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Variation formulas for harmonic modules of domains in $\mathbf{R}^3$

Dedicated to Professor Yukio Kusunoki on his 70th birthday

By

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1. Introduction

Let D be a domain spread over the complex plane $\mathbf{C}$ with C^ω smooth boundary ∂D. Suppose that D has a nono-trivial cycle γ. Then there exists a unique L^2 harmonic differential σ on D such that $\int_\gamma \omega = (\omega, *\sigma)_D$ for all C^∞ closed differentials ω on $\bar{D}$. We put $\mu = \|\sigma\|_D^2$. Then $*\sigma$ and μ are called *the reproducing differential* and *the harmonic module for* (D, γ) (see L. V. Ahofors [2]). The geometric meaning of μ was originally studied by Y. Kusunoki [6] and R. Accola [1]. We now let the domain $D(t)$ over $\mathbf{C}$ and the cycle $\gamma(t) \subset D(t)$ vary C^ω smoothly with a complex parameter t in a disk $B = \{|t| < r\}$, where $D(0) = D$ and $\gamma(0) = \gamma$. For any $t \in B$, we have the reproducing differential $*\sigma(t, z)$ and the harmonic module $\mu(t)$ for $(D(t), \gamma(t))$, so $\mu(t)$ is a function on B. We put $\omega(t, z) = \sigma(t, z) + i*\sigma(t, z) = f(z)\,dz$, $\|\omega\|(t, z) = |f(t, z)|$, and $\frac{\partial \omega}{\partial t} = \frac{\partial f}{\partial t} dz$ for $z \in D(t)$. We here put $\mathfrak{D} = \cup_{t\in B}(t, D(t))$ and $\partial\mathfrak{D} = \cup_{t \in B}(t, \partial D(t))$. Thus $\mathfrak{D}$ is a complex 2 dimensional domain spread over $B \times \mathbf{C}$. Let $\varphi(t, z)$ be a defining function of $\partial\mathfrak{D}$, that is, $\varphi(t, z)$ is a C^ω function in a neighborhood $\mathcal{V}$ of $\partial\mathfrak{D}$ over $B \times \mathbf{C}$ such that $\mathfrak{D} \cap \mathcal{V}$ (resp. $\partial\mathfrak{D}$) $= \{\varphi < 0$ (resp. $= 0)\}$ and $\frac{\partial \varphi}{\partial z} \neq 0$ on $\partial\mathfrak{D}$. We define, for $(t, z) \in \partial\mathfrak{D}$,

$$k_1(t, z) = \frac{\partial \varphi}{\partial t} \Big/ \left|\frac{\partial \varphi}{\partial z}\right|$$

$$k_2(t, z) = \left\{\frac{\partial^2 \varphi}{\partial t \partial \bar{t}}\left|\frac{\partial \varphi}{\partial z}\right|^2 - 2\Re\left\{\frac{\partial^2 \varphi}{\partial \bar{t} \partial z}\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial \bar{z}}\right\} + \left|\frac{\partial \varphi}{\partial t}\right|^2 \frac{\partial \varphi}{\partial z \partial \bar{z}}\right\} \Big/ \left|\frac{\partial \varphi}{\partial z}\right|^3. \quad (1.1)$$

Note that neither $k_1(t, z)$ nor $k_2(t, z)$ on $\partial\mathfrak{D}$ depends on the choice of $\varphi(t, z)$. In [4] we call $k_2(t, z)$ *the Levi curvature of* $\partial\mathfrak{D}$ *at* (t, z), and proved the following variation formulas:

$$\frac{\partial \mu(t)}{\partial t} = \frac{1}{2}\int_{\partial D(t)} k_1(t, z)\|\omega\|^2(t, z)\,|dz|$$

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$$\frac{\partial^2\mu(t)}{\partial t\partial\bar{t}}=\left\|\frac{\partial\omega}{\partial\bar{t}}(t,\cdot)\right\|^2_{D(t)}+\frac{1}{2}\int_{\partial D(t)}k_2(t,z)\|\omega\|^2(t,z)|dz|.$$

(See also F. Maitani [8], M. Taniguchi [9], and [12]). So, if $\mathfrak{D}$ is pseudoconvex, then $\frac{1}{\mu(t)}$ is a superharmonic function on B.

In this paper we study the case of $\mathbf{R}^3$. Let D be a domain in $\mathbf{R}^3$ bounded by a finite number of C^ω smooth boundary surfaces ∂D. Suppose that D has a non-trivial i-cycle γ_i ($i=1$ or 2). By H. Weyl [11], there exists a unique L^2 harmonic i-form $*\Omega_{3-i}$ on D such that

$$\int_{\gamma i}\omega=(\omega,*\Omega_{3-i})_D \quad \text{for all } C^\infty \text{ closed } i\text{-forms } \omega \text{ on } \bar{D}. \tag{1.2}$$

We call $*\Omega_{3-i}$ and $\mu_i=\|\Omega_{3-i}\|^2_D$ *the reproducing* i*-form* and *the harmonic* i*-module for* (D,γ_i). Note that Ω_{3-i} is C^ω smoothly extended up to ∂D. We write, on $\bar{D}$,

$$\begin{aligned}&\text{Case } i=1: \quad \Omega_2=\alpha_1\,dy\wedge dz+\alpha_2\,dz\wedge dx+\alpha_3\,dx\wedge dy\equiv\boldsymbol{\alpha}(x)\cdot *dx\\ &\text{Case } i=2: \quad \Omega_1=a_1\,dx+a_2\,dy+a_3\,dz\equiv\mathbf{a}(x)\cdot dx\end{aligned}$$

where $dx=(dx,dy,dz)$. By (1.2), $\mathbf{a}(x)$ and $\boldsymbol{\alpha}(x)$ restricted on ∂D are normal and tangential, respectively. At any $x\in\partial D$ such that $\boldsymbol{\alpha}(x)\neq 0$ (where the set $\{x\in\partial D|\boldsymbol{\alpha}(x)=0\}$ is real one dimensional at most), we shall use notation:

$$\boldsymbol{e}_{\Omega_2}(x)=\frac{\boldsymbol{\alpha}(x)}{\|\boldsymbol{\alpha}(x)\|}, \tag{1.3}$$

which is called *the tangent vector field on* ∂D *associated with* Ω_2.

Now let $D(t)\subset\subset\mathbf{R}^3$ and $\gamma_i(t)\subset D(t)$ vary C^ω smoothly with a real parament t in an interval $I=(-\rho,\rho)$, where $D(0)=D$ and $\gamma_i(0)=\gamma_i$. For any $t\in I$, we have the reproducing i-form $*\Omega_{3-i}(t,x)$ and the harmonic i-module $\mu_i(t)$ for $(D(t),\gamma_i(t))$. When we write $\Omega_1(t,x)=\mathbf{a}(t,x)\cdot dx$, we define $\|\Omega_1\|^2(t,x)=\|\mathbf{a}(t,x)\|^2(\geq 0)$, and $\frac{\partial\Omega_1}{\partial t}(t,x)=\frac{\partial\mathbf{a}}{\partial t}\cdot dx$. Analogously, we define $\|\Omega_2\|^2(t,x)$ and $\frac{\partial\Omega_2}{\partial t}(t,x)$. We consider the real 4 dimensional domain $\mathfrak{D}=\cup_{t\in I}(t,D(t))$ in the product space $I\times\mathbf{R}^3$, and put $\partial\mathfrak{D}=\cup_{t\in I}(t,\partial D(t))$. Let $\varphi(t,x)$ be a C^ω defining function of $\partial\mathfrak{D}$ in $I\times\mathbf{R}^3$. Instead of the Levi curvature $k_2(t,x)$ in (1.1), we introduce two kinds of curvatures $K_2(t,x)$ and $\widetilde{K}_2(\boldsymbol{e},t,x)$ of $\partial\mathfrak{D}$ as follows: First let $\boldsymbol{e}\in\mathbf{R}^3$ with $\|\boldsymbol{e}\|=1$. For $(t,x)\in\partial\mathfrak{D}$, we put

$$K_1(t,x)=\frac{1}{\|\nabla\varphi\|}\frac{\partial\varphi}{\partial t} \tag{1.4}$$

$$L_{\boldsymbol{e}}(t,x)=\frac{1}{\|\nabla\varphi\|^3}\left\{\frac{\partial^2\varphi}{\partial t^2}\left|\frac{\partial\varphi}{\partial\boldsymbol{e}}\right|^2-2\frac{\partial^2\varphi}{\partial t\partial\boldsymbol{e}}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\boldsymbol{e}}+\left|\frac{\partial\varphi}{\partial t}\right|^2\frac{\partial^2\varphi}{\partial\boldsymbol{e}^2}\right\}, \tag{1.5}$$

where $\nabla=(\partial/\partial x_i)_{i=1,2,3}$ and $\partial^j\varphi/\partial\boldsymbol{e}^j=[\partial^j\varphi(t,x+s\boldsymbol{e})/\partial s^j]_{s=0}$ $(j=1,2)$. We note that neither $K_1(t, x)$ nor $L_{\boldsymbol{e}}(t, x)$ on $\partial\mathfrak{D}$ depends on the choice of $\varphi(t, x)$. Next, let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form an orthonormal base of $\mathbf{R}^3$. We put

$$K_2(t, x) = L_{\boldsymbol{e}_1}(t, x) + L_{\boldsymbol{e}_2}(t, x) + L_{\boldsymbol{e}_3}(t, x)$$
$$= \frac{1}{\| \nabla \varphi \|^3} \left\{ \frac{\partial^2 \varphi}{\partial t^2} \| \nabla \varphi \|^2 - 2 \sum_{i=1}^{3} \left\{ \frac{\partial^2 \varphi}{\partial t \partial x_i} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x_i} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \Delta \varphi \right\}, \quad (1.6)$$

where $\Delta = \sum_{i=1}^{3} \partial^2 / \partial x_i^2$. Thus, $K_2(t, x)$ is independent of the choice of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. In [7] we call $K_2(t, x)$ *the (real) Levi curvature of* $\partial\mathfrak{D}$ at (t, x). Finally, let $\boldsymbol{e}$ be a unit *tangent* vector of the surface $\partial D(t)$ in $\mathbf{R}^3$ at x, and denote by $\boldsymbol{n}$ the unit outer normal vector of $\partial D(t)$ at x. We put $\boldsymbol{e}' = \boldsymbol{n} \times \boldsymbol{e}$ and define

$$\widetilde{K}_2(\boldsymbol{e}, t, x) = L_{\boldsymbol{e}}(t, x) - L_{\boldsymbol{e}'}(t, x) + L_{\boldsymbol{n}}(t, x). \quad (1.7)$$

We denote by dS_x the Euclidean surface area element of $\partial D(t)$ at x. Then we shall show the following variation formulas for $t \in I$:

Theorem I.

$$\frac{d\mu_1(t)}{dt} = \int_{\partial D(t)} K_1(t, x) \| \Omega_2 \|^2 (t, x) dS_x \quad (1.8)$$

$$\frac{d^2 \mu_1(t)}{dt^2} = 2 \left\| \frac{\partial \Omega_2}{\partial t}(t, \cdot) \right\|_{D(t)}^2 + \int_{\partial D(t)} \widetilde{K}_2(\boldsymbol{e}_{\Omega_2}, t, x) \| \Omega_2 \|^2 (t, x) dS_x. \quad (1.9)$$

Theorem II.

$$\frac{d\mu_2(t)}{dt} = \int_{\partial D(t)} K_1(t, x) \| \Omega_1 \|^2 (t, x) dS_x \quad (1.10)$$

$$\frac{d^2 \mu_2(t)}{dt^2} = 2 \left\| \frac{\partial \Omega_1}{\partial t}(t, \cdot) \right\|_{D(t)}^2 + \int_{\partial D(t)} K_2(t, x) \| \Omega_1 \|^2 (t, x) dS_x. \quad (1.11)$$

Since Theorem II can be proved by the combination of the ideas in papers [4] and [7], we give its brief proof in §4. On the other hand, to prove Theorem I, we need a new idea (relevant to the notion of equilibrium surface current density introduced in [13]), which will be precisely discussed in §5. In §6 we shall apply Theorems I and II for the z-axially symmetric domains to show the variation formulas related to the norm of functions which satisfy the following Stokes-Beltrami partial differential equations (see E. Beltrami [3]):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \pm \frac{1}{x} \frac{\partial u}{\partial x} = 0.$$

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2. Electromagnetic meaning of harmonic modules

Let D be a bounded domain with C^{ω} smooth surfaces $\Sigma(=\partial D)$ in $\mathbf{R}^3$. We put $D'=\mathbf{R}^3\setminus\bar{D}$, where $\bar{D}=D\cup\partial D$. For $i=1, 2$, we write

$C_i^{\infty}(D)$ (resp. $C_{i,0}^{\infty}(D)$) $=$the space of C^{∞} (resp. C_0^{∞}) i-forms in D
$Z_i^{\infty}(\bar{D})$ $=$the space of C^{∞} closed i-forms on $\bar{D}$
$H_i(D)$ $=$the space of L^2 harmonic i-forms in D.

We also denote by $B_i(D)$ or $Z_i(D)$ the closure of $dC_{i-1,0}^{\infty}(D)$ or $Z_i^{\infty}(\bar{D})$ in the space $L_i^2(D)$ of L^2 i-forms in D. Then Weyl's orthogonal decomposition theorems in [11] hold:

$$L_i^2(D)=Z_i(D)\dot{+}*B_{3-i}(D), \quad Z_i(D)=H_i(D)\dot{+}B_i(D). \tag{2.1}$$

Let $\omega_i\in C_i^{\infty}(U)$, where $U\supset\supset\Sigma$. If all theree coefficients of ω_i vanish on Σ, we write $\omega_i=0$ *on* Σ. If the restriction $\omega_i|_{\Sigma}$ of ω_i to the surface Σ is 0 as an i-form on Σ, we write $\omega_i=0$ *along* Σ. As an analogue to Ahlfor's definition [2], we put

$$H_{i0}(D)=\{\,\omega\in H_i(D)\,|\,\omega \text{ is of class } C^{\omega} \text{ on } \bar{D} \text{ and } \omega=0 \text{ along } \Sigma\,\}.$$

Concerning the reproducing $(3-i)$-form $*\,\Omega_i$ for (D, γ_{3-i}), we see from (1.2) and (2.1) that $\Omega_i\in H_{i0}(D)$.

Let us study the static electromagnetic meaning of Ω_i and μ_i by simple examples:

[I] For $b>a>0$, let Σ be a solenoid of torus type. That is, consider a circle $C=\{(x, z)=(b-a\cos\phi, a\sin\phi)\,|\,0\le\phi\le 2\pi\}$ in the (x, z)-plane with $x>0$. We rotate C around the z-axis to obtain the torus Σ. We use cylindrical coordinates $[r, \theta, z]$ of $\mathbf{R}^3$. Then the solenoid is the torus Σ equipped with equilibrium surface current density on Σ

$$J(x)\,dS_x=\frac{1}{2\pi ar}(z\cos\theta, z\sin\theta, b-r)\,dS_x,$$

where dS_x denotes the surface area element of Σ at x (see [13] in detail). We denote by D the solid torus bounded by Σ in $\mathbf{R}^3$. From Biot-Savart's law, the solenoid Σ induces the static magnetic field in $\mathbf{R}^3\setminus\Sigma$:

$$B(x)=\mathrm{rot}\Big(\frac{1}{4\pi}\int_{\Sigma}\frac{J(y)}{\|y-x\|}\,dS_y\Big)=\frac{1}{4\pi}\int_{\Sigma}\frac{y-x}{\|y-x\|^3}\times J(y)\,dS_y.$$

By use of the symmetry of Σ, we obtain

$$B(x)=\begin{cases}\dfrac{1}{2\pi r}(-\sin\theta, \cos\theta, 0) & \text{for } x\in D\\ 0 & \text{for } x\in D'.\end{cases}$$

The magnetic energy of B is defined by

$$\| B(x) \|_{\mathbf{R}^3}^2 = \int_D \left(\frac{1}{2\pi r}\right)^2 \{(-\sin\theta)^2 + (\cos\theta)^2\}\, dv_x = b - \sqrt{b^2 - a^2}.$$

Now consider a circle $\gamma_1 = \{(b\cos\theta, b\sin\theta, 0) \mid 0 \le \theta \le 2\pi\}$ in D. We thus have the reproducing 1-form $*\Omega_2 = \boldsymbol{\alpha}(x) \cdot dx$ and the harmonic 1-module μ_1 for (D, γ_1). Then we have the relationship between the harmonic 2-form Ω_2 and the magnetic field B:

Proposition 2. 1. $\boldsymbol{\alpha}(x) = \dfrac{B(x)}{\| B(x) \|_{\mathbf{R}^3}^2}$ *in* D, $\mu_1 = \dfrac{1}{\| B(x) \|_{\mathbf{R}^3}^2}$.

Proof. We put $\tau(x) = (2\pi r)^{-1}(-\sin\theta\, dy \wedge dz + \cos\theta\, dz \wedge dx)$ and $p(x) = -(2\pi)^{-1}(\log r)\, dz$ in D. Hence, $\tau(x) = dp(x)$, $d * p(x) = 0$ in D, and $*\tau = d\theta/2\pi \in Z_1^\infty(\overline{D})$. Let $\forall \omega \in Z_1^\infty(\overline{D})$. We put $C_0 := \{(r, z) = (b - a\cos\phi, a\sin\phi) \mid 0 \le \phi \le 2\pi\}$. For $\forall (r, z) \in C_0$, we take a circle on ∂D: $\gamma_\theta = \{(r\cos\theta, r\sin\theta, a\sin\phi) \in \partial D \mid 0 \le \theta \le 2\pi\}$. Since $\int_{\gamma_\theta} = \int_{\gamma_1} \omega$ by $\gamma_\theta \sim \gamma_1$ on $\overline{D}$, we have

$$(\omega, *\tau)_D = -\int_\Sigma \omega \wedge p = \frac{1}{2\pi}\int_{C_0} \left\{\int_{\gamma_\theta} \omega\right\} \log r\, dz = \left\{\int_{\gamma_1} \omega\right\} (b - \sqrt{b^2 - a^2}).$$

Therefore, $*\Omega_2 = *\tau / \| B(x) \|_{\mathbf{R}^3}^2$, by which Proposition 2.1 follows.

[**II**] For $b > a > 0$, let D be a condenser of shell type. That is, $D \subset\subset \mathbf{R}^3$ is a domain between two concentric electric conductors $K_a = \{\| x \| \le a\}$ and $K_b = \{\| x \| \ge b\}$ with charge $+1$ and -1, respectively. Hence, $\partial D = C_b - C_a$ where C_a, $C_b = \{\| x \| = a, b\}$. By Coulomb's law, their equilibrium density distribution

$$\rho(x)\, dS_x = \begin{cases} \dfrac{1}{4\pi a^2}\, dS_x & \text{on } C_a \\[2ex] \dfrac{1}{4\pi b^2}\, dS_x & \text{on } C_b \end{cases}$$

induces the static electric field E in $\mathbf{R}^3 \backslash \Sigma$ such that

$$E(x) = \nabla\left(\frac{1}{4\pi}\int_{C_b - C_a} \frac{1}{\| y - x \|} \rho(y)\, dS_y\right) = \frac{1}{4\pi}\int_{C_b - C_a} \frac{y - x}{\| y - x \|^3} \rho(y)\, dS_y.$$

By simple calculation we obtain

$$E(x) = \begin{cases} \dfrac{x}{4\pi \| x \|^3} & \text{for } x \in D \\[2ex] 0 & \text{for } x \in D'. \end{cases}$$

The electric energy of E is defined by

$$\| E(x) \|_{\mathbf{R}^3}^2 = \int_D \left\| \frac{x}{4\pi \| x \|^3} \right\|^2 dv_x = \frac{1}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Now consider the positively oriented sphere $\gamma_2=\{\|x\|=(a+b)/2\}$ in D. Then we have the reproducing 2-form $*\Omega_1=\mathbf{a}(x)\cdot *dx$. Then we have the relationship between the harmonic 1-form Ω_1 and the electric field E:

Proposition 2. 2. $\mathbf{a}(x)=\dfrac{E(x)}{\|E(x)\|^2_{\mathbf{R}^3}}$ *in* D, $\mu_2=\dfrac{1}{\|E(x)\|^2_{\mathbf{R}^3}}$.

Proof. We put $\tau(x)=(4\pi\|x\|^3)^{-1}\sum_{i=1}^3 x_i\,dx_i$ and $u(x)=-(4\pi\|x\|)^{-1}$. Then $\tau(x)=du(x)\in H_1(\bar{D})$. Let $\forall\,\omega\in Z_2^\infty(\bar{D})$. Since $\gamma_2\sim\{\|x\|=a\}\sim\{\|x\|=b\}$ on $\bar{D}$, we have

$$(\omega,\ *\tau)_D=\int_D d(u\omega)=-\frac{1}{4\pi}\int_{\partial D}\frac{1}{\|x\|}\,\omega=\left(\frac{1}{a}-\frac{1}{b}\right)\int_{\gamma_2}\omega.$$

Therefore, $*\Omega_1=*\tau/\|E(x)\|^2_{\mathbf{R}^3}$, by which Proposition 2.2 follows.

3. Smooth variatios and Levi curvatures

Let $I=(-\rho,\rho)\subset\mathbf{R}$. Given any set $\mathcal{G}$ in $I\times\mathbf{R}^3$, we put $G(t):=\{x\in\mathbf{R}^3|\,(t,x)\in\mathcal{G}\}$ for each $t\in I$. We call $G(t)$ *the fiber of* $\mathcal{G}$ *at* t. Now consider a 4 dimensional domain $\mathcal{D}$ in $I\times\mathbf{R}^3$ such that $D(t)\neq 0$ for any $t\in I$. We denote by $\partial\mathcal{D}$ the boundary of $\mathcal{D}$ in $I\times\mathbf{R}^3$. We regard $\mathcal{D}$ as a variation of domains $D(t)$ in $\mathbf{R}^3$ with parament $t\in I$, and write

$$\mathcal{D}:\ t\longrightarrow D(t),\ t\in I.$$

Assume that there exists a C^ω-function $\varphi(t,x)$ defined in a neighborhood $\mathcal{V}$ of $\partial\mathcal{D}$ in $I\times\mathbf{R}^3$ such that (1) $\mathcal{D}\cap\mathcal{V}=\{(t,x)\in\mathcal{V}|\varphi(t,x)<0\}$, $D(t)\cap V(t)=\{x\in V(t)|\varphi(t,x)<0\}$ for $t\in I$, (2) $\nabla\varphi(t,x):=\left(\dfrac{\partial\varphi}{\partial x_i}\right)_{i=1,2,3}(t,x)\neq 0$ for any $x\in\partial D(t)$. Then we say that $\mathcal{D}$ is a C^ω *smooth variation*, and $\varphi(t,x)$ is *a* C^ω defining *function of* $\partial\mathcal{D}$. By (3), $\partial D(t)$ for each $t\in I$ is C^ω smooth in $\mathbf{R}^3$. We thus have

$$\mathcal{D}=\bigcup_{t\in I}(t,D(t)),\qquad \partial\mathcal{D}=\bigcup_{t\in I}(t,\partial D(t)).$$

In Introduction we defined quantities $L_{\boldsymbol{e}}(t,x)$, $K_2(t,x)$, $\widetilde{K}_2(\boldsymbol{e},t,x)$ on $\partial\mathcal{D}$. We shall represent these by means of usual normal curvatures. Let $P=(t,x)\in\partial\mathcal{D}$. First, we denote by $\boldsymbol{n}_P$ the unit outer normal vector of the 3 dim. surface $\partial\mathcal{D}$ at the point P in $I\times\mathbf{R}^3$. We consider the 2 dim. plane $\pi_{t,\boldsymbol{n}_P}$ in $I\times\mathbf{R}^3$ which passes through P and is generated by the 2 vectors $\{(1,(0,0,0)),\boldsymbol{n}_P\}$. We denote by $\boldsymbol{v}_t$ the unit tangent vector of the 1 dim. curve $\pi_{t,\boldsymbol{n}_P}\cap\partial\mathcal{D}$. Thus,

$$\frac{1}{\rho_t}:=\text{the normal curvature of } \partial\mathcal{D} \text{ for } \boldsymbol{v}_t \text{ at the point } P \tag{3.1}$$

is deternined, which is called *the t-normal curvature of* $\partial\mathcal{D}$ *at* P. Next, from $x\in\partial D(t)\subset\mathbf{R}^3$, we denote by $\boldsymbol{n}_x$ the unit outer normal vector of the 2 dim. surface

$\partial\mathfrak{D}(t)$ in $\mathbf{R}^3$ at the point x and, by $\boldsymbol{T}_x(=\boldsymbol{T}(t)_x)$ the set of all unit tangent vectors of $\partial D(t)$ at x. Thus, for any $\boldsymbol{e}=(e_1, e_2, e_3)\in\boldsymbol{T}_x$,

$$\frac{1}{\rho_e}:=\text{the normal curvature of } \partial D(t) \text{ for } \boldsymbol{e} \text{ at the point } x$$

$$=\sum_{i,j=1}^{3}\left(\frac{1}{\|\nabla\varphi\|}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right)_{(t,x)} e_i e_j \tag{3.2}$$

is determined. Finally, we denote by H and K the mean and the Gaussian curvatures of $\partial D(t)$ at x:

$$H=\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right), \qquad K=\frac{1}{\rho_1}\frac{1}{\rho_2},$$

wehre $1/\rho_i$ $(i=1, 2)$ are the principal curvatures of $\partial D(t)$ at x such that $1/\rho_1\geq 1/\rho_2$.

Proposition 3. 1. *It holds for* $(t, x)\in\partial\mathfrak{D}$,

$$L_e(t, x)=\begin{cases} K_1(t, x)^2\dfrac{1}{\rho_e} & \text{for } \boldsymbol{e}\in\boldsymbol{T}_x \\ (1+K_1(t, x)^2)^{3/2}\dfrac{1}{\rho_t} & \text{for } \boldsymbol{e}=\boldsymbol{n}_x. \end{cases} \tag{3.3}$$

Proof. Since both sides are invariant under the Euclidean motions, we may assume that $(t, x)=(0, 0)$ and $\boldsymbol{n}_x(=\boldsymbol{n})=(0, 0, 1)$. Hence, $\partial\mathfrak{D}$ near $(t, (x, y, z))=(0, 0)$ is represented in the form $z=\phi(t, (x, y))$ where $\phi(0, (x, y))=O(x^2+y^2)$, so that $\varphi(t, \boldsymbol{x})=z-\phi(t, (x, y))$ is a defining function of $\partial\mathfrak{D}$ near $(0, 0)$. In case $\boldsymbol{e}\in\boldsymbol{T}_x$, we may assume $\boldsymbol{e}=(1, 0, 0)$. By direct calculation, we have

$$K_1(0, 0)=-\frac{\partial\phi}{\partial t},\quad \frac{1}{\rho_e}=-\frac{\partial^2\phi}{\partial x^2},\quad \frac{1}{\rho_t}=-\frac{\partial^2\phi}{\partial t^2}\Big/\left(1+\left(\frac{\partial\phi}{\partial t}\right)^2\right)^{3/2}$$

$$L_e=-\left(\frac{\partial\phi}{\partial t}\right)^2\frac{\partial^2\phi}{\partial x^2},\quad L_n=-\frac{\partial^2\phi}{\partial t^2}$$

evaluated at $(0, 0)$. Proposition 3.1 follows by these formulas.

Let $(t, x)\in\partial\mathfrak{D}$ and $\boldsymbol{e}\in\boldsymbol{T}_x$. We put $\boldsymbol{n}:=\boldsymbol{n}_x$ and $\boldsymbol{e}':=\boldsymbol{n}_x\times\boldsymbol{e}\in\boldsymbol{T}_x$. We can consider the normal curvatures $1/\rho_e$ and $1/\rho_{e'}$ of the surface $\partial D(t)$ for $\boldsymbol{e}$ and $\boldsymbol{e}'$ at x, respectively. Concerning $K_2(t, x)$ and $\widetilde{K}_2(\boldsymbol{e}, t, x)$ defined by (1.6) and (1.7) we have from (3.3).

$$K_2(t, x)=(1+K_1(t, x)^2)^{3/2}\ \frac{1}{\rho_t}+K_1(t, x)^2\left(\frac{1}{\rho_e}+\frac{1}{\rho_{e'}}\right) \tag{3.4}$$

$$\widetilde{K}_2(\boldsymbol{e}, t, x)=((1+K_1(t, x)^2)^{3/2}\ \frac{1}{\rho_t}+K_1(t, x)^2\left(\frac{1}{\rho_e}-\frac{1}{\rho_{e'}}\right) \tag{3.5}$$

$$=K_2(t, x)-2K_1(t, x)^2\frac{1}{\rho_{e'}}. \tag{3.6}$$

We shall study the geometric meaning of $K_1(t, x)$ and $K_2(t, x)$. Let $x_0 \in \partial D(0)$ and let C_{x_0}: $x = \boldsymbol{x}(t)$ for $t \in I$ be the orthogonal trajectory passing through x_0 of the family of surfaces $\{\partial D(t)\}_{t\in I}$. Namely, $x = \boldsymbol{x}(t)$ is the solution of the following differential equation in I:

$$\dot{\boldsymbol{x}} = -K_1(t, \boldsymbol{x})\boldsymbol{n}_x \quad \text{with} \quad \boldsymbol{x}(0) = x_0, \tag{3.7}$$

where we put $\boldsymbol{n}_x = \boldsymbol{n}_{x(t)}$ and $\dot{\boldsymbol{x}} = d\boldsymbol{x}(t)/dt$. Therefore, if we put $\widehat{\partial D(t)} = (t, \partial D(t))$ and $\widehat{C_{x_0}} = \bigcup_{t\in I}(t, C_{x_0}(t))$ in $I \times \mathbf{R}^3$, then we have the following two coordinations of $\partial\mathfrak{D}$:

$$\partial\mathfrak{D} = \bigcup_{t\in I}\widehat{\partial D(t)} = \bigcup_{x_0\in\partial D(0)}\widehat{C_{x_0}} \quad \text{such that} \quad \widehat{C_{x_0}} \perp \widehat{\partial D(t)}$$

for $\forall t \in I$ and $\forall x_0 \in \partial D(0)$. By simple calculation we have

$$L_{\boldsymbol{n}}(t, \boldsymbol{x}(t)) = \frac{d}{dt}K_1(t, \boldsymbol{x}(t)) \quad \text{for } t \in I,$$

so that $\ddot{\boldsymbol{x}} = -L_{\boldsymbol{n}}(t, \boldsymbol{x})\boldsymbol{n}_x - K_1(t, \boldsymbol{x})(\partial\boldsymbol{n}_x/\partial t)$ on I. Since $\boldsymbol{n}_x \perp (\partial\boldsymbol{n}_x/\partial t)$ on C_{x_0}, it follows from (3.7) that

$$K_1(t, \boldsymbol{x}) = -\dot{\boldsymbol{x}}(t) \cdot \boldsymbol{n}_x, \qquad L_{\boldsymbol{n}}(t, \boldsymbol{x}) = -\ddot{\boldsymbol{x}}(t) \cdot \boldsymbol{n}_x.$$

We assume $\dot{\boldsymbol{x}}(t) \neq 0$, and denote by s the arc length of C_{x_0} such that $ds/dt > 0$. We put $\boldsymbol{x}^{(i)} = d^i\boldsymbol{x}/ds^i$ $(i = 1, 2)$, and define $\varepsilon := \pm 1$ according to $\boldsymbol{x}' = \mp\boldsymbol{n}_x$. In general, ± 1 changes to ∓ 1 along the envelope of the family of surfaces $\{\partial D(t)\}_{t\in I}$. Since $\dot{\boldsymbol{x}} = (ds/dt)\,\boldsymbol{x}'$ and $\ddot{\boldsymbol{x}} = (ds^2/dt^2)\,\boldsymbol{x}' + (ds/dt)^2\,\boldsymbol{x}''$, it follows from $\boldsymbol{x}'' \perp \boldsymbol{n}_x$ that, for $\forall \boldsymbol{e} \in \boldsymbol{T}_x$,

$$L_{\boldsymbol{e}}(t, x) = \left(\frac{ds}{dt}\right)^2\frac{1}{\rho_e}, \quad L_{\boldsymbol{e}'}(t, x) = \left(\frac{ds}{dt}\right)^2\frac{1}{\rho_{e'}}, \quad L_{\boldsymbol{n}}(t, x) = \varepsilon\frac{d^2s}{dt^2},$$

$$K_1(t, x) = \varepsilon\frac{ds}{dt}, \quad K_2(t, x) = \varepsilon\frac{d^2s}{dt^2} + 2\left(\frac{ds}{dt}\right)^2 H(t, x).$$

We give sufficient conditions for which $K_2(t, x)$ or $\widetilde{K}_2(t, x) \geq 0$ on $\partial\mathfrak{D}$.

Proposition 3. 2. 1. *If $\mathfrak{D}$ is a convex domain in $\mathbf{R}^4$, then $K_2(t, x) \geq 0$ on $\partial\mathfrak{D}$.*

2. (*a*) *If $\frac{1}{\rho_t} \geq \frac{4}{3\sqrt{3}}|\mathrm{H}|$ on $\partial\mathfrak{D}$, then $K_2(t, x) \geq 0$.*

(*b*) *If $\frac{1}{\rho_t} \geq \frac{4}{3\sqrt{3}}\sqrt{H^2 - K}$ on $\partial\mathfrak{D}$, then $\widetilde{K}_2(\boldsymbol{e}, t, x) \geq 0$ for all $\boldsymbol{e} \in \boldsymbol{T}_x$.*

Proof. If $\mathfrak{D}$ is convex in $\mathbf{R}^4$, then $L_{\boldsymbol{e}}(t, x)$, $L_{\boldsymbol{e}'}(t, x)$, and $L_{\boldsymbol{n}}(t, x) \geq 0$ on $\partial\mathfrak{D}$, by which 1 follows. By (3.4), we have

$$K_2 = \frac{1}{\rho_t}(1 + K_1^2)^{3/2} + 2K_1^2 H \geq (1 + K_1^2)^{3/2}\left\{\frac{1}{\rho_t} - \frac{4}{\sqrt{3}}|H|\right\},$$

by which 2 (a) follows. By (3.5), we have

$$\widetilde{K}_2 \geq (1+K_1^2)^{3/2}\left\{\frac{1}{\rho_t}-\frac{2}{3\sqrt{3}}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right\},$$

by which 2 (b) follows.

The following proposition will be useful in this paper:

Proposition 3.3. *Let $u(t, x)$ be a C^ω function in a neighborhood $\mathcal{V}$ of $\partial\mathfrak{D}$ in $I\times\mathbf{R}^3$ such that $\partial\mathfrak{D}(t)\subset\subset V(t)\subset\mathbf{R}^3$ for each $t\in I$. Assume that*
(1) *$u(t, x)=$const. c on each component of $\partial\mathfrak{D}$,*
(2) *For any fixed $t\in I$, $u(t, x)$ is harmonic for $x\in V(t)$.*
Then it holds for $(t, x)\in\partial\mathfrak{D}$ such that $\frac{\partial u}{\partial n_x}(t, x)\neq 0$,

$$\frac{\partial u}{\partial t}=K_1(t, x)\frac{\partial u}{\partial n_x} \tag{3.8}$$

$$\frac{\partial^2 u}{\partial t^2}=\left\{K_2(t, x)\left(\frac{\partial u}{\partial n_x}\right)^2+\frac{\partial}{\partial n_x}\left(\frac{\partial u}{\partial t}\right)^2\right\}\Big/\left(\frac{\partial u}{\partial n_x}\right). \tag{3.9}$$

Proof. Let $(t_0, x_0)\in\partial\mathfrak{D}$ at which $\partial u/\partial n_x\neq 0$. Say, $(\partial u/\partial n_x)(t_0, x_0)>0$. Then, from (1), $(u(t, x)-c)$ in $\mathcal{V}$ is a C^ω defining function of $\partial\mathfrak{D}$ near (t_0, x_0), and $\frac{\partial u}{\partial n_x}=\|\nabla u\|$ at (t_0, x_0). So, definition (1.4) of $K_1(t_0, x_0)$ implies (3.8). Further, since $\frac{\partial u}{\partial x_i}\Big/\frac{\partial u}{\partial n_x}=\cos\theta_i$, where θ_i is the angle between $\boldsymbol{n}_x$ and the x_i-axis, we have

$$2\sum_{i=1}^{3}\left\{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t\partial x_i}\frac{\partial u}{\partial x_i}\Big/\frac{\partial u}{\partial n_x}\right\}=\frac{\partial}{\partial n_x}\left(\frac{\partial u}{\partial t}\right)^2 \quad \text{at } (t_0, x_0).$$

So, formula (1.6) of $K_2(t, x)$ under condition (2) implies

$$K_2(t_0, x_0)=\left\{\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial n_x}-\frac{\partial}{\partial n_x}\left(\frac{\partial u}{\partial t}\right)^2\right\}\Big/\left(\frac{\partial u}{\partial n_x}\right)^2 \qquad \text{at } (t_0, x_0),$$

by which (3.9) follows.

4. Proof of Theorem II

Let $D\subset\subset\mathbf{R}^3$ be a domain bounded by C^ω smooth boundary surfaces ∂D. We denote by $\{C_j\}_{j=1,\cdots,q}$ the boundary components of D, so that $\partial D=\sum_{j=1}^q C_j$. Then D carries the harmonic function $u_j(x)$ such that

$$u_j(x)=\begin{cases}1 & \text{on } C_j\\ 0 & \text{on } \partial D\setminus C_j.\end{cases}$$

We call $u_j(x)$ *the harmonic measure for* (D, C_j). Let γ_j be a 2-cycle in D such that $\gamma_j \sim C_j$ (homologous) on $\overline{D}$, and denote by $*\Omega_j(x)$ and $\mu_2(t)$ the reproducing 2-form and the harmonic 2-module for (D, γ_j). By Stokes formula we then have $\Omega_j(x)=du_j(x)$ on $\overline{D}$.

Let $\mathfrak{D}$: $t \rightarrow D(t)$, $t \in I$ be a C^{ω} smooth variation. For each $t \in I$, we denote by $\{C_j(t))\}_{j=1,\cdots,q}$ the boundary components of the domain $D(t)$ such that $\partial D(t)=\sum_{j=1}^{q} C_j(t)$, and by $u_j(t, x)$ the harmonic measure for $(D(t), C_j(t))$. Let $\gamma_2(t)$ be a 2-cycle in $D(t)$ which varies smoothly with $t \in I$ in $\mathfrak{D}$. Therefore, $\gamma_2(t) \sim \sum_{j=1}^{q} n_j C_j(t)$ on $\overline{D(t)}$, where n_j are integers independent of $t \in I$. We denote by $*\Omega_1(t, x)$ the reproducing 2-form for $(D(t), \gamma_2(t))$. We have $\Omega_1(t, x)=dU(t, x)$, where $U(t, x)=\sum_{j=1}^{q} n_j u_j(t, x)$. Let us prove (1.10) and (1.11). It suffices to prove these at $t=0$. Since $\partial\mathfrak{D}$ is C^{ω} smooth, we find a small interval $I_0 (\subset I)$ centered at 0 such that, for any $t \in I_0$, $U(t, x)$ is harmonic on $\overline{D(0)}$ and $\gamma_2(t) \sim \gamma_2(0)$ in $D(0) \cup D(t)$. Then

$$\mu_2(t)=\int_{\gamma_2(0)} *\Omega_1(t, x)=(\Omega_1(t, \cdot), \Omega_1(0, \cdot))_{D(0)}=\int_{\partial D(0)} U(t, x) *dU(0, x). \quad (4.2)$$

After differentiating both sides with respect to t, $k(=1, 2)$ times, we put $t=0$ to obtain

$$\frac{\partial^k \mu_2}{dt^k}(0)=\left(\frac{\partial^k \Omega_1}{\partial t^k}(0, \cdot), \Omega_1(0, \cdot)\right)_{D(0)}=\int_{\partial D(0)} \frac{\partial^k U}{\partial t^k}(0, x) *dU(0, x). \quad (4.3)$$

Since $U(t, x)$ is const. on each component of $\partial\mathfrak{D}$, it follows by (3.8) that

$$\frac{\partial U}{\partial t}=K_1(t, x)\frac{\partial U}{\partial n_x} \quad \text{on } \partial\mathfrak{D}.$$

Note that $*dU(0, x)=\frac{\partial U(0, x)}{\partial n_x} dS_x$ along $\partial D(0)$. Applying (3.8) for $k=1$, we thus obtain

$$\frac{\partial \mu_2}{\partial t}(0)=\int_{\partial D(0)} K_1(0, x)\left(\frac{\partial U}{\partial n_x}(0, x)\right)^2 dS_x.$$

Since $\| \Omega_1 \|^2(0, x)=\left(\frac{\partial U(0, x)}{\partial n_x}\right)^2$ on $\partial D(0)$, we have (1.10). To prove (1.11), we get by (3.9)

$$\frac{\partial^2 U}{\partial t^2}=\left\{K_2(t, x)\left(\frac{\partial U}{\partial n_x}\right)^2+\frac{\partial}{\partial n_x}\left(\frac{\partial U}{\partial t}\right)^2\right\}\Big/\left(\frac{\partial U}{\partial n_x}\right) \quad \text{on } \partial\mathfrak{D}.$$

Applying (4.3) for $k=2$, we obtain by Stokes formula

$$\begin{aligned}\frac{d^2 \mu_2}{dt^2}(0)&=\int_{\partial D(0)} K_2(0, x)\left(\frac{\partial U}{\partial n_x}(0, x)\right)^2 dS_x+\int_{\partial D(0)} \frac{\partial}{\partial n_x}\left(\frac{\partial U}{\partial t}(0, x)\right)^2 dS_x \\ &=\int_{\partial D(0)} K_2(0, x)\| \Omega_1 \|^2(0, x) dS_x+\int_{D(0)} \Delta\left(\frac{\partial U}{\partial t}(0, x)\right)^2 dv_x.\end{aligned}$$

Since $\Delta\left(\frac{\partial U}{\partial t}\right)(0, x)=0$ and $\frac{\partial \Omega_1}{\partial t}(0, x)=d\left(\frac{\partial U}{\partial t}\right)(0, x)$ on $\overline{D(0)}$, the last integral is equal to

$$2\int_{D(0)}\left\{\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2+\left(\frac{\partial^2 U}{\partial t\partial y}\right)^2+\left(\frac{\partial^2 U}{\partial t\partial z}\right)^2+\frac{\partial U}{\partial t}\Delta\left(\frac{\partial U}{\partial t}\right)\right\}_{(0,x)} dv_x=2\left\|\frac{\partial \Omega_1}{\partial t}(0, x)\right\|^2_{D(0)},$$

which proves (1.11).

Corollary 4. 1. *If $K_2(t, x)\geq 0$ on $\partial\mathfrak{D}$, then $\frac{1}{\mu_2(t)}$ is a concave function on I.*

Proof. Assume $K_2(t, x)\geq 0$ *on* $\partial\mathfrak{D}$. Then, (1.11) implies $\mu''(t)\geq 2\|\frac{\partial \Omega_1}{\partial t}\|^2_{D(t)}$, and (4.3) implies $|\mu_1'(t)|\leq \mu_1(t)\left\|\frac{\partial \Omega_1}{\partial t}\right\|^2_{D(t)}$. Hence, $\left(\frac{1}{\mu_2(t)}\right)''\geq 0$.

5. Proof of Theorem I

Let $\mathfrak{D}$: $t\rightarrow D(t)$, $t\in I$ be a C^ω smooth variation and let a 1-cycle $\gamma_1(t)$ in $D(t)$ vary smoothly with parament $t\in I$. For $t\in I$, we denote by $*\Omega_2(t, \cdot)$ and $\mu_1(t)$ the reproducing 1-form and the harmonic 1-module for $(D(t), \gamma_1(t))$. Let us prove (1.8) and (1.9). It suffices to prove these at $t=0$. We may assume that each $\gamma_1(t)$ is a C^∞ closed curve in $D(t)$. Like in [13] we need a rather concrete costruction of the 2-form $\Omega_2(t, x)$. We first take the u-axially symmetric solid torus $G:=L\times A$ in the (u, v, w)-space $\mathbf{R}^3$ such that $L=\{|u|<1\}$, and $A=\{1/2<\sqrt{v^2+w^2}<2\}$. In G, we take the circle $C_0=\{(0, \cos\theta, \sin\theta)\,|\,0\leq\theta\leq 2\pi\}$ and the rectangle $S_0=L\times\{(v, 0)\in A\,|\,1/2<v<2\}$, so that $S_0\times C_0$ (intersection number) $=1$. We here construct C^∞ functions $\chi(u)$ on $\overline{L}$ and $\varphi(v, w)$ on $\overline{A}$ such that

$$\chi(u)=\begin{cases}0 & \text{on } [-1, -1/2]\\ 1 & \text{on } [1/2, 1]\end{cases}\qquad \varphi(v, w)=\begin{cases}0 & \text{on } 1/2\leq\sqrt{v^2+w^2}\leq 2/3\\ 1 & \text{on } 3/2\leq\sqrt{v^2+w^2}\leq 2,\end{cases}$$

and put $\sigma_0=d\chi(u)\wedge d\varphi(v, w)\in Z^\infty_{20}(G)$. We next take a tubular neiborgood $\widetilde{G}$ of $\gamma_1(0)$ in $D(0)$. We find an interval I_0 centered at 0 such that $\gamma_1(t)\subset\widetilde{G}\subset\subset D(t)$ for all $t\in I_0$. So, we may assume $\gamma_1(t)=\gamma_1(0)$ for any $t\in I_0$. We may also assume that $\widetilde{G}$ admits a C^∞ (orientation preseving) transformation T: $\widetilde{G}\mapsto G$ with $T(\gamma_1(0))=C_0$. We denote by $T\#\sigma_0$ the pull back of the above σ_0 by T, so that $T\#\sigma_0\in Z^\infty_{20}(\widetilde{G})$. If we set $\tilde{\sigma}(x):=T\#\sigma_0$ (resp. 0) in $\widetilde{G}$ (resp. $\mathbf{R}^3\setminus\widetilde{G}$), then $\tilde{\sigma}(x)\in Z^\infty_{20}(\mathbf{R}^3)$. Note that $\tilde{\sigma}(x)$ is independent of $t\in I_0$. Fix $t\in I_0$. Then we obtain

$$(\omega, *\tilde{\sigma})_{D(t)}=\int_{\gamma_1(t)}\omega \quad \text{for } \forall\in Z^\infty_1(\overline{D(t)}).$$

Therefore, when we regard $\tilde{\sigma}$ as an element of $Z_2(D(t))$, the harmonic 2-form $\Omega_2(t, \cdot)$ on $\overline{D(t)}$ is the orthogonal projection of $\tilde{\sigma}(x)$ to $H_2(D(t))$ in the second formula of (2.1):

$$\tilde{\sigma}(x) = \Omega_2(t, x) \dot{+} \tau(t, x), \tag{5.1}$$

where $\Omega_2(t, x) \in H_2(D(t))$ and $\tau(t, x) \in B_2(D(t))$. Note that $\Omega_2(t, x) + \tau(t, x) = 0$ in $D(t) \setminus \widetilde{G}$. Since $\Omega_2(t, \cdot) \in H_{20}(D(t))$ for each $t \in I_0$, we have from Theorem 5.1 and Lemma 5.2 in [13] the following fact: We find a neighborhood $V(t)$ of $\partial D(t)$ in $\mathbf{R}^3$ such that

1. $\Omega_2(t, \cdot) \in H_2(D(t) \cup V(t))$ and there esists a unique $\mathscr{A}(t, \cdot) \in C_1^\omega(V(t))$ such that

(i) $d\mathscr{A}(t, \cdot) = \Omega_2(t, \cdot)$ in $D(t) \cup V(t)$, (ii) $\delta\mathscr{A}(t, \cdot) = 0$ in $V(t)$,

(iii) $\mathscr{A}(t, \cdot) = 0$ on $\partial D(t)$.

We call $\mathscr{A}(t, \cdot)$ *the vector potential of* $\Omega_2(t, \cdot)$ *with boundary values* 0 *in* $V(t)$.

2. There exists an element $\sigma_1(t, \cdot) \in C_1^\infty(D(t)) \subset C_1^\omega(V(t))$ such that

$$\tilde{\sigma}(\cdot) = \Omega_2(t, \cdot) + d\sigma_1(t, \cdot) \text{ in } D(t) \cup V(t) \tag{5.2}$$

$$\mathscr{A}(t, \cdot) + \sigma_1(t, \cdot) = 0 \text{ in } V(t). \tag{5.3}$$

Since $\partial\mathscr{D}$ is C^ω smooth, we may assume that the neighborhood $V(t)$ of $\partial D(t)$ is independent of $t \in I_0$ and so is $D(t) \cup V(t)$ (if necessary, take a smaller interval I_0 centered at 0). We thus put $V = V(t)$ and $\widetilde{D} = D(t) \cup V(t)$ for $t \in I_0$. Hence, $\Omega_2(t, x)$ is of class C^ω for $(t, x) \in I_0 \times \widetilde{D}$. Let $k = 1, 2$. Since $\tilde{\sigma}(x)$ does not depend on $t \in I_0$, we have from (5.2) and (5.3)

$$\frac{\partial^k \Omega_2}{\partial t^k}(t, \cdot) + d\left(\frac{\partial^k \sigma_1}{\partial t^k}\right)(t, \cdot) = 0 \text{ in } \widetilde{D}(\supset \overline{D(0)}) \tag{5.4}$$

$$\frac{\partial^k \mathscr{A}}{\partial t^k}(t, \cdot) + \frac{\partial^k \sigma_1}{\partial t^k}(t, \cdot) = 0 \text{ in } V(\supset \partial D(0)). \tag{5.5}$$

It follows from (i) and (ii) for $\mathscr{A}(t, \cdot)$ that

$$\delta\left(\frac{\partial^k \sigma_1}{\partial t^k}\right)(t, \cdot) = -\left(\frac{\partial^k}{\partial t^k}\delta\mathscr{A}\right)(t, \cdot) = 0 \text{ in } V \tag{5.6}$$

$$\delta d\left(\frac{\partial^k \sigma_1}{\partial t^k}\right)(t, \cdot) = -\left(\frac{\partial^k}{\partial t^k}\delta\Omega_2\right)(t, \cdot) = 0 \text{ in } \widetilde{D}. \tag{5.7}$$

We put

$$\mathscr{A}(t, \cdot) = \sum_{i=1}^{3} A_i(t, \cdot)dx_i \text{ in } V, \quad \sigma_1(t, \cdot) = \sum_{i=1}^{3} a_i(t, \cdot)dx_i \text{ in } \widetilde{D}. \tag{5.8}$$

Then conditions (i), (ii) and (iii) of $\mathscr{A}(t, \cdot)$ are written into

$$\Omega_2(t,\cdot)=\sum_{1\le i<j\le 3}(A_j^i-A_i^j)(t,\cdot)dx_i\wedge dx_j \text{ in } V \tag{5.9}$$

$$\sum_{j=1}^{3}\frac{\partial A_j}{\partial x_j}(t,\cdot)=0 \text{ in } V \tag{5.10}$$

$$A_i(t,\cdot)=0 \text{ on } \partial D(t), \tag{5.11}$$

where $A_i^j(t,\cdot)=\frac{\partial A_i}{\partial x_j}(t,\cdot)$ $(1\le i,j\le 3)$. Note that $\Delta\mathscr{A}=(d\delta-\delta d)\mathscr{A}=-\delta\Omega_2=0$, so that *each* $A_i(t,x)$, $i=1,2,3$ *is a harmonic function for* $x\in V$. Given any C^∞ 1-form $\omega=\sum_{i=1}^3\alpha_i dx_i$ in a domain of $\mathbf{R}^3$, we conveniently put $\nabla\omega:=\sum_{i=1}^{3}(\nabla\alpha_i)dx_i$ and $\|\nabla\omega\|^2(x):=\sum_{i=1}^{3}\|\nabla\alpha_i(x)\|^2=\sum_{i,j=1}^{3}\left(\frac{\partial\alpha_i}{\partial x_j}\right)^2(x)$. By direct calculation we have

$$\Delta(\|\omega\|^2(x))=2\Big(\|\nabla\omega\|^2(x)+\sum_{i=1}^{3}(\alpha_i\Delta\alpha_i)(x)\Big) \tag{5.12}$$

$$\|\nabla\omega\|^2(x)=\|d\omega\|^2(x)+\|\delta\omega\|^2(x)+2\sum_{1\le i<j\le 3}(\alpha_j^j\alpha_i^i-\alpha_i^j\alpha_j^i)(x), \tag{5.13}$$

where $\alpha_i^j=\frac{\partial\alpha_i}{\partial x_j}$ $(1\le i,j\le 3)$. By (5.9), (5.10), and (5.11) for $\mathscr{A}$, we also have

$$\|\Omega_2\|^2(t,x)=\|\nabla\mathscr{A}\|^2(t,x) \quad \text{on } \partial D(t). \tag{5.14}$$

We shall show the following foumula:

$$\frac{d^k\mu_1}{dt^k}(0)=\int_{\partial D(0)}\Big\{\sum_{i=1}^{3}\frac{\partial^k A_i}{\partial t^k}\frac{\partial A_i}{\partial n_x}\Big\}_{(0,x)}dS_x. \tag{5.15}$$

In fact, since $\gamma_1(t)\sim\gamma_1(0)$ in $\widetilde{D}$ and $*\Omega_2(t,\cdot)\in H_1(\widetilde{D})$ $(\subset Z_1^\infty(\overline{D(0)}))$ for any $t\in I_0$, we have

$$\mu_1(t)=\int_{\gamma_1(0)}*\Omega_2(t,\cdot)=\int_{D(0)}\Omega_2(t,\cdot)\wedge*\Omega_2(0,\cdot).$$

Differentiate both sides with respect to t, k times, and put $t=0$. Then we have

$$\begin{aligned}\frac{d^k\mu_1}{dt^k}(0)&=\int_{D(0)}\frac{\partial^k\Omega_2}{\partial t^k}(0,\cdot)\wedge*\Omega_2(0,\cdot) \text{ by (5.16).}\\ &=-\int_{D(0)}d\Big(\frac{\partial^k\sigma_1}{\partial t^k}(0,\cdot)\wedge*\Omega_2(0,\cdot)\Big) \text{ by (5.4)}\\ &=\int_{\partial D(0)}\frac{\partial^k\mathscr{A}}{\partial t^k}(0,\cdot)\wedge*\Omega_2(0,\cdot) \text{ by (5.5).}\end{aligned}$$

On the other hand, from (5.8) and (5.9) the integrand is written into

$$\frac{\partial^k \mathscr{A}}{\partial t^k}(0,\cdot)\wedge *\Omega_2(0,\cdot)\equiv\sum_{i=1}^{3}\frac{\partial^k A_i}{\partial t^k}(0,\cdot)S_i \quad \text{on } \partial D(0),$$

where

$$S_1=-\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)dz\wedge dx+\left(\frac{\partial A_1}{\partial z}-\frac{A_3}{\partial x}\right)dx\wedge dy \quad \text{etc.} \quad \text{on } \partial D(0).$$

Since (5.11) implies $dA_j(0,\cdot)=\sum_{i=1}^{3}\frac{\partial A_j}{\partial x_i}dx_i=0$ along $\partial D(0)$ for $j=2, 3$, we have

$$S_1=\frac{\partial A_1}{\partial y}dz\wedge dx+\frac{\partial A_1}{\partial z}dx\wedge dy-\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)dy\wedge dz=\frac{\partial A_1}{\partial n_x}dS_x$$

for $x\in\partial D(0)$. Similar results hold for S_2 and S_3. We thus obtain the desired (5.15).

By applying (3.8) to $A_i(t,x)$, we have

$$\frac{\partial A_i}{\partial t}\frac{\partial A_i}{\partial n_x}\bigg|_{(0,x)}=K_1(0,x)\|\nabla A_i(0,x)\|^2 \quad \text{on } \partial D(0).$$

Consequently, (5.15) for $k=1$ and (5.14) imply foumula (1.8) at $t=0$.

Let us prove formula (1.9) at $t=0$. Since $A_i(t,x)$, $i=1, 2, 3$, is harmonic for $x\in D(t)$, we can apply (3.9) to $A_i(t,x)$ and obtain

$$\frac{\partial^2 A_i}{\partial t^2}\frac{\partial A_i}{\partial n_x}=K_2(t,x)\left(\frac{\partial A_i}{\partial n_x}\right)^2+\frac{\partial}{\partial n_x}\left(\frac{\partial A_i}{\partial t}\right)^2 \quad \text{on } \partial\mathscr{D}.$$

Formulas (5.15) for $k=2$ and (5.14) imply

$$\frac{d^2\mu_1}{dt^2}(0)=\int_{\partial D(0)}K_2(0,\cdot)\|\Omega_2\|^2(0,x)dS_x+\int_{\partial D(0)}\frac{\partial}{\partial n_x}\left\{\left\|\frac{\partial\mathscr{A}}{\partial t}\right\|^2(0,x)\right\}dS_x$$
$$\equiv I+J. \tag{5.17}$$

For the sake of simplicity, given any function $f(t,x)$ or any i-form $\omega(t,x)$ of class C^1 for $(t,x)\in I_0\times G$, where I_0 is an interval centered at 0 and G is a domain in $\mathbf{R}^3$, we write

$$f^{\cdot}=\frac{\partial f}{\partial t}(0,x), \quad \omega^{\cdot}=\frac{\partial\omega}{\partial t}(0,x), \quad \|\omega^{\cdot}\|^2=\|\frac{\partial\omega}{\partial t}\|^2(0,x).$$

By (5.4) we replace $\mathscr{A}^{\cdot}$ in J by $-\sigma_1^{\cdot}$. Since $\sigma_1^{\cdot}\in C^\infty(\widetilde{D})$, it follows from Stokes formula that

$$J=\int_{\partial D(0)}\frac{\partial\|\sigma_1^{\cdot}\|^2}{\partial n_x}dS_x=\int_{D(0)}\Delta\|\sigma_1^{\cdot}\|^2 dv_x$$

$$=2\Big(\int_{D(0)}\|\nabla\sigma_1^{\cdot}\|^2 dv_x+\int_{D(0)}\Big\{\sum_{i=1}^{3}a_i^{\cdot}\Delta a_i^{\cdot}\Big\}dv_x\Big)\quad\text{by (5.8) and (5.12)}$$

$$\equiv 2(J_1+J_2). \tag{5.18}$$

Note that the surface integral J is uniquely determined by $\mathscr{A}(t,\cdot)$ but the volume integrals J_1 and J_2 depend on the choice of extension $\sigma_1(t,\cdot)$ into $D(t)$ (determined by (5.3)). Since $\Delta\sigma_1^{\cdot}=(d\delta-\delta d)\sigma_1^{\cdot}=d\delta\sigma_1^{\cdot}$ from (5.7), the integral J_2 (involving derivatives of the second order for x, y and z of $\sigma_1^{\cdot}$) is written by means of derivatives of the first order of $\sigma_1^{\cdot}$ as follows:

$$\begin{aligned}J_2&=\int_{D(0)}\Delta\sigma_1^{\cdot}\wedge*\sigma_1^{\cdot}=\int_{D(0)}(d\delta)\sigma_1^{\cdot}\wedge*\sigma_1^{\cdot}\\&=\int_{D(0)}\{d(\delta\sigma_1^{\cdot}\wedge*\sigma_1^{\cdot})-\delta\sigma_1^{\cdot}\wedge d*\sigma_1^{\cdot}\}\\&=\int_{\partial D(0)}\delta\sigma_1^{\cdot}\wedge*\sigma_1^{\cdot}-\int_{D(0)}\|\delta\sigma_1^{\cdot}\|^2 dv_x\\&=-\int_{D(0)}\|\delta\sigma_1^{\cdot}\|^2 dv_x\ \text{by (5.6)}.\end{aligned}$$

By (5.18), we thus have

$$\begin{aligned}J_1+J_2&=\int_{D(0)}\{\|\nabla\sigma_1^{\cdot}\|^2-\|\delta\sigma_1^{\cdot}\|\}dv_x\\&=\int_{D(0)}\Big\{\|d\sigma_1^{\cdot}\|^2+2\sum_{1\le i<j\le 3}((a_j^i)^{\cdot}(a_i^j)^{\cdot}-(a_i^i)^{\cdot}(a_j^j)^{\cdot})\Big\}dv_x\ \text{by (5.13)}\\&=\|\Omega_2^{\cdot}\|_{D(0)}^2+2\sum_{1\le i<j\le 3}\int_{D(0)}((a_j^i)^{\cdot}(a_i^j)^{\cdot}-(a_i^i)^{\cdot}(a_j^j)^{\cdot})dv_x\ \text{by (5.4)}\\&\equiv\|\Omega_2^{\cdot}\|_{D(0)}^2+2\sum_{1\le i<j\le 3}L_{ij}.\end{aligned}$$

If we put $k=\{1,2,3\}\setminus\{i,j\}$, then we have the following representation of the volume integral L_{ij} by menas of the surface integral of A_i on $\partial D(0)$:

$$L_{ij}=-\operatorname{sgn}(i,j,k)\int_{\partial D(0)}K_1(0,x)^2\Big\{\frac{\partial A_i}{\partial n_x}d\Big(\frac{\partial A_j}{\partial n_x}\Big)-\frac{\partial A_j}{\partial n_x}d\Big(\frac{\partial A_i}{\partial n_x}\Big)\Big\}\wedge dx_k. \tag{5.19}$$

In fact, since $\int_{\partial D(0)}A_i^{\cdot}(dA_j^{\cdot})\wedge dx_k+A_j^{\cdot}(dA_i^{\cdot})\wedge dx_k=0$, it follows that

$$\begin{aligned}L_{ij}&=-\operatorname{sgn}(i,j,k)\int_{D(0)}da_i^{\cdot}\wedge da_j^{\cdot}\wedge dx_k\\&=-\operatorname{sgn}(i,j,k)\int_{\partial D(0)}a_i^{\cdot}(da_j^{\cdot})\wedge dx_k\ \text{by Stokes formula}\\&=-\frac{1}{2}\operatorname{sgn}(i,j,k)\int_{\partial D(0)}\{A_i^{\cdot}(dA_j^{\cdot}))-A_j^{\cdot}(dA_i^{\cdot})\}\wedge dx_k\ \text{by (5.5)}.\end{aligned}$$

From (3.8) and (5.11) we have

$$A_i^{\cdot}=K_1(0,x)\frac{\partial A_i}{\partial n_x} \text{ on } \partial D(0)$$
$$dA_i^{\cdot}=(dK_1(0,x))\frac{\partial A_i}{\partial n_x}+K_1(0,x)d\left(\frac{\partial A_i}{\partial n_x}\right) \text{ along } \partial D(0).$$

By substituting these into the above formula, we immediately obtain (5.19).

We put, for $x\in\partial D(0)$,

$$\Xi(0,x)=\sum_{1\le i<j\le 3}\operatorname{sgn}(i,j,k)\left\{\frac{\partial A_i}{\partial n_x}d\left(\frac{\partial A_j}{\partial n_x}\right)-\frac{\partial A_i}{\partial n_x}d\left(\frac{\partial A_i}{\partial n_x}\right)\right\}\wedge dx_k, \quad (5.20)$$

which is a 2-form on $\partial D(0)$ such that $L_{ij}=-\int_{\partial D(0)}K_1(0,x)^2\,\Xi(0,x)$. From (5.17) it turns out

$$\begin{aligned}\frac{d^2\mu_1}{dt^2}&=I+2\left\{\left\|\frac{\partial\Omega_2}{\partial t}(0,x)\right\|^2_{D(0)}-2\int_{\partial D(0)}K_1(0,x)^2\,\Xi(0,x)\right\}\\&=2\left\|\frac{\partial\Omega_2}{\partial t}(0,x)\right\|^2_{D(0)}+\int_{\partial D(0)}\{K_2(0,x)\|\Omega_2\|^2(0,x)dS_x-2K_1(0,x)^2\,\Xi(0,x)\}\end{aligned}$$

.

By (3.6), it now suffices for (1.9) to prove

$$\Xi(0,x)=\frac{1}{\rho_{e'}}\|\Omega_2\|^2(0,x)dS_x \text{ for } x\in\partial D(0), \quad (5.21)$$

where $1/\rho_{e'}$ is the normal curvature of the surface $\partial D(0)$ in $\mathbf{R}^3$ for $\boldsymbol{e}'_{\Omega_2}(=\boldsymbol{e}_{\Omega_2}\times\boldsymbol{n}_x)$ at x.

To verify (5.21), let $x_0\in\partial D(0)$. We many assume $x_0=0\in\partial D(0)$ and $\boldsymbol{n}_{x_0}=(0,0,1)$. Thus, $\partial D(0)$ near 0 in $\mathbf{R}^3$ is given by

$$z=\phi(x,y) \text{ where } \phi(x,y)=O(x^2+y^2). \quad (5.22)$$

To avoid the ambiguity we write $\boldsymbol{x}=(x,y,z)=(x_1,x_2,x_3)$ and $\mathbf{0}=(0,0,0)$ in $\mathbf{R}^3$. We simply put $\Omega_2(0,\boldsymbol{x})=\Omega_2(\boldsymbol{x})$, $\Xi(0,\boldsymbol{x})=\Xi(\boldsymbol{x})$, and $A_i(0,\boldsymbol{x})=A_i(\boldsymbol{x})$. By (5.11), we have

$$A_i(\boldsymbol{x})=f_i(\boldsymbol{x})\,(z-\phi(x,y)) \quad \text{for } \boldsymbol{x}\in U, \quad (5.23)$$

where U is a neighborhood of $\mathbf{0}$ in $\mathbf{R}^3$ and $f_i\in C^\omega(U)$. It follows from (5.9) and (5.10) that

$$\begin{aligned}&\nabla A_i(\mathbf{0})=(0,0\,f_i(\mathbf{0})) \text{ where } f_3(\mathbf{0})=0\\&\Omega_2(\mathbf{0})=-f_2(\mathbf{0})dy\wedge dz+f_1(\mathbf{0})dz\wedge dx.\end{aligned}$$

Hence, $\|\Omega_2\|^2(\mathbf{0})=f_1(\mathbf{0})^2+f_2(\mathbf{0})^2$ and

$$\begin{aligned}&\boldsymbol{e}_{\Omega_2}(=\boldsymbol{e})=\frac{1}{\|\Omega_2\|(\mathbf{0})}(-f_2(\mathbf{0}),f_1(\mathbf{0}),0)\\&\boldsymbol{e}'_{\Omega_2}(=\boldsymbol{e}')=(e'_1,e'_2,e'_3)=\frac{1}{\|\Omega_2\|(\mathbf{0})}(f_1(\mathbf{0}),f_2(\mathbf{0}),0). \quad (5.24)\end{aligned}$$

By (5.22), $\frac{\partial}{\partial n_x}(z-\phi(x,y))=1$ at $\boldsymbol{x}=\mathbf{0}$. By (5.23), $\frac{\partial A_i}{\partial n_x}(\mathbf{0})=f_i(\mathbf{0})$. We carefully have

$$dz=0,\ d\left\{\frac{\partial}{\partial n_x}(z-\phi(x,y))\right\}=0,\ \left(\frac{\partial A_i}{\partial n_x}\right)=df_i$$

along $\partial D(0)$ at $\boldsymbol{x}=\mathbf{0}$. Since $f_3(\mathbf{0})=0$, it follows from (5.20) that

$$\begin{aligned}\Xi(\mathbf{0}) &= \sum_{1\le i<j\le 3} \operatorname{sgn}(i,j,k)\,(f_i\,df_j-f_j\,df_i)|_{x=0}\wedge dx_k \\ &= -\left(f_2\frac{\partial f_3}{\partial y}+f_1\frac{\partial f_3}{\partial x}\right)\Big|_{x=0} dx\wedge dy. \qquad (5.25)\end{aligned}$$

On the other hand, equations (5.10), (5.11), and (5.23) imply

$$\left(\sum_{j=1}^{3}\frac{\partial f_j}{\partial x_j}\right)(z-\phi(x,y))+f_1(\boldsymbol{x})\left(-\frac{\partial\phi}{\partial x}\right)+f_2(\boldsymbol{x})\left(-\frac{\partial\phi}{\partial y}\right)+f_3(\boldsymbol{x})=0 \text{ for } \boldsymbol{x}\in U.$$

After defferentiating both sides with respect to x or y, we put $\boldsymbol{x}=\mathbf{0}$. It follows from $\phi(0,0)=\frac{\partial\phi}{\partial x}(0,0)=\frac{\partial\phi}{\partial y}(0,0)=0$ that

$$\frac{\partial f_3}{\partial x}=f_1\frac{\partial^2\phi}{\partial x^2}+f_2\frac{\partial^2\phi}{\partial x\partial y},\quad \frac{\partial f_3}{\partial y}=f_1\frac{\partial^2\phi}{\partial x\partial y}+f_2\frac{\partial^2\phi}{\partial y^2}$$

evaluated at $\boldsymbol{x}=\mathbf{0}$. We substitute these into (5.25) and obtain

$$\begin{aligned}\Xi(\mathbf{0}) &= -\left\{f_1^2\frac{\partial^2\phi}{\partial x^2}+2f_1f_2\frac{\partial^2\phi}{\partial x\partial y}+f_2^2\frac{\partial^2\phi}{\partial y^2}\right\}_{x=0} dx\wedge dy \\ &= -(f_1(\mathbf{0})^2+f_2(\mathbf{0})^2)\left\{(e_1')^2\frac{\partial^2\phi}{\partial x^2}+2e_1'e_2'\frac{\partial^2\phi}{\partial x\partial y}+(e_2')^2\frac{\partial^2\phi}{\partial y^2}\right\}_{(0,0)} dx\wedge dy \text{ by (5.24)} \\ &= \|\Omega_2\|^2(\mathbf{0})\frac{1}{\rho_{e'}}dx\wedge dy \text{ by (3.2).}\end{aligned}$$

Since $dS_x=dx\wedge dy$ at $\boldsymbol{x}=0$, (5.21) is proved. Formula (1.9) is completely proved.

By (5.16) for $k=1$, it holds $|\mu_1'(0)|^2\le\mu_1(0)\left\|\frac{\partial\Omega_2}{\partial t}(0,\cdot)\right\|^2_{D(0)}$. Thus, (1.9) implies

Corollary 5. 1. *If* $\widetilde{K}_2(\boldsymbol{e},t,x)\ge 0$ *on* ∂D *for all* $\boldsymbol{e}\in\boldsymbol{T}_x(=\boldsymbol{T}(t)_x)$, *then* $\frac{1}{\mu_1(t)}$ *is a concave function on* I.

6. Examples related to $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\pm\frac{1}{x}\frac{\partial u}{\partial x}=0$

We use the cylindrical coordinates $x=[r,\theta,z]$ in $\mathbf{R}^3$ so that

$$*dr = r\,d\theta\wedge dz,\quad *d\theta = \frac{1}{r}\,dz\wedge dr,\quad *dz = r\,dr\wedge d\theta \tag{6.1}$$

and $dz\wedge dr = drdz > 0$. We consider the half-plane Π and its boundary $\partial\Pi$:

$$\Pi = \{\zeta = (r, z)\,|\,0<r<+\infty,\ -\infty<z<+\infty\}$$
$$\partial\Pi = \{(0, z)\,|\,-\infty<z<+\infty\}.$$

We identify Π with the half (x, z)-plane π_+ in $\mathbf{R}^3$ with $x>0$ by $(r, z) = (x, z)$, and use the simple notation $x = [r, \theta, z] = [\zeta, \theta] \in \mathbf{R}^3$. Given a set $K\subset\pi_+ (=\Pi)$, we denote by $\ll K\gg$ the z-axially symmetric set in $\mathbf{R}^3$ obtained by rotating K around the z-axis, namely, $\ll K\gg = \{[\zeta, \theta]\,|\,\zeta\in K,\ 0\leq\theta\leq 2\pi\}$.

We shall give explicit formulas of the reproducing i-form $*\,\Omega_{3-i}(x)$ for some examples (D, γ_i), where D is a z-axially symmetric domain. Let $K\subset\subset\Pi$ be a double connected domain bounded by two C^ω smooth closed curves C_0 and C_1 such that $\partial K = C_1 - C_0$. We set $K' = \Pi\setminus\overline{K}$, which consists of the bounded component K_0' such that $\partial K_0' = C_0$ and the unbounded one K_1' such that $\partial K_1' = -C_1$ in Π. For $j=0, 1$, we define the z-axially symmetric sets:

$$D = \ll K\gg,\quad \Sigma_j = \ll C_j\gg,\quad \Sigma = \partial D = \Sigma_1 - \Sigma_0,$$

so that $D' (=\mathbf{R}^3\setminus\overline{D})$ consists of a bounded solid torus $D_0' = \ll K_0'\gg$ with $\partial D_0' = \Sigma_0$ and an unbounded domain $D_1' = \ll K_1'\gg \cup$ {the z-axis} with $\partial D_1' = -\Sigma_1$. We draw a closed cycle γ_1 in K such that $\gamma_1\sim C_1$ on $\overline{K}$, and make a closed surface $\gamma_2 := \ll\gamma_1\gg$, which is homologous to Σ_1 on $\overline{D}$. For $i=1, 2$, we have the reproducing i-form $*\,\Omega_{3-i}(x)$ and the harmonic i-module μ_i for (D, γ_i).

We here consider the following two differential operators $\Delta^\pm$ in Π:

$$\Delta^\pm = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} \pm \frac{1}{r}\frac{\partial}{\partial r}$$

and construct two C^ω functions $v^\pm(\zeta) = v^\pm(r, z)$ on $\overline{K}$ which satisfy

$$\Delta^\pm v^\pm = 0 \text{ in } K,\qquad v^\pm(\zeta) = \begin{cases} 0 & \text{on } C_0 \\ 1 & \text{on } C_1. \end{cases} \tag{6.2}$$

Such functions $v^\pm(r, z)$ are uniquely determined. Differential equations in (6.2) are called *Stokes-Beltrami equations* and studied in E. Beltrami [3], A. Weinstein [10], R. Gilbert [5], etc..

Remark 6.1. The operator Δ^+ is associated with Δ^- in the sense that, if a C^2 function $u(\zeta)$ satisfies $\Delta^+ u = 0$ in a simply connected domain X in Π, then there exists a $v(\zeta)\in C^2(X)$ satisfying $\Delta^- v = 0$ in X such that

$$\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial z},\quad \frac{\partial u}{\partial z} = \frac{1}{r}\frac{\partial v}{\partial r}.$$

Remark 6.2 Let $X\subset\subset\Pi$ be a domain with smooth boundary and let

$f(\zeta), g(\zeta) \in C^2(\overline{X})$. If we define

$$\langle f, g\rangle_{\pm,X} := \int_X r^{\pm 1}\left\{\frac{\partial f}{\partial r}\frac{\partial g}{\partial r}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right\}drdz, \quad \|f\|^2_{\pm,X} := \langle f, f\rangle_{\pm,X},$$

then we have

$$\langle f, g\rangle_{\pm,X} = \int_{\partial X} r^{\pm 1} f \frac{\partial g}{\partial n_\zeta} ds_\zeta - \int_X r^{\pm 1} f\, \Delta^{\pm} g\, drdz. \tag{6.3}$$

Using notation (6.2), we have the following expressions of the above $*\Omega_i$ and $\mu_i\,(i=1, 2)$:

Thorem 6. 1. *It holds for any* $x=[r, \theta, z] \in D$

$$\begin{cases} *\Omega_2(x) = \dfrac{1}{2\pi r}\left(\dfrac{\partial v^-}{\partial z}dr - \dfrac{\partial v^-}{\partial r}dz\right) \\ \mu_1 = \dfrac{1}{2\pi}\| v^- \|^2_{-,K} \end{cases} \tag{6.4}$$

$$\begin{cases} *\Omega_1(x) = r\left(\dfrac{\partial v^+}{\partial z}dr - \dfrac{\partial v^+}{\partial r}dz\right)\wedge d\theta \\ \mu_2 = 2\pi \| v^+ \|^2_{+,K}. \end{cases} \tag{6.5}$$

Proof. We put $*\omega_2(x) := r^{-1}(v^-_z dr - v^-_r dz)$ on $\overline{D}$. By simple calculation we have $d*\omega_2 = -r^{-1}(\Delta^- v^-)dr\wedge dz = 0$, so that $*\omega_2 \in Z_1^\infty(\overline{D})$. By (6.1) we have $\omega_2 = v^-_z d\theta\wedge dz - v^-_r dr\wedge d\theta = -d(v^- d\theta)$. For any θ_0: $0\le\theta_0<2\pi$, we put $C(\theta_0) := \Sigma_1 \cap \{\theta=\theta_0\}$, which is a 1-cycle homologous to γ_1 on $\overline{D}$. Let $\forall\,\sigma\in Z_1^\infty(\overline{D})$. Then we have

$$\begin{aligned}(\sigma, *\omega_2)_D &= \int_D -d(v^- d\theta\wedge\sigma) = \int_{\partial D} v^-(\sigma\wedge d\theta) = \int_{\Sigma 1}\sigma\wedge d\theta \\ &= \int_0^{2\pi}\Big(\int_{C(\theta)}\sigma\Big)d\theta = 2\pi\int_{\gamma 1}\sigma.\end{aligned}$$

Hence, $*\Omega_2 = *\omega_2/2\pi$, which proves (6.4).

To prove (6.5), we put $*\omega_1 = r(v^+_z dr - v^+_r dz)\wedge d\theta$ on $\overline{D}$. We thus have $d*\omega_1 = (\Delta^+ v^+)\,dr\wedge d\theta\wedge dz = 0$, so that $*\omega_1 \in Z_2^\infty(\overline{D})$. Note that $\omega_1 = dv^+$ by (6.1). Let $\forall\,\sigma\in Z_2^\infty(\overline{D})$. Since $\Sigma_1 \sim \gamma_2$ on $\overline{D}$, we have

$$(\sigma, *\omega_1)_D = \int_{\partial D} v^+\sigma = \int_{\Sigma 1}\sigma = \int_{\gamma 2}\sigma.$$

Hence, $*\omega_1 = *\Omega_1$, which proves (6.5).

Now let $I=(-\rho, +\rho)\subset \mathbf{R}^3$. To each $t\in I$, we let correspond a domain $K(t)\subset\subset\Pi$ bounded by two C^ω smooth curves $C_1(t)$ and $C_0(t)$ such that $\partial K(t) = C_1(t) - C_0(t)$. We assume that $\partial K(t)$ varies C^ω smoothly with $t\in I$ in Π. In the 3 dimensional space $I\times\Pi$ we put

$$\mathcal{K}=\bigcup_{t\in I}(t, K(t)), \quad \partial\mathcal{K}=\bigcup_{t\in I}(t, \partial K(t)).$$

We thus have a variation $\mathcal{K}$ of domains $K(t)$ in Π with parameter $t\in I$ such that

$$\mathcal{K}: t\to K(t), t\in I.$$

For each $t\in I$ and $j=0, 1$, we consider the z-axially symmetric sets in $\mathbf{R}^3$:

$$D(t)=\ll K(t)\gg, \quad \Sigma_j(t)=\ll C_j(t)\gg, \quad \Sigma(t)=\partial D(t)=\Sigma_1(t)-\Sigma_0(t).$$

In the 4 dimensional space $I\times\mathbf{R}^3$ we put

$$\mathcal{D}=\bigcup_{t\in I}(t, D(t)), \quad \partial\mathcal{D}=\bigcup_{t\in I}(t, \partial D(t)).$$

We thus have a variation of domains $D(t)$ in $\mathbf{R}^3$ with parameter $t\in I$ such that

$$\mathcal{D}: t\to D(t), t\in I.$$

Now take a closed curve $\gamma_1(t)$ in $K(t)$ such that $\gamma_1(t)\sim C_1(t)$ on $\overline{K(t)}$ and $\gamma_1(t)$ varies smoothly with $t\in I$ in Π. We consider the 2-cycle $\gamma_2(t):=\ll\gamma_1(t)\gg$, which is homologous to $\Sigma_1(t)$ on $\overline{D(t)}$. For any $t\in I$ we have the reproducing i-form $*\Omega_{3-i}(t, x)$ $(i=1, 2)$ and the harmonic i-module $\mu_i(t)$ for $(D(t), \gamma_i(t))$. By Theorem 6.1, it holds for any $x=[\zeta, \theta]=[r, \theta, z]\in\overline{D(t)}$

$$\begin{cases} *\Omega_2(t, x)=\dfrac{1}{2\pi r}(v_z^- dr-v_r^- dz) \\ \mu_1(t)=\dfrac{1}{2\pi}\|v^-\|^2_{-,K(t)} \end{cases} \quad \begin{cases} *\Omega_1(t, x)=r(v_z^+ dr-v_r^+ dz)\wedge d\theta \\ \mu_2(t)=2\pi\|v^+\|^2_{+,K(t)} \end{cases} \tag{6.6}$$

where $v^{\pm}(t, \zeta)$ are C^{ω} functions for $\zeta\in\overline{K(t)}$ such that

$$\Delta^{\pm}v^{\pm}(t, \zeta)=0 \text{ in } K(t), \quad v^{\pm}(t, \zeta)=\begin{cases} 0 \text{ on } C_0(t) \\ 1 \text{ on } C_1(t). \end{cases} \tag{6.7}$$

Let us apply (1.9) and (1.11) for $\mu_1(t)$ and $\mu_2(t)$, and study what these formulas are reduced to in this special case. We take a C^{ω} defining function $\varphi(t, \zeta)=\varphi(t, (r, z))$ of $\partial\mathcal{K}$ defined in a neighborhood $\mathcal{U}$ of $\partial\mathcal{K}$ in $I\times\Pi$. Then $\varphi(t, \zeta)$ necessarily becomes a C^{ω} defining function of $\partial\mathcal{K}$ (independent of θ). Fix any point $p_0=(t_0, \zeta_0)=(t_0, (r_0, z_0))\in\partial\mathcal{K}$. We denote by $\boldsymbol{n}_{p0}$ the unit outer normal vector of the 2 dim. surface $\partial\mathcal{K}$ at p_0. We consider the 2 dim. plane $\widehat{\pi}_{t,n_{p_0}}$ which passes through the point p_0 and is generated by the 2 vectors $\{(1, (0, 0)), \boldsymbol{n}_{p_0}\}$ in $I\times\Pi$, and denote by $\widehat{\boldsymbol{v}}_t$ the unit tangent vector of the 1 dim. curve $\widehat{\pi}_{t,n_{p_0}}\cap\partial\mathcal{K}$ at p_0. We thus have

$\dfrac{1}{\widehat{\rho}_t}:=$ the normal curvature of the surface $\partial\mathcal{K}$ for $\widehat{\boldsymbol{v}}_t$ at the point p_0, which is called *the t-normal curvature of the surface $\partial\mathcal{K}$ at p_0*. In the half plane Π we denote by $\widehat{\boldsymbol{n}}=(\xi,\eta)$ the unit outer normal vector of the 1 dim. curve $\partial K(t_0)$ at the point ζ_0, namely,

$$(\xi,\eta)=\left(\frac{\nabla\varphi}{\|\nabla\varphi\|}\right)_{(t_0,\zeta_0)}\quad\text{where } \nabla\varphi=\left(\frac{\partial\varphi}{\partial r},\frac{\partial\varphi}{\partial z}\right).$$

Thus, $\widehat{\boldsymbol{s}}:=(\eta,-\xi)$ is the unit tangent vector of $\partial K(t_0)$ at ζ_0. Therefore,

$\frac{1}{\widehat{\rho}_s}:=$ the normal curvature of the curve $\partial K(t_0)$ for $\widehat{\boldsymbol{s}}$ at the point ζ_0 is determined. By simple calculation, we have

$$\frac{1}{\widehat{\rho}_s}=\frac{1}{\|\nabla\varphi\|}(\varphi_{rr}\eta^2-2\varphi_{rz}\xi\eta+\varphi_{zz}\xi^2) \tag{6.8}$$

$$\frac{1}{\widehat{\rho}_t}=\frac{1}{(\varphi_t^2+\|\nabla\varphi\|^2)^{3/2}}\times\begin{pmatrix}(\varphi_{tt}\|\nabla\varphi\|^2-2\varphi_t(\varphi_{tr}\xi+\varphi_{tz}\eta))\|\nabla\varphi\|+\\ +\varphi_t^2(\varphi_{rr}\xi^2-2\varphi_{rz}\xi\eta+\varphi_{zz}\eta^2)\end{pmatrix} \tag{6.9}$$

where the right hand sides are evaluated at (t_0,ζ_0). By (1.3) we defined the tangent vector field $\boldsymbol{e}_{\Omega_2}(t_0,x)$ on $\Sigma(t_0)$ associated with $\Omega_2(t_0,x)$. We consider the particular points $x\in\Sigma(t_0)$ such that $x=x_0=[\zeta_0,0]=(r_0,0,z_0)\in\Sigma(t_0)\cap\Pi(=\partial K(t_0))$. We simply put $\{\boldsymbol{e}_{\Omega_2}(t_0,x_0),\boldsymbol{e}'_{\Omega_2}(t_0,x_0),\boldsymbol{n}_{x_0}\}\equiv\{\boldsymbol{e},\boldsymbol{e}',\boldsymbol{n}\}$, where $\boldsymbol{n}_{x_0}$ denotes the unit outer normal vector of the surface $\Sigma(t_0)$ at the point x_0 in $\mathbf{R}^3$, and $\boldsymbol{e}'_{\Omega_2}(t_0,x_0)=\boldsymbol{n}_{x_0}\times\boldsymbol{e}_{\Omega_2}(t_0,x_0)$. It follows from (6.6) and (6.7) for v^- that

$$\boldsymbol{e}=(\eta,0,-\xi),\quad \boldsymbol{e}'=(0,1,0),\quad \boldsymbol{n}=(\xi,0,\eta).$$

Since $\boldsymbol{e}$ and $\boldsymbol{e}'$ are unit tangent vectors of the surface $\Sigma(t_0)$ in $\mathbf{R}^3$ at x_0, we have the normal curvatures $1/\rho_{e'}$ of $\Sigma(t_0)$ for $\boldsymbol{e}$ and $\boldsymbol{e}'$ at x_0, respectively. By (3.1), we also have the t-normal curvature $1/\rho_t$ of the surface $\partial\mathcal{D}$ in $I\times\mathbf{R}^3$ at the point $P_0:=(t_0,x_0)$. Since each $\Sigma(t)$, $t\in I$ is obtained by rotating $\partial K(t)$ around the z-axis, we have by direct dalculation

$$\frac{1}{\rho_e}=\frac{1}{\widehat{\rho}_s},\quad \frac{1}{\rho_t}=\frac{1}{\widehat{\rho}_t},\quad \frac{1}{\rho_{e'}}=\frac{\xi}{r_0},\quad K_1(t_0,x_0)=\left(\frac{1}{\|\nabla\varphi\|}\frac{\partial\varphi}{\partial t}\right)_{(t_0,\zeta_0)} \tag{6.10}$$

By use of (6.8) and (6.9) we substitute these into (3.5) and (3.6) and obtain

$$K_2(t_0,x_0)=\boldsymbol{k}_2^+(t_0,\zeta_0),\quad \widetilde{K}_2(\boldsymbol{e}_1,t_0,x_0)=\boldsymbol{k}_2^-(t_0,\zeta_0),$$

where

$$\boldsymbol{k}_2^\pm(t_0\ \zeta_0):=\frac{1}{\|\nabla\varphi\|^3}\left\{\frac{\partial^2\varphi}{\partial t^2}\|\nabla\varphi\|^2-2\left\{\sum_{i=1}^{2}\frac{\partial^2\varphi}{\partial t\partial r_i}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial r_i}\right\}+\left|\frac{\partial\varphi}{\partial t}\right|^2\Delta^\pm\varphi\right\}.$$

We here put $(r_1,r_2)=(r,z)$ and evaluate the right hand side at (t_0,ζ_0). Futher, let $\forall x=[\zeta_0,\theta]=(r_0,\theta,z_0)\in\partial D(t_0)$, where $0\leq\forall\theta\leq2\pi$. Namely, x is the point in $\mathbf{R}^3$ obtained by rotaiting $x_0 = [\zeta_0, 0] \in \partial K(t_0)$ positively with quantity θ around the z-axis. Then, using again the symmetry of $D(t)$ with respect to the z-axis, we see that

$$K_2(t_0, x) = K_2(t_0, x_0), \quad \widetilde{K}_2(\boldsymbol{e}_{\Omega_2}, t_0, x) = \widetilde{K}(\boldsymbol{e}_{\Omega_2}, t_0, x_0).$$

It follows from (6.6) that the variation formulas (1.9) and (1.11) are reduced to

Corollary 6. 1.

$$\frac{d^2}{dt^2}\left\{\|v^\pm(t, \cdot)\|^2_{\pm, K(t)}\right\} = 2\left\|\frac{\partial v^\pm}{\partial t}(t, \cdot)\right\|^2_{\pm, K(t)} + \int_{\partial K(t)} \boldsymbol{k}_2^\pm(t, \zeta) r^{\pm 1} \|\nabla v^\pm\|^2(t, \zeta)\, |d\zeta|.$$

This concrete corollary will be useful in future for the study to find the view point from which the variation formulas (1.9) and (1.11) are unified.

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