k, because automorphism group of $\boldsymbol{P}^1$ fixing two points is of 1-dimension as stated just after 8. 1 in [Mo2]. Moreover by virtue of the deformation theory of Grothendieck ZT_v is isomorphic to $H^0(\boldsymbol{P}^1, v^*T_X\otimes\mathcal{O}(-2))$. Thus we get the former.

Next if v is β or γ-type, ZT_v is a 2-dimensional vector space by the above argument. Hence g is not an isomorphism at $\Gamma_x(v)$.

Thus we complete the proof of Proposition 7. 9.

Since x is contained in $X-S\mathscr{C}$, x is a smooth point or a nodal point of a rational curve $\ell_{q(z)}$ for z in W.

Hence we finally show that

(7. 10) g is a finite morphism.

In fact assume that g is not finite. By 7. 9, we see that $A-\mathring{A}$ consist of β or γ-type and it is of at least one dimension. First since $\dim\mathscr{C}\leq n-1$ by Proposition 7. 1, there are at most finite rational curves of γ-type passing through a general point in X. Thus we infer that $A-\mathring{A}$ contains a point of β-type. Now we claim that $\operatorname{codim}_A(A-\mathring{A})\leq 2$, namely $\dim(A-\mathring{A})\geq n-2$. In fact the deformation theory says that $\operatorname{codim}_A(A-\mathring{A})\leq\dim H^1(P^1, F\otimes F')=2$ with $F=\mathcal{O}(2)^{\oplus 2}\oplus\mathcal{O}(1)^{\oplus n-3}\oplus\mathcal{O}$ as stated in the argument in 7. 2. On the other hand since the set

$\{z\in A-\mathring{A}\mid\ell_{q(z)}$ has a nodal point $x\}$

is at most finite from 2. B, there is a projective curve W in $A-\mathring{A}$ so that each curve $\ell_{q(z)}$ $(z\in W)$ is smooth at the point x. But this contradicts Claim 7. 4. Hence we get 7. 10.

Therefore we see that

$p^{-1}(x)\longrightarrow\boldsymbol{P}(\Omega^1{}_{X,x})$ $(\simeq\boldsymbol{P}^{n-1})$ is a finite birational morphism and therefore an isomorphism, which means that for each point y in Y_x, ℓ_y is of α-type. By virtue of the proof of [Mo2] we have $X\simeq\boldsymbol{P}^n$.

Hence we get

Theorem 7.11. *Let X be a smooth projective variety defined over the algebraically closed field whose characteristic is arbitrary. Assume that $\overset{2}{\wedge}T_X$ is ample and length $(X)=\dim X+1\geq 6$. Then $X\simeq\boldsymbol{P}^n$.*

Combining 6. 12 and Theorem 7. 11, we can show 1) of Main Theorem.

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