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Root lattices and pencils of varieties

By

H. A. VERRILL

1. Introduction

In this paper a particular pencil of K3 surfaces is investigated. The Picard-Fuchs equation, solutions, and the monodromy group are found. The paper [21] gives the Picard-Fuchs equation for several families of elliptic curves. Similar methods are also used in [14] and [15] to obtain the Picard-Fuchs equation for the case of certain pencils of K3 surfaces. In these papers finding the Picard-Fuchs equation comes down to finding a recurrence relation for some sequence of combinatorially defined terms, e.g., in [5], recurrence relations are found for a_n in the following cases:

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \sum_{k=0}^{n} \binom{n}{k}^3, \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

In these examples, the symmetry of the defining equations of the varieties leads to the determination of the Picard-Fuchs equations from the combinatorial data. This gives motivation to try and find further examples, by considering other pencils of varieties with a good degree of symmetry.

The pencil of K3 surfaces in [15] is acted on by the Weyl group of the root lattice $A_1 \times A_1 \times A_1$, and the pencil can be viewed as being constructed from this lattice. This construction is described in section 2, for a general root lattice.

In this paper, the A_3 case is investigated. The corresponding pencil of K3 surfaces is denoted by $\mathcal{X}_{A3}$.

To find the Picard-Fuchs equation for $\mathcal{X}_{A_3}$, one has to find a recurrence relation for

$$a_n = \sum_{p+q+r+s=n} \binom{n}{pqrs}^2,$$

where $\binom{n}{pqrs} = \dfrac{n!}{p!q!r!s!}$.

The following result is obtained (cf. 4.3, Proposition 7):

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Proposition. For $n \geq 2$ there is a recurrence relation:

$$n^3 a_n = 2(2n-1)(5n^2-5n+2)a_{n-1} - 64(n-1)^3 a_{n-2}\,.$$

This recurrence relation is one of the keys to proving the following

Theorem 1 (cf. 4.4, Theorem 1). *The Picard-Fuchs equation for the family* $\mathcal{X}_{A_3}$ *is given by*

$$\mathcal{F} := \lambda(\lambda+4)(\lambda-12)y''' + 6(\lambda^2-7\lambda-12)y'' + \frac{(7\lambda^2-12\lambda-96)}{(\lambda+4)}y' + \frac{\lambda}{\lambda+4}y = 0\,.$$

Theorem 2. *The monodromy group for* $\mathcal{X}_{A_3}$ *is isomorphic to*

$$\overline{\Gamma_0(6)^+3}\,.$$

The group $\Gamma_0(6)^+3$ is one of the groups associated to the Monster group, given in the paper [6]. Explicitly,

$$\Gamma_0(6)^+3 = \left\{\begin{pmatrix} a & b \\ 6c & d \end{pmatrix}, \sqrt{3}\begin{pmatrix} a & b/3 \\ 2c & d \end{pmatrix} \in SL_2(\mathbf{R}) | a, b, c, d \in \mathbf{Z}\right\}\,.$$

The group $\overline{\Gamma_0(6)^+3}$ is the image of $\Gamma_0(6)^+3$ after quotienting out by scalars.

Finally (in 6.2, theorem 3), the solutions of the above Picard-Fuchs equation are given:

Theorem 3. *If* $\lambda = \lambda(\tau) = -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 - 4$, *where* η *is the Dedekind eta function, then the Picard-Fuchs equation for* $\mathcal{X}_{A_3}$ *has solution space*

$$G(\tau)(\mathbf{C} \oplus \tau\mathbf{C} \oplus \tau^2\mathbf{C})$$

where

$$G(\tau) = \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}\,.$$

Although this paper only deals with the A_3 case, it is shown in section 2 that the A_n case gives an $n-1$ dimensional pencil of Calabi-Yau manifolds. In the A_4 case, we obtain a pencil of Calabi-Yau 3-folds, with Picard-Fuchs equation

$$\begin{aligned}
&(x-20)(x+4)(x-4)(5+x)^2\frac{d^4}{dx^4}g(x) \\
&+ 2(5+x)(5x^3-65x^2-248x+560)\frac{d^3}{dx^3}g(x) \\
&+ (25x^3-143x^2-1466x-1080)\frac{d^2}{dx^2}g(x) \\
&+ (3x+10)(5x-32)\frac{d}{dx}g(x) + g(x)x\,.
\end{aligned}$$

The proof of this is not given in this paper, but it is the same method as for the A_3 case. At the end of the paper, a combinatorial application is given.

2. Preliminaries

2.1. Notation and Definitions. This notation follows [8] and [9]. Let R be a root system (of rank n), and let M_R be the root lattice generated by R. Moreover the lattice dual to M_R is denoted N_R. The Weyl Chambers in $N_R \otimes \mathbf{Q}$ can be defined as follows:

for $r \in R$, let $H_r = \{s \in N_R \otimes \mathbf{Q} | \langle s, r \rangle = 0\}$. A Weyl chamber is the closure of any connected component of

$$N_R \otimes \mathbf{Q} \setminus \bigcup_{r \in R} H_r \, .$$

Let Σ_R be the fan in $N_R \otimes \mathbf{Q}$ consisting of the Weyl chambers, together with all their sub-faces. We denote by $X(\Sigma_R)$ the toric variety associated to the fan Σ. Let Δ_R be the polyhedron in $M_R \otimes \mathbf{Q}$ with vertices given by $r \in R$, and let $L(\Delta_R)$ be the space of Laurent polynomials with support in Δ_R. The notation e^x denotes the passing from M_R to $\mathbf{C}[M_R]$, $x \mapsto e^x$, so that each $r \in R$ gives a monomial $e^r \in \mathbf{C}[M_R]$. The character of the adjoint representation means the polynomial

$$\chi_R := \sum_{r \in R} e^r \, .$$

Then we have $\chi_R \in L(\Delta_R)$, so we have a rational function

$$\chi_R : X(\Sigma_R) \longrightarrow \mathbf{P}^1 \, .$$

For $\lambda \in \mathbf{P}^1$, X_λ is defined to be the closure in the ambient toric variety $X(\Sigma_R)$, of the inverse image of λ under χ_R. I.e., set $X_\lambda := \overline{\chi_R^{-1}(\lambda)} \subset X(\Sigma_R)$.

Let $\mathscr{B}$: $X_R \to X(\Sigma_r)$ be the blow up of the base locus, so that we obtain the following commutative diagram:

$$\begin{array}{ccc} & & X_R \\ & \swarrow^{\mathscr{B}} & \downarrow \overline{\chi R} \\ X(\Sigma_r) & \xrightarrow[\chi_R]{} & \mathbf{P}^1 \, , \end{array}$$

For simplicity, we denote by $\mathscr{X}_R$ the family

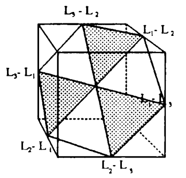
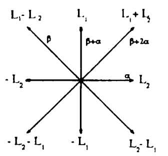
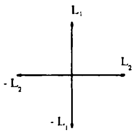
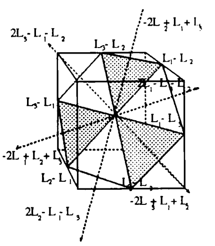
$$\overline{\chi_R} \colon X_R \to \mathbf{P}^1 \, ,$$

with fibres $X_\lambda := \overline{\mathscr{B}^{-1}(\overline{\chi_R} = \lambda)}$.

The family $\mathscr{X}_R$ is the object under consideration in this paper.

2.2. Examples. The two dimensional cases are shown in Table 1. The family $\mathscr{X}_{A_3}$ is the main concern of this paper. The $\mathscr{X}_{A_1^3}$ case is the family in [15].

Table 1: The 2-dimensional lattices

Name and Dynkin diagram	Σ_L and $X(\Sigma_L)$	the fiber X_λ, and reference
A_2 •——•	 $Bl_3(\mathbf{P}^2)$	$(X + Y + Z)(XY + YZ + ZX) = (\lambda + 3)XYZ$, which appears in [SB] as $\mathscr{C}$
B_2 •═•	 $Bl_4(\mathbf{P}^1 \times \mathbf{P}^1)$	X_λ: $(1 + x + x^2)(1 + z + z^2) = zx(1 + \lambda)$ where x comes from L_1, and z from L^2
$A_1 \times A_1$ • •	 $\mathbf{P}^1 \times \mathbf{P}^1$	$(x + y)(xy + 1) = \lambda xy$, This is example $\mathscr{A}'$ in [SB].
G_2 •≡•	 $Bl_3(Bl_6(\mathbf{P}^2))$	X_λ: $\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 1\right)\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 1\right) = \lambda + 1$

Lemma 1. *The family $\mathscr{X}_{A_n}$ is a pencil of Calabi-Yau manifolds.*

Proof. In this case, M_{A_n} can be given by

$$M_{A_n} = \mathbf{C}\{L_1, L_2, \ldots, L_{n+1}\}/(L_1 + L_2 + \cdots + L_{n+1} = 0)\,,$$

and the roots are the differences $L_i - L_j$ ([9], § 15.1). In the toric construction, let $X_i = e^{L_i}$, with $\prod X_i = 1$, so the roots correspond to $X_i X_j^{-1}$, and

$$\chi_{A_n}((X_1 : \cdots : X_{n+1})) = \sum_{\substack{i \neq j \\ 1 \leq i, j \leq n+1}} X_i X_j^{-1}\,.$$

The fibre X_λ is given by the closure of the locus of points where this has value λ.

Now we shall apply Theorem 4.1.9, of [2]. In [2], X_{A_n} is called a MPCP-desingularization of P_Δ. Here $\Delta = \Delta_{A_n}$ is an integral polyhedron in M_{A_n}, with vertices at $L_i - L_j$. This desingularization corresponds to a maximal projective triangulation of the dual polyhedron Δ^*. The only integral point in the interior of Δ is 0.

We claim that: Faces of Δ are given by the hyperplanes

$$H_J = \{x \in M_{A_n} | \langle x, E_J \rangle = 1\} ,$$

where

$$E_J = \frac{n-k}{n} \sum_{j \in J} L_j - \frac{k}{n} \sum_{j \in J'} L_j$$

and

$$J \subset \{1, \dots, n\},\ J' = \{1, \dots, n\} \backslash J,\ k = \# J .$$

(Note that $H_\emptyset = H_{1,2,\dots,n} = \emptyset$)

This claim means that (Δ, M_{A_n}) is a reflexive pair, so Theorem 4.1.9 of [2] says that the fibre X_λ is a Calabi-Yau manifold.

Proof of Claim: It is easy to show that:

a) Δ is contained in the intersection of the

$$\{x \in M_{A_n} | \langle x, E_J \rangle \leq 1\} ,$$

and that:

b) each H_i is spanned by vertices of Δ.

This gives the result.

Remark. 1) $A_{n_1} \times \cdots \times A_{n_k}$ $(n_1, \dots n_k \in \mathbf{N})$ also gives a pencil of Calabi-Yau manifolds.

2) In general this construction does not give rise to Calabi-Yau manifolds. In A_n cases, X_∞ is linearly equivalent to the anti-canonical divisor of $X(A_n)$, so by adjunction $K_{X_\lambda} \cong 0$. This does not happen in general.

3. The A_3 case

From now on, only the A_3 case is considered. By Proposition 1, $\mathscr{X}_{A_3}$ is a pencil of K3 surfaces. For theory and definitions of K3 surfaces, see [1] chapter VIII.

3.1. $X(A_3)$. Figures 1, 2, 3, show the construction of the fan for X_{A_3}.

Define local coordinates as follows: Set $X_i := e^{L_i}$ $(i = 1, \dots 4)$. Let $\sigma_{(1,2,3,4)}$ be the cone in N spanned by $L_1 - L_2$, $L_2 - L_3$, $L_3 - L_4$. Then $A_{(1,2,3,4)}$, the affine piece corresponding to $\sigma^\vee_{(1,2,3,4)}$, has local coordinates

$$x = X_1/X_2,\ y = X_2/X_3,\ z = X_3/X_4 .$$

Similarly for any permutation of the indices.

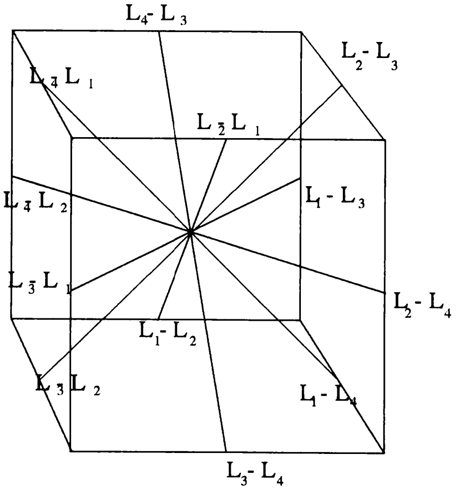


Figure 1: The root system A_3

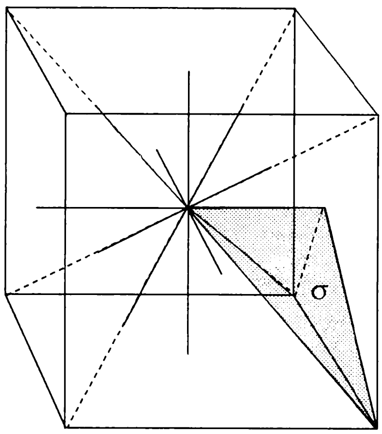


Figure 2: The dual lattice N and a Weyl chamber, σ.

3.2. X_λ. In the A_3 case, we have

$$\chi_{A_3}(X_1, X_2, X_3, X_4) = (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}) - 4\,.$$

Therefore the fibre X_λ restricted to $A_{(1,2,3,4)}$ is given by the equation

$$(1) \qquad (1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)xyz\,.$$

In terms of these local coordinates, χ_{A_3} is replaced by

$$\phi(x, y, z) := \chi_{A_3}(xyz : zy : y : 1) = \frac{(1 + x + xy + xyz)(1 + z + zy + zyx) - 4xyz}{xyz}\,.$$

3.3. X_∞. The fibre X_∞ at $\lambda = \infty$ is given in terms of local coordinates by $xyz = 0$.

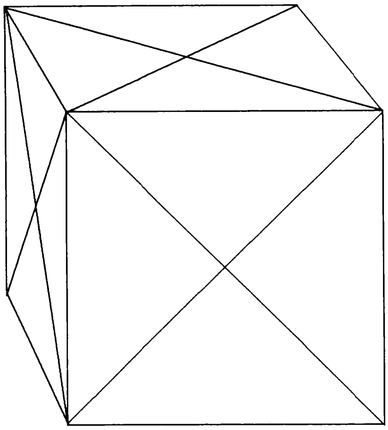


Figure 3: The fan defining X

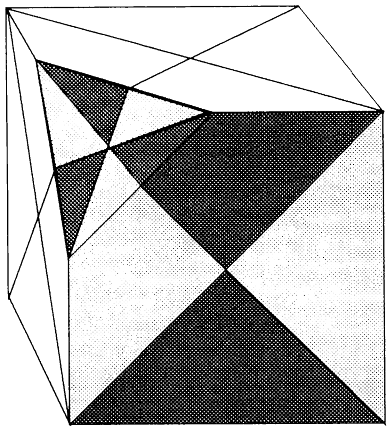


Figure 4: Some stars

Proposition 1. *The fibre X_∞ consists of eight copies of $Bl_3(\mathbf{P}^2)$ and six copies of $\mathbf{P}^1 \times \mathbf{P}^1$, configured as in Figure 5.*

Proof. These facts follow from toric geometry (cf § 5.7, [8]). Figure 4 shows the configuration of X_∞ as desired the result.

3.4. The base locus. Locally the base locus is defined by

$$(1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)xyz = 0 .$$

It consists of 20 rational curves, in configuration as indicated in Figure 6.

3.5. The Picard group of X_λ. In the section we show that the family is non-constant and that the Picard group of general X_λ has rank 19.

Lemma 2. *We have $\rho(X_\lambda) \leq 19$ for general λ.*

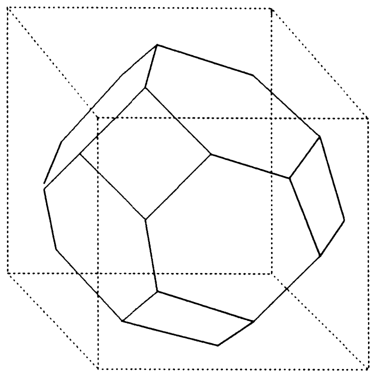


Figure 5: X_∞

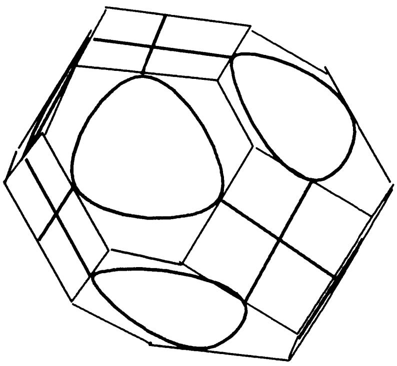


Figure 6: The base locus-rational curves on X_∞

Proof. X_λ is a K3 surface, so $\rho(X_\lambda) \leq 20$. Shioda and Inose [19] show that K3 surfaces with $\rho = 20$ from a discrete countable family, so if $\rho = 20$, X_λ would have to be locally constant, and the monodromy would be finite. After some base change, which we can assume to have been made, the monodromy would be trivial.

The kind of degeneration that occurs shows that the monodromy is infinite. At ∞, the degeneration is good in the sense of Kulikov; the degeneration is semistable (all fibres are reduced with normal crossings), and the canonical class is trivial in a neighborhood of each fibre. Clemens—Schmidt exact sequence implies that the local monodromy is trivial on cohomology if and only if p_g (general fibre) $= \sum p_g$ (component), where the sum is over all components of the degenerate fibre. (cf. e.g. [Theorem 2.7.5, 15])

In this case, p_g of the general fibre, a K3 surface, is 1, and the components of X_∞ are all rational surfaces, and so have $p_g = 0$. Hence the monodromy is not trivial, which implies the assertion.

Proposition 2. *We have $\rho(X_\lambda) = 19$ for general λ, and the discriminant of the Picard lattice is* 6.

Proof. Since $\rho(X_\lambda) \leq 19$ (Lemma 2), we just have to find a set of lines which have an intersection matrix of rank 19.

The base locus consists of 20 rational curves, and the corresponding intersection matrix has rank 16.

For some more lines, take the intersection of

$$X_\lambda\colon (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}) = (\lambda + 4) \tag{2}$$

with the hyperplane $X_1 = -X_2$, to give a quadratic curve, with

$$X_3/X_4 = \frac{\lambda + 2 \pm \sqrt{(\lambda(\lambda + 4))}}{2}.$$

By the action of the Weyl group there are 12 such lines. The resulting intersection matrix has rank 19 and discriminant 6.

Since 6 is square free, these 32 lines generate the Picard group over $\mathbf{Z}$.

Proposition 3. *With respect to some basis, the intersection matrix of the lattice of transcendental cycles, $T = T(X_\lambda) := Pic(X_\lambda)^\perp$ is*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Proof. For a K3 surface X, $H^2(X, \mathbf{Z})$ is an even unimodular lattice, with signature (3, 19), ([1] Chapter VIII).

Since the Picard lattice is even, with rank 19, signature (1, 18) (Hodge index theorem), discriminant 6 (Proposition 2), the lattice of transcendental cycles, $T(X_\lambda) = Pic(X_\lambda)^\perp$ is even, rank 3, signature (2, 1), discriminant ± 6.

Two quadratic forms are defined to have the same genus if they are equivalent over the p-adic integers for all primes p (cf. [§7, 7]).

In [7] table 15.4, genera of forms with $|\det| \leq 11$ are listed. For determinant ± 6, the genera are $I_{r,s}(2 \times 3^{\pm})$ and $II_{r,s}(2 \times 3^{\pm})$. (This notation is described in §7.8 of [7].)

Since $(r, s) = (2, 1)$, the determinant is -6 and we are in the $2 \times 3^-$ case. The lattice is even, so of type II. Hence the genus is $II_{2,1}(2 \times 3^-)$.

The above matrix is also of this genus. By Theorem 21 in [7], there is only one genus in this class, hence we obtain the assertion.

4. The Picard-Fuchs equation

In this section we determine the Picard-Fuchs equation for the family $\mathcal{X}_{A_3}$.

4.1. Notation and Definitions. As we see from the equation (1) in 3.2, the fibre X_λ is a smooth K3 surface for

$$\lambda \in B := \mathbf{C}\backslash\{0, -4, 12, \infty\}.$$

Therefore, up to scalar multiplication, there is a unique 2-form $\omega_\lambda \in H^{2,0}(X_\lambda, \mathbf{C})$, and for a fixed value λ_0 of λ, we have a monodromy representation

$$\pi_1(B, \lambda_0) \to \mathrm{Aut}\,(\mathbf{P}H_2(X_\lambda, \mathbf{Z})).$$

The image of this map, Γ, is the monodromy group of the family $\mathscr{X}_{A_3}$.

Let $\gamma_{1_\lambda}, \ldots, \gamma_{22_\lambda}$ be a basis for $H_2(X_\lambda, \mathbf{Z})$, which is flat with respect to parameter λ.

We can define the map

$$\varphi_m \colon B \to \mathbf{P}^{21}/\Gamma$$

by

$$\lambda \mapsto (\gamma_{i_\lambda}, \omega_\lambda) = \left(\int_{\gamma_{1_\lambda}} \omega_\lambda : \cdots : \int_{\gamma_{22_\lambda}} \omega_\lambda\right), \tag{3}$$

which we call the period map. (Here, $\mathbf{P}^{21} = \mathbf{P}(H_2(X_\lambda, \mathbf{Z}))$.) Each function $\int_{\gamma_{i_\lambda}} \omega_\lambda$ is a period.

The Picard-Fuchs equation is a differential equation for the periods, with the same monodromy.

Throughout this section, $I(\lambda)$ is defined by

$$I(\lambda) := \frac{1}{(2\pi i)^3} \int_{|x|=|y|=|z|=1} \frac{\Omega}{\phi - \lambda},$$

where Ω is the 3-form on X_{A_3} lifted from the 3-form $\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}$ on $X(\Sigma_{A_3})$, and $\phi(x, y, z) = \chi_{A_3}(xyz : yz : z : 1)$.

For the rest of the paper, the differential equations $\mathscr{F}$ and $\mathscr{D}$ are defined as follows:

$$\mathscr{F} := \lambda(\lambda + 4)(\lambda - 12)\frac{d^3}{d\lambda^3} + 6(\lambda^2 - 7\lambda - 12)\frac{d^2}{d\lambda^2} + \frac{(7\lambda^2 - 12\lambda - 96)}{(\lambda + 4)}\frac{d}{d\lambda} + \frac{\lambda}{\lambda + 4},$$

$$\mathscr{D} := \lambda(\lambda + 4)(\lambda - 12)y'' + 2(\lambda^2 - 7\lambda - 12)y' + \tfrac{1}{4}(\lambda - 4)y.$$

4.2. The existence of the Picard-Fuchs equation

Proposition 4. *The Picard-Fuchs equation is a third order linear differential equation.*

Proof. We have $H^2_{DR}(X_\lambda, \mathbf{C}) \cong H^2(X_\lambda, \mathbf{C}) \cong \mathbf{C}^{22}$, and

$$H^2(X_\lambda, \mathbf{C}) \supseteq Pic\,(X_\lambda) \otimes \mathbf{C} \cong \mathbf{C}^{19}.$$

So if ω_λ is a 2-form on X_λ, then

$$[\omega_\lambda],\ [\partial\omega_\lambda/\partial\lambda],\ [\partial^2\omega_\lambda/\partial\lambda^2],\ [\partial^3\omega_\lambda/\partial\lambda^3] \in H^2_{DR}/Pic(X_\lambda) \otimes \mathbf{C}$$

must satisfy a linear relation in $H^2_{DR}/Pic(X_\lambda)\otimes \mathbf{C}$, since this has dimension $22-19=3$. So for some C^∞ functions of λ, $f_0, \dots, f_3$, we have

$$f_0\omega_\lambda + \cdots + f_3\partial^3\omega_\lambda/\partial\lambda^3 \in Pic(X_\lambda)\otimes \mathbf{C}\,.$$

Integrating round a cycle in $Pic(X_\lambda)$ gives zero, this relation becomes a third order differential equation for the period $\int_\gamma \omega_\lambda$, via differentiating under the integral. Hence we have a third order differential equation for the periods.

Proposition 5. *The integral $I(\lambda)$ is a period for the family $\mathscr{X}_{A_3}$.*

Proof. By the Poincaré residue theorem, we obtain

$$I(\lambda) = \frac{1}{(2\pi i)^2}\int_{\Gamma_\lambda} Res_{X_\lambda}\left(\frac{\Omega}{\phi-\lambda}\right),$$

where Γ_λ is a 2-cycle on X_λ. Hence $I(\lambda)$ is a period.

Proposition 6. *If $v=\dfrac{1}{\lambda+4}$, then there is a power series expansion for $I(\lambda)$ given by*

$$-\sum v^{n+1}a_n\,,$$

where

$$a_n = \sum_{p+q+r+s=n}\binom{n}{pqrs}^2,$$

and $\binom{n}{pqrs}$, is the binomial coefficient, $\dfrac{n!}{p!q!r!s!}$.

Proof. We have

$$I(\lambda) = \frac{1}{(2\pi i)^3}\int_{|x|=|y|=|z|=1}\frac{dx\wedge dy\wedge dz}{xyz((\phi+4)-(\lambda+4))}\,.$$

In terms of the X_is ($x=X_1/X_2$, $y=X_2/X_3$, $z=X_3/X_4$),

$$\phi+4=(X_1+X_2+X_3+X_4)(X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1}),$$

so

$$\Omega\wedge\frac{dX_4}{X_4}=\frac{dX_1}{X_1}\wedge\frac{dX_2}{X_2}\wedge\frac{dX_3}{X_3}\wedge\frac{dX_4}{X_4},$$

and

$$(2\pi i)^4 I(\mu)$$
$$=\int_{\substack{|X_i|=1\\1\le i\le 4}}\frac{dX_1\wedge dX_2\wedge dX_3\wedge dX_4}{X_1X_2X_3X_4(X_1+X_2+X_3+X_4)(X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1})-\mu},$$

where $\mu = \lambda + 4$. If $\nu = 1/\mu$ then

$$\begin{aligned}
&-(2\pi i)^4 I(\nu) \\
&= \nu \int \Omega \wedge \frac{dX_4}{X_4} \sum_{n \geq 0} \nu^n (X_1 + X_2 + X_3 + X_4)^n (X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1})^n \\
&= -\sum \nu^{n+1} a_n ,
\end{aligned}$$

where $a_n =$ constant term in $(X_1 + X_2 + X_3 + X_4)^n (X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1})^n$. So we get

$$a_n = \sum_{p+q+r+s=n} \binom{n}{pqrs}^2 .$$

To find a differential equation satisfied by $I(\lambda)$, we just have to find a recurrence relation for the a_ns.

4.3. A recurrence relation

Proposition 7. *For $n \geq 2$, there is a recurrence relation*

$$n^3 a_n = 2(2n-1)(5n^2 - 5n + 2)a_{n-1} - 64(n-1)^3 a_{n-2} .$$

Proof. Since $n = p + q + r + s$, $n^m a_n$ can be expressed in terms of sums of the form

$$\sum_{p+q+r+s=n} \binom{n}{pqrs}^2 \mathcal{M}(p, q, r, s) ,$$

where $\mathcal{M}$ is a monomial in p, q, r, s. For example, we have

$$n^3 a_n = \sum \binom{n}{pqrs}^2 (p+q+r+s)^3 = \sum \binom{n}{pqrs}^2 (4p^3 + 36p^2 q + 24pqr) .$$

Squares in the monomial $\mathcal{M}$ can be cancelled with squares in the denominator of $\binom{n}{pqrs}^2$, e.g.:

$$\begin{aligned}
\sum_{p+q+r+s=n} \binom{n}{pqrs}^2 p^2 = \sum_{p=1}^{n} \sum_{q+r+s=n-p} \binom{n}{pqrs}^2 p^2 &= \sum_{p=1}^{n} \sum_{q+r+s=n-p} \left(\frac{n!}{(p-1)!q!r!s!} \right)^2 \\
&= \sum_{p=0}^{n-1} \sum_{q+r+s=n-1-p} \binom{n-1}{pqrs}^2 n^2 \\
&= n^2 a_{n-1} .
\end{aligned}$$

Define

$$b_n = \sum_{p+q+r+s=n} \binom{n}{pqrs}^2 p ,$$

$$c_n = \sum_{p+q+r+s=n} \binom{n}{pqrs}^2 pq ,$$

$$d_n = \sum_{p+q+r+s=n} \binom{n}{pqrs}^2 pqr .$$

We obtain

$$na_n = 4b_n ,$$

$$n^2 a_n = 4n^2 a_{n-1} + 12c_n ,$$

$$n^3 a_n = 40n^2 b_{n-1} + 4n^2 a_{n-1} + 24d_n ,$$

$$n^5 a_n = n^2(4a_{n-1} + 136(n-1)^2 a_{n-2} + 72b_{n-1} + (n-1)^2 544b_{n-2} + 360c_{n-1} + 480d_{n-1}) .$$

So, using the first three to eliminate the b_ns, c_ns, and d_ns of the fourth relation, we obtain the result.

Proposition 8. *We obtain*

$$\mathscr{F}(I(\lambda)) = 0 .$$

Proof. From the recurrence relation of Proposition 7, $I(v)$ satisfies the differential equation

$$(\Theta - 1)^3 - v2(2\Theta - 1)(5\Theta^2 - 5\Theta + 2) + v^2 64\Theta^3 ,$$

where

$$\Theta = v\frac{d}{dv} .$$

After a change of variables, we see that $I(\lambda)$ satisfies the differential equation $\mathscr{F} = 0$.

4.4. $\mathscr{F}$ is the Picard-Fuchs equation. The argument is the same as in [5].

Lemma 3. i) *$\mathscr{F}$ is irreducible.*

ii) *The* **C***-linear space spanned by all branches obtained by analytic continuation of a non-trivial solution of $\mathscr{F}y = 0$ has dimension* 3.

Proof. i) $\mathscr{F}$ is a Fuchsian differential equation, with the local exponents $(0, 0, 0)$ at $\lambda = -4$, $(0, -\frac{1}{2}, 1)$ at 0 and 12, and $(1, 1, 1)$ at ∞. Using Fuchs' formula, as in the lemma of [5] gives the result.

ii) This follows from (i), as in [5].

Theorem 1. *The Picard-Fuchs equation for the family $\mathscr{X}_{A_3}$ is given by*

$$\mathscr{F} := \lambda(\lambda+4)(\lambda-12)y''' + 6(\lambda^2 - 7\lambda - 12)y'' + \frac{(7\lambda^2 - 12\lambda - 96)}{(\lambda+4)}y' + \frac{\lambda}{\lambda+4}y = 0 .$$

Proof. This follows from the above lemmas, as for Corollary 1 in [5].

5. The monodromy group

The first few results of this section aim at showing that the monodromy group is a discrete subgroup of $PSL(2, \mathbf{R})$. The cusps and elliptic points are determined by studying the Picard-Fuchs equation. This allows the group to be given explicitly.

5.1. Notation. In this section, the following notation is used, and remains for the rest of the paper.

We denote by T the transcendental lattice of X_λ, and we set:

$$SO(T) = \left\{ A \in PSL(3, \mathbf{R}) \,\middle|\, A^T \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \right\},$$

$$SO(T, \mathbf{Z}) = SO(T) \cap PSL(3, \mathbf{Z}).$$

$$\Gamma_0(6) = \left\{ \begin{pmatrix} a & b \\ 6c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \,\middle|\, a, b, c, d \in \mathbf{Z} \right\},$$

$$\Gamma_0(3)^+ = \left\{ \begin{pmatrix} a & b \\ 3c & d \end{pmatrix}, \sqrt{3}\begin{pmatrix} a & b/3 \\ c & d \end{pmatrix} \in SL_2(\mathbf{R}) \,\middle|\, a, b, c, d \in \mathbf{Z} \right\},$$

$$\Gamma_0(6)^+3 = \left\{ \begin{pmatrix} a & b \\ 6c & d \end{pmatrix}, \sqrt{3}\begin{pmatrix} a & b/3 \\ 2c & d \end{pmatrix} \in SL_2(\mathbf{R}) \,\middle|\, a, b, c, d \in \mathbf{Z} \right\}.$$

$\Gamma^0(6)$, $\Gamma^0(3)^+$, $\Gamma^0(6)^+3$ are obtained from $\Gamma_0(6)$, $\Gamma_0(3)^+$, $\Gamma_0(6)^+3$ by conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For G a subgroup of $SL(2, \mathbf{R})$, $\bar{G}$ denotes the images of G in $SL(2, \mathbf{R})/\mathbf{R}^\times$, and using square brackets for a matrix in G means the class of the matrix $\bar{G}$.

These groups act naturally on $\mathscr{H}$, the upper half plane,

Proposition 9. *The monodromy group Γ is isomorphic to a subgroup $\tilde{\Gamma}$ of $SO(T, \mathbf{Z})$. The period map reduces to*

$$\varphi_{\tilde{\Gamma}}: B \to \mathbf{P}^2/\tilde{\Gamma},$$

Proof. First the restriction from $\mathbf{P}^{21}$ to $\mathbf{P}^2$ is because $\int_{\gamma_i} \omega = 0$ for $i > 3$, so $\tilde{\Gamma}$ is isomorphic to a subgroup of $PSL_3(\mathbf{R})$.

Now, $\gamma \in \tilde{\Gamma}$ takes an integral basis to an integral basis since moving round a loop on B takes curves to other curves, so $\tilde{\Gamma} \in SL(T, \mathbf{Z})$. The inner product is preserved, since everything varies smoothly, and the intersections are in $\mathbf{Z}$ and so constant. Hence $\tilde{\Gamma} \subseteq SO(T, \mathbf{Z})$.

Proposition 10. *If $\omega \in H^{2,0}$, with $\omega = a\gamma_1^* + b\gamma_2^* + c\gamma_3^*$, with a, b, c functions of λ, then a, b, c satisfy*

$$ab + 3c^2 = 0$$

and for all $\lambda \in B$,

$$\operatorname{Im}(b(\lambda)/c(\lambda)) \neq 0 .$$

Proof. This is from the facts that $\omega \wedge \omega = 0$ and $\omega \wedge \overline{\omega} > 0$ ([1] chapter VIII).

Because of this relation between the roots of $\mathscr{F}$, the problem of solving $\mathscr{F}$ can be reduced to a problem of solving a second order differential equation.

If Λ is a differential equation, then its symmetric square, $S^2\Lambda$ is a differential equation whose solution space consists of the products of solutions of Λ.

Proposition 11.

$$\mathscr{F} = S^2\mathscr{D}.$$

Proof. Proposition 10 gives a quadratic relation between a certain choice of roots of this equation, $ab + 3c^2 = 0$, so we can find ζ, ξ with

$$\zeta^2 = a ,$$

$$-3\xi^2 = b ,$$

$$\zeta\xi = c .$$

Now we can find a differential equation with roots ζ and ξ, and the Picard-Fuchs equation will be its symmetric square.

Using the lemma in 6.5 of [14], we find that the Picard-Fuchs equation is the symmetric square of $\mathscr{D}$.

Proposition 12. *The group $\tilde{\Gamma}$ is isomorphic to a subgroup Δ of $\overline{SL_2(\mathbf{R})}$, and the period map becomes*

$$\varphi_\Delta : B \to \mathscr{H}/\Delta . \tag{4}$$

Proof. By the above result, a is determined by b, c, and the map φ_m induces

$$\tilde{\varphi}' : B \to \mathbf{P}^1/\tilde{\Gamma}|_{\mathbf{P}^1}$$

$$\lambda \mapsto (c : b) = \left(\int_{\gamma_3} \omega : \int_{\gamma_2} \omega \right) .$$

Hence an action on a, c, b is determined by an action on b, c, so $\tilde{\Gamma}$ is isomorphic to some subgroup of $\overline{SL_2(\mathbf{R})}$.

φ_Δ is the quotient of two integrals, $\int_{\gamma_{\lambda_3}} \omega_\lambda$ and $\int_{\gamma_{\lambda_2}} \omega_\lambda$.

By Proposition 10, $\operatorname{Im}\left(\frac{c}{b}\right) \neq 0$, so by changing γ_3 to $-\gamma_3$ if necessary, we can take φ_Δ to map to $\mathscr{H}/\Delta$.

By considering the conditions imposed on the matrix entries, it can be shown that:

Lemma 4. *Corresponding to the map* $[\xi:\zeta] \mapsto [\xi^2 : -3\zeta^2 : \xi\zeta]$, *there is an isomorphism*

$$j\colon \overline{SL(2,\mathbf{C})} \to SO(T \otimes \mathbf{C})$$

given by

$$j\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d^2 & -\frac{1}{3}c^2 & 2dc \\ -3b^2 & a^2 & -6ba \\ db & -\frac{1}{3}ca & ad+cb \end{bmatrix}.$$

This restricts to an isomorphism $j\colon \Delta \to \tilde{\Gamma}$, *and* $j^{-1}(SO(T,\mathbf{Z})) = \overline{\Gamma_0(3)^+}$.

From the above lemmas we obtain:

Corollary 1. *The monodromy group is isomorphic to a subgroup of* $\overline{\Gamma_0(3)^+}$.

Proposition 13. *The period map* $\varphi_\Delta\colon B \to \mathscr{H}/\Delta$ *is given by* $\frac{\xi}{\zeta}$, *where* ξ, ζ *is a certain basis for the space of solutions to the differential equation* $\mathscr{D}$, *and* φ_Δ *can be extended to*

$$\varphi_\Delta^*\colon \mathbf{P}^1 \to \mathscr{H}^*/\Delta\,.$$

The singularities of the equation, 0, -4, 12, ∞ *will be mapped to elliptic fixed points or to cups by* φ_Δ^* *(see Table* 2).

Proof. Let r_1, r_2 be roots of the indicial equation at a singular point λ.

If $r_1 - r_2 \in \mathbf{Z}$, then either λ is mapped to a cusp, or this is only an apparent singularity.

However, when $r_1 = r_2$ then there is a solution involving logarithms ([§ 15.31, 10]), so the singularity is not apparent; the universal cover will have infinite order here, and so the point must correspond to a cusp.

If $r_1 - r_2 = p/q$, p, $q \in \mathbf{Z}$, $(p,q) = 1$, $g \geq 1$, then λ will be mapped to an elliptic point of order q. (For more details about the indicial equations, see [10].)

Table 2: Orders of monodromy at singularities

λ	roots of indicial equation	image of λ
0	$0, \frac{1}{2}$	elliptic point, order 2
-4	$0, 0$	cusp
12	$0, \frac{1}{2}$	elliptic point, order 2
∞	$\frac{1}{2}, \frac{1}{2}$	cusp

The results of inspection of the equation are given in Table 2, which gives the result.

There is a formula for the 'area' of the fundamental domain of subgroups in $SL_2(\mathbf{Z})$, which will also give an area of our subgroups:

Proposition 14. *We have*

$$\frac{1}{2\pi}\int_{\mathscr{H}/\Delta} y^{-2}\,dx\,dy = 2g - 2 + m + \sum_{v=1}^{r}(1 - 1/e_v)$$

where g is the genus of $\mathscr{H}^*/\Delta$, m is the number of inequivalent cusps, and $e_1, \ldots, e_r$ are the orders of the inequivalent elliptic points of Δ.

Proof. [17], §2.5, Theorem 2.20.

Proposition 15. *The group Δ has 2 cusps and 2 elliptic points of order 2, and*

$$[\overline{\Gamma_0(3)^+} : \Delta] = 3\,.$$

Proof. From Corollary 1, $\Delta \subseteq \Gamma_0(3)^+$.
There is a map

$$\varphi_\Delta^*\colon \mathbf{P}^1 \to \mathscr{H}^*/(\Delta)\,,$$

so the genus g of $\mathscr{H}^*/\Delta$ is 0 (Riemann-Hurwitz). We do not know what the order of φ_Δ^* is, so we can as yet only say that Δ has one or two cusps, and one of two elliptic points.

From Proposition 14, we have the following possibilities for $\mathscr{H}/\Delta$.

G	possibilities for Δ				$\overline{\Gamma_0(3)^+}$
g	0	0	0	0	0
m	2	2	1	1	1
r	2	1	2	1	2
e_1	2	2	2	2	6
e_2	2	—	2	—	2
area ($\mathscr{H}/G$)	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{3}$

The ratios of the areas is equal to the index of the subgroups, hence the only possibility is as stated.

Lemma 5. *The group $\overline{\Gamma^0(6)}$ is generated by*

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & -18 \\ 2 & -5 \end{bmatrix}.$$

Proof. This can be shown from the results of [11].

Lemma 6. *If V is a subgroup of index 3 in $\overline{\Gamma^0(3)^+}$ with 2 cusps and 2 elliptic points, then up to conjugation by an element of $\Gamma^0(3)^+$, V has a fundamental domain as in Figure 8.*

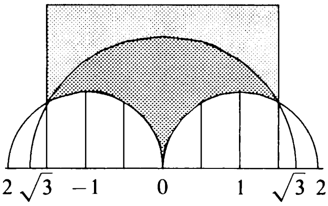


Figure 7: The fundamental region for $\Gamma^0(3)$ divided into 2 fundamental regions for $\Gamma^0(3)^+$

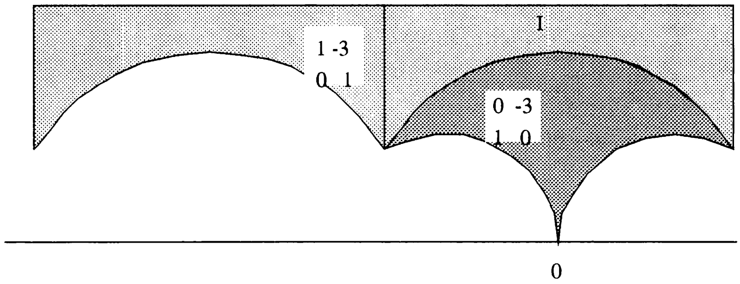


Figure 8: The fundamental domain of V

Proof. The fundamental domain for V in $\mathcal{H}$ will be the union of 3 copies of the fundamental domain for $\overline{\Gamma^0(3)^+}$, corresponding to 3 coset representatives for V in $\overline{\Gamma^0(3)}$. The identity matrix I can be taken as one coset representative, and the others can be chosen so that the union is connected. (See [16] Theorem 2.4.3(ii).) Up to translation, there are only 5 connected regions formed by the union of three fundamental domains for $\overline{\Gamma^0(3)^+}$. Using that fact that V has 2 cusps, it can be shown that its fundamental domain must be as stated. In the figure, coset representatives in $\overline{\Gamma^0(3)^+}$ are labeled.

Lemma 7. *If V is a subgroup of index 3 in $\overline{\Gamma^0(3)^+}$ with 2 cusps and 2 elliptic points, with a fundamental domain* **F** *as in Figure 8, then $\overline{\Gamma^0(6)} \subseteq V$.*

Proof. Because translates of **F** cover $\mathcal{H}$, V must contain some element fixing ∞ and mapping $\frac{-3}{2}$ to $\frac{9}{2}$. The only possibility is $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$. Similarly, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in V$.

The group V has two elliptic points. These must be elliptic points of $\overline{\Gamma^0(3)^+}$. So $3 + i\sqrt{3}$ is an elliptic point of V.

A calculation shows that the only non-trivial elements of $\overline{\Gamma^0(3)^+}$ fixing $3 + i\sqrt{3}$ are $\sqrt{3}\begin{bmatrix} 1 & -4 \\ 1/3 & -1 \end{bmatrix}$ and its inverse. So these elements are in V, and so is

$$\begin{bmatrix} 7 & -18 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1/3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -4 \\ 1/3 & -1 \end{bmatrix}. \tag{5}$$

Hence by Lemma 5 we have $\overline{\Gamma^0(6)} \subseteq V$.

Corollary 2. *Up to conjugation by an element of* $\Gamma_0(3)^+$, *we have*

$$\overline{\Gamma_0(6)} \subseteq \Delta,$$

with index 2.

Proof. This is just from conjugation by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ of Lemma 6 and Proposition 15.

Theorem 2. *The monodromy group is isomorphic to*

$$\overline{\Gamma_0(6)^+3}.$$

Proof. By Proposition 12, the monodromy group is isomorphic to $\Delta \subset SL(2, \mathbf{R})$. By Corollary 2 we may assume that $\overline{\Gamma_0(6)} \subseteq \Delta$, and the index in two, so Δ must be in the normaliser for $\overline{\Gamma_0(6)}$. The normalising quotient is the four-group, so there are 5 groups of genus zero between $\overline{\Gamma_0(6)}$ and its normaliser: $\overline{\Gamma_0(6)^-}$, $\overline{\Gamma_0(6)^+2}$, $\overline{\Gamma_0(6)^+3}$, $\overline{\Gamma_0(6)^+6}$, $\overline{\Gamma_0(6)^+}$. The only one with the right index and the right number of cusps (listed in [6], table 2), is $\overline{\Gamma_0(6)^+3}$.

6. Solutions to the Picard-Fuchs equation

In this section we shall find the solution space for the Picard-Fuchs equation $\mathscr{F}$. This is done by making a series of transformations, starting from an equation $\mathscr{E}$, with known solutions, and finishing with $\mathscr{F}$.

6.1. Notation and Definitions. The Dedekind eta function η is given by

$$\eta(\tau) = e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n}), \qquad \tau \in \mathscr{H}.$$

We write $q = e^{\pi i\tau/12}$.

The functions $s(\tau)$, $V(\tau)$ and $\lambda(\tau)$ are defined by

$$s(\tau) = \frac{\eta(6\tau)^8 \eta(\tau)^4}{\eta(2\tau)^8 \eta(3\tau)^4}, \tag{6}$$

$$V(\tau) = \frac{\eta(2\tau)^6 \eta(3\tau)}{\eta(t)^3 \eta(6\tau)^2}, \tag{7}$$

$$\lambda(\tau) = -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 - 4. \tag{8}$$

Lemma 8. *With* λ *and* s *as above,*

$$\lambda = -(1 - 3s)^2/s.$$

Table 3: Stabilizers of the cusps and elliptic points

τ	stabilizer	$\lambda(\tau)$
0	$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$	-4
∞	$\begin{pmatrix}1 & 6\\ 0 & 1\end{pmatrix}$	∞
$(3+i\sqrt{3})$	$\sqrt{3}\begin{pmatrix}-1 & 4\\ \frac{1}{3} & 1\end{pmatrix}$	0
$(3+i\sqrt{3})/2$	$\sqrt{3}\begin{pmatrix}1 & -2\\ \frac{2}{3} & -1\end{pmatrix}$	12

The value of $\lambda(\tau)$ at elliptic points and cusps are as in Table 3.

Proof. Lines 9 and 12 of table 3, in [6], say that

$$t_{6E} = \frac{\eta(2\tau)^8\eta(3\tau)^4}{\eta(6\tau)^8\eta(\tau)^4} + c_1$$

is a Hauptmodul for $\Gamma_0(6)$, that

$$t_{5C} = \left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 + c_2$$

is a Hauptmodul for $\Gamma_0(6)^+3$, and that

$$t_{6C} = t_{6E} + 9/t_{6E}\,. \tag{9}$$

The constants c_1, c_2 can be determined by expanding in terms of q, and comparing coefficients: we can take

$$c_1 = 0\,, \qquad c_2 = 10\,.$$

Then from Equation 9, and the relations

$$s = 1/t_{6E}\,, \qquad \text{and} \qquad \lambda = -t_{6C} + 6\,,$$

we obtain the required relation between λ and s.

The q expansion of λ is given by

$$\lambda = -q^{-1} + 2 - 15q + 32q^2 - 87q^3 + 192q^4 - 343q^5 + 672q^6\cdots$$

Hence $\lambda(i\infty) = \infty$. From the above relations for η,

$$\left[\frac{\eta\left(\frac{1}{\tau}\right)\eta\left(3\frac{1}{\tau}\right)}{\eta\left(2\frac{1}{\tau}\right)\eta\left(6\frac{1}{\tau}\right)}\right]^6 = 2\left(\frac{\eta(r)\eta(3r)}{\eta(2r)\eta(6r)}\right)^{-6},$$

where $r = -\frac{\tau}{6}$, so $\lambda(0) = -4$.

The branch points of $s(\tau)$ must map to the branch points of $\lambda(\tau)$ (i.e., cusps are mapped to cusps). The branch points of $-(1-3s)^2/s$ are at $s=\pm\frac{1}{3}$, and here $-(1-3s)^2/s=0$ or 12. By a computer estimation we can determine which cusp takes which value.

6.2. The solutions of the Picard-Fuchs equation

Lemma 9. *The differential equation*

$$\mathscr{E}(v) = s(s-1)(9s-1)\frac{d^2v}{ds^2} + (27s^2-20s+1)\frac{dv}{ds} + (9s-3)v = 0$$

has solution space

$$V(\tau)(\mathbf{C}\oplus\tau\mathbf{C}).$$

Proof. In [3] § 1, it is shown that $\mathscr{E}$ has a solution

$$f(x) = \sum_{n=0}^{\infty} v_n s^n, \qquad v_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n},$$

with

$$f(s(\tau)) \in M_1(\Gamma_1(6)).$$

In [21] § 10, § 11 and § 14, it is shown that $\mathscr{E}$ is the Picard-Fuchs equation of a family $\mathscr{C}$ of elliptic curves, the elliptic modular family associated to $\Gamma_1(6)$, and that the $\mathscr{E}$, has solution space

$$h(\tau)(\mathbf{C}\oplus\tau\mathbf{C}),$$

where $h(\tau)\in M_1(\Gamma_1(6))$. We can take $h(\tau)=f(s(\tau))$.

To show that $f(s(\tau))=V(\tau)$, the q expansions are compared. Computation shows that these agree to high enough order to imply the equality.

Lemma 10. *The differential equation*

$$\mathscr{D} = \lambda(\lambda+4)(\lambda-12)\frac{d^2y}{d\lambda^2} + 2(\lambda^2-7\lambda-12)\frac{dy}{d\lambda} + \frac{1}{4}(\lambda-4)y$$

has solution space

$$s^{1/2}(\tau)V(\tau)(\mathbf{C}\oplus\tau\mathbf{C}).$$

Proof. This is because $\mathscr{D}$ is obtained from $\mathscr{L}$ by a change of variables

$$y = s^{1/2}v$$

$$\lambda = -(1-3s)^2/s.$$

Theorem 3. *The Picard-Fuchs equation*

$$\mathscr{F} = \lambda(\lambda+4)(\lambda+12)\frac{d^3y}{d\lambda^3} + 6(\lambda^2-7\lambda-12)\frac{d^2y}{d\lambda^2} + \frac{(7\lambda^2-12\lambda-96)}{(\lambda+4)}\frac{dy}{d\lambda} + \frac{\lambda}{\lambda+4}y,$$

has solution space

$$G(\tau)(\mathbf{C} \oplus \tau\mathbf{C} \oplus \tau^2\mathbf{C})$$

where

$$G(\tau) = \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}.$$

Proof. Since $\mathcal{F} = S^2\mathcal{D}$ (Proposition 11), from Lemma 10,

$$sol\,(\mathcal{F}) = (sol\,(\mathcal{D}))^2 = s(\tau)V^2(\tau)(\mathbf{C} \oplus \tau\mathbf{C} \oplus \tau^2\mathbf{C}).$$

Remark. It can be shown that $\mathcal{F}$ is also the Picard-Fuchs equation for a pencil of abelian surfaces $\mathcal{A}$. $\mathcal{A}$ is constructed from the family $\mathcal{C}$, using the method of [14] §5. $\mathcal{C}$ in our notation is the family $\mathcal{X}_{A_2}$. It can also be shown that fibres of these two families have the same trnascendental lattice, so [12] gives a geometrical relation between the fibres; but we do not know if this can be extended to a global relationship.

7. A combinatorial application

We remark in this section that the a_n are related to something purely combinatorial. For more details, see [20], chapter 1.

Consider a random walk on the root lattice A_3, starting from the origin. This means that we take the state space to be the elements of $A_3 \subset \mathbf{R}^4$, and for $z \in A_3$, the transition function P is given as follows:

$$P(z, z + r) = \frac{1}{12}, \qquad \text{if } r \text{ is a root},$$

$$= 0 \qquad \text{otherwise}.$$

At time $t = 0$, the position is at the origin of A_3, and at time $t > 0$, the possible position is determined by the position at time $t - 1$, and the transition function P. Set $b_n =$ the probability of getting back to the origin in n steps, and denote the generating function for the b_ns by

$$\mathcal{B}(v) = \sum_{n\geq 0} v^n b_n.$$

Lemma 11.

$$(-3)^n b_n = n! \sum_{m=0}^{n} \frac{(-4)^{-m} a_m}{m!} \frac{1}{(n-m)!}.$$

Theorem 4. *If*

$$v(\tau) = \frac{(\eta(2\tau)\eta(6\tau))^6}{12(\eta(\tau)\eta(3\tau))^6 - 4(\eta(2\tau)\eta(6\tau))^6},$$

then

$$\mathscr{B}(v(\tau)) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2} .$$

Proof. In 4.1, we defined $I(\lambda) = \sum_{n\geq 0} v^{n+1} a_n$, where $v = -1/(\lambda + 4)$. For convenience, define $I_v(v) = I(\lambda)$. Now by considering the relationship between a_ns and b_ms, $I_v(v)$ and $\mathscr{B}$ are related as follows:

$$\mathscr{B}(v) = \frac{1 + 4v}{12v} I_v\left(\frac{12v}{1 + 4v}\right) .$$

The function $I(\lambda)$ is a solution of the Picard-Fuchs equation $\mathscr{F}$, and so by Theorem 3 can be expressed as $G(\tau)(a + b\tau + c\tau^2)$. Considering coefficients of the q expansions ($q = e^{2\pi i\tau}$) shows that as a function of τ, $I(\tau) = -G(\tau)$. A change of variables gives the result.

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DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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