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Ergodic decomposition of probability measures on the configuration space

Dedicated to Professor Takeshi Hirai on his 60th birthday

By

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Introduction

Let X be a locally compact space which satisfies the second countable axiom. Any locally finite subset of X is called a configuration in X, that is a subset $\gamma \subset X$ such that $\gamma \cap K$ is finite for any compact set $K \subset X$. Let us denote by Δ_X the space of all infinite and by B_X the space of all finite configurations in X, and set $\Gamma_X := \Delta_X \cup B_X$. We introduce a measurable structure $\mathscr{C}$ on Γ_X such that $\mathscr{C}$ is a minimal σ-algebra with which all the functions, $\gamma \in \Gamma_X \to |\gamma \cap B| \in \mathbf{R}$ are measurable, where B runs through all the Borel sets in X and $|\gamma \cap B|$ is the number of the set $\gamma \cap B$. It is known that $(\Gamma_X, \mathscr{C})$ is a standard space (See, theorem 1.2 in [3]) and hence any probability measure μ on $(\Gamma_X, \mathscr{C})$ is decomposed into conditional probability measures with resect to any sub-σ-field of $\mathscr{C}$. The subject of this paper are two kinds of measures on $(\Gamma_X, \mathscr{C})$ with well known properties and their ergodic decompositions. The first one is a $\mathrm{Diff}_0 X$-quasi-invariant probability measure μ, where X is a connected para-compact C^∞-manifold and $\mathrm{Diff}_0 X := \{\psi \mid \psi\text{: diffeomorphism on } X \text{ with compact support}\}$. In 1975, Vershick-Gel'fand-Graev introduced elementary representations U_μ generated by these μ's and discussed fully their interesting properties in [5]. In particular they showed that U_μ is irreducible if and only if μ is ergodic. Thus our subject correspondes to an irreducible decomposition of U_μ. It will be shown in section 1 that an ergodic decomposition of $\mathrm{Diff}_0 X$-quasi-invariant probability measure is actually possible.

The second one is a consideration of Gibbs measures μ having been discussed in great detail in statistical mechanics. An ergodic decomposition of such μ relative to the tail-σ-field leads us to a remarkable fact that there exist typical extremal measures which are regarded as a base on a convex set formed by such μ's. These contents will be discussed in section 2. In both of section 1 and section 2, we denote a σ-finite non atomic Borel measure on X by m. The direct product m^n of n copies of m is naturally regarded as a measure on $\tilde{X}^n := \{(x_1, \cdots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}$ and thus an image measure $p_n m^n$ is obtained by the natural map $p_n\colon (x_1, \cdots, x_n) \in \tilde{X}^n \to \{x_1, \cdots, x_n\} \in B_X^n := \{\gamma \in \Gamma_X \mid |\gamma| = n\}$. We denote it by $m_{X,n}$.

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1. Ergodic decomposition of $\mathrm{Diff}_0 X$-quasi-invariant measures

1.1. Basic notion and result. As before let X be a d-dimensional C^∞-manifold and m be a locally Euclidean Borel measure on X with smooth densities. And we associate with each $\psi \in \mathrm{Diff}_0 X$ a transformation T_ψ on Γ_X such that $T_\psi\{x_1,\cdots,x_n,\cdots\} = \{\psi(x_1),\cdots,\psi(x_n),\cdots\}$. A probability measure μ on $(\Gamma_X, \mathscr{C})$ is said to be $\mathrm{Diff}_0 X$-quasi-invariant, if and only if $T_\psi \mu \simeq \mu$ for all $\psi \in \mathrm{Diff}_0 X$, where the symbol $\simeq$ means the equivalence relation of measures. Moreover μ is said to be $\mathrm{Diff}_0 X$-ergodic, if $\mu(A) = 1$ or 0 provided that $\mu(T_\psi(A) \ominus A) = 0$ for all $\psi \in \mathrm{Diff}_0 X$. It is an aim of the present section that after suitably setting a measure space (Λ, λ) we decompose μ such as $\mu = \int_\Lambda \mu_l \lambda(dl)$ with $\mathrm{Diff}_0 X$-ergodic measures μ_l. Besides it is to be desired that μ_l's are mutually singular.

Now let μ be a $\mathrm{Diff}_0 X$-quasi-invariant probability measure and put $\alpha := \mu(B_X)$ and $\beta := \mu(\Delta_X)$. Then we have $\mu = \alpha\mu_1 + \beta\mu_2$, where $\mu_1(E) := \mu(E \cap B_X)/\alpha$ and $\mu_2(E) := \mu(E \cap \Delta_X)/\beta$ for all $E \in \mathscr{C}$. Furthermore we put $\alpha_n := \mu_1(B_X^n)$. Then μ_1 is decomposed as

$$\mu_1 = \sum_{n=0}^{\infty} \alpha_n \mu_{1,n}, \tag{1.1}$$

where $\mu_{1,n}(E) = \mu_1(E \cap B_X^n)/\alpha_n$ for all $E \in \mathscr{C}$. Since B_X^n $(n = 0, 1, \cdots)$ is a $\mathrm{Diff}_0 X$-invariant set, so $\mu_{1,n}$ is a $\mathrm{Diff}_0 X$-quasi-invariant measure. Here we give the following theorem.

Theorem 1.1 ([5]). *Any non zero σ-finite* $\mathrm{Diff}_0 X$*-quasi-invariant measure on B_X^n is equivalent to $m_{X,n}$.*

The proof will be seen in a discussion for the proof of Lemma 1.2 which will be stated later on.

In anyway, it follows immediately from the above theorem that any non zero σ-finite $\mathrm{Diff}_0 X$-quasi-invariant measure on B_X^n is $\mathrm{Diff}_0 X$-ergodic. Hence $\mu_{1,n}$ is ergodic and (1.1) is actually an ergodic decomposition of μ_1. Next let us observe μ_2, so we shall assume that $\mu(\Delta_X) = 1$ from now on. Here we introduce a set $\tilde{X}^\infty := \{(x_1,\cdots,x_n,\cdots) \in X^\infty \mid x_i \neq x_j$ for all $i \neq j$ and the set $\{x_1,\cdots,x_n,\cdots\}$ has no accumulation points$\}$ and consider a cross section s of the natural map

$$p: (x_1,\cdots,x_n,\cdots) \in \tilde{X}^\infty \to \{x_1,\cdots,x_n,\cdots\} \in \Delta_X.$$

Let us take and fix an increasing sequence $\{Y_n\}$ of connected open sets with compact closure such that $\bar{Y}_n \subset Y_{n+1}$ and $Y_n \uparrow X$. Then there exists a measurable section s possessing the following property (P) with this $\{Y_n\}$.

(P) *If we have $|\gamma \cap Y_1| = k_1$, $|\gamma \cap (Y_2 \setminus Y_1)| = k_2, \cdots, |\gamma \cap (Y_n \setminus Y_{n-1})| = k_n, \cdots$ for $\gamma \in \Delta_X$, then the first k_1 elements of $s(\gamma)$ are in $\gamma \cap Y_1$, the next k_2 element of $s(\gamma)$ are in $\gamma \cap (Y_2 \setminus Y_1)$ and so on.*

We call it to be admissible. Notice that $s(E)$ is a Borel set in $\tilde{X}^{\infty}$ for any $E \in \mathscr{C} \cap \Delta_X$, because s is one to one and measurable, and the space $(\Gamma_X, \mathscr{C})$ is standard.

For measures on the natural measurable space $(\tilde{X}^{\infty}, \mathfrak{B}(\tilde{X}^{\infty}))$ we also obtain the notion of $\mathrm{Diff}_0 X$-quasi-invariance and ergodicity with maps $\tilde{T}_{\psi}: (x_1, \cdots x_n, \cdots) \in \tilde{X}^{\infty} \to (\psi(x_1), \cdots, \psi(x_n), \cdots) \in \tilde{X}^{\infty}$ for $\psi \in \mathrm{Diff}_0 X$. Here we shall define a new measure $\tilde{\mu}$ on $\mathfrak{B}(\tilde{X}^{\infty})$ from a given probability measure μ taking the above measurable admissible cross section s:

$$\tilde{\mu}(E) := \sum_{\sigma \in \mathfrak{S}_{\infty}} c(\sigma)(s\mu)\sigma(E) \tag{1.2}$$

for all $E \in \mathfrak{B}(\tilde{X}^{\infty})$, where $\mathfrak{S}_{\infty}$ is the set of all finite permutations on $\mathbf{N}$, $\{c(\sigma)\}_{\sigma \in \mathfrak{S}_{\infty}}$ is a fixed positive sequence such that $\sum_{\sigma \in \mathfrak{S}_{\infty}} c(\sigma) = 1$ and $(s\mu)\sigma$ is a image measure of μ by the map,

$$\gamma \in \Delta_X \xrightarrow{s} s(\gamma) = (x_1, \cdots, x_n, \cdots) \xrightarrow{\sigma} s(\gamma)\sigma := (x_{\sigma(1)}, \cdots, x_{\sigma(n)}, \cdots) \in \tilde{X}^{\infty}.$$

Theorem 1.2 (section 2 in [5]). *Under the above notations,*

(*a*) *μ is $\mathrm{Diff}_0 X$-quasi-invariant if and only if so is $\tilde{\mu}$.*

(*b*) *μ is $\mathrm{Diff}_0 X$-ergodic if and only if so is $\tilde{\mu}$.*

(*c*) *If a Borel probability measure μ_1 on $\tilde{X}^{\infty}$ is $\mathfrak{S}_{\infty}$-quasi-invariant and $\mu_1(\bigcup_{\sigma \in \mathfrak{S}_{\infty}} s(\Gamma_X)\sigma) = 1$, then $\tilde{p\mu_1}$ is equivalent to μ_1.*

(*d*) *A $\mathrm{Diff}_0 X$-quasi-invariant probability measure $\tilde{\mu}$ on $(\tilde{X}^{\infty}, \mathfrak{B}(\tilde{X}^{\infty}))$ is $\mathrm{Diff}_0 X$-ergodic if and only if $\tilde{\mu}(A) = 1$ or 0 for any $A \in \mathfrak{B}_{\infty}$, where $\mathfrak{B}_{\infty} = \bigcap_{n=1}^{\infty} q_n^{-1}(\mathfrak{B}(\tilde{X}^{\infty}))$ and $q_n: (x_1, \cdots, x_n, \cdots) \in \tilde{X}^{\infty} \to (x_{n+1}, \cdots, x_m, \cdots) \in \tilde{X}^{\infty}$. $\mathfrak{B}_{\infty}$ is called the tail-σ-field.*

1.2. $\mathrm{Diff}_0 Y$-quasi-invariant measure on B_Y and one parameter group of $\mathrm{Diff}_0 X$. In this paragraph the letter Y stands for connected open subset in X with compact closure, and we observe subgroups of diffeomorphisms on X whose support is contained in Y which will be denoted by $\mathrm{Diff}_0 Y$. Using the sequence $\{Y_n\}$ already stated in the admissible cross section, we have,

$$\mathrm{Diff}_0 X = \bigcup_{n=1}^{\infty} \mathrm{Diff}_0 Y_n. \tag{1.3}$$

Now from a trivial equality, $\gamma = (\gamma \cap Y) \cup (\gamma \cap Y^c)$ we can identify Γ_X with a product space B_Y and Γ_{Y^c}. Put $\pi_Y: \gamma \in \Gamma_X \to \gamma \cap Y \in B_Y$ and $\pi_{Y^c}: \gamma \in \Gamma_X \to \gamma \cap Y^c \in \Gamma_{Y^c}$. Since B_Y and Γ_{Y^c} are naturally regarded as subspaces of Γ_X so the measurable structure $\mathscr{C}_Y$ and $\mathscr{C}_{Y^c}$ are induced from $\mathscr{C}$ respectively. It is easy to see that the above identification $\Gamma_X \simeq B_Y \times \Gamma_{Y^c}$ is an isomorphism with the measurable structure $\mathscr{C}$ and $\mathscr{C}_Y \times \mathscr{C}_{Y^c}$. By the way probability measures ν on $(B_Y, \mathscr{C}_Y)$ naturally arises, if we decompose $\mathrm{Diff}_0 Y$-quasi-invariant probability measures with respect to sub-σ-field $\pi_{Y^c}^{-1}(\mathscr{C}_{Y^c})$. So we shall observe such ν's, especially with $\nu(B_Y^n) = 1$ for a while. As before we define

$$\tilde{\nu}(E) = \sum_{\sigma \in \mathfrak{S}_n} (s_n \nu)\sigma(E) \tag{1.4}$$

for all $E \in \mathfrak{B}(\tilde{Y}^n)$, taking a measurable cross section s_n of the natural map

$p_n: (y_1, \cdots, y_n) \in \tilde{Y}^n \to \{y_1, \cdots, y_n\} \in B_Y^n$. Then ν and $\tilde{\nu}$ have the same kind of quasi-invariance. Now $\tilde{Y}^n$ is covered by countable sets of the form $O_1^m \times \cdots \times O_n^m$ $(m = 1, \cdots)$, where O_i^m is an open set with compact closure which is diffeomorphic to $\mathbf{R}^d$ by a map ψ_i^m and $O_i^m \cap O_j^m = \phi$ for $i \neq j$. Since $\mathrm{Diff}_0 Y$ acts on $\tilde{Y}^n$ transitively, it follows that there exists an at most countable set $\{\varphi_k\}$ such that $\tilde{Y}^n = \bigcup_{k=1}^{\infty} \varphi_k(O_1^m \times \cdots \times O_n^m)$ for each m. Hence $\tilde{\nu}(O_1^m \times \cdots \times O_n^m) > 0$, if $\tilde{\nu}$ is quasi-invariant under a group H_n generated by such φ_k's. Here we shall prepare some basic lemma.

Lemma 1.1. *There exists a one parameter group* π_i^l $(i = 1, \cdots, d, l = 1, \cdots)$ *of* $\mathrm{Diff}_0 \mathbf{R}^d$ *which satisfies* $\pi_i^l(t)(\xi) = (\xi_1, \cdots, \overset{i}{\check{\xi_i + t}}, \cdots, \xi_d)$ *for all* $\xi = (\xi_1, \cdots, \xi_d) \in \mathbf{R}^d$ *and* $t \in \mathbf{R}$ *such that* $\mathrm{Max}_{1 \leq t \leq d} |\xi_i| < l$ and $|t| < l$.

Proof. For the existence of such one parameter group, we solve the following differential equation (1.5) with a function f_l of C^∞-class on $\mathbf{R}$ such that $f_l(s) = 1$ on $|s| \leq 2l$ and $f_l(s) = 0$ on $|s| \geq 3l$.

$$
(1.5) \qquad \begin{cases} \dfrac{d\mathbf{x}}{dt} = f_l(x_1) \cdots f_l(x_d) \mathrm{e}_i \\ \\ \mathbf{x}(0) = \xi, \end{cases}
$$

where $\mathrm{e}_i = (0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$. Then the solution $\mathbf{x}(t, \xi)$ of (1.5) gives directly a desired diffeomorphism.

Let G_d^0 be a group generated by the one parameter groups π_i^l $(i = 1, \cdots, d, l = 1, \cdots)$. Then it is easily seen that

Proposition 1.1. *For any* $l \in \mathrm{N}$ *and for any* $\tau = (t_1, \cdots, t_d) \in \mathbf{R}^d$, *there exists* $\psi \in G_d^0$ *such that*

$$
\psi(\xi) = \xi + \tau \text{ for all } \xi \text{ with } \mathrm{Max}_{1 \leq i \leq d} |\xi_i| < l.
$$

Proposition 1.2. *Any σ-finite* G_d^0*-quasi-invariant Borel measure on* $\mathbf{R}^d$ *is equivalent to the Lebesgue measure.*

Now let us pull back each element of G_d^0 by the maps ψ_i^m $(i = 1, \cdots, d, m = 1, \cdots)$ and extend it to the element of $\mathrm{Diff}_0 Y$. Then considering the restriction of $\tilde{\nu}$ to $O_1^m \times \cdots \times O_n^m$, we deduce that

Lemma 1.2. *In* $\mathrm{Diff}_0 Y$ *there exist one parameter groups* $\pi_{i,n}$ $(i = 1, \cdots)$ *and a countable subgroup* H_n *such that the following are equivalent for any Borel probability measure* ν *on* B_Y^n.

(*a*) ν *is quasi-invariant under the groups* $\pi_{i,n}$ $(i = 1, \cdots)$ *and* H_n.

(*b*) ν *is equivalent to* $m_{X,n}$.

(*c*) ν *is* $\mathrm{Diff}_0 Y$*-quasi-invariant.*

The following lemma is an immediate consequence of the above lemma by letting n run from 0 to ∞.

Lemma 1.3. *In* $\mathrm{Diff}_0 Y$ *there exists one parameter subgroups* $\pi_{i,Y}$ $(i = 1,\cdots)$ *and a countable group* H_Y *such that the following are equivalent for any Borel probability measure* ν *on* B_Y.
(*a*) ν *is quasi-invariant under the groups* $\pi_{i,Y}$ $(i = 1,\cdots)$ *and* H_Y.
(*b*) ν *is* $\mathrm{Diff}_0 Y$*-quasi-invariant.*

Next let us decompose a probability measure μ on $(\Gamma_X, \mathscr{C})$ into the regular conditional probability measures $\{\mu^\gamma\}_{\gamma\in\Gamma_{Y^c}}$ on $(\mathfrak{B}_Y, \mathscr{C}_Y)$ with respect to the map π_{Y^c} which satisfy

(1.6) $\mu^\gamma(A)$ *is a* $\mathscr{C}_{Y^c}$*-measurable function of* $\gamma\in\Gamma_{Y^c}$ *for each fixed* $A\in B_Y$, *and*

(1.7) $$\mu(A \times B) = \int_B \mu^\gamma(A)\pi_{Y^c}\mu(d\gamma)$$

for all $A\in\mathscr{C}_Y$ and $B\in\mathscr{C}_{Y^c}$.

Lemma 1.4. *Under the above notations, the following are equivalent.*
(a) μ *is* $\mathrm{Diff}_0 Y$*-quasi-invariant.*
(b) μ^γ *is* $\mathrm{Diff}_0 Y$*-quasi-invariant for* $\pi_{Y^c}\mu$*-a.e.* γ.

Proof. There is nothing to prove "(b) implies (a)". Let us see the converse relation. For this we calculate $T_\psi\mu(A \times B)$, $\psi\in\mathrm{Diff}_0 Y$ in two ways. The first one is,

$$T_\psi\mu(A \times B) = \mu(T_\psi^{-1}(A) \times B) = \int_B T_\psi\mu^\gamma(A)\pi_{Y^c}\mu(d\gamma). \tag{1.8}$$

And the other one is,

$$T_\psi\mu(A \times B) = \int_B\int_A \frac{dT_\psi\mu}{d\mu}(\gamma')\mu^\gamma(d\gamma')\pi_{Y^c}\mu(d\gamma). \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$T_\psi\mu^\gamma(\cdot) = \int_{(\cdot)} \frac{dT_\psi\mu}{d\mu}(\gamma')\mu^\gamma(d\gamma') \tag{1.10}$$

for $\pi_{Y^c}\mu$-*a.e.* γ, and thus we have

$$T_\psi\mu^\gamma \simeq \mu^\gamma \tag{1.11}$$

for $\pi_{Y^c}\mu$-*a.e.* γ. Here we take an arbitrary one parameter group $\{\psi_t\}_{t\in\mathbf{R}}$ of $\mathrm{Diff}_0 Y$ and set

$$\Pi := \{(t, \gamma)\in\mathbf{R} \times \Gamma_{Y^c} \mid T_{\psi_t}\mu^\gamma \simeq \mu^\gamma\} \text{ and}$$
$$\Pi_0 := \{(t, \gamma)\in\mathbf{R} \times \Gamma_{Y^c} \mid T_{\psi_t}\mu^\gamma(\cdot) = \int_{(\cdot)} \frac{dT_{\psi_t}\mu}{d\mu}(\gamma')\mu^\gamma(d\gamma')\}.$$

Then Π_0 is a $\mathfrak{B}(\mathbf{R}) \times \mathscr{C}_{Y^c}$-measurable subset of Π and for any fixed t the $\mathbf{R}$-section Π_0^t determined by t has full measure for $\pi_{Y^c}\mu$. Thus by virtue of Fubini's theorem the Γ_{Y^c}-section Π_0^γ determined by γ has full Lebesgue measure for $\pi_{Y^c}\mu$-a.e. γ. So the Γ_{Y^c}-section Π^γ is Lebesgue measurable and it is a subgroup of $\mathbf{R}$ with positive measure for $\pi_{Y^c}\mu$-a.e. γ. This implies that $\Pi^\gamma = \mathbf{R}$ for $\pi_{Y^c}\mu$-a.e. γ. Now consider groups $\pi_{i,Y}$ $(i = 1, \cdots)$ and H_Y stated in Lemma 1.3. Applying the above arguments to these subgroups, we conclude that μ^γ is $\mathrm{Diff}_0 Y$-quasi-invariant for $\pi_{Y^c}\mu$-a.e. γ.

From (1.3), Lemma 1.3 and Lemma 1.4 we have the following theorem.

Theorem 1.3. *In* $\mathrm{Diff}_0 X$, *there exist one parameter groups* π_i $(i = 1, \cdots)$ *which are subgroups of* $\mathrm{Diff}_0(Y_{k_i})$ *and a countable group* G_0 *such that the following are equivalent for any probability measure* μ *on* $(\Gamma_X, \mathscr{C})$.
(*a*) μ *is* $\mathrm{Diff}_0 X$*-quasi-invariant.*
(*b*) μ *is quasi-invariant under the groups* π_i $(i = 1, \cdots)$ *and* G_0.

1.3. Ergodic decomposition of $\mathrm{Diff}_0 X$-quasi-invariant measure. Let μ be a $\mathrm{Diff}_0 X$-quasi-invariant probability measure on $(\Gamma_X, \mathscr{C})$ with $\mu(\Delta_X) = 1$ and $\tilde{\mu}$ be the Borel measure on $\tilde{X}^\infty$ defined by (1.2). We decompose $\tilde{\mu}$ into conditional probability measures $\{\tilde{\mu}^x\}_{x \in \tilde{X}^\infty}$ with respect to the tail-σ-field $\mathfrak{B}_\infty$. Namely,

(1.12) $\tilde{\mu}^x(B)$ *is a* $\mathfrak{B}_\infty$*-measurable function of* $x \in \tilde{X}^\infty$ *for each fixed* $B \in \mathfrak{B}(\tilde{X}^\infty)$, *and*

(1.13) $$\tilde{\mu}(A \cap B) = \int_A \tilde{\mu}^x(B)\tilde{\mu}(dx)$$

for all $A \in \mathfrak{B}_\infty$ and $B \in \mathfrak{B}(\tilde{X}^\infty)$. Since the measurable space $(\tilde{X}^\infty, \mathfrak{B}(\tilde{X}^\infty))$ is standard, and $\mathfrak{B}_\infty$ is an intersection of a decreasing sequence of the countably generated σ-fields $q_n^{-1}(\mathfrak{B}(\tilde{X}^\infty))$, so by the well known fact, (For example see theorem 2.3 in [2])

(1.14) $$\exists A_1 \in \mathfrak{B}_\infty \textit{ with } \tilde{\mu}(A_1) = 1 \textit{ s.t.}, \forall x \in A_1,\ \tilde{\mu}^x(\cdot) = 1 \textit{ or } 0 \textit{ on } \mathfrak{B}_\infty.$$

Furthermore it follows easily from the construction of $\tilde{\mu}$,

(1.15) $\exists A_2 \in \mathfrak{B}_\infty$ *with* $\tilde{\mu}(A_2) = 1$ *s.t.*, $\forall x \in A_2$, $\tilde{\mu}^x$ *is* $\mathfrak{S}_\infty$*-quasi-invariant and* $\tilde{\mu}^x(\bigcup_{\sigma \in \mathfrak{S}_\infty} s(\Gamma_X)\sigma) = 1$.

Consequently putting $\mu^{[x]} := p\tilde{\mu}^x$, we have $\mu^{[x]} \simeq \tilde{\mu}^x$ for all $x \in A_2$ by virtue of (c) in Theorem 1.2. Next we have for each fixed $\psi \in \mathrm{Diff}_0 X$, $T_\psi \tilde{\mu}^x \simeq \tilde{\mu}^x$ for $\tilde{\mu}$-a.e. x, because every set in $\mathfrak{B}_\infty$ is $\mathrm{Diff}_0 X$-invariant. So using Theorem 1.3 and proceeding similar manner with the proof of Lemma 1.4, we deduce that

(1.16) $$\exists A_3 \in \mathfrak{B}_\infty \textit{ with } \tilde{\mu}(A_3) = 1 \textit{ s.t.}, \forall x \in A_3,\ \mu^{[x]} \textit{ is } \mathrm{Diff}_0 X\textit{-quasi-invariant.}$$

Thus we have,

(1.17) $$\forall x \in A_1 \cap A_2 \cap A_3,\ \mu^{[x]} \textit{ is } \mathrm{Diff}_0 X\textit{-ergodic},$$

by virtue of (c) and (d) in Theorem 1.2. Since

$$\mu(s^{-1}(A_1 \cap A_2 \cap A_3)) = \textstyle\sum_{\sigma \in \mathfrak{S}_\infty} c(\sigma)(s\mu)\sigma(A_1 \cap A_2 \cap A_3) = \tilde{\mu}(A_1 \cap A_2 \cap A_3) = 1$$

so the following result is obtained.

Theorem 1.4. *Let μ be a* $\mathrm{Diff}_0 X$*-quasi-invariant probability measure on $(\Gamma_X, \mathscr{C})$ with $\mu(\Delta_X) = 1$. Then there exists a family of* $\mathrm{Diff}_0 X$*-ergodic probability measures $\{\mu^\gamma\}_{\gamma \in \Delta(X)}$ such that*

(*a*) *$\mu^\gamma(B)$ is a $s^{-1}(\mathfrak{B}_\infty)$-measurable function of $\gamma \in \Delta_X$ for each fixed $B \in \mathscr{C}$, and*

(*b*) $\mu(B \cap s^{-1}(A)) = \int_{s^{-1}(A)} \mu^\gamma(B)\mu(d\gamma)$ *for all $B \in \mathscr{C}$ and $A \in \mathfrak{B}_\infty$.*

Proof. For it we have only to put $\mu^\gamma := \mu^{[s(\gamma)]}$ if $\gamma \in s^{-1}(A_1 \cap A_2 \cap A_3)$ and $\mu^\gamma := \theta$, otherwise, where θ is some definite $\mathrm{Diff}_0 X$-ergodic probability measure on $(\Gamma_X, \mathscr{C})$.

We wish to rewrite this decomposition in a somewhat elegant style which is independent of the admissible sections. For this let us put

$$\mathfrak{A}_\infty := \{B \in \mathscr{C} \mid T_\psi B = B \text{ for all } \psi \in \mathrm{Diff}_0 X\}.$$

Then we have $s^{-1}(\mathfrak{B}_\infty) \subset \mathfrak{A}_\infty$, as is easily seen. Moreover for the μ in Theorem 1.4

$$\mu(A \ominus \tilde{A}) = 0 \text{ for all } A \in \mathfrak{A}_\infty, \text{ where } \tilde{A} := \{\gamma \in \Delta_X \mid \mu^\gamma(A) = 1\}.$$

Because,

$$\mu(B \cap \tilde{A}) = \int_{\tilde{A}} \mu^\gamma(B)\mu(d\mu)$$

for all $B \in \mathscr{C}$ by virtue of Theorem 1.4, while

by virtue of Theorem 1.4, while

$$\mu(B \cap A) = \int_{\Delta_X} \mu^\gamma(B)\mu^\gamma(A)\mu(d\gamma) = \int_{\tilde{A}} \mu^\gamma(B)\mu(d\gamma).$$

Theorem 1.5. *Let μ be a* $\mathrm{Diff}_0 X$*-quasi-invariant probability measure on $(\Gamma_X, \mathscr{C})$. Then there exists a family of probability measures $\{\mu^\gamma\}_{\gamma \in \Gamma_X}$ on $(\Gamma_X, \mathscr{C})$ such that*

(*a*) *μ^γ is* $\mathrm{Diff}_0 X$*-ergodic for each $\gamma \in \Gamma_X$,*

(*b*) *$\mu^\gamma(B)$ is an $\mathfrak{A}_\infty$-measurable function of γ for each fixed $B \in \mathscr{C}$*

(*c*) $\mu(A \cap B) = \int_A \mu^\gamma(B)\mu(d\gamma)$ *for all $A \in \mathfrak{A}_\infty$ and $B \in \mathscr{C}$.*

Proof. First we divide μ into μ_1 and μ_2 as in the first place of this section and decompose μ_1 into $\mu_{1,n}$ according to (1.1). Further we decompose μ_2 into $\{\mu_2^\gamma\}_{\gamma \in \Delta_X}$ as in Theorem 1.4. Next we define $\{\mu^\gamma\}_{\gamma \in \Gamma_X}$ such that $\mu^\gamma = \mu_2^\gamma$ for $\gamma \in \Delta_X$ and $\mu^\gamma = \mu_{1,n}$ for $\gamma \in B_X^n$. Then the result easily follows from what we stated.

Lemma 1.5. *For any* $\mathrm{Diff}_0 X$-quasi-*invariant probability measures* μ *and* ν, *the following are equivalent.*
(*a*) ν *is absolutely continuous with* μ.
(*b*) *There exists* $A \in \mathfrak{A}_\infty$ *such that* "$\nu(B) = 0$ *if and only if* $\mu(A \cap B) = 0$".

Proof. We have only to check the implication "(a) implies (b)". From the assumption there exists $A_0 \in \mathscr{C}$ such that

$$\text{“}\nu(B) = 0 \text{ if and only if } \mu(A_0 \cap B) = 0\text{”}.$$

Thus A_0 must satisfy $\mu(A_0 \ominus T_\psi(A_0)) = 0$ for all $\psi \in \mathrm{Diff}_0 X$. It follows from the above theorem that $\mu^\gamma(A_0 \ominus T_\psi(A_0)) = 0$ for μ-a.e. γ. Here let us take an arbitrary one parameter group $\{\psi_t\}_{t \in \mathbf{R}}$ contained in some $\mathrm{Diff}_0 Y$. Then $\mu^\gamma(A_0 \ominus T_{\psi_t}(A_0))$ is a $\mathfrak{B}(\mathbf{R}) \times \mathfrak{A}_\infty$-measurable function of $(t, \gamma) \in \mathbf{R} \times \Gamma_X$, which is easily checked, so by virtue of Fubini's theorem the Lebesgue measure of $Q_\gamma := \{t \in \mathbf{R} \mid \mu^\gamma(A_0 \ominus T_{\psi_t}(A_0)) = 0\}$ is full for μ-a.e. γ. As Q_γ is a group, so $Q_\gamma = \mathbf{R}$ for μ-a.e. γ. It follows from Theorem 1.3 that a measure ν^γ defined by the restriction of μ^γ to the set A_0 is $\mathrm{Diff}_0 X$-quasi-invariant for μ-a.e. γ. Since μ^γ is $\mathrm{Diff}_0 X$-ergodic, so $\mu^\gamma \simeq \nu^\gamma$ unless $\mu^\gamma(A_0) = 0$. That is $\mu^\gamma(A_0) = 1$ or 0 for μ-a.e. γ. Thus we have $\mu(A_0 \ominus A) = 0$ for an A defined by $A := \{\gamma \in \Gamma_X \mid \mu^\gamma(A_0) = 1\} \in \mathfrak{A}_\infty$.

Theorem 1.6. *For any* $\mathrm{Diff}_0 X$-*quasi-invariant probability measures* μ *and* ν,
(*a*) $\nu \lesssim \mu$ *if and only if* $\nu \lesssim \mu$ *on* $\mathfrak{A}_\infty$.
(*b*) μ *is* $\mathrm{Diff}_0 X$-*ergodic if and only if* $\mu(\cdot) = 1$ *or* 0 *on* $\mathfrak{A}_\infty$.
(*c*) *If* μ *and* ν *are* $\mathrm{Diff}_0 X$-*ergodic, then* $\mu \simeq \nu$ *or* $\mu \perp \nu$.

Proof. (a) Suppose that $\nu \lesssim \mu$ on $\mathfrak{A}_\infty$ and put $\lambda = (\mu + \nu)/2$. Then by virtue of the above lemma, there exists $A \in \mathfrak{A}_\infty$ such that "$\mu(B) = 0$ if and only if $\lambda(A \cap B) = 0$". Especially we have $\mu(A^c) = 0$ and thus $\nu(A^c) = 0$. Consequently, $\nu(B) = \nu(B \cap A) \leq 2\lambda(A \cap B)$, which implies $\nu(B) = 0$ if $\mu(B) = 0$. The converse relation is obvious. (b) and (c) easily follow from (a).

If we wish to be that factor measures $\{\mu^\gamma\}_{\gamma \in \Gamma_X}$ appearing in Theorem 1.5 are mutually singular, then the following technique will be useful. First notice that a minimal σ-algebra $\mathscr{D}$ with which all the functions, $\gamma \in \Gamma_X \to \mu^\gamma(B) \in \mathbf{R}$, where B runs through $\mathscr{C}$, are measurable is countably generated and thus $\mathscr{D} = g^{-1}(\mathscr{B}(\mathbf{R}))$ with a suitable map $g: \Gamma_X \to \mathbf{R}$. It is not difficult to see,

$$g(\gamma) = g(\gamma'), \text{ if and only if } \mu^\gamma = \mu^{\gamma'}. \tag{1.18}$$

Further by virtue of (c) in Theorem 1.5 we have,

$$\int_{g^{-1}(F)} \mu^\gamma(g^{-1}(E))\mu(d\gamma) = \int_{g^{-1}(F)} \chi_E(g(\gamma))\mu(d\gamma) \tag{1.19}$$

for all $E, F \in \mathfrak{B}(\mathbf{R})$, and hence for μ-a.e. γ,

$$\mu^\gamma(g^{-1}(E)) = \chi_E(g(\gamma)) \tag{1.20}$$

for all $E \in \mathfrak{B}(\mathbf{R})$. Especially we have,

$$\mu^{\gamma}(g^{-1}\{g(\gamma)\}) = 1 \tag{1.21}$$

for μ-a.e. γ. Now define $\mu_t' = \mu^{\gamma}$, if $t = g(\gamma)$ and $\mu_t' = \theta$, otherwise, where θ is some definite $\mathrm{Diff}_0 X$-ergodic probability measure on $(\Gamma_X, \mathscr{C})$. Then we have

(1.22) *For each fixed* $B \in \mathscr{C}$, $\mu_t'(B)$ *is a universally measurable function of* $t \in \mathbf{R}$.

Because $g\{\gamma \mid \mu^{\gamma}(B) \leq a\}$ is an analytic set for every $a \in \mathbf{R}$. And further we have,

$$\mu(B \cap g^{-1}(E)) = \int_E \mu_t'(B) g\mu(dt) \tag{1.23}$$

for all $B \in \mathscr{C}$ and $E \in \mathfrak{B}(\mathbf{R})$. Compairing μ_t' with regular conditional probability measure $p(t, \cdot)$ given $g = t$, we deduce that

$$^{\exists}T_1 \in \mathfrak{B}(\mathbf{R}) \text{ with } g\mu(T_1) = 1 \text{ such that } {}^{\forall}t \in T_1,\ \mu_t' = p(t, \cdot). \tag{1.24}$$

Finally we put $\mu_t(\cdot) = p(t, \cdot)$, if $t \in T_1$ and $\mu_t = \theta$, otherwise. Then

Theorem 1.7. *Let* μ *be a* $\mathrm{Diff}_0 X$*-quasi-invariant probability measure. Then there exist a map* g *and a family of probability measures* $\{\mu_t\}_{t \in \mathbf{R}}$ *on* $(\Gamma_X, \mathscr{C})$ *such that*
(*a*) *g is a measurable map from* $(\Gamma_X, \mathfrak{A}_{\infty})$ *to* $(\mathbf{R}, \mathfrak{B}(\mathbf{R}))$,
(*b*) μ_t *is* $\mathrm{Diff}_0 X$*-ergodic for every* $t \in \mathbf{R}$,
(*c*) $\mu_t(B)$ *is a Borel measurable function of* $t \in \mathbf{R}$ *for each fixed* $B \in \mathscr{C}$,
(*d*) *there exists a Borel set* T_0 *with* $g\mu(T_0) = 1$ *such that* $\mu_t(g^{-1}\{t\}) = 1$ *for all* $t \in T_0$, *especially* $\mu_t (t \in T_0)$ *are mutually singular, and*
(*e*) $\mu(B \cap g^{-1}(E)) = \int_E \mu_t(B) g\mu(dt)$ *for all* $B \in \mathscr{C}$ *and* $E \in \mathfrak{B}(\mathbf{R})$.

2. Ergodic decomposition of Gibbs measure

2.1. Basic properties. In this section X is a general locally compact topological space which satisfies the second countable axiom and m stands for non atomic Radon measures on $\mathfrak{B}(X)$ which is the natural Borel σ-field on X. A function $U(x \mid \gamma) \in (-\infty, \infty]$ defined on $(x, \gamma) \in X \times \Gamma_X$ is said to be a potential if it satisfies

(2.1) $U(x \mid \gamma)$ *is a* $\mathfrak{B}(\mathbf{R}) \times \mathscr{C}$*-measurable function, and*

$$U(x \mid \gamma \cup \{y\}) + U(y \mid \gamma) = U(y \mid \gamma \cup \{x\}) + U(x \mid \gamma) \tag{2.2}$$

for all $x, y \in X$ and $\gamma \in \Gamma_X$. We shall extend the domain of definition of the potential to B_x^n such that
$U(\phi \mid \gamma) := 0$ for $n = 0$, $U(\{x_1, x_2\} \mid \gamma) := U(x_1 \mid \gamma \cup \{x_2\}) + U(x_2 \mid \gamma)$ for $n = 2$, and $U(\underline{x} \mid \gamma) = U(\{x_1, \cdots, x_{n-1}\} \mid \gamma \cup \{x_n\}) + U(x_n \mid \gamma)$ for $\underline{x} := \{x_1, \cdots, x_n\} \in B_X^n$ inductively. These are well defined by the property (2.2).
Now let μ be a probability measure on $(\Gamma_X, \mathscr{C})$ and denote the conditional expectation of a $\mathscr{C}$-measurable function f on Γ_X with respect to the σ-field $\pi_{Y^c}^{-1}(\mathscr{C}_{Y^c})$ by $\mathrm{Exp}(f \mid \mathscr{C}_{Y^c})$. Let us proceed to the definition of Gibbs measure. A

probability measure μ on $(\Gamma_X, \mathscr{C})$ is said to be (U, m)-Gibbsian or simply Gibbsian (in a sense of Dobrushin, Ruelle, Lanford) for a potential U and a measure m if and only if it satisfies,

$$\Xi_K := \sum_{n=0}^{\infty} n!^{-1} \int_{B_K^n} \exp(-U(\underline{x} \mid \gamma \cap K^c))\, m_{K,n}(d\underline{x}) < \infty \tag{2.3}$$

for μ-a.e. γ, and

$$\begin{aligned}&\operatorname{Exp}(f \mid \mathscr{C}_{K^c})(\gamma) = \\ &\quad \Xi_K(\gamma)^{-1} \sum_{n=0}^{\infty} n!^{-1} \int_{B_K^n} \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cup (\gamma \cap K^c))\, m_{K,n}(d\underline{x})\end{aligned} \tag{2.4}$$

for each non negative bounded $\mathscr{C}$-measurable function f on Γ_X. Notice that we always have $\Xi_K(\gamma) \geq 1$. And it is fairly easy to see that a set of all (U, m)-Gibbsian measure is closed under the convex combination. From now on we shall write

$$\int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cdot \gamma \cap K^c) m(d\underline{x})$$

instead of

$$\sum_{n=0}^{\infty} n!^{-1} \int_{B_K^n} \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cup (\gamma \cap K^c))\, m_{K,n}(d\underline{x})$$

according to [4].

Lemma 2.1. (2.3) *and* (2.4) *is equivalent to the following condition* (2.5).

$$\int_{\Gamma_X} f(\gamma)\mu(d\gamma) = \int_{\{\gamma \mid |\gamma \cap K| = 0\}} \mu(d\gamma) \int^K \exp(-U(\underline{x} \mid \gamma)) f(\underline{x} \cdot \gamma) m(d\underline{x}) \tag{2.5}$$

for each compact set K and non negative bounded $\mathscr{C}$-measurable function f.

Proof. Suppose that (2.3) and (2.4) are satisfied and let χ_{N_K} be the indicator function of the set $N_K := \{\gamma \mid |\gamma \cap K| = 0\}$. Then for $f = \chi_{N_K}$ (2.4) gives

$$\operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K^c})(\gamma) = \Xi_K(\gamma)^{-1}. \tag{2.6}$$

Thus,

$$\begin{aligned}\int_{\Gamma_X} f(\gamma)\mu(d\gamma) &= \int_{\Gamma_X} \operatorname{Exp}(f \mid \mathscr{C}_{K^c})(\gamma)\mu(d\gamma) \\ &= \int_{\Gamma_X} \mu(d\gamma) \operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K^c})(\gamma) \int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cdot \gamma \cap K^c) m(d\underline{x}) \\ &= \int_{N_K} \mu(d\gamma) \int^K \exp(-U(\underline{x} \mid \gamma)) f(\underline{x} \cdot \gamma) m(d\underline{x}).\end{aligned}$$

Conversely, Put $F_K := \{\gamma \mid \Xi_K(\gamma) < \infty\}$ and substitute the indicator function $\chi_{F_{K^c}}$ for f in (2.5). Then it yields that

$$\mu(F_{K^c}) = \int_{N_K} \chi_{F_K^c}(\gamma)\,\Xi_K(\gamma)\mu(d\gamma),$$

and thus $\mu(F_K^c) = 0$. The rest of the proof easily follows from (2.6) which is easily derived from (2.5).

Let us look quickly how the Gibbsian property implies $\mathrm{Diff}_0 X$-quasi-invariance. So let μ be a Gibbs measure and Y be any open subset with compact closure. Then as is easily seen, (2.4) also holds for such Y provided that $m(\bar{Y} \setminus Y) = 0$. Thus the conditional probability measure μ^γ with respect to $\pi_{Y^c}^{-1}(\mathscr{C}_{Y^c})$ is,

$$\mu^\gamma(A) = \Xi_Y(\gamma)^{-1} \textstyle\sum_{n=0}^{\infty} n!^{-1} \displaystyle\int_{A \cap B_Y^n} \exp\left(-U(\underline{x} \mid \gamma \cap Y^c)\right) m_{Y,n}(d\underline{x}) \tag{2.7}$$

for all $A \in \mathscr{C}_Y$. Hence we have,

Theorem 2.1. *Let X be a connected σ*-compact *C^∞-manifold and m be a locally Euclidean Radon measure on X. Then under the assumption that the potential function $U(x \mid \gamma)$ is always finite, any (U, m)-Gibbs measure μ is* $\mathrm{Diff}_0 X$*-quasi-invariant.*

Proof. Take a sequence $\{Y_n\}$ of connected open sets with compact closure such that $m(\bar{Y}_n \setminus Y_n) = 0$ and apply Lemma 1.4.

Let $\{U_n\}$ be a countable open base of X such that $\bar{U}_n$ is compact for all n, $\mathscr{K}$ be a collection of all the sets being finite union of U_n $(n = 1, \cdots)$, and $\mathscr{F}$ be a countable field generatig $\mathscr{C}$.

Lemma 2.2. *In order that a probability measure μ on $(\Gamma_X, \mathscr{C})$ is Gibbsian, it is necessary and sufficient that* (2.5) *is satisfied for all $K \in \mathscr{K}$ and $\chi_B (= f)$, where χ_B is the indicator function of a set $B \in \mathscr{F}$.*

Proof. We have only to check the sufficiency. Now it is immediate from the assumption that (2.5) holds for all $K \in \mathscr{K}$ and for all non negative bounded $\mathscr{C}$-measurable functions. Hence proceeding in the same way with the proof of Lemma 2.1, we have for each $K \in \mathscr{K}$

$$\Xi_K(\gamma) < \infty \tag{2.8}$$

for μ-a.e. γ, and for each $K \in \mathscr{K}$

$$\Xi_K^{-1}(\gamma) = \mathrm{Exp}\,(\chi_{N_K} \mid \mathscr{C}_{K^c})(\gamma) \tag{2.9}$$

for μ-a.e. γ. Take any compact set K. Then there exists a sequence $\{K_n\} \subset \mathscr{K}$ such that $K_n \downarrow K$. It gives that

$$\{\gamma \mid |\gamma \cap K_n| = 0\} \uparrow \{\gamma \mid |\gamma \cap K| = 0\} \text{ and } \{\gamma \mid \gamma \cap K_n^c = \gamma \cap K^c\} \uparrow \Gamma_X.$$

Here we notice that if $\gamma \cap K^c = \gamma \cap K_n^c$ for some n, then

$$\Xi_K(\gamma) \le \int^{K_n} \exp(-U(\underline{x} \mid \gamma \cap K_n^c))m(d\underline{x}) = \Xi_{K_n}(\gamma).$$

Thus (2.8) and the above relation show that (2.3) holds for all compact sets K. By the assumption (2.5) holds for all K_n, so by virtue of Lebesgue-Fatou's lemma we have,

$$\text{(2.10)} \qquad \int_{\Gamma_X} f(\gamma)\mu(d\gamma) \ge \int_{N_K} \mu(d\gamma) \int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cdot \gamma \cap K^c)) m(d\underline{x}).$$

And hence,

$$\text{(2.11)} \qquad \operatorname{Exp}(f \mid \mathscr{C}_{K^c})(\gamma) \ge \operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K^c})(\gamma) \cdot \int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cdot \gamma \cap K^c) m(d\underline{x})$$

for μ-a.e. γ. Especially,

$$\text{(2.12)} \qquad \Xi_K(\gamma)^{-1} \ge \operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K^c})(\gamma)$$

for μ-a.e. γ. Now let us consider the relation (2.9) for $K = K_n$. As is easily seen, $\pi_{K_n^c}^{-1}(\mathscr{C}_{K_n^c}) \uparrow \pi_{K^c}^{-1}(\mathscr{C}_{K^c})$, so the right hand side of (2.9) is

$$\text{(2.13)} \qquad \begin{cases} \operatorname{Exp}(\chi_{N_{K_n}} \mid \mathscr{C}_{K_n^c})(\gamma) \le \operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K_n^c})(\gamma) \\ \operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K_n^c})(\gamma) \to \operatorname{Exp}(\chi_K \mid \mathscr{C}_{K^c})(\gamma) \quad \text{as } n \to \infty. \end{cases}$$

While for the left hand side, first we put $F_\infty := \bigcap_{n=1}^\infty F_{K_n}$. Then (2.8) gives $\mu(F_\infty) = 1$. And if $\gamma \in F_\infty$ and $\gamma \cap K_N^c = \gamma \cap K^c$ for some N, then for all $n \ge N$,

$$\Xi_{K_n}(\gamma) = \sum_{l=0}^L l!^{-1} \int_{B_{K_n}^l} \exp(-U(\underline{x} \mid \gamma \cap K^c)) m_{K_n,l}(d\underline{x}) + \varepsilon_{L,n},$$

where

$$\varepsilon_{L,n} := \sum_{l=L+1}^\infty l!^{-1} \int_{B_{K_n}^l} \exp(-U(\underline{x} \mid \gamma \cap K^c)) m_{K_n,l}(d\underline{x}) \le \sum_{l=L+1}^\infty l!^{-1} \int_{B_{K_N}^l} \exp(-U(\underline{x} \mid \gamma \cap K^c)) m_{K_N,l}(d\underline{x}).$$

And if we take a sufficiently large L, the last term becomes smaller than ε for a given $\varepsilon > 0$. Consequently for such an L,

$$\varlimsup_n \Xi_{K_n}(\gamma) \le \varepsilon + \sum_{l=0}^L l!^{-1} \int_{B_K^l} \exp(-U(\underline{x} \mid \gamma \cap K^c)) m_{K,l}(d\underline{x}) \le \varepsilon + \Xi_K(\gamma).$$

So we have,

$$\overline{\lim_{n}}\, \Xi_{K_n}(\gamma) \leq \Xi_K(\gamma) \tag{2.14}$$

for μ-a.e. γ. It follows from (2.13) and easy calculations that

$$\Xi_K(\gamma)^{-1} \leq \operatorname{Exp}(\chi_{N_K} \mid \mathscr{C}_{K^c})(\gamma) \tag{2.15}$$

for μ-a.e. γ. This and (2.12) show that (2.6) holds for all compact sets K. Now the inequality (2.11) becomes,

$$\operatorname{Exp}(f \mid \mathscr{C}_{K^c})(\gamma) \geq \Xi_K(\gamma)^{-1} \int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) f(\underline{x} \cdot \gamma \cap K^c) m(d\underline{x}) \tag{2.16}$$

for μ-a.e. γ. By the way (2.16) becomes an equality for $f = const$, thus it is actually an equality for any $f \geq 0$.

Let us take and fix an increasing sequence $\{K_n\}$ of compact sets such that $K_n \uparrow X$, and consider the tail σ-field $\mathscr{C}_\infty := \bigcap_{n=1}^{\infty} \pi_{K_n^c}^{-1}(\mathscr{C}_{K_n^c})$. $\mathscr{C}_\infty$ does not depend on a particular choice of $\{K_n\}$.

Theorem 2.1. *Let μ be a (U, m)-Gibbs measure and $\{\mu_\infty^\gamma\}_{\gamma \in \Gamma_X}$ be a family of conditional probability measure of μ with respect to $\mathscr{C}_\infty$. Then μ_∞^γ is (U, m)-Gibbsian for μ-a.e.γ.*

Proof. For $A \in \mathscr{C}_\infty$ and $B \in \mathscr{F}$ we calculate $\mu(A \cap B)$ in two ways. The first one is,

$$\mu(A \cap B) = \int_A \mu_\infty^\gamma(B) \mu(d\gamma),$$

and the other one is,

$$\mu(A \cap B) = \int_{N_K} \mu(d\gamma) \chi_A(\gamma) \int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) \chi_B(\underline{x} \cdot \gamma \cap K^c) m(d\underline{x}),$$

where K is taken from $\mathscr{K}$. These show that

$$\mu_\infty^\delta(B) = \int_{N_K} \mu_\infty^\delta(d\gamma) \int^K \exp(-U(\underline{x} \mid \gamma \cap K^c)) \chi_B(\underline{x} \cdot \gamma \cap K^c) m(d\underline{x}) \tag{2.17}$$

for μ-a.e.δ. Since $\mathscr{K}$ and $\mathscr{F}$ are countable, so the assertion directly follows from Lemma 2.2.

Here we introduce a notion of ergodicity. A Gibbs measure μ on $(\Gamma_X, \mathscr{C})$ is said to be ergodic if and only if $\mu(\cdot) = 1$ or 0 on $\mathscr{C}_\infty$. As $(\Gamma_X, \mathscr{C})$ is a standard space and $\mathscr{C}_\infty$ is an intersection of a decreasing sequence of countably generated σ-fields $\pi_{K_n^c}^{-1}(\mathscr{C}_{K_n^c})$, so μ_∞^γ is ergodic for almost all γ by a well known result (For example see theorem 2.3 in [2]) and the ergodic decomposition seems to be settled.

2.2. Ergodic measures as a base. However we will have a stronger result that factor measures $\mu_\infty^\gamma(\gamma\in\Gamma_X)$ do not depend on each μ. From now on we take and fix a countable field $\mathscr{F}_0$ generating $\mathscr{C}$ such that *any finitely additive finite measure on $\mathscr{F}_0$ has a σ-additive extension on $\mathscr{C}$.* The existence of such $\mathscr{F}_0$ is assured by the fact that $(\Gamma_X, \mathscr{C})$ is standard. Now set

$$\Omega_1 := \{\gamma \mid \Xi_{K_n}(\gamma) < \infty \text{ holds except finitely many } n\text{'s}\}$$

$$\Omega_2 := \{\gamma\in\Omega_1 \mid \lim_n \Xi_{K_n}(\gamma)^{-1}\int^{K_n} \exp(-U(\underline{x}\mid\gamma\cap K_n^c))\chi_B(\underline{x}\cdot\gamma\cap K_n^c)m(d\underline{x}) \text{ exists for every } B\in\mathscr{F}_0\}.$$

Then $\Omega_1, \Omega_2\in\mathscr{C}_\infty$ and for any (U, m)-Gibbs measure μ, $\mu(\Omega_2)=1$ by virtue of the martingale convergence theorem. And by the nice property of $\mathscr{F}_0$, we can define a probability measure $\omega_\gamma^0(\gamma\in\Omega_2)$ on $(\Gamma_X, \mathscr{C})$ as the extension of a finitely additive measure:

$$B\in\mathscr{F}_0 \to \lim_n \Xi_{K_n}(\gamma)^{-1}\int^{K_n} \exp(-U(\underline{x}\mid\gamma\cap K_n^c))\chi_B(\underline{x}\cdot\gamma\cap K_n^c)m(d\underline{x}).$$

Let us make up a definition ω_γ^0 as $\omega_\gamma^0=\zeta$ for $\gamma\in\Omega_2^c$, where ζ is some definite (U, m)-Gibbs ergodic measure. Then

(2.18) $\omega_\gamma^0(B)$ *is a $\mathscr{C}_\infty$-measurable function of $\gamma\in\Gamma_X$ for each fixed $B\in\mathscr{C}$.*

Further by virtue of the martingale convergence theorem we have for any Gibbs measure μ,

$$\mu(A\cap B) = \int_A \omega_\gamma^0(B)\mu(d\gamma) \tag{2.19}$$

for all $A\in\mathscr{C}_\infty$ and $B\in\mathscr{C}$. Because (2.19) is first valid for $B\in\mathscr{F}_0$ and holds in general by the extension property. It follows from (2.19) that

$$\omega_\gamma^0 = \mu_\infty^\gamma \tag{2.20}$$

for μ-a.e. γ. Here we shall put

$$\Omega_3 := \{\delta\in\Gamma_X \mid \omega_\delta^0(B) = \int_{N_K}\omega_\delta^0(d\gamma)\int^K \exp(-U(\underline{x}\mid\gamma\cap K^c))\chi_B(\underline{x}\cdot\gamma\cap K^c)m(d\underline{x}) \text{ for all } B\in\mathscr{F}_0 \text{ and } K\in\mathscr{K}\}.$$

Then $\Omega_3\in\mathscr{C}_\infty$ and (2.20) gives $\mu(\Omega_3)=1$ for any Gibbs measure μ. And it follows from Lemma 2.2 that

(2.21) ω_δ^0 *is (U, m)-Gibbsian for each $\delta\in\Omega_3$.*

Thus (2.19) derives that for $\delta\in\Omega_3$

$$\omega_\delta^0(A \cap B) = \int_A \omega_\gamma^0(B)\omega_\delta^0(d\gamma) \tag{2.22}$$

for all $A \in \mathscr{C}_\infty$ and $B \in \mathscr{C}$. Finally we put

$$\Omega_4 := \left\{\delta \in \Omega_3 \mid \int_{\Gamma_X} \{\omega_\gamma^0(B) - \omega_\delta^0(B)\}^2 \omega_\delta^0(d\gamma) = 0 \quad \text{for all} \quad B \in \mathscr{F}_0\right\}.$$

Then we have $\Omega_4 \in \mathscr{C}_\infty$ and for any (U, m)-Gibbs measure μ,

$$\int_{\Gamma_X}\int_{\Gamma_X} \{\omega_\gamma^0(B) - \omega_\delta^0(B)\}^2 \omega_\delta^0(d\gamma)\mu(d\gamma) = \int_{\Gamma_X}\int_{\Gamma_X} \{\mu_\infty^\gamma(B) - \mu_\infty^\delta(B)\}^2 \mu_\infty^\delta(d\gamma)\mu(d\gamma) = 0$$

Thus we have $\mu(\Omega_4) = 1$. Moreover it follows from (2.22) that $\omega_\delta^0(A \cap B) = \omega_\delta^0(A)\omega_\delta^0(B)$ for each $\delta \in \Omega_4$, which implies $\omega_\delta^0(A) = 1$ or 0 for all $A \in \mathscr{C}_\infty$. Thus

$$\omega_\delta^0 \text{ is ergodic for each } \delta \in \Omega_4. \tag{2.23}$$

Define $\omega_\delta = \omega_\delta^0$, if $\delta \in \Omega_4$ and $\omega_\delta = \zeta$, otherwise. Then we have,

Theorem 2.2. *As for a convex set formed by all (U, m)-Gibbs measures, there exists a family of probability measures $\{\omega_\gamma\}_{\gamma \in \Gamma_X}$ on $(\Gamma_X, \mathscr{C})$ such that*
(a) ω_γ is a (U, m)-Gibbs ergodic measure for each $\gamma \in \Gamma_X$,
(b) $\omega_\gamma(B)$ is a $\mathscr{C}_\infty$-measurable function of $\gamma \in \Gamma_X$ for each fixed $B \in \mathscr{C}$ and
(c) for any (U, m)-Gibbs measure μ

$$\mu(A \cap B) = \int_A \omega_\gamma(B)\mu(d\gamma) \quad \textit{for all} \quad A \in \mathscr{C}_\infty \textit{ and } B \in \mathscr{C}.$$

Corollary. *For any (U, m)-Gibbs measures μ and ν,*
(a) $\mu = \nu$ if and only if $\mu = \nu$ on $\mathscr{C}_\infty$.
(b) $\nu \lesssim \mu$ if and only if $\nu \lesssim \mu$ on $\mathscr{C}_\infty$.
(c) If μ and ν are ergodic, then $\mu = \nu$ or $\mu \perp \nu$.

Let us take and fix an above family $\{\omega_\gamma\}_{\gamma \in \Gamma_X}$ and consider a minimal σ-field $\mathscr{C}_{\infty,\omega}$ with which all the functions, $\gamma \to \omega_\gamma(B)$ $(B \in \mathscr{C})$ are measurable. Since $\mathscr{C}_{\infty,\omega}$ is countably generated, so there exists a map $h: \Gamma_X \to \mathbf{R}$ such that $\mathscr{C}_{\infty,\omega} = h^{-1}(\mathfrak{B}(\mathbf{R}))$. As before it is easily checked that

$$\omega_\gamma = \omega_{\gamma'} \textit{ if and only if } h(\gamma) = h(\gamma'). \tag{2.24}$$

Further we claim that

$$h\omega_\gamma = \delta_{h(\gamma)} \tag{2.25}$$

for all $\gamma \in \Gamma_X$, where δ_s is the Dirac measure at $s \in \mathbf{R}$. For, put $S := \{\gamma \in \Gamma_X \mid h\omega_\gamma = \delta_{h(\gamma)}\}$. Then we have $S \in \mathscr{C}_{\infty,\omega}$ and for any Gibbs measure μ,

$$\mu(h^{-1}(E \cap F)) = \int_{h^{-1}(F)} \omega_\gamma(h^{-1}(E))\mu(d\gamma) = \int_{h^{-1}(F)} \chi_E(h(\gamma))\mu(d\gamma)$$

for all E, $F\in\mathfrak{B}(\mathbf{R})$. Since both the integrants are $\mathscr{C}_{\infty,\omega}$-measurable, so it follows that $\omega_\gamma(h^{-1}(E))=\chi_E(h(\gamma))$ for μ-a.e. γ, and thus $\mu(S)=1$. Especially we have,

$$\omega_\gamma(S)=1 \tag{2.26}$$

for all $\gamma\in\Gamma_X$. Now for any fixed $\gamma\in\Gamma_X$ let us take $\sigma\in\{\delta\,|\,\omega_\delta=\omega_\gamma\}\cap S$. Notice that the last set is not empty because we have $\omega_\delta=\omega_\gamma$ for ω_γ-a.e.δ by virtue of the ergodicity of ω_γ. Then,

$$\omega_\gamma(h^{-1}(E))=\omega_\sigma(h^{-1}(E))=\chi_E(h(\sigma))=\chi_E(h(\gamma)).$$

We settle these arguments as the following theorem.

Theorem 2.3. *Under the notation in Theorem 2.2, let $\mathscr{C}_{\infty,\omega}$ be the minimal σ-field with which all the functions, $\gamma\to\omega_\gamma(B)$ $(B\in\mathscr{C})$ are measurable. Then there exists a map $h\colon\Gamma_X\to\mathbf{R}$ such that $\mathscr{C}_{\infty,\omega}=h^{-1}(\mathfrak{B}(\mathbf{R}))$ and we have*
(*a*) *$\omega_\gamma=\omega_{\gamma'}$ if and only if $h(\gamma)=h(\gamma')$, and*
(*b*) *$h\omega_\gamma=\delta_{h(\gamma)}$ for all $\gamma\in\Gamma_X$.*

Remark 2.1. As for the uniqueness of such a family $\{\omega_\gamma\}_{\gamma\in\Gamma_X}$, it is desirable to state it without exceptional set instead of with exceptional set of measure 0. The following is an answer for this question. Namely, in order that such families $\{\omega_\gamma\}_{\gamma\in\Gamma_X}$ and $\{\omega'_\gamma\}_{\gamma\in\Gamma_X}$ coincide, it is necessary and sufficient that the σ-fields generated by them are the same one, *i.e.*, $\mathscr{C}_{\infty,\omega}=\mathscr{C}_{\infty,\omega'}$.

Theorem 2.4. *A map: $\lambda(\cdot)\to\int_{\Gamma_X}\omega_\gamma(\cdot)\lambda(d\gamma)$ is a bijection from a space of all probability measures on $(\Gamma_X,\mathscr{C}_{\infty,\omega})$ to the space of all (U, m)-Gibbs measures.*

The proof is obvious from what we have stated.

We conclude this paragraph with the following theorem.

Theorem 2.5. *As for the convex set formed by all (U, m)-Gibbs measures, there exist a map $h\colon\Gamma_X\to\mathbf{R}$ and a family of probability measures $\{\beta_\tau\}_{\tau\in h(\Gamma_X)}$ on $(\Gamma_X,\mathscr{C})$ such that*
(*a*) *h is a measurable map from $(\Gamma_X,\mathscr{C}_\infty)$ to $(\mathbf{R},\mathfrak{B}(\mathbf{R}))$,*
(*b*) *β_τ is a (U, m)-Gibbs ergodic measure for each $\tau\in h(\Gamma_X)$,*
(*c*) *$\beta_\tau(B)$ is a $h(\Gamma_X)\cap\mathfrak{B}(\mathbf{R})$-measurable function of $\tau\in h(\Gamma_X)$ for each fixed $B\in\mathscr{C}$, and hence it is universally measurable,*
(*d*) *$\beta_\tau(h^{-1}(\tau))=1$ for all $\tau\in h(\Gamma_X)$, especially $\beta_\tau(\tau\in h(\Gamma_X))$ are mutually singular, and*
(*e*) *for any (U, m)-Gibbs measure μ,*

$$\mu(B\cap h^{-1}(E))=\int_E\beta_\tau(B)h\mu(d\tau)\quad \textit{for all } B\in\mathscr{C} \textit{ and } E\in\mathfrak{B}(\mathbf{R}).$$

Proof. Let us put $\beta_\tau=\omega_\gamma$, if $\tau=h(\gamma)$. Then the well-definedness and (d) come from Theorem 2.3. Next we shall show (c). Notice that for each fixed

$a \in \mathbf{R}$, there exists some $E_a \in \mathfrak{B}(\mathbf{R})$ such that $\{\gamma \in \Gamma_X \mid \omega_\gamma(B) \leq a\} = h^{-1}(E_a)$. Hence

$$\{\tau \in h(\Gamma_X) \mid \beta_\tau(B) \leq a\} = h\{\gamma \in \Gamma_X \mid \omega_\gamma(B) \leq a\} = E_a \cap h(\Gamma_X).$$

Since an image of a Borel set in a standard space by a Borel map is an analytic set, so $h(\Gamma_X)$ is universally measurable. The rest of the proof easily follows from Theorem 2.2.

We remark that $\beta_\tau(\tau \in h(\Gamma_X))$ runs all over the set of all (U, m)-Gibbs ergodic measures.

2.3. Specific Gibbs measures. First we shall characterize Gibbs measures with total mass on B_X.

Theorem 2.6. *If a (U, m)-Gibbs measure μ on $(\Gamma_X, \mathscr{C})$ have total mass on B_X, then it follows that*

(*a*) $S := \int^X \exp(-U(\underline{x} \mid \phi)) m(d\underline{x}) < \infty$,

and the explicit form of μ is given by

(*b*) $\int_{\Gamma_X} f(\gamma)\mu(d\gamma) = S^{-1} \int^X \exp(-U(\underline{x} \mid \phi)) f(\underline{x}) m(d\underline{x})$ *for all non negative bounded measurable function f.*

Conversely if (*a*) *holds, then a measure μ given by* (*b*) *is (U, m)-Gibbsian with total mass on B_X.*

Proof. As is easily seen, B_X is an atom of $\mathscr{C}_\infty$, so the measure μ with total mass on B_X must be ergodic. It follows from the martingale convergence theorem that

$$\int_{B_X} f(\delta)\mu(d\delta) = \lim_n \frac{\int^{K_n} \exp(-U(\underline{x} \mid \gamma \cap K_n^c)) f(\underline{x} \cdot \gamma \cap K_n^c) m(d\underline{x})}{\int^{K_n} \exp(-U(\underline{x} \mid \gamma \cap K_n^c)) m(d\underline{x})} \tag{2.27}$$

for μ-a.e. γ. However $\gamma \in B_X$ implies $\gamma \cap K_n^c = \phi$ for sufficiently large n, so (2.27) is actually,

$$\int_{B_X} f(\gamma)\mu(d\mu) = \lim_n \frac{\int^{K_n} \exp(-U(\underline{x} \mid \phi)) f(\underline{x}) m(d\underline{x})}{\int^{K_n} \exp(-U(\underline{x} \mid \phi)) m(d\underline{x})}. \tag{2.28}$$

By the assumption we have $\mu(B_{X,k,l}) > 0$ for some k and l, where $B_{X,k,l} = \{\gamma \in B_X^k \mid \gamma \subset K_l\}$. Let us put the indicator function of $B_{X,k,l}$ for f. Then the numerator under the limit sign in (2.28) becomes

$$k!^{-1} \int \cdots \int_{K_l^k} \exp(-U(\{x_1, \cdots, x_l\} \mid \phi)) m^l(dx)$$

for all $n \geq l$ which is independent of n. So we have

$$\lim_n \int^{K_n} \exp(-U(\underline{x}\mid\phi))m(d\underline{x}) = S < \infty,$$

and (b) follows directly from (2.28).
Conversely, we claim that (2.5) holds for μ defined by (b) under the assumption (a). For it we have only to check it for functions $f(\gamma)\chi_{B_X^n}(\gamma)$ $(n = 0, 1, \cdots)$. It is obvious for $n = 0$, so let $n > 0$. Then,

$$\int_{N_K} \mu(d\gamma) \int^K \exp(-U(\underline{x}\mid\gamma)f(\underline{x}\cdot\gamma)\chi_{B_X^n}(\underline{x}\cdot\gamma)m(d\underline{x})$$

$$= \sum_{l=0}^n \int_{\{\gamma \mid |\gamma\cap K|=0, |\gamma\cap K^c|=l\}} \mu(d\gamma)(n-l)!^{-1} \int\cdots\int_{x\in K^{n-l}} \exp(-U(\{x_1,\cdots,x_{n-l}\}$$

$$|\gamma\cap K^c))\cdot f(\{x_1,\cdots,x_{n-l}\}\cup(\gamma\cap K^c))m^{n-l}(dx)$$

$$= S^{-1}\sum_{l=0}^n \{l!(n-l)!\}^{-1} \int\cdots\int_{x\in K^{n-l}} \int\cdots\int_{y\in(K^c)^l} \exp(-U(\{x_1,\cdots,x_{n-l}\}|\{y_1,\cdots,y_l\})$$

$$\cdot \exp(-U(\{y_1,\cdots,y_l\}\mid\phi))f(\{x_1,\cdots,x_{n-l}, y_1,\cdots,y_l\})m^{n-l}(dx)m^l(dy)$$

$$= S^{-1}n!^{-1}\sum_{l=0}^n {}_nC_l \int\cdots\int_{x\in K^{n-l}} \int\cdots\int_{y\in(K^c)^l} \exp(-U(\{x_1,\cdots,x_{n-l}, y_1,\cdots,y_l\}\mid\phi))\cdot$$

$$f(\{x_1,\cdots,x_{n-l}, y_1,\cdots,y_l\})m^{n-l}(dx)m^l(dy)$$

$$= S^{-1}n!^{-1} \int\cdots\int_{X^n} \exp(-U(\{z_1,\cdots,z_n\}\mid\phi))f(\{z_1,\cdots,z_n\})m^n(dz)$$

$$= \int_{B_X} f(\gamma)\chi_{B_X^n}(\gamma)\mu(d\gamma).$$

Theorem 2.7. *If the potential U is constant, say $U(x\mid\gamma) = -\log a$, then the convex set of all (U, m)-Gibbs measures consists of only a Poisson measure P_{am} with intesity am. (P_{am} is of course ergodic by virtue of 0–1 law.)*

Proof. Let μ be any (U, m)-Gibbs measure. Then for each compact set K we have,

$$\pi_K\mu(B) = \int_{N_K} \mu(d\gamma) \int^K \exp(-U(\underline{x}\mid\gamma))\chi_B(\underline{x})m(d\underline{x})$$
$$= \mu(N_K)\sum_{n=0}^\infty n!^{-1}a^n m_{K,n}(B\cap B_K^n)$$

for all $B\in\mathscr{C}_K$. So by virtue of Obata's result [1], there exists a Borel probability measure λ on $[0, \infty)$ such that $\mu = \int_0^\infty P_{cm}\lambda(dc)$ in the case $m(X) = \infty$, or μ is a convex combination of $m_{X,n}/m(X)^n$ in the case $m(X) < \infty$. First we shall consider the infinite case. So let us take a set $E\in\mathfrak{B}(\mathbf{R})$ and a function

$$\rho(\gamma) = \begin{cases} \lim_n \frac{1}{n} \sum_{l=1}^n \frac{|\gamma \cap (K_{l+1} \setminus K_l)|}{m(K_{l+1} \setminus K_l)}, & \text{if the limit exists.} \\ 0 \qquad\qquad\qquad\qquad\qquad , & \text{otherwise.} \end{cases} \tag{2.29}$$

And we calculate $\int_{\Gamma_X} \chi_E(\rho(\gamma))\mu(d\gamma)$ in two ways, noting that $\rho(\gamma) = c$ for P_{cm}-a.e. γ. the first one is,

$$\int_{\Gamma_X} \chi_E(\rho(\gamma))\mu(d\gamma) = \int_{\Gamma_X} \chi_E(\rho(\gamma))P_{cm}(d\gamma)\lambda(dc) = \lambda(E),$$

and the other one is,

$$\int_{\Gamma_X} \chi_E(\rho(\gamma))\mu(d\gamma) = \int_0^\infty \lambda(dc) \int_{N_K} P_{cm}(d\gamma) \int^K a^{|\underline{x}|} \chi_E(\rho(\underline{x} \cdot \gamma \cap K^c))m(d\underline{x})$$

$$= \exp(am(K)) \int_E \exp(-cm(K))\lambda(dc).$$

These show that $c = a$ for λ-a.e.c and hence $\mu = P_{am}$. Next we shall consider the finite case. So there exists a non negative sequence $\{c_n\}$ with $\sum_{n=0}^\infty c_n = 1$ such that

$$\mu = \sum_{n=0}^\infty c_n m(X)^{-n} m_{X,n}. \tag{2.30}$$

Then for each compact set K,

$$c_n = \mu(B_X^n) = \int_{N_K} \mu(d\gamma) \int^K a^{|\underline{x}|} \chi_{B_X^n}(\underline{x} \cdot \gamma \cap K^c)m(d\underline{x})$$

$$= \sum_{l=0}^\infty c_l m(X)^{-l} \int_{N_K} m_{X,l}(d\gamma) \int^K a^{|\underline{x}|} \chi_{B_X^n}(\underline{x} \cdot \gamma \cap K^c)m(d\underline{x}).$$

So we have,

$$c_n = \sum_{l=0}^n \{(n-l)!\, m(X)^l\}^{-1} a^{n-l} m(K)^{n-l} m(K^c)^l c_l. \tag{2.31}$$

It follows from the mathematical induction for n that

$$c_n = n!^{-1} a^n m(X)^n c_0, \quad \text{and} \quad c_0 = \exp(-am(X)),$$

which follows from the normalizing condition. Thus we have

$$\mu = \exp(-am(X)) \sum_{n=0}^\infty n!^{-1} a^n m_{X,n} = P_{am}.$$

Corollary. *Let U be a potential defined by $U(x \mid \gamma) = -\log \rho(\gamma)$, where ρ is a function defined by* (2.29). *If a Borel Radon measure m on X is infinite, then the extremal points of the convex set of all (U, m)-Gibbs measures consists of $\{P_{cm}\}_{c \in [0,\infty)}$. That is, for any (U, m)-Gibbs measure μ,*

$$\mu(B\cap\rho^{-1}(E))=\int_E P_{cm}(B)\rho\mu(dc)\quad \textit{for all}\ \ B\in\mathscr{C}\ \ \textit{and}\ \ E\in\mathfrak{B}([0,\infty)).$$

Proof. Let $\{\omega_\tau\}_{\tau\in\Gamma_X}$ be as in Theorem 2.2. Then the $\mathscr{C}_\infty$-measurabilty of ρ implies that for each fixed $\gamma\in\Gamma_X$, there exists a constant $a(\gamma)$ such that

$$\rho(\sigma)=a(\gamma) \tag{2.32}$$

for ω_γ-a.e. σ. Thus by the above theorem we have

$$\omega_\gamma=P_{a(\gamma)m} \tag{2.33}$$

for all $\gamma\in\Gamma_X$. Especially for each compact set K,

$$\omega_\gamma(N_K)=P_{a(\gamma)m}(N_K)=\exp(-a(\gamma)m(K)).$$

It follows that $a(\gamma)$ is also a $\mathscr{C}_\infty$-measurable function of γ, and therefore (2.32) implies that for any (U, m)-Gibbs measure μ,

$$\rho(\gamma)=a(\gamma) \tag{2.34}$$

for μ-a.e. γ. Thus we have,

$$\mu(B\cap\rho^{-1}(E))=\int_{\rho^{-1}(E)}\omega_\gamma(B)\mu(d\gamma)=\int_{\rho^{-1}(E)}P_{\rho(\gamma)m}(B)\mu(d\gamma)=\int_E P_{cm}(B)\rho\mu(dc).$$

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