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Pontrjagin rings of the Morava K-theory for finite H-spaces

By

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Introduction

In this note, we study Morava K-theory of finite homotopy associative H-spaces X with p-torsion. By using a result of Ravenel-Wilson [R-W] we compute the Pontrjagin product structure for $K(2)_*(X)$ or $K(3)_*(X)$. As its application, we give very short proofs of the Kane's theorems [K] about the relations between H-spaces with p-torsion and the exceptional Lie groups. In particular, we show that the Pontrjagin ring $K(3)_*(E_8)$ for $p=3$ is extremely simple, e.g., it is generated by only two elements as a $K(3)_*$-algebra. Moreover this $K(3)_*$-algebra structure deduces the Hopf algebra structure of the ordinary mod 3 cohomology $H^*(E_8; Z/3)$ without using any theories of classification of simple Lie algebras. These arguments are some analogue for the proof of non homotopy nilpotency of exceptional Lie groups in [R], [Y3].

§ 1. H-spaces with one even degree generator

Let X be a simply connected homotopy associative H-space. By the Borel theorem, $H^*(X, Z/p)$ is a tensor algebra of truncated polynomial and exterior algebras generated by elements of even and odd dimensional respectively. In this section we consider the case that the polynomial algebra is generated by only one element y. From Kane [K] we have $|y| = 2(p^i + p^{i-1} + \cdots + p + 1)$ for some i and $y^{p^2} = 0$. However all known examples satisfy the case $i = 1$ and $y^p = 0$. Hence we assume here

$$(1.1) \qquad H^*(X; Z/p) \cong Z/p[y]/(y^p) \otimes \Lambda\,, \qquad |y| = 2p+2$$

where Λ is an exterior algebra generated by odd degree elements. Then it is also well known (see [K]) that there are elements $x, x' \in \Lambda$ such that

$$Q_1 x = Q_0 x' = y\,, \qquad |x| = 3,\ |x'| = 2p+1,\ |y| = 2p+2$$

where Q_i is the Milnor primitive operator, i.e., $Q_0 = \beta$, $Q_1 = \beta\mathscr{P}^1 - \mathscr{P}^1\beta$.

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Lemma 1.2. *Let X be an H-space in* (1.1). *Then*

$$K(n)^*(X) \cong K(n)^* \otimes H^*(X; Z/p) \qquad \text{for } n \geqslant 2 .$$

Proof. We consider the Atiyah-Hirzebruch spectral sequence

$$(1) \qquad E_2^{*,*} = H^*(X; Z/p) \otimes K(n)^* \Rightarrow K(n)^*(X) .$$

It is known (Theorem 4.9 in [Y2])

$$y = Q_1 x \in \text{Image}\,(BP^*(X) \to H^*(X; Z/p)) .$$

Hence y is permanent in the Atiyah-Hirzebruch spectral sequence for BP, hence so for $P(n)$. Here $P(n)^*(-)$ is the cohomology theory with the coefficient $P(n)^* = Z/p[v_n, v_{n+1}, \dots]$. Since $K(n)^*(X) \cong K(n)^* \otimes_{P(n)^*} P(n)^*(X)$, y is also permanent in the spectral sequence (1) for $K(n)^*$-theory.

Considering the biprimitive spectral sequence ([B], [K], confer the proof of Lemma 3.1 in [Y1]) we see that the differential in the spectral sequence (1) maps generators to generators. Since there is no even degree generator of degree $> 2p^2 - 2 = -|v_2|$, all odd generators are also permanent. q.e.d.

Here we recall a result of Ravenel-Wilson (Theorem 12.4 in [R-W]) for the Morava K-theory of the Eilenberg-MacLane space.

Theorem 1.3 ([R-W]). *Let $\bar{K}(n)^* = K(n)^*/(v_n = 1) \cong Z/p$. Then*

$$\bar{K}(2)^*(K(Z, 3)) \cong Z/p[[s]] \qquad |s| = 2p + 2$$

and the vershiebung map V (the dual of the Frobenius map $x \mapsto x^p$) is given by $V(s) = -s$.

Since $H^4(X; Z/p) = 0$, the mod p reduction $H^3(X; Z) \to H^3(X; Z/p)$ is an epimorphism. We choose a class in $H^3(X; Z)$ which map to x by the mod p reduction. This class is represented by the map $f: X \to K(Z, 3)$. We also know from [R-W] the element $s \in K(2)^*(K(Z, 3))$ corresponds $Q_1 \iota_3$ in $H^*(K(Z, 3); Z/p)$ for the fundamental class ι_3. Since $f^*(\iota_3) = x$ in $H^*(X; Z/p)$, we get $f^*(s) = y$ in $K(2)^*(X)$. Therefore we know

$$V(y) = -v_2 y$$

Let us write by z, z', y the dual of x, x', y in $K(2)_*(X) \cong \text{Hom}_{k(2)^*}(K(2)^*(X), K(2)^*)$. Hence

$$(1.4) \qquad y^p = -v_2 y .$$

Recall that Q_i ($i \neq 2$) is also defined in $K(2)_*(X)$ such that it is a derivation [S-Y]. From (1.4) and $Q_1 y = z$, we get

$$\begin{aligned} -v_2 z &= Q_1 y^p \\ &= zy \dots y + yzy \dots y + \cdots + y \dots yz , \end{aligned}$$

here note the non commutativity of y and z. Let us write $\mathrm{ad}(y)(z) = [y, z] = yz - zy$. Then it is well known

$$\mathrm{ad}^i(y)(z) = \underbrace{[y, [y, \ldots, [y, z]]..]}_{i} = \sum \binom{i}{k} (-1)^k \underbrace{y \ldots y}_{k} zy \ldots y .$$

In particular the case $i = p - 1$ shows

$$-v_2 z = \mathrm{ad}^{p-1}(y)(z) .$$

Similarly, we have $-v_2 z' = \mathrm{ad}^{p-1}(y)(z')$.

Here we consider the connective Morava K-theory $k(2)_*(-)$. By [S-Y], we can also define the Pontrjagin product in $k(2)_*(X) \cong k(2)_* \otimes H_*(X; Z/p)$.

By the dimensional reason, $k(2)_*(X) \cong H_*(X, Z/p)$ for $0 < * < |v_2| = 2(p^2 - 1)$. Since $|\mathrm{ad}^i(y)(z)| < 2(p^2 - 1)$ for $i < p - 1$, we have $\mathrm{ad}^i(y)(z) \neq 0$ in $H_*(X; Z/p)$ for $i < p - 1$. These elements are primitive since so are z, y. Thus there are ring generators x_i in $H^*(X; Z/p)$ which are dual of $\mathrm{ad}^i(y)(z)$. Therefore we get the following theorem (for the case $H^*(X, Z/p)$, more general results are given by Kane [K]).

Theorem 1.5. *Let X be an H-space with* (1.1). *Then $H^*(X, Z/p)$ (resp. $K(2)^*(X)$) has a quotient Hopf algebra*

$$K[y]/(y^p) \otimes \Lambda(x_i, x_i' \mid 0 \leqslant i < p - 1) \qquad \text{with } K = Z/p \text{ or } K(2)^*$$

$|x_i| = 2(p + 1)i + 3$, $|x_i'| = 2(p + 1)(i + 1) - 1$ *such that its dual Hopf algebra is generated by three elements z, z' and y with the relations*

$$\mathrm{ad}^{p-1}(y)(z) = 0 \quad (\text{resp.} = -v_2 z) , \qquad \mathrm{ad}^{p-1}(y)(z') = 0 \quad (\text{resp.} = -v_2 z')$$

$$y^p = 0 \quad (\text{resp.} = -v_2 y) \qquad [z, z'] = 0 .$$

The following facts on the cohomology of the exceptional Lie groups are known ([K-M], [Ko])

$$H^*(F_4, Z/3) \cong Z/3[y_8]/(y_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}) \qquad \text{and}$$

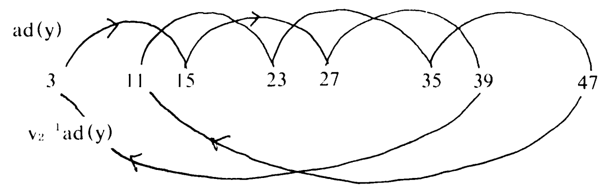
$$H^*(E_8, Z/5) \cong Z/5[y_{12}]/(y_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) .$$

Hence we get

Corollary 1.5. *Let X be an H-space in* (1.1). *Then there are epimorphisms*

$$K(2)^*(X) \twoheadrightarrow K(2)^*(F_4) , \qquad H^*(X; Z/3) \twoheadrightarrow H^*(F_4; Z/3) \qquad \text{for } p = 3$$

$$K(2)^*(X) \twoheadrightarrow K(2)^*(E_8) , \qquad H^*(X; Z/5) \twoheadrightarrow H^*(E_8; Z/5) \qquad \text{for } p = 5 .$$



§ 2. Two even degree generators

In this section, we consider a simply connected homotopy associative H-space X such that

(2.1) $\quad H^*(X, Z/p) \cong Z/p[y, u]/(y^p, u^p) \otimes \Lambda, \qquad |y| \neq |u|, \qquad |y| = 2p + 2$

where Λ is an exterior algebra generated by odd dimensional elements. Then by a theorem of Kane [K], there are x, x', w, w' such that

$$Q_1 x = Q_0 x' = y, \qquad Q_2 x = Q_1 w = Q_0 w' = u.$$

By the arguments similar to the proof of Lemma 1.2, we have

(2.2) $$K(2)^*(X) \cong K(2)^*[y]/(y^p) \otimes \Lambda' \otimes \Lambda(xu^{p-1})$$

where $\Lambda = \Lambda' \otimes \Lambda(x)$. We also see

(2.3) $$\mathrm{ad}^{p-1}(y)(z') = -v_2 z' \qquad \text{for the dual } z' \text{ of } x'.$$

We study $K(3)^*(X)$ now. Also by the arguments similar to the section 2, we can prove

$$K(3)^*(X) \cong K(3)^* \otimes H^*(X, Z/p).$$

The Ravenel-Wilson theorem for this case is stated as

(2.4) $\quad \bar{K}(3)^*(K(Z, 3)) \cong Z/p[[s, t]], \qquad |s| = 2p + 2, \qquad |t| = 2p^2 + 2$

and $V(s) = -t$, $V(t) = -s^p$. Let z, z', v, v' be the dual elements in $K(3)_*(X)$ of x, x', w, w' respectively. Then the vershiebung map shows

(2.5) $$u^p = -v_3 y, \qquad y^p = 0.$$

This induces $\mathrm{ad}^{p-1}(u)(v) = -Q_1(v_3 y) = -v_3 z$. Similarly, $\mathrm{ad}^{p-1}(u)(v') = -v_3 z'$. Moreover from (2.3),

$$\mathrm{ad}^p(u)(z') = [u^p, z'] = -v_3[y, z'] \neq 0.$$

Taking x_j', w_i, w_i' as dual of $\mathrm{ad}^j(u)(z')$, $\mathrm{ad}^i(u)(v)$, $\mathrm{ad}^i(u)(v')$ respectively, we get;

Proposition 2.6. *Let X be an H-space in* (2.1). *Then $H^*(X, Z/p)$ (resp. $K(3)^*(X)$) has a quotient algebra*

$$K[y, u]/(y^p, u^p) \otimes \Lambda(x, x_j', w_i, w_i' \mid 0 \leqslant j < p, 0 \leqslant i < p - 1) \qquad K = Z/p \text{ or } K(3)^*$$

with $|x_j'| = 2(p^2 + 1)j + 2p + 1$, $|w_i| = 2(p^2 + 1)(i + 1) - (2p - 1)$, $|w_i'| = 2(p^2 + 1)(i + 1) - 1$.

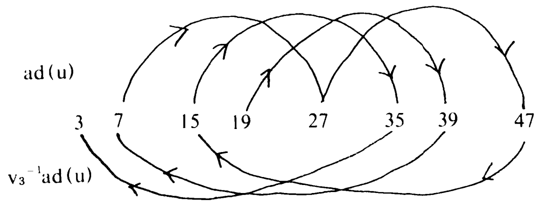
The following fact is also known ([K-M])

$$H^*(E_8, Z/3) \cong Z/3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}).$$

Corollary 2.7. *The Pontrjagin ring $K(3)_*(E_8)$ for $p = 3$ is generated by two elements u, z' with relations $u^9 = 0$, $\mathrm{ad}^8(u)(z') = 0$, $z'^2 = 0$. Hence*

$$K(3)^* \otimes H^*(E_8, Z/3) \cong K(3)^*[u^*, u^{3*}]/(u^{*3}, (u^{3*})^3) \otimes \Lambda(\mathrm{ad}^i(u)(z')^* \mid 0 \leqslant i \leqslant 7)$$

where a^* is the dual of a.



Proof. For the dimensional reason, $[y, z'] = v$. From the formulas after (2.5),

$$\mathrm{ad}^2(u)(v') = -v_3 z'$$

$$\mathrm{ad}^3(u)(z') = \mathrm{ad}(u^3)(z') = -v_3[y, z'] = -v_3 v$$

$$\mathrm{ad}^2(u)(v) = -v_3 z$$

$$\mathrm{ad}(u)(z) = \mathrm{ad}(y)(v) = \mathrm{ad}^2(y)(z') = 0\,.$$

The last equation is induced from (2.3) and

$$-v_3\,\mathrm{ad}(y)(v) = \mathrm{ad}(u^3)(v) = \mathrm{ad}^3(u)(v) = \mathrm{ad}(u)\,\mathrm{ad}^2(u)(v)$$
$$= \mathrm{ad}(u)(-v_3 z) = -v_3\,\mathrm{ad}(u)(z)\,.$$

By the same reason, $\mathrm{ad}(y)(v') = \mathrm{ad}(u)(z')$. q.e.d.

The Pontrjagin product structuure for $H_*(E_8, Z/3)$ is easily reduced from the arguments in the above proof, indeed, $y^3 = u^3 = 0$ and

$$\mathrm{ad}^2(u)(v') = \mathrm{ad}^3(u)(z') = \mathrm{ad}(u)(z) = 0$$

$$\mathrm{ad}(y)(v') = \mathrm{ad}(u)(z')\,, \qquad \mathrm{ad}^2(y)(z') = \mathrm{ad}(y)(z) = 0\,.$$

Corollary 2.8. *Let X be a H-space in* (2.1) *and suppose* $\mathrm{ad}(y)(z') = v$. *Then for $p = 3$, there are epimorphisms of Hopf-algebras.*

$$H^*(X, Z/3) \twoheadrightarrow H^*(E_8, Z/3)\,, \qquad K(3)^*(X) \twoheadrightarrow K(3)^*(E_8)\,.$$

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