J. Math. Kyoto Univ. (JMKYAZ)
35-3 (1995) 341-355

Regularized determinants of Laplacians for hermitian line bundles over projective spaces

By

LIN WENG

Let $\mathbf{P}^n(\mathbf{C})$ be an n-dimensional projective space. If we put the Fubini-Study metric on the tangent bundle, we know that the eigenvalues of the Laplacians for any $(0, q)$ C^∞-forms could be given by using the method from representation theory, as we know that there is a connection between the Laplacians and the Casimir operators [4]. In other words, this means that we know the eigenvalues of Laplacians for the $(0, q)$ C^∞-forms with coefficients being the sections of the structure sheaf of $\mathbf{P}^n$. But there are other kinds of line sheaves on $\mathbf{P}^n$, say, the tautological line sheaf $\mathcal{O}_{\mathbf{P}^n}(1)$. So we also want to know what should be the eigenvalues of corresponding Laplacians. In fact, this is an open problem, at least, to me. In this paper, we will give the regularized determinant of the Laplacians for line sheaves, such as $\mathcal{O}_{\mathbf{P}^n}(m)$, i.e., we will give the regularized infinite product of the non-zero eigenvalues of the Laplacians. Put this in a more fashionable language, we will give the analytic torsions associated to hypersurface line sheaves with the induced metric from the Fubini-Study one.

The point here is that we do not know the eigenvalues. How could we give the associated analytic torsions? The idea goes as follows: We consider arithmetic model $\mathbf{P}^n_{\mathbf{Z}}$ over $\mathbf{Z}$ first. Then use the arithmetic Riemann-Roch formula of Gillet and Soulé to find the value of the first arithmetic Chern class for the associated determinant line sheaf with the Quillen metric. Thus, finally, after deleting the L^2-norm contribution, we could get the analytic torsions in question.

This process looks very good. But the point is that it contains very delicate calculation for the terms involved. Thus, finally, we could use the Stirling numbers to express such a quantity. Here we will mainly follow the work given by Gillet, Soulé and Zagier [3].

1. The arithmetic Riemann-Roch formula

Let $f : X \longrightarrow Y$ be a smooth morphism of arithmetic varieties, then we have the following

Communicated by Prof. K. Ueno, July 26, 1993
Revised April 20, 1995

Arithmetic Riemann-Roch formula ([2]). *For any hermitian vector sheaf* $(\mathscr{E}, \rho)$ *on* X, *if we put a hermitian metric* ρ_f *on the relative tangent sheaf* $\mathscr{T}_f$, *we have*

$$c_1^{\mathrm{Ar}}(\det(R^{\cdot}f_*\mathscr{E}), h_{\mathrm{Q}}(\rho, \rho_f)) = (\mathrm{f}_*(\mathrm{ch}_{\mathrm{Ar}}(\mathscr{E}, \rho)\,\mathrm{Td}_{\mathrm{Ar}}(\mathscr{T}_f, \rho_f)))^{(1)},$$

where $h_{\mathrm{Q}}(\rho, \rho_f)$ *denotes the associated Quillen metric on the determinant line sheaf with respect to the metrics* ρ *and* ρ_f.

For our purpose, let $f: \mathbf{P}^n_{\mathbf{Z}} \longrightarrow \mathrm{Spec}(\mathbf{Z})$ be the projective space of dimension n over $\mathbf{Z}$. On $\mathbf{P}^n(\mathbf{C})$, we shall take the Fubini-Study metric ρ_{FS}. Hence, we are given not only a hermitian metric on the tautological line sheaf, but a hermitian metric on the tangent sheaf, and all of them are invariant under SU $(n+1)$.

Obviously, we have the following results for $\mathscr{O}_{\mathbf{P}^n}(m)$ with $m \geq 0$:

$$R^i f_* \mathscr{O}_{\mathbf{P}^n}(m) = \begin{cases} H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(m)), & \text{if } i=0, \\ 0, & \text{if } i>0. \end{cases}$$

Thus, the arithmetic Riemann-Roch formula above gives the following relation:

$$c_1^{\mathrm{Ar}}(\det H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(m)), h_{\mathrm{Q}}(\rho_{\mathrm{FS}})) = (\mathrm{f}_*(\mathrm{ch}_{\mathrm{Ar}}(\mathscr{O}_{\mathbf{P}^n}(m), \rho_{\mathrm{FS}})\,\mathrm{Td}_{\mathrm{Ar}}(\mathscr{T}_{\mathrm{F}}, \rho_f)))^{(1)},$$

In the sequel, we will calculate both sides separeterly.

2. The left hand side

By definition, in our case the Quillen metric may be given as

$$h_{\mathrm{Q}} = h_{\mathrm{L}^2} \exp(-\tau(\mathscr{O}_{\mathbf{P}^n}(m), \rho_{\mathrm{FS}})),$$

where h_{L^2} denotes the natural L^2-metric on the determinant $\det H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(m))$, which, in turn, is defined as follows [5]: First, on $A^{0,q}(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(m))$, we have the L^2-metric defined by

$$\langle \eta, \eta' \rangle_{L^2} = \int_{\mathbf{P}^n} \langle \eta(x), \eta'(x) \rangle \frac{\omega^n}{n!},$$

where ω is the curvature form associated to the Fubini-Study metric. Then the space $H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(m))$ is canonically isomorphic to the corresponding harmonic forms, hence it has a natural induced L^2-metric. Finally, we take the determinant metric; on the other hand, $\tau(\mathscr{O}_{\mathbf{P}^n}(m), \rho_{\mathrm{FS}})$ is usually called the analytic torsion associated to the line bundle $\mathscr{O}_{\mathbf{P}^n}(m)$, with respect to the Fubini-Study metric, which is what we try to compute, and may be defined as follows [6]: With respect to the Fubini-Study metric, the operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $A^{0,q}(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(m))$ has non-negative eigenvalues. In particular, if we only consider the non-negative eigenvalues, we could have strictly positive eigenvalues, say,

$$\lambda(n, m, q)_1 \leq \lambda(n, m, q)_2 \leq \lambda(n, m, q)_3 \ldots,$$

indexed in increasing order, and taking into account multiplicaties. It is well-known that the associated zeta function

$$\zeta_{n,m,q}(s) := \sum_{i\geq 1} \lambda(n, m, q)_i^{-s}$$

is intitially defined for Re $(s) > n$ and could be meromorphically extended to the whole complex plane. In particular, the resulting function is holomorphic at $s=0$. Hence, it makes sense for us to talk about $\zeta'_{n,m,q}(0)$. With this, the analytic torsion is given by

$$\sum_{q=0}^{n} (-1)^q q \zeta'_{n,m,q}(0).$$

Therefore, if we choose the coordinates $z_0, \ldots, z_n$, then we have

$$\begin{aligned} &c_1^{\mathrm{Ar}}(\det H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)), h_{\mathrm{Q}}(\rho_{\mathrm{FS}})) \\ &\quad = -\frac{1}{2} \log h_{L^2}(\alpha(p_1) \wedge \ldots \wedge \alpha(p_N)) + \frac{1}{2} \tau(\mathcal{O}_{\mathbf{P}^n}(m), h_{\mathrm{Q}}(\rho_{\mathrm{FS}})), \end{aligned}$$

where p_i denotes all monic polymonials of degree m of z_j,

$$\alpha : S^m \mathscr{E} \longrightarrow f_* \mathcal{O}_{\mathbf{P}^n}(m)$$

is the canonical isomorphism, and N is the dimension of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m))$. Thus note that z_j may be chosen as orthonormal basis with respect to L^2 metric, we see that if $x = z_0^{\alpha_0} \otimes \cdots \otimes z_n^{\alpha_n}$, then

$$\| x \|^2 = \frac{\alpha_0! \cdots \alpha_n!}{m!}$$

In particular, we know that

$$\| \alpha(x) \|_{L^2}^2 = \frac{(\alpha_0! \cdots \alpha_n!)}{(n+m)!},$$

as we choose the density to be $\frac{\omega^n}{n!}$. Therefore, we see that

$$\begin{aligned} &c_1^{\mathrm{Ar}}(\det H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)), h_{\mathrm{Q}}(\rho_{\mathrm{FS}})) \\ &\quad = -\frac{1}{2} \log \prod_{(\alpha_0, \cdots, \alpha_n) \in \mathbf{Z}_{>0}^{n+1}, \Sigma_j \alpha_j = m} \frac{\alpha_0! \cdots \alpha_n!}{(n+m)!} + \frac{1}{2} \tau(\mathcal{O}_{\mathbf{P}^n}(m), h_{\mathrm{Q}}(\rho_{\mathrm{FS}})). \end{aligned}$$

3. The right hand side

We need to compute

$$(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,\mathrm{Td}_{\mathrm{Ar}}(\mathcal{T}_f,\ \rho_f)))^{(1)}.$$

For doing so, we divide it into several pieces. First, by definition, we know that

$$\mathrm{Td}_{\mathrm{Ar}}(\mathcal{E},\ \rho)=\mathrm{td}_{\mathrm{Ar}}(\mathcal{E},\ \rho)\ (1+a(R(\mathcal{E}))),$$

where $R(\mathcal{E})$ is an additive characteristic class in even cohomology, which, in turn, has a harmonic representation, and then $a(R)$ means $(0,\ R)$ in the arithmetic Chow ring. More precisely, R is associated with the following power series

$$R(x)=\sum_{k\mathrm{odd},\geq 1}\left(2\zeta'(-k)+\zeta(-k)\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)\right)\frac{x^k}{k!},$$

with $\zeta(s)$ the Riemann zeta function and $\zeta'(s)$ its derivative.

On the other hand, we have the canonical exact sequence $\mathbf{E}_n$

$$\mathbf{E}_n: 0\longrightarrow \mathcal{O}_{\mathbf{P}^n}\longrightarrow \mathcal{O}_{\mathbf{P}^n}(1)^{n+1}\longrightarrow \mathcal{T}_{\mathbf{P}^n}\longrightarrow 0.$$

With the Fubini-Study metrics for the terms in $\mathbf{E}_n$, by the property of $\mathrm{td}_{\mathrm{Ar}}$, we see that

$$\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f,\ \rho_{\mathrm{FS}})=\mathrm{td}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(1),\ \rho_{\mathrm{FS}})^{n+1}-a(\mathrm{td}_{\mathrm{BC}}(\mathbf{E}_n,\ \rho_{\mathrm{FS}})).$$

Therefore, we see that

$$\begin{aligned}
&(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,\mathrm{Td}_{\mathrm{Ar}}(\mathcal{T}_f,\ \rho_f)))^{(1)}\\
&\quad=(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(1),\ \rho_{\mathrm{FS}})^{n+1}))^{(1)}\\
&\qquad-(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,a(\mathrm{td}_{\mathrm{BC}}(\mathbf{E}_n,\ \rho_{\mathrm{FS}}))))^{(1)}\\
&\qquad+(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f,\ \rho_f)\,a(R(\mathcal{T}_f))))^{(1)}\\
&\quad=(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(1),\ \rho_{\mathrm{FS}})^{(n+1)}))^{(1)}\\
&\qquad-(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,a(\mathrm{td}_{\mathrm{BC}}(\mathbf{E}^n,\ \rho_{\mathrm{FS}}))))^{(1)}\\
&\qquad+(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m))\,\mathrm{td}(\mathcal{T}_f)\,R(\mathcal{T}_f))).
\end{aligned}$$

Here, in the last equation, we also use the property of arithmetic intersection theory [7].

Lemma. *We have the following relations:*

1. *Let* $t_{n,m}:=-2(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,\mathrm{td}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(1),\ \rho_{\mathrm{FS}})^{n+1}))^{(1)}$. *Then* $t_{n,m}$ *is the product of* $-\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j}$ *and the coefficient of* x^{n+1} *in* $e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}$;
2. *Let* $s_{n,m}:=-2(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m),\ \rho_{\mathrm{FS}})\,a(\mathrm{td}_{\mathrm{BC}}(\mathbf{E}_n,\ \rho_{\mathrm{FS}}))))^{(1)}$. *Then* $s_{n,m}$ *is the coefficient of* x^n *in* $e^{mx}\int_0^1\frac{\phi(t)-\phi(0)}{t}dt$, *where*

$$\phi(t)=\left(\frac{1}{tx}-\frac{e^{-tx}}{1-e^{-tx}}\right)\left(\frac{x}{1-e^{-x}}\right)^{n+1}.$$

3. *Let* $r_{n,m} := -2\,(f_*(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(m))\,\mathrm{td}(\mathcal{T}_f)\,R(\mathcal{T}_f)))$. *Then* $r_{n,m}$ *is the coefficient of* x^n *in*

$$(n+1)e^{\mathrm{mx}}\left(\frac{x}{1-e^{-x}}\right)^{n+1}R(x).$$

Rroof. Let $x=c_1(\mathcal{O}_{\mathbf{P}^n}(1))\in H^2(\mathbf{P}^n)$, and $x_{\mathrm{Ar}}=c_{\mathrm{Ar}}(\mathcal{O}_{\mathbf{P}^n}(1),\rho_{\mathrm{FS}})$ be the first Chern class and the first arithmetic Chern class of the tautological line bundle with the Fubini-Study metric. Then the result 1 and 3 come from the facts that

a
$$\int_{\mathbf{P}^n(\mathbf{C})} x^k=\begin{cases}1, & \text{if } k=n,\\ 0, & \text{otherwise;}\end{cases}$$

b
$$f_*(x_{\mathrm{Ar}}^k)^{(1)}=\begin{cases}\frac{1}{2}\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j}, & \text{if } k=n+1,\\ 0, & \text{otherwise;}\end{cases}$$

For 2, we need to apply the method of Bott and Chern to calculate the classical Bott-Chern secondary characteristic form associated with $\mathbf{E}_n$ with respect to the Fubini-Study metric [1], which was first carried out in [3] with the following form: If we have a short exact sequence

$$\mathcal{E}.:0\longrightarrow\mathcal{E}_1\longrightarrow\mathcal{E}_2\longrightarrow\mathcal{E}_3\longrightarrow 0$$

with hermitian metric ρ_i on $\mathcal{E}_i$, We may write $\mathcal{E}_2$ as the orthogonal direct sum of $\mathcal{E}_1$ and $\mathcal{E}_1^{\perp}$, which is C^∞ isomorphic to $\mathcal{E}_3$. The curvature of $\mathcal{E}_2$ $\left(\text{multiplied by } \frac{i}{2\pi}\right)$ decomposes as a 2 by 2 matrix $K=(K_{ij})$. Let $K_{\mathcal{E}_i}$ be the curvature of $\mathcal{E}_i$ multiplied by $\frac{i}{2\pi}$. Let $\mathrm{Td}(A)=\det\frac{A}{1-e^{-A}}$ for any square matrix A. For every $t\in[0,1]$, let $\phi(t)$ be the coefficient of λ in

$$\mathrm{Td}\begin{pmatrix} tK_{11}+(1-t)K_{\mathcal{E}_1}+\lambda & tK_{12}\\ K_{21} & tK_{22}+(1-t)K_{\mathcal{E}_3}\end{pmatrix}.$$

and

$$I=\int_0^1\frac{\phi(t)-\phi(0)}{t}dt.$$

Then we know that I gives the classical Bott-Chern secondary characteristic from associated with $(\mathcal{E}.,\rho.)$. In particular, in our case, K is equal to the product of ω and the identity matrix. Furthermore, $\mathcal{E}_1$ has rank 1 and $K_{\mathcal{E}_1}=0$. Therefore, we see that $\phi(t)$ is the coefficient of λ in

$$\mathrm{Td}\begin{pmatrix} t\omega+\lambda & 0 \\ 0 & t\boldsymbol{\omega}+(1-t)K_{g_3} \end{pmatrix}.$$

Thus, by the fact that Td is multiplicative, we have

$$\phi(t)=\frac{d}{d\lambda}\left(\frac{t\omega+\lambda}{1-e^{-t\omega-\lambda}}\right)\Big|_{\lambda=0}\mathrm{Td}\,(t\omega+(1-t)K_{g_3}).$$

From here, note that

$$\int_{\mathbf{P}^n(\mathbf{C})}\omega^k=\begin{cases}1, & \text{if } k=n,\\ 0, & \text{otherwise,}\end{cases}$$

we easily have 2.

4. The expression for analytic torsions

In this section, we will use the results in the previous two sections to give the final expression of analytic torsions in question by some technical combinatorial equalities.

For later use, we introduce the following notation.

Let

$$R_1(x)=\sum_{m\mathrm{ood},m\geq 1}2\zeta'(-m)\frac{x^m}{m!},$$

$$R_2(x)=\sum_{m\mathrm{ood},m\geq 1}\zeta(-m)\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)\frac{x^m}{m!},$$

$$\phi_1(t,x)=\frac{1}{tx}-\frac{e^{-tx}}{1-e^{-tx}},$$

$A_{n,m}$ the coefficients of x^{n+1} in

$$-\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j}e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1},$$

$B_{n,m}$ the coefficients of x^n in

$$e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}\int_0^1\frac{\phi_1(t)-\phi_1(0)}{t}dt,$$

$C_{n,m}$ the coefficient of x^n in

$$(n+1)e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}R_2(x),$$

and $D_{n,m}$ the coefficient of x^n in

$$(n+1)e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}R_1(x).$$

With this, the results in the previous section may be stated as a

Lemma. *The right hand side of the arithmetic Riemann-Roch formula for $\mathcal{O}_{\mathbf{P}^n}(m)$ with respect to the Fubini-Study metric is given by* $-\frac{1}{2}(A_{n,m}-B_{n,m}+C_{n,m}+D_{n,m})$.

Next we study the properties of $A_{n,m}$, $B_{n,m}$, $C_{n,m}$, $D_{n,m}$ respectively.

First, we consider $D_{n,m}$. Let $P_{n,m,1}(k)$ be the coefficient of x^n in

$$2(n+1)e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}\sum_{m\text{odd},m\geq 1}k^m\frac{x^m}{m!},$$

If for any power series

$$P(x)=\sum_{n\geq 0}c_nx^n,$$

we define

$$\zeta P=\sum_{n\geq 0}c_n\zeta'(-n),$$

we easily have the following

Lemma. $D_{n,m}=\zeta P_{n,m,1}$.

Therefore, we need to discuss the property of $P_{n,m,1}$.

By definition, for any power series $f(x)$, we define $2f^{\text{odd}}(x)=f(x)-f(-x)$, then $P_{n,m,1}(z)$ is the coefficient of x^n in

$$2(n+1)e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}(e^{zx})^{\text{odd}(z)}$$

which is the odd function of the coefficient of x^n in

$$2(n+1)e^{(m+z)x}\left(\frac{x}{1-e^{-x}}\right)^{n+1}$$

with respect to z. Thus we have $P_{n,m,1}(k)$ is the odd part of the residue of

$$2(n+1)\frac{e^{(m+k)x}}{(1-e^{-x})^{n+1}}dx,$$

which, in turn, is the odd part of the residue of

$$2(n+1)\frac{1}{y^n(1-y)^{m+k+1}}dy$$

with $y=1-e^{-x}$. Therefore, we see that

$$P_{n,m,1}(k)=2(n+1)\left(\frac{(n+m+k)!}{n!(m+k)!}\right)^{\text{odd}(k)}.$$

Hence, we have

Theorem. $D_{n,m}=2(n+1)\zeta\left(\frac{(n+m+k)!}{n!(m+k)!}\right)^{\mathrm{odd}(k)}.$

Next, let us study the term $A_{n,m}$.

First, note that

$$\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j}=(n+1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-n=(n+1)(\sigma_{n+1}-1)$$

with

$$\sigma_n=1+\frac{1}{2}+\cdots+\frac{1}{n}.$$

So $A_{n,m}$ is the coefficient of x^{n+1} in

$$(n+1)(1-\sigma_{n+1})e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1},$$

which is the residue of

$$(n+1)(1-\sigma_{n+1})\frac{e^{mx}}{x(1-e^{-x})^{n+1}}dx.$$

Thus by taking the transform $y=1-e^{-x}$, we see that $A_{n,m}$ is the residue of

$$(n+1)(1-\sigma_{n+1})\frac{1}{x(1-y)^{m+1}}dy,$$

which, in turn, is the coefficient of y^n in

$$(n+1)(1-\sigma_{n+1})\frac{1}{x(1-y)^{m+1}}.$$

Thus if we define $\alpha_{n,m}$ by the generating function

$$\sum_{n\geq 0}\alpha_{n,m}y^n=\frac{y}{x(1-y)^{m+1}}$$

with $y=1-e^{-x}$, we have the following

Theorem. $A_{n,m}=(n+1)(1-\sigma_{n+1})\alpha_{n,m}.$

Now we study the terms $B_{n,m}$ and $C_{n,m}$. We start with the following

Lemma. *Let $a_{n,m}(f)$ be the coefficient of x^n in*

$$e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}f(x),$$

and $y=1-e^{-x}$. Then

$$\sum_{n\geq 1} a_{n,m}(f) y^n = \frac{f(x)}{(1-y)^{m+1}}.$$

Proof. This is a direct consequence of a standard technique from generating functions.

From this, we easily see that the following theorem holds.

Theorem. *With the notation as above,*

a.
$$\sum_{n\geq 1} B_{n,m} y^n = \frac{1}{(1-y)^{m+1}} \int_0^1 \frac{\phi_1(t)-\phi_1(0)}{t} dt;$$

b.
$$\sum_{n\geq 1} \frac{C_{n,m}}{n+1} y^n = \frac{1}{(1-y)^{m+1}} R_2(x).$$

In particular, we have

$$\sum_{n\geq 1} \frac{B_{n,m}}{n+1} y^{n+1} = \int_0^y \frac{1}{(1-z)^m} \int_0^1 \frac{\phi_1(t)-\phi_1(0)}{t} dt\, dz.$$

Thus, if we define the coefficients β_l, σ_l and α_n by the generating functions

$$\sum_{l\geq 0} \beta_l x^l = x \frac{1-y}{y},$$

$$\sum_{n\geq 0} \sigma_n y^n = \frac{x}{1-y},$$

$$\sum_{n\geq 0} \alpha_n y^n = \frac{1}{x} \frac{y}{1-y}.$$

Then we first have

$$\sum_{n\geq 1} \frac{B_{n,m}}{n+1} y^{n+1} = \frac{1}{(1-y)^m} \int_0^y \varphi(z)\, dz,$$

where

$$\varphi(x) = -\int_0^1 \sum_{l\geq 2} \zeta_l t^{l-2} x^{l-1} dt.$$

In particular, we may restate the theorem above as the following

Theorem′. *With the notation as above, we have*

a.
$$\sum_{n\geq 1} \frac{B_{n,m}}{n+1} y^{n+1} = -\frac{1}{(1-y)^m} \sum_{k\geq 2} \frac{\beta_k}{k(k-1)} x^k;$$

b.
$$\sum_{n\geq 1} \frac{C_{n,m}}{n+1} y^n = \frac{1}{(1-y)^{m+1}} \sum_{k\geq 2} \sigma_{k-1} \beta_k x^{k-1};$$

c.
$$\frac{A_{n,m}}{n+1} = \alpha_{n+1,m} (\sigma_{n+1} - 1), \text{ where}$$

$$\sum_{n\geq 0}\alpha_{n,m}y^n=\frac{y}{(1-y)^{m+1}}\frac{1}{x}.$$

Next, we need to get rid of the factor $\frac{1}{(1-y)^m}$. For this, we introduce the following

Theorem″. *With the notation as above, we have*

a. $$\sum_{n\geq 1}\frac{B_{n,m}}{(n+1)(n+2)\cdots(n+m+1)}y^{n+m+1}=-\sum_{k\geq 2}\frac{\beta_k}{(k-1)k(k+1)\cdots(k+m)}x^{k+m};$$

b. $$\sum_{n\geq 1}\frac{C_{n,m}}{(n+1)(n+1)\cdots(n+m+1)}y^{n+m+1}=\sum_{k\geq 2}\frac{\sigma_{k-1}\beta_k}{k(k+1)\cdots(k+m)}x^{k+m};$$

c. $$\sum_{n\geq 0}\frac{\alpha_{n,m}}{(n+1)\cdots(n+m)}y^{n+m}=\sum_{k\geq 0}\frac{x^{k+m}}{(k+m)!}.$$

The proof can be given by taking the integration.

Now we introduce the Stirling number by the following process: Let

$$y=1-e^{-x}=\sum_{l\geq 1}\frac{(-1)^{l-1}}{l!}x^l,\quad x=\log(1-y)=\sum_{n\geq 1}\frac{1}{n}y^n.$$

Define coefficients $s_1(n,l)$, $s_2(l,n)$, n, $l\geq 0$ by generating functions

$$x^l=\sum\nolimits_{n=0}^{\infty}s_1(n,l)y^n,\quad y^n=\sum_{l=0}^{\infty}s_2(l,n)x^l.$$

Therefore $\{s_1(n,l)\}_{n,l\geq 0}$, $\{s_2(l,n)\}_{l,n\geq 0}$ are mutually inverse infinite triangular metrices, and usually we call $\frac{n!}{l!}s_1(n,l)$ and $\frac{l!}{n!}s_2(l,n)$ the Stirling numbers of the first and the second kind (up to sign), which is the number of permutations of $\{1,2,\ldots,n\}$ having exactly l cycles and the number of partition of $\{1,2,\ldots,n\}$ into exactly n non-empty subsets, respectively. Then we have

$$\begin{aligned}&\sum_{n\geq 1}\frac{B_{n,m}}{(n+1)(n+2)\cdots(n+m+1)}y^{n+m+1}\\&=-\sum_{n\geq 2}\frac{\beta_k}{(k-1)k(k+1)\cdots(k+m)}\sum_{n=0}^{\infty}s_1(n,k+m)y^n\\&=-\sum_{n=1}^{\infty}\sum_{k\geq 2}\frac{\beta_k}{(k-1)k(k+1)\cdots(k+m)}s_1(n,k+m)y^n\end{aligned}$$

Therefore,

$$\begin{aligned}B_{n,m}=&-(n+1)(n+2)\cdots(n+m+1)\\&\sum_{k=2}^{n+1}\frac{\beta_k}{(k-1)k(k+1)\cdots(k+m)}s_1(n+m+1,k+m).\end{aligned}$$

Similarly,

$$C_{n,m}=(n+1)(n+1)\cdots(n+m+1)\sum_{k=2}^{n+1}\frac{\sigma_{k-1}\beta_k}{k(k+1)\ldots(k+m)}s_1(n+m+1,k+m).$$

And

$$\alpha_{n,m}=(n+1)\cdots(n+m)\sum_{k\geq 1}^{n}\frac{1}{(k+m+1)!}s_1(n+m,k+m).$$

Thus, put all terms together, we have the following

Main Theorem. *Put the Fubini-Study metric on the tangent bundle of* $\mathbf{P}^n(\mathbf{C})$ *and the associated metric on the tautological line bundle. Then for the induced metric on the supersurface line bundle* $\mathcal{O}_{\mathbf{P}^n(\mathbf{C})}(m)$, $m\geq 0$, *the associated analytic torsion is given as follows:*

$$\begin{aligned}&\tau(\mathcal{O}_{\mathbf{P}^n(\mathbf{C})}(m),\rho_{FS})\\&=(\sigma_{n+1}-1)(n+1)(n+1)\cdots(n+m)\sum_{k\geq 1}^{n}\frac{1}{(k+m+1)!}s_1(n+m,k+m)\\&\quad-(n+1)(n+2)\cdots(n+m+1)\sum_{k=2}^{n+1}\frac{\beta_k}{(k-1)k(k+1)\cdots(k+m)}\\&\quad-(n+1)(n+1)\cdots(n+m+1)\sum_{k=2}^{n+1}\frac{\sigma_{k-1}\beta_k}{k(k+1)\cdots(k+m)}\\&\quad-2(n+1)\zeta\left(\frac{(n+m+k)!}{n!(m+k)!}\right)^{odd(k)}\\&\quad+\sum_{(\alpha_0,\ldots,\alpha_n)\in\mathbf{Z}_{\geq 0}^{n+1},\Sigma_j\alpha_j=m}\log\frac{(\alpha_0!\cdots\alpha_n!)}{(n+m)!},\end{aligned}$$

where

$$\sigma_n=1+\frac{1}{2}+\cdots+\frac{1}{n},$$

and $s_1(n,l)$ is the twisted Stirling number given by the generating function

$$x^l=\sum_{n=0}^{\infty}s_1(n,l)y^n$$

with $y=1-e^{-x}$.

We end this general discussion by the following remarks: From the computation above, we actually could make a guess about the eigenvalues and multiplicities for the associated representations of $SU(n+1)$ on $C^{0,q}(\mathbf{P}^n,\mathcal{O}_{\mathbf{P}^n}(m))$, by further studying the meanning of the quantities from both sides of the arithmetic Riemann-Roch formula stated above. But such a guess should depend on a certain highly non-trivial combinatorial calculation. We will discuss this matter elsewhere.

5. Some examples

In this part, we give some down-to-earth examples.

5.1. The situation for projective line. We will use two methods to give the associated analytic torsion.

5.1.1. A classical method. On $\mathbf{P}^1$, we have the tautological line bundle $\mathcal{O}(1)$. We put the Fubini-Study metric on it. Then it naturally induces metrics on $\mathcal{O}(m)$ for all integer m. $\mathbf{P}^1$ may be realized as the quotient space $SU(2)/S(U(1)\times U(1))$. Let Λ be the fundamental weight of the Lie group of $SU(2)$. Let I_k be the irreducible representation of $SU(2)$ with highest weight $k\Lambda$ for $k\in\mathbf{Z}_{\geq 0}$. Then by the Peter-Weyl theorem, we know that $\Gamma(T^{*(0,1)}\mathbf{P}^1\otimes\mathcal{O}(m))$ contains the L^2-dense subspace

$$\bigotimes_{k\geq 0}I_{|m+2|+2k}.$$

Furthermore, the eigenvalues of the complex Laplacian for $\mathcal{O}(m)$ are given by $k(k+m+1)$ on I_{m+2k} for $k\geq 1$, $m\geq -1$ and $k(k-m-1)$ on $I_{-m-2+2k}$ for $k>0$, $k<-1$.

Note that by the Serre duality, it is suffcient to give the analytic torsion for m positive. So in the sequel, we always assume that m is positive.

Note that the dimension of I_{m+2k} is nothing but $m+2k+1$, we know that the associated zeta function is given by

$$\sum_{k\geq 1}\frac{2k+m+1}{k^s(k+m+1)^s}.$$

Therefore, by a standard technique, we see that the analytic torsion is given by the following

Theorem. *Let $\mathcal{O}(m)$, $m\geq 0$ be a line bundle over the projective line $\mathbf{P}^1$. Put the Fubini-Study metric on them. Then the analytic torsion for the complex Laplacian is given by*

$$-2\sum_{k=0}^{m}(m-k)\log(k+1)+(m+1)\log(m+1)!+4\zeta'(-1)-(m+1)^2/2.$$

The proof may be given as follows: We first separate $2k+m+1$ as $2k$ and $m+1$. Then we calculate the corresponding contributions to the analytic torsion. Thus by using the Riemann zeta function and its generalized version — the Hurwitz zeta function, we may easily have the result.

5.1.2. The arithmetic geometry method. Now we use the arithmetic method to calculate the above analytic torsion.

By the result in Chapter 4, we see that

$$A_{1,m}=\text{the coefficient of } x^2 \text{ in } -e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{1+1}$$

$=$the coefficient of x^2 in

$$-\left(1+mx+\frac{m^2}{2}x^2+\text{higher degree terms}\right)\left(1+x+\frac{5}{12}x^2+\text{higher degree terms}\right)$$

$=-(m^2/2+m+5/12)$.

$B_{1,m}=$the coefficient of x in

$$-e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{1+1}\int_0^1\left(\sum_{l\geq 1}\beta_l t^{l-1}x^{l-1}-\beta_1\right)\frac{dt}{t}$$

$=$the coefficient of x in

$-(1+(m+1)x+\text{higher degree terms})(\beta_2 x+\text{higher degree terms})$

$=-1/12$,

where if $T=1-e^{-y}$, we define β_l be the generating function

$$y\frac{1-T}{T}=\sum_{l\geq 0}\beta_l y^l.$$

$C_{1,m}=$the coefficient of x in $e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{1+1}R_2(x)$

$=$the coefficient of x in

$(1+(m+1)x+\text{higher degree terms})(\zeta(-1)x+\text{higher degree terms})$

$=2\zeta(-1)=-1/6$.

$D_{1,m}=$the coefficient of x in $2e^{mx}\left(\frac{x}{1-e^{-x}}\right)^{1+1}R_1(x)$

$=$the coefficient of x in

$2(1+(m+1)x+\text{higher degree terms})(2\zeta'(-1)x+\text{higher degree terms})$

$=4\zeta'(-1)$.

Therefore, the analytic torsion is given by

$$-\log\frac{\prod_{a=0}^m a!(m-a)!}{(m+1)!}-\frac{1}{2}(m+1)^2+4\zeta'(-1)$$

which is the same as what we obtained from the classical method. In fact, it is sufficient to show that

$$-2\sum_{k=0}^m(m-k)\log(k+1)+(m+1)\log(m+1)!=-\log\frac{\prod_{a=0}^m a!(m-a)!}{(m+1)!}.$$

So we need to prove that $2\sum_{k=0}^m(m-k)\log(k+1)=\log\prod_{a=0}^m a!(m-a)!$. Now the right hand side is just $2\sum_{a=0}^m\log a!$, so we may use the induction on m to give the result.

5.2. The situation for $\mathbf{P}^2$. From now on, we assume that m is a positive integer. We want to give the analytic torsion for all line bundles over $\mathbf{P}^2$ with the metrics induced from the Fubini-Study metric. For this, as above, by using the result in Chapter 4, we have the following

Proposition. *With the same notation as above, we have*

$$A_{2,m}=-\frac{5}{2}\left(\frac{3}{8}+m+\frac{3}{4}m^2+\frac{1}{6}m^3\right);$$

$$B_{2,m}=-\frac{1}{12}\left(\frac{3}{2}+m\right);$$

$$C_{2,m}=-\frac{1}{4}\left(\frac{3}{2}+m\right);$$

$$D_{2,m}=6\left(\frac{3}{2}+m\right)\zeta'(-1).$$

Thus, we get the following

Theorem. *The analytic torsion for $(\mathcal{O}(m),\rho_m)$, $m\geq 0$, over $\mathbf{P}^2$ with respect to the Fubini-Study metric is given by*

$$-\sum_{(a,b,c)\in\mathbf{Z}^3_{\geq 0},a+b+c=m}\log\frac{a!b!c!}{(2+m)!}-\frac{19}{16}-\frac{8}{3}m-\frac{15}{8}m^2-\frac{5}{12}m^3+(9+6m)\zeta'(-1).$$

5.3. The situation for $\mathbf{P}^3$. Similarly, by using the result in Chapter 4 with a direct calculation, we get the following

Proposition. *With the same notation as above, we have*

$$A_{3,m}=-\frac{13}{3}\left(\frac{251}{720}+m+\frac{11}{12}m^2+\frac{1}{3}m^3+\frac{1}{24}m^4\right);$$

$$B_{3,m}=-\left(\frac{329}{2160}+\frac{1}{6}m+\frac{1}{24}m^2\right);$$

$$C_{3,m}=-\left(\frac{649}{1080}+\frac{2}{3}m+\frac{1}{6}m^2\right);$$

$$D_{3,m}=\frac{4}{3}\zeta'(-3)+\left(4m^2+16m+\frac{44}{3}\right)\zeta'(-1).$$

Hence, we have the following

Theorem. *The analytic torsion for $(\mathcal{O}(m),\rho_m)$ with $m\geq 0$ over $\mathbf{P}^3$ is given by*

$$-\sum_{(a,b,c,d)\in\mathbf{Z}^4_{\geq 0},a+b+c+d=m}\log\frac{a!b!c!d!}{(3+m)!}$$
$$-\frac{13}{72}m^4-\frac{13}{9}m^3-\frac{295}{72}m^2-\frac{29}{6}m-\frac{2116}{1080}$$
$$+\frac{4}{3}\zeta'(-3)+\left(4m^2+16m+\frac{44}{3}\right)\zeta'(-1).$$

Department of Mathematics
National University of Singapore

Refferences

[1] R. Bott and S. S. Chern, Hermitian vector bundles and the equidistribution of the zeros of their holomorphic cross-sections, Acta Math., **114** (1968), 71-112.

[2] H. Gillet and CH. Soulé, An arithmetic Riemann-Roch theorem, Invent. Math., 1992.

[3] H. Gillet and CH. Soulé and D. Zagier, Analytic torsion and the arithmetic Todd genus, Topology, **30** (1991).

[4] A. Ikeda and Y. Taniguchi, Spectra and eigenforms of the Laplacian on S^n and $P^n(C)$, Osaka J. Math., **15**-3 (1978), 515-546.

[5] D. Quillen, Determinants of Cauchy-Riemann operators over Riemann surfaces, Funct. Anal. Appl. (1985), 31-34.

[6] D. B. Ray and I. M. Singer, Analytic torsion for complex manifolds, Ann. of Math., **98** (1973), 154-177.

[7] L. Weng, Bott-Chern secondary characteristic objects and arithmetic Riemann-Roch theorem, manuscript, Singapore, 1992.