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Note on reflection maps and self maps of $U(n)$, $Sp(n)$ and $U(2n)/Sp(n)$

Dedicated to Professor Teiichi Kobayashi on his sixtieth birthday

By

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1. Statement of result

Let $U(n)$ and $Sp(n)$ be the n-th unitary and symplectic group, respectively. We denote the complex numbers by $\mathbf{C}$, and the quaternions by $\mathbf{H}$. Let $\mathbf{F}$ be $\mathbf{C}$, $\mathbf{H}$ or $(\mathbf{C}, \mathbf{H})$. In order to describe uniformly for three cases, we write

$$G_n(\mathbf{F}) = \begin{cases} U(n) & \text{if } \mathbf{F}=\mathbf{C} \\ Sp(n) & \text{if } \mathbf{F}=\mathbf{H} \\ U(2n)/Sp(n) & \text{if } \mathbf{F}=(\mathbf{C}, \mathbf{H}). \end{cases}$$

When $\mathbf{F}$ is $\mathbf{C}$ or $\mathbf{H}$, we denote by $P(\mathbf{F}^n)$ and $Q_n(\mathbf{F})$ the projective space and the quasi-projective space, respectively. We write $Q_n(\mathbf{C}, \mathbf{H}) = \Sigma P(\mathbf{H}^n)_+$, the suspension of the union of $P(\mathbf{H}^n)$ and a point space. Recall from [1, 6, 8] (cf. §2 and §4 of this paper) that there is a map, called the *reflection map*,

$$r: Q_n(\mathbf{F}) \to G_n(\mathbf{F})$$

which induces an epimorphism on cohomology. Our result is

Theorem. *For any integer k, there exist maps $c_k : Q_n(\mathbf{F}) \to Q_n(\mathbf{F})$ and m_k: $G_n(\mathbf{F}) \to G_n(\mathbf{F})$ such that*

(1) *the following diagram commutes*

$$(1.1) \qquad \begin{array}{ccc} Q_n(\mathbf{F}) & \xrightarrow{r} & G_n(\mathbf{F}) \\ \downarrow{\scriptstyle c_k} & & \downarrow{\scriptstyle m_k} \\ Q_n(\mathbf{F}) & \xrightarrow{r} & G_n(\mathbf{F}); \end{array}$$

(2) *c_k induces the homomorphism of k-multiple on the integral cohomology;*

(3) *m_k induces the homomorphism of k-multiple on the ring basis of the integ-*

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ral cohomology which will be given in Lemmas 2.1 and 4.1 below.

When $\mathbf{F}$ is $\mathbf{C}$ or $\mathbf{H}$, setting m_k to be the k-times multiplication map, Theorem may be well-known for experts. We give its proof in §2 for completeness, though. Since $G_n(\mathbf{C}, \mathbf{H})$ is not an H-space for $n\geq 2$ (cf. [4]), the existence of the map m_k is not obvious when $\mathbf{F}=(\mathbf{C}, \mathbf{H})$.

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2. The cases C and H

In this section $\mathbf{F}$ is $\mathbf{C}$ or $\mathbf{H}$. All vector spaces are considered as right vector spaces. The standard inner product $\langle , \rangle$ in $\mathbf{F}^n$ is defined so that if $x=(x_1, \ldots, x_n)$, $y=(y_1, \ldots, y_n)$ are elements of $\mathbf{F}^n$ then $\langle x, y\rangle=\sum_k \bar{x}_k y_k$. We identify $\mathbf{H}^n$ with $\mathbf{C}^{2n}$ as follows: every quaternion n-vector $(z_1, \ldots, z_n)=x+jy$ determines a complex $2n$-vector $(x_1, \ldots, x_n, y_1, \ldots, y_n)=x\oplus y$ where $z_r=x_r+jy_r$. Let $\iota : Sp(n)\to U(2n)$ be the inclusion, $c: U(m) \to U(m)$ the complex conjugation, and $S(\mathbf{F}^n)$ the unit sphere of $\mathbf{F}^n$. Recall from [6] that the quasi-projective space is defined by

$$Q_n(\mathbf{F})=S(\mathbf{F}^n)\times_{S(\mathbf{F})}S(\mathbf{F})/S(\mathbf{F}^n)\times_{S(\mathbf{F})}\{1\}$$

where $S(\mathbf{F})$ acts on $S(\mathbf{F}^n)$ by the multiplication from the right and acts on $S(\mathbf{F})$ by the inner automorphism. Note that $Q_n(\mathbf{C})=\Sigma P(\mathbf{C}^n)_+$. Let $x\in S(\mathbf{F}^n)$, $y\in\mathbf{F}^n$ and $\lambda\in S(\mathbf{F})$. Then the reflection map $r: Q_n(\mathbf{F})\to G_n(\mathbf{F})$ is defined by

$$r[x, \lambda](y)=y+x(\lambda-1)\langle x, y\rangle.$$

Now, for $k\in\mathbf{Z}$, define $c_k: Q_n(\mathbf{F})\to Q_n(\mathbf{F})$ and $m_k: G_n(\mathbf{F})\to G_n(\mathbf{F})$ by

$$c_k[x, \lambda]=[x, \lambda^k] \text{ and } m_k(x)=x^k.$$

Then, as is easily seen, c_k is well-defined and the diagram (1.1) commutes. The following is well-known (cf. (3.8) in [6] or [7]).

Lemma 2.1 *We have*

$$H^*(U(n); \mathbf{Z})=\Lambda_z(x_1, \ldots, x_n), \quad \deg(x_i)=2i-1,$$
$$H^*(Sp(n); \mathbf{Z})=\Lambda_z(y_1, \ldots, y_n), \quad \deg(y_i)=4i-1$$

such that

(1) *x_i and y_i are primitive;*

(2) *$r^*(x_i)$ and $r^*(y_i)$ are generators of $H^{2i-1}(Q_n(\mathbf{C});\mathbf{Z})$ and $H^{4i-1}(Q_n(\mathbf{H});\mathbf{Z})$, respectively;*

(3) $c^*(x_i)=(-1)^i x_i$;

(4) $\iota^*(x_i)=\begin{cases}(-1)^{i/2}y_{i/2} & \text{if } i \text{ is even}\\ 0 & \text{if } i \text{ is odd.}\end{cases}$

From this lemma, (2) and (3) of Theorem follow. This ends the proof of Theorem for $\mathbf{F}=\mathbf{C}, \mathbf{H}$.

As is well-known [8, 9], there is a map $T: Q_n(\mathbf{H}) \to Q_{2n}(\mathbf{C})$ such that the following square is commutative up to homotopy

$$\begin{array}{ccc} Q_n(\mathbf{H}) & \xrightarrow{T} & Q_{2n}(\mathbf{C}) \\ \downarrow r & & \downarrow r \\ Sp(n) & \xrightarrow{\iota} & U(2n) \end{array}$$

and there is a cofibre sequence

$$P(\mathbf{C}^{2n})_+ \xrightarrow{p_+} P(\mathbf{H}^n)_+ \longrightarrow Q_n(\mathbf{H}) \xrightarrow{T} Q_{2n}(\mathbf{C}) \xrightarrow{\Sigma p_+} \cdots$$

where p is the canonical map.

3. Sele map of a symmetric space

Let G be a topological group having an involutive automorphism $\sigma: G \to G$. Let H be a subgroup of G such that $H \subset G^\sigma = \{x \in G;\ \sigma(x) = x\}$. Notice that if G is a Lie group and H contains a path-component of G^σ, then G/H is a symmetric space. Lep $p: G \to G/H$ be the projection, and define a map $\xi: G \to G$ by

$$\xi(x) = x\sigma(x^{-1}).$$

Let $k \in \mathbf{Z}$. We define self maps μ_k, f_k of G by

$$\mu_k(x) = x^k \quad \text{and} \quad f_k(x) = \begin{cases} \xi(x)^l & \text{if } k = 2l \\ \xi(x)^l x & \text{if } k = 2l+1. \end{cases}$$

They induce maps $g_k: G/H \to G/H$ and $\tilde{\xi}: G/H \to G$ such that $g_k \circ p = p \circ f_k$ and $\tilde{\xi} \circ p = \xi$. If follows easily that $\tilde{\xi} \circ g_k \circ p = \mu_k \circ \tilde{\xi} \circ p$ so that $\tilde{\xi} \circ g_k = \mu_k \circ \tilde{\xi}$.

Lemma 3.1 (1) *The following diagram is commutative.*

$$\begin{array}{ccccc} G & \xrightarrow{p} & G/H & \xrightarrow{\tilde{\xi}} & G \\ \downarrow f_k & & \downarrow g_k & & \downarrow \mu_k \\ G & \xrightarrow{p} & G/H & \xrightarrow{\tilde{\xi}} & G \end{array}$$

(2) *If $G^\sigma = H$, then $\tilde{\xi}$ is injective.*

Proof. The assertion (1) has already proved, and (2) is true, since $\xi(x) = \xi(y)$ if and only if $x^{-1}y \in G^{\sigma}$.

We would like to determine the cohomology induced homomorphism of g_k. However it seems difficult in general. We will treat this for some nice cases in the following sections.

4. The case (C, H)

We use the notations in §2. Let J: $\mathbf{C}^{2n} \to \mathbf{C}^{2n}$ be the conjugate linear map defined by the multiplication of $j \in \mathbf{H}$ from the right. Define σ: $U(2n) \to U(2n)$ by $\sigma(h) = J \circ h \circ J^{-1}$. In words of matrices,

$$\sigma(A) = J_n\, c(A)\, J_n^{-1}, \quad J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Where I_n is the unit matrix of dimension n. Then, as is well-known, $U(2n)^{\sigma} = Sp(n)$. Thus, from §3, we get a self map g_k of $U(2n)/Sp(n)$, which from now on is denoted by

$$m_k : U(2n)/Sp(n) \to U(2n)/Sp(n)\,.$$

We will show that m_k is the desired map.

Lemma 4.1. *Under the notations of Lemma 2.1. we have*

(1) $H^*(U(2n)/Sp(n)\,;\mathbf{Z}) = \Lambda_{\mathbf{Z}}(z_1, \ldots, z_n)$, $\deg(z_i) = 4i-3$, $p^*(z_i) = x_{2i-1}$

and, for any $k \in \mathbf{Z}$,

(2) $$m_k^*(z_i) = kz_i\,.$$

Proof. We refer the proof of (1) to [7]. We prove (2) as follows. As we described above, σ is the composition of the inner automorphism by J_n and conjugation, and since x_i is primitive, by Lemma 2.1, we see that

$$\xi^*(x_i) = x_i - c^*(x_i) = (1+(-1)^{i+1})x_i$$

thus

$$f_k^*(x_i) = \begin{cases} l(1+(-1)^{i+1})x_i & \text{if } k = 2l \\ \{l(1+(-1)^{i+1})+1\}x_i & \text{if } k = 2l+1. \end{cases}$$

Therefore, using Lemmas 2.1 and 3.1, we have the desired result.

To define the map r: $\Sigma P(\mathbf{H}^n)_+ \to U(2n)/Sp(n)$, we recall some constructions. For $\mathbf{F} = \mathbf{C}, \mathbf{H}$, let $V_m(\mathbf{F}^n)$ be the Stiefel manifold of orthonormal m-frames in $\mathbf{F}^n$, $G_m(\mathbf{F}^n)$ the Grassmann manifold of m-dimensional subspaces in $\mathbf{F}^n$, and $q\colon V_m(\mathbf{F}^n) \to G_m(\mathbf{F}^n)$ the natural projection. Define $\tilde{\phi}\colon G_m(\mathbf{C}^n)\times I \to U(n)$ by

$$\tilde{\phi}(q(v_1, \dots, v_m);\, t)(x) = x + \sum_k v_k (e^{\pi i t} - 1)\langle v_k, x\rangle .$$

Notice that $\tilde{\phi}(q(v_1, \dots, v_m);\, t)(x \oplus y) = x e^{\pi i t} \oplus y$ for $x \in q(v_1, \dots, v_m) = W$, $y \in W^\perp$. There exists a map $\phi\colon \Sigma G_m(\mathbf{H}^n)_+ \to U(2n)/Sp(n)$ which makes the following diagram commutative.

$$\begin{array}{ccccc}
G_m(\mathbf{H}^n)\times I & \xrightarrow{\iota\times id} & G_{2m}(\mathbf{C}^{2n})\times I & \xrightarrow{\tilde{\phi}} & U(2n) \\
\downarrow q & & & & \downarrow p \\
G_m(\mathbf{H}^n)\times I/G_m(\mathbf{H}^n)\times\{0,1\} & = & \Sigma G_m(\mathbf{H}^n)_+ & \xrightarrow{\phi} & U(2n)/Sp(n)
\end{array}$$

Here ι is the inclusion map. Write

$$r = \phi : \Sigma G_1(\mathbf{H}^n)_+ = \Sigma P(\mathbf{H}^n)_+ \to U(2n)/Sp(n) .$$

Proposition 4.2. *Let $p : U(2n) \to U(2n)/Sp(n)$ is the canonical projection. Then, the following square is commutative up to homotopy.*

$$\begin{array}{ccc}
\Sigma P(\mathbf{C}^{2n})_+ & \xrightarrow{r} & U(2n) \\
\downarrow \Sigma p_+ & & \downarrow p \\
\Sigma P(\mathbf{H}^n)_+ & \xrightarrow{r} & U(2n)/Sp(n).
\end{array}$$

Proof. Define $\tilde{H} : I\times G_1(\mathbf{C}^{2n})\times I \to U(2n)$ by

$$\tilde{H}(s, q(v), t)(x) = x + v(e^{\pi i(1+s)t} - 1)\langle v, x\rangle + vj(e^{\pi i(1-s)t} - 1)\langle vj, x\rangle .$$

This induces the map H which makes the following commutative.

$$\begin{array}{ccc}
I\times G_1(\mathbf{C}^{2n})\times I & \xrightarrow{\tilde{H}} & U(2n) \\
\downarrow id\times q & & \downarrow p \\
I\times \Sigma G_1(\mathbf{C}^{2n})_+ & \xrightarrow{H} & U(2n)/Sp(n)
\end{array}$$

Then H is a homotopy between $r\circ \Sigma p_+$ and $p\circ r$. This completes the proof.

Given $k \in \mathbf{Z}$, let $c_k : \Sigma P(\mathbf{H}^n)_+ \to \Sigma P(\mathbf{H}^n)_+$ be defined by

$$c_k[v, t] = [v, \widetilde{kt}]$$

where $kt-\widetilde{kt} \in \mathbf{Z}$ and $0 \leq \widetilde{kt} < 1$. Note that $c_k[v, \lambda] = [v, \lambda^k]$ under the identification $\Sigma P(\mathbf{H}^n)_+ = P(\mathbf{H}^n) \times S^1 / P(\mathbf{H}^n) \times \{1\}$.

Proposition 4.3. *For any $k \in \mathbf{Z}$, the following diagram is commutative.*

$$\begin{array}{ccc} \Sigma P(\mathbf{H}^n)_+ & \xrightarrow{r} & U(2n)/Sp(n) \\ \downarrow{\scriptstyle c_k} & & \downarrow{\scriptstyle m_k} \\ \Sigma P(\mathbf{H}^n)_+ & \xrightarrow{r} & U(2n)/Sp(n) \end{array}$$

Proof. Take $[W, t] \in \Sigma P(\mathbf{H}^n)_+ = \Sigma G_1(\mathbf{H}^n)_+$. Let $W^\perp$ be the orthogonal complement of W in $\mathbf{C}^{2n}$. By definitions, $r[W, t] = (p \circ \tilde{\phi})(\iota(W), t)$ and $\tilde{\phi}(\iota(W), t)(x \oplus y) = xe^{\pi i t} \oplus y$ for $x \in W, y \in W^\perp$. It follows that

$$\tilde{\xi}(r[W, t]) = \tilde{\xi} \circ p \circ \tilde{\phi}(\iota(W), t) = \xi(\tilde{\phi}(\iota(W), t)) = \tilde{\phi}(\iota(W), t)\sigma(\tilde{\phi}(\iota(W), t)^{-1})$$

which is the multiplication by $j^{-1}e^{-\pi i t}je^{\pi i t} = e^{2\pi i t}$ on W and the identity on $W^\perp$, respectively. Then, for $x \in W$ and $y \in W^\perp$, we have

$$\begin{aligned} ((\tilde{\xi} \circ r \circ c_k)[W, t])(x \oplus y) &= ((\tilde{\xi} \circ r)[W, \widetilde{kt}])(x \oplus y) \\ &= xe^{2\pi i \widetilde{kt}} \oplus y \\ &= xe^{2\pi i k t} \oplus y \end{aligned}$$

and

$$\begin{aligned} ((\tilde{\xi} \circ m_k \circ r)[W, t])(x \oplus y) &= ((\mu_k \circ \tilde{\xi} \circ r)[W, t])(x \oplus y) \\ &= xe^{2\pi i k t} \oplus y. \end{aligned}$$

Hence $\tilde{\xi} \circ r \circ c_k = \tilde{\xi} \circ m_k \circ r$, therefore $r \circ c_k = m_k \circ r$ by Lemma 3.1(2). This completes the proof.

Therefore Theorem for the case $\mathbf{F} = (\mathbf{C}, \mathbf{H})$ follows from Lemma 4.1, Propositions 4.2, 4.3 and Theorem for $\mathbf{F} = \mathbf{C}$. This completes the proof of Theorem.

5. M_k-structure

For a given path-connected space X and an integer k, if there exists a self map h_k of X such that $h_k^*(x) = kx$ for all $x \in QH^*(X; \mathbf{Q})$, then we say [5] that X has an M_k-structure or that X is an M_k-space, where $QH^*(X; \mathbf{Q})$ is the indecomposable module of the rational cohomology ring $\tilde{H}^*(X; \mathbf{Q})$.

Note that any connected, finite (co-) H-space is an M_k-space for any non-negative integer k. So we think that if the space X is an M_k-space for all $k \in \mathbf{Z}$, then X must have some structure near (co-) H-space.

As a corollary of Theorem, we have

Corollary 5.1. *$U(2n)/Sp(n)$ and $Q_n(\mathbf{H})$ are M_k-spaces for any $k \in \mathbf{Z}$.*

Proposition 5.2. *$U(2n+1)/O(2n+1)$ and E_6/F_4 are M_k-space for any $k \in \mathbf{Z}$.*

Proof. According to Harris [3], in the above cases, the map $\psi: H \times G/H \rightarrow G$ defined by $\psi(h, gH) = h\xi(gH) = hg\sigma(g^{-1})$ is a rational equivalence. Therefore ξ induces an epi-morphism: $QH^*(G; \mathbf{Q}) \rightarrow QH^*(G/H; \mathbf{Q})$. Thus, from the commutative diagram of the right hand side in Lemma 3.1, it is clear that the map g_k in Lemma 3.1 gives the desired M_k-structure of G/H.

Hence $U(2n)/Sp(n)$, $U(2n+1)/O(2n+1)$ and E_6/F_4 are near H-spaces, and $Q_n(\mathbf{H})$ is a near co-H-space.

There are many symmetric spaces which can be considered far from H-spaces. An example is the following result of Glover and Homer [2].

Example 5.3. If $\mathbf{F}$ is $\mathbf{C}$ or $\mathbf{H}$ and $k \neq 0, \pm 1$, then $G_m(\mathbf{F}^n)$ is not an M_k-space for the following cases:

(1) $2 \leq m \leq 3$ *and* $n \geq 2m+1$,

(2) $m \geq 4$ *and* $n \geq 2m^2 - 1$.

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