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The Picard group of the moduli space of stable sheaves on a ruled surface

By

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0. Introduction

Let (X, H) be a pair consisting of a smooth projective surface X defined over $\mathbf{C}$ and an ample divisor on X. Let $M_H(r, c_1, c_2)$ be the moduli space of stable sheaves of rank r on (X, H) with Chern classes $(c_1, c_2) \in H^2(X, \mathbf{Z}) \times H^4(X, \mathbf{Z})$, and $\overline{M}_H(r, c_1, c_2)$ the Gieseker-Maruyama compactification of $M_H(r, c_1, c_2)$. Let C be a smooth projective curve of genus g and $\pi: X \to C$ a ruled surface over $\mathbf{C}$. Let C_0 be a minimal section of π and f a fibre of π. Let H be an ample divisor with $(K_X, H) < 0$, where K_X is the canonical divisor on X. If $(c_1, f) = -1$, Nakashima [N] and Qin [Q] computed the Picard group of $M_H(2, c_1, c_2)$. In the rational ruled surface case, Ellingsrud and Strømme essentially computed the Picard group of $M_H(2, c_1, c_2)$ if $M_H(2, c_1, c_2)$ is compact ([Y, 5]). But no other results are known for general c_2. In this paper, we shall treat the case where $(c_1, f) = 0$ under suitable conditions on H. Let $H_n = C_0 + nf$ be an ample divisor on X. For a fixed triplet (r, c_1, c_2), there is an integer N and for all $n, n' \geq N$, $\overline{M}_{H_n}(r, c_1, c_2) = \overline{M}_{H_{n'}}(r, c_1, c_2)$ (Lemma 1.2). We denote this space by $\overline{M}(r, c_1, c_2)$ and $M_{H_n}(r, c_1, c_2)$ by $M(r, c_1, c_2)$. If $g \geq 1$, $r = 2$ and $(K_X, H) < 0$, then we proved that $Pic(M_H(2, c_1, c_2)) = Pic(M(2, c_1, c_2))$ for $c_2 \geq 2$ ([Y, Lemma 3.6]). Hence it is sufficient to treat $M(2, c_1, c_2)$. To compute the Picard groups of these spaces, it is not necessary to restrict ourselves to the case that $r = 2$. So we shall treat $M(r, c_1, c_2)$. If $g \geq 1$ and $(K_X + f, H) < 0$, then we can prove that $Pic\,(M_H(r, c_1, c_2)) = Pic\,(M(r, c_1, c_2))$ (Proposition 5.1).

The author was also motivated by the work of Drezet [D1] on the computation of $Pic\,(\overline{M}_H(r, c_1, c_2))$ in the case where $X = \mathbf{P}^2$. Let $K(X)$ be the Grothendieck group of a surface X and $K(r, c_1, c_2) := \{\alpha \in K(X) | \chi(\alpha \otimes E) = 0,\ E \in M_H(r, c_1, c_2)\}$. Drezet constructed a homomorphism $\kappa: K(r, c_1, c_2) \to Pic\,(M_H(r, c_1, c_2))$ and proved that κ is surjective for $X = \mathbf{P}^2$. For a ruled surface with $g \geq 1$, we cannot expect κ to the surjective. In this paper, we shall construct a morphism α: $\overline{M}(r, c_1, c_2) \to Alb\,(X) \times Pic^0\,(X)$ and prove that $Pic\,(\overline{M}(r, c_1, c_2))/\alpha^*(Pic\,(Alb\,(X) \times Pic^0\,(X)))$ is generated by the image of κ.

Let $\overline{M}(r, c_1)$ be the Seshadri compactification of the moduli space of stable

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vector bundles on C. Then we have $\overline{M}(r, c_1, 0) \cong \overline{M}(r, c_1)$ which was treated by Drezet and Narasimhan [D-N]. So we may assume that $c_2 \geq 1$. In this paper, we shall consider a general member E of $M(r, c_1, c_2)$ (and hence $E_{|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r}$) as a sheaf defined by an exact sequence $0 \to L^{\oplus(r-1)} \to E \to M \to 0$, where L is the pull-back of a line bundle on C and M is a torsion free sheaf of rank 1. Then with slight modifications, we can use the same argument as in [D-N]. Another idea to compute the Picard group is to consider the divisor on C determined by jumping lines. In order to analyse moduli spaces of stable sheaves on $\mathbf{P}^2$, the notion of the jumping line plays important roles. In this paper, by using the divisor of jumping lines, we define a morphism $\lambda: \overline{M}(r, c_1, c_2) \to S^{c_2}C$, where $S^{c_2}C$ is the symmetric product of C. Since $Alb(X) \cong J$ (Jacobian of C), by using this morphism and the Jacobian map, we can construct a morphism $\alpha: \overline{M}(r, c_1, c_2) \to Alb(X) \times Pic^0(X)$. Then we obtain the following theorem.

Theorem 0.1. *Assume that* $g \geq 1$ *and* $c_2 \geq 2$. *Then the following holds.*

(i) $\alpha^*: Pic(Alb(X) \times Pic^0(X)) \to Pic(\overline{M}(r, c_1, c_2))$ *is injective.*

(ii) κ *is injective.*

(iii) $\operatorname{im} \kappa \cap \operatorname{im} \alpha^* \cong Pic^0(X) \times Alb(X)$.

(iv) $Pic(\overline{M}(r, c_1, c_2))/Pic(Alb(X) \times Pic^0(X))$ *is generated by the image of* κ. *In particular,* $Pic(\overline{M}(r, c_1, c_2)) \cong Pic(Alb(X) \times Pic^0(X)) \oplus \mathbf{Z}^{\oplus 3}$.

In Proposition 3.14, we shall treat the case where $c_2 = 1$. In section 4, we shall treat the case where $g = 0$.

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1. The existence of stable sheaves

1.1. Notation. Let Y be a smooth projective variety defined over $\mathbf{C}$ and F a coherent sheaf on Y. Let $A^*(Y)$ be the Chow ring of Y. Grothendieck defined the i-th Chern class of F as an element of $A^i(Y)$. We denote it by $\tilde{c}_i(F)$ and its image to $H^{2i}(Y, \mathbf{Z})$ by $c_i(F)$.

Let C be a smooth projective curve of genus g and $\pi: X \to C$ a ruled surface defined over $\mathbf{C}$. Let f be a fibre of π and C_0 a minimal section of π with $(C_0^2) = -e$. Let H be an ample divisor on X. Note that the Neron-Severi group $NS(X)$ is isomorphic to $H^2(X, \mathbf{Z})$.

We denote the moduli of stable sheaves of rank r on (X, H) with Chern classes $(c_1, c_2) \in H^2(X, \mathbf{Z}) \times H^4(X, \mathbf{Z})$ by $M_H(r, c_1, c_2)$ and the Gieseker-Maruyama compactification by $\overline{M}_H(r, c_1, c_2)$. Throughout this paper, we assume that

$$(c_1, f) = 0 . \tag{1.1}$$

For a torsion free sheaf F on X, we set

$$\mu(F)=\frac{c_1(F)}{\mathrm{rk}\,(F)}\in H^2(X,\mathbf{Q}),$$

$$\Delta(F)=\frac{1}{\mathrm{rk}\,(F)}\left(c_2(F)-\frac{\mathrm{rk}\,(F)-1}{2\,\mathrm{rk}\,(F)}(c_1(F)^2)\right)\in H^4(X,\mathbf{Q}),$$

and $\deg(F)=(c_1(F),C_0)$.

For a scheme S, we denote the projection $S\times X\to S$ by p_S and $S\times X\to S\times C$ by π_S. Let $\mathscr{E}$ be a family of coherent sheaves on X parametrized by S. For a divisor D on X, we denote $\mathscr{E}\otimes r^*\mathscr{O}_X(D)$ by $\mathscr{E}[D]$, where $r\colon S\times X\to X$ is the projection. In this paper, we also denote a divisor defining the canonical line bundle K_X by K_X.

1.2. We shall first prove the following lemma which is due to [L] or [Mr2].

Lemma 1.1. *Let E be a torsion free sheaf of rank r with Chern classes c_1, c_2 and assume that $E_{|\pi^{-1}(\eta)}\cong\mathscr{O}^{\oplus r}_{\pi^{-1}(\eta)}$, where η is the generic point of C. Then E is obtained by successive elementary transformations from $\pi^*\pi_*E$ along sheaves on fibres.*

Proof. Since $E_{|\pi^{-1}(\eta)}\cong\mathscr{O}^{\oplus r}_{\pi^{-1}(\eta)}$, the natural homomorphism $\pi^*\pi_*E\to E$ is injective. We assume that $\pi^*\pi_*E\subsetneq E$. Then there is a fibre f_1 such that $E_{|f_1}\not\cong\mathscr{O}^{\oplus r}_{f_1}$. Let T_1 be the torsion part of $E_{|f_1}$ and set $F_1=E_{|f_1}/T_1$. Since F_1 is torsion free, $F_1\cong\bigoplus_{j=1}^r\mathscr{O}_{f_1}(a_j)$, $a_j\in\mathbf{Z}$. We set $F_{1+}:=\bigoplus_{a_j\geq 0}\mathscr{O}_{f_1}(a_j)$ and $F_{1-}:=F_1/F_{1+}$. Let E_1 be the kernel of $E\to F_{1-}$. Then E_1 is obtained by an elementary transformation from E along F_{1-} and E is obtained by an elementary transformation from $E_1(f_1)$ along $F_{1+}\oplus T_1$. Since $\pi_*F_{1-}=0$ and $F_{1-}\neq 0$, there is an inclusion $\pi^*\pi_*E\subset E_1\subsetneq E$. Applying this argument successively, we obtain a sequence of inclusions $\pi^*\pi_*E\subset E_i\subsetneq E_{i-1}\subsetneq\cdots\subsetneq E$. Since $\deg(E_k)\leq\deg(E_{k-1})-1$, there is an integer s with $\pi^*\pi_*E=E_s$. Therefore E is obtained by successive elementary transformations from $\pi^*\pi_*E$ along sheaves on fibres.

Remark 1.1. From this proof, we obtain that E is a subsheaf of $\pi^*(\pi_*E\otimes\mathscr{O}_C(\sum_{i=1}^s f_i))$. Since $c_2(E_{i-1})=c_2(E_i)+\deg(F_{i-})$, $c_2(E)=-\sum_{i=1}^s\deg(F_{i-})\geq s$. Hence E is a subsheaf of $\pi^*(\pi_*E\otimes L)$ where L is a line bundle of degree c_2 on C.

Lemma 1.2. *Let $H_n=C_0+nf$ be an ample divisor on X. Let m_0 be an integer such that $|C_0+m_0f|$ is base point free, $m_1=2rc_2-r^2(1-g)+r^2+1$ and $m_2=[((K_X,C_0)+m_1)/2+1]$. We set $N=\max\{m_0+r^2c_2+1,m_1,m_2\}$. Then we obtain that $\overline{M}_{H_n}(r,c_1,c_2)\cong\overline{M}_{H_{n'}}(r,c_1,c_2)$ for n, $n'\geq N$. We denote this space by $\overline{M}(r,c_1,c_2)$ and $M_{H_n}(r,c_1,c_2)$ by $M(r,c_1,c_2)$.*

Proof. We shall first prove that if E is μ-semi-stable with respect to H_n $(n\geq N)$, then $E_{|\pi^{-1}(\eta)}\cong\mathscr{O}^{\oplus r}_{\pi^{-1}(\eta)}$. Replacing E by $E^{\vee\vee}$, we may assume that E is locally free. Let $f_1,f_2,\ldots,f_{m_1}$ be general fibres of π. Then there is an exact sequence

$$\mathrm{Ext}^1(E,E)\to\prod_{i=1}^{m_1}\mathrm{Ext}^1(E_{|f_i},E_{|f_i})\to\mathrm{Ext}^2\left(E,E\left(-\sum_{i=1}^{m_1}f_i\right)\right).$$

The Serre duality implies that $\operatorname{Ext}^2(E, E(-\sum_{i=1}^{m_1} f_i)) \cong \operatorname{Hom}(E, E(K_X + \sum_{i=1}^{m_1} f_i))^\vee$. From the choice of n, we obtain that $(K_X + \sum_{i=1}^{m_1} f_i, H_n) = (K_X, C_0) + n(K_X, f) + m_1 < 0$. Since E is μ-semi-stable, this implies that $\operatorname{Ext}^2(E, E(-\sum_{i=1}^{m_1} f_i)) = 0$. Assume that $E_{|\pi^{-1}(\eta)} \not\cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r}$, and then $\operatorname{Ext}^1(E_{|f_i}, E_{|f_i}) \neq 0$ for all i. Thus $\dim(\prod_{i=1}^{m_1} \operatorname{Ext}^1(E_{|f_i}, E_{|f_i})) \geq m_1 > 2rc_2 - r^2(1-g) + r^2 = -\chi(E, E) + r^2 \geq \dim \operatorname{Ext}^1(E, E)$, which is a contradiction. Therefore $E_{|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r}$.

Let $n, n' \geq N$ be integers and assume that there is a torsion free sheaf E which is not semi-stable with respect to H_n but semi-stable with respect to $H_{n'}$. By this assumption, there is a subsheaf F such that $(\mu(F), H_n) \geq (\mu(E), H_n)$, $(\mu(F), f) < (\mu(E), f)$ and E/F is torsion free. In fact, the first paragraph of this proof implies that $(\mu(F), f) \leq (\mu(E), f)$, and if $(\mu(F), f) = (\mu(E), f)$, then $(\mu(F), H_{n'}) = (\mu(F), H_n)$. Thus F is a distabilizing subsheaf of E with respect to $H_{n'}$, which is a contradiction. Let C' be a member of $|H_n - (n - m_0)f|$ which does not meet pinch points of E. Then $(\mu(F) - \mu(E), C') \geq (n - m_0)(\mu(E) - \mu(F), f) \geq (n - m_0)/r^2$. By Remark 1.1, we get $\pi^*\pi_*E \subset E \subset \pi^*\pi_*E \otimes \pi^*L$ where L is a line bundle of degree c_2 on C. We set $F' = \pi_* F_{|C'}$. Then F' is a subsheaf of $\pi_*E \otimes L$, and $\pi^*F' \subset \pi^*\pi_*E \otimes \pi^*L \subset E \otimes \pi^*L$. Since $(\mu(\pi^*F'), H_{n'}) = (\mu(F), C')$ and $(\mu(E), H_{n'}) = (\mu(E), C')$, we get $(\mu(\pi^*F'), H_{n'}) \geq (\mu(E), H_{n'}) + (n' - m_0)/r^2$. Therefore, $(\mu(\pi^*(F' \otimes L^\vee)), H_{n'}) - (\mu(E), H_{n'}) = (\mu(\pi^*F'), H_{n'}) - (\mu(E), H_{n'}) - c_2 \geq (n' - m_0)/r^2 - c_2 > 0$. This contradicts the assumption that E is semi-stable with respect to $H_{n'}$. Hence every $H_{n'}$-semi-stable sheaf is H_n-semi-stable. Replacing the role of n and n', we get that every H_n-semi-stable sheaf is $H_{n'}$-semi-stable. Thus the notion of the semi-stability does not depend on H_n, $n \geq N$. By this proof, we can also show that the notion of the stability does not depend on H_n, $n \geq N$. By the definition of the coarse moduli scheme, we get our claim.

Remark 1.2. Let E be a torsion free sheaf such that $E_{|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r}$ and F a subsheaf of E such that E/F is torsion free, $(\mu(F), H_n) \geq (\mu(E), H_n)$ and $(\mu(F), f) < (\mu(E), f)$. By this proof, we see that $(\mu(\pi^*(F' \otimes L^\vee)) - \mu(F), H_n) = (\mu(F), C') - c_2 - (\mu(F), C') - (n - m_0)(\mu(F), f) \geq (n - m_0)/r^2 - c_2 > 0$. Let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ be the Harder-Narasimhan filtration or a Jordan-Hölder filtration of E with respect to H_n. Then this implies $(c_1(F_i), f) = 0$, that is, $F_{i|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus \operatorname{rk} F_i}$.

Remark 1.3. Since $(K_X, H_n) < 0$ for $n \geq N$, $M(r, c_1, c_2)$ is smooth of dimension $2rc_2 + (r^2 - 1)(g - 1) + g$ and $\overline{M}(r, c_1, c_2)$ is normal ([Mr1]).

From now on, we shall fix a polarization H_n with $n \gg c_2$ and denote it by $\mathcal{O}_X(1)$. By using the same method as in [D-L], we shall show the following proposition.

Proposition 1.3. *Assume that* $g \geq 1$. *Then* $M(r, c_1, c_2)$ *is not empty for* $c_2 \geq 1$.

Proof. Let I be a torsion free sheaf of rank 1 with Chern classes c_1, c_2, and set $E = \mathcal{O}_X^{\oplus(r-1)} \oplus I$. Assume that $E(m)$ is generated by global sections and $h^i(E(m)) = 0$, $i > 0$. We set $V = H^0(X, E(m)) \otimes \mathcal{O}_X(-m)$, then $V \to E$ defines a point x of $\mathcal{Q}uot_{V/X}$. Let Q^x be the connected component which contains x. Let $V \otimes_{\mathbf{C}} \mathcal{O}_{Q^x} \to \mathcal{E}$ be the universal quotient and $Q_1 = \{y \in Q^x | h^i(X, \mathcal{E}_y(m)) = 0, i > 0\}$. Let $\mathcal{H}$ be the set of sequences of polynomials $h = (h_1, h_2, \dots, h_s)$ which is the Hilbert polynomial of the Harder-Narasimhan filtration of $\mathcal{E}_y$, $y \in Q_1$: if $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \mathcal{E}_y$ is the Harder-Narasimhan filtration of $\mathcal{E}_y$, then $h_i(m) = \chi(F_i(m))$. By the above remark, $(c_1(F_i), f) = 0$. Let $f_h: \mathcal{F}lag^h_{\mathcal{E}/Q_1 \times X} \to Q_1$ be the flag-scheme whose point F corresponds to a filtration $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \mathcal{E}_y$, $y = f_h(F)$ with $\chi(F_i(m)) = h_i(m)$ and $HN(h)$ the open subscheme of $\mathcal{F}lag^h_{\mathcal{E}/Q_1 \times X}$ such that F is the Harder-Narasimhan filtration of $\mathcal{E}_y$. Let $HN(h)^c$ be the complement of $HN(h)$. Since $HN(h)^c$ is closed and f_h is proper, $f_h(HN(h)^c)$ is a closed subset of Q_1. Let Q_h be the open subscheme of Q_1 which is the complement of $f_h(HN(h)^c)$. By using analogous arguments to the proof of the uniqueness of the Harder-Narasimhan filtration, we can easily show that $f_h(HN(h)) \cap f_h(HN(h)^c) = \emptyset$. Hence we obtain a proper morphism $f_{h|HN(h)}: HN(h) \to Q_h$. We set $f_h' := f_{h|HN(h)}$.

For simplicity, we denote $\mathcal{E}_y$ by E and $gr_i(E)$ by E_i. Since F is the Harder-Narasimhan filtration of E, we get

$$\text{(1.2)} \qquad \operatorname{Hom}(E_i, E_j) = 0 \qquad \text{for } i < j .$$

Since $(c_1(E_i), f) = 0$, we get

$$\text{(1.3)} \qquad \operatorname{Ext}^2(E_i, E_j) \cong \operatorname{Hom}(E_j, E_i \otimes K_X)^\vee = 0 \qquad \text{for any } i, j .$$

In the notation of [D-L, 1.5], there is an exact sequence $\operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1_{F,+}(E, E) \to \operatorname{Ext}^2_{F,-}(E, E)$. By using (1.2), (1.3) and the spectral sequence in [D-L, Proposition 1.3], we see that

$$\text{(1.4)} \qquad \operatorname{Ext}^2_{F,-}(E, E) = 0 , \qquad \operatorname{Ext}^i_{F,+}(E, E) = 0 \quad \text{for } i \neq 1 .$$

Let J be the kernel of the quotient $V \to E$, and let T_F be the Zariski tangent space of $\mathcal{F}lag^h_{\mathcal{E}/Q_h \times X}$ at F. By virtue of [D-L, Proposition 1.5], there is an exact sequence

$$\text{(1.5)} \qquad 0 \to T_F \to \operatorname{Hom}(J, E) \xrightarrow{\omega_+} \operatorname{Ext}^1_{F,+}(E, E) \to 0 ,$$

where ω_+ is the composition $\operatorname{Hom}(J, E) \to \operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1_{F,+}(E, E)$. By [D-L, Proposition 1.7], $\mathcal{F}lag^h_{\mathcal{E}/Q_h \times X}$ is smooth at F. Since f_h' is proper and one to one, (1.5) implies f_h' is a closed immersion and the normal space of $\operatorname{im}(f_h')$ at F is $\operatorname{Ext}^1_{F,+}(E, E)$. The spectral sequence in [D-L, Proposition 1.3] and the Riemann-Roch theorem imply that

$$\text{(1.6)} \qquad \dim \operatorname{Ext}^1_{F,+}(E, E) = -\sum_{i<j} \chi(E_i, E_j)$$
$$= -\sum_{i<j} \{r_i d_j - r_j d_i + r_i r_j (1 - g) - r_i e_j - r_j e_i\} ,$$

where $r_i = \mathrm{rk}(E_i)$, $d_i = \deg E_i$ and $e_i = c_2(E_i)$. Since $g \geq 1$, we get

$$(1.7) \qquad \dim \mathrm{Ext}^1_{F,+}(E, E) \geqq -\sum_{i<j} \{r_i d_j - r_j d_i - r_i e_j - r_j e_i\} .$$

Since $r_i d_j - r_j d_i$, $e_i \geq 0$ and $c_2 = \sum_i e_i \geq 1$, we obtain that $\dim \mathrm{Ext}^1_{F,+}(E, E) > 0$. Thus $\mathrm{codim}(\mathrm{im}\, f_h') > 0$. By the boundedness theorem of Grothendieck, $\mathscr{H}$ is a finite set. Hence $Q_2 := Q_1 \backslash \bigcup_{h \in \mathscr{H}} \mathrm{im}(f_h')$ is an non-empty open subset of Q_1. By the definition of Q_2, for any point y of Q_2, $\mathscr{E}_y$ is semi-stable. Therefore $\overline{M}(r, c_1, c_2)$ is not empty for $c_2 \geq 1$. The existence of stable sheaves follows from the following lemma.

Lemma 1.4. *Let Q be an open set of $\mathscr{Q}uot_{\mathcal{O}_X(-m)^{\oplus N}/X}$ which satisfies the following*:

For a quotient $\mathcal{O}_X(-m)^{\oplus N} \to E$ which belongs to Q,

(i) *$\mathcal{O}_X^{\oplus N} \to E(m)$ induces an isomorphism $H^0(X, \mathcal{O}_X^{\oplus N}) \cong H^0(X, E(m))$ and $H^i(X, E(m)) = 0$ for $i > 0$.*

(ii) *E is semi-stable.*

We set $Q^s := \{\mathcal{O}_X(-m)^{\oplus N} \to E | E$ is a stable sheaf$\}$. Then $\mathrm{codim}(Q \backslash Q^s) \geq 2$ for $g \geq 1$.

Proof. Let $\mathcal{O}_{Q \times X}[-m]^{\oplus N} \to \mathscr{E}$ be the universal qoutient on $Q \times X$. Let $\mathscr{H}'$ be the set of sequences of polynomials $h = (h_1, h_2, \dots, h_s)$ whose element is the Hilbert polynomial of a Jordan-Hölder filtration of $\mathscr{E}_y$, $y \in Q$: there is a Jordan-Hölder filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \mathscr{E}_y$ such that $h_i(m) = \chi(F_i(m))$. By the above remark, $(c_1(F_i), f) = 0$. Let f_h: $\mathscr{F}lag^h_{\mathscr{E}/Q \times X} \to Q$ be the flag-scheme whose point F corresponds to a filtration F: $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \mathscr{E}_y$, $y = f_h(F)$ with $\chi(F_i(m)) = h_i(m)$ and $JH(h)$ the open subscheme of $\mathscr{F}lag^h_{\mathscr{E}/Q \times X}$ such that F is a Jordan-Hölder filtration of $\mathscr{E}_y$. Although f_h: $JH(h) \to Q$ is not an immersion, in the same way as above, we can show that ω_+ is surjective and $\mathrm{codim}(f_h(JH(h))) \geq -\sum_{i<j} \chi(E_i, E_j)$. By the definition of the filtration, we get $\chi(E_i, E_j) = r_i r_j(1 - g) - 2r_i e_j \leq -2$. Therefore we obtain that $\mathrm{codim}(Q/Q^s) \geq 2$.

Remark 1.4. We can easily prove that $JH(h) = \mathscr{F}lag^h_{\mathscr{E}/Q \times X}$.

Remark 1.5. We consider a Jordan-Hölder filtration of a μ-semi-stable sheaf with respect to the μ-stability. Then, in the same way as in the proof of Lemma 1.4, we can also prove that there is a μ-stable sheaf for $c_2 \geq 1$. Let M^μ be the open subscheme of $M(r, c_1, c_2)$ consisting of μ-stable sheaves and D the closed subset of M^μ consisting of non-locally free sheaves. Then, by using the proof of Lemma 3.1, we can show that $\mathrm{codim}\, D \geq r - 1$. In particular, there is a μ-stable vector bundle for $c_2 \geq 1$.

Remark 1.6. In **3.1** and **3.2**, we see that $\overline{M}(r, c_1, c_2)$ is irreducible (see Lemma 3.1 and (3.9)).

Corollary 1.5. *$\overline{M}(r, c_1, c_2)$ is locally factorial for $g \geq 1$.*

Proof. The proof is completely the same as that in [D-N, Theorem A] and hence we omit the proof.

2. Some perparations on the structure of $Pic\,(M(1, c_1, c_2))$

2.1. We shall first construct a morphism $\overline{M}(r, c_1, c_2) \to S^{c_2}C \times Pic^0(X)$ by using the notion of jumping lines. Let Q be the open subscheme of $\mathcal{Q}uot_{\mathcal{O}_X(-m)^{\oplus N}/X}$ parametrizing all quotients $\mathcal{O}_X(-m)^{\oplus N} \to E$ such that quotient sheaves E are semi-stable sheaves of rank r with Chern classes c_1, c_2 and $h^0(X, E(m)) = N$. If m is sufficiently larger than r, c_1 and c_2, then $\overline{M}(r, c_1, c_2) = Q/PGL(N)$ (good quotient) for some N. We choose such m and N. We denote the universal family of quotients by $\mathcal{O}_{Q\times X}[-m]^{\oplus N} \to \mathcal{E}$. Let $\Gamma_\pi \subset X \times C$ be the graph of the projection $\pi: X \to C$ and $\overline{\Gamma}$ the pull-back of Γ_π to $Q \times X \times C$. Let $r: Q \times X \times C \to Q \times X$ be the projection. By using the base change theorem, we see that $R^1p_{Q\times C*}(r^*\mathcal{E}[-C_0] \otimes \mathcal{O}(-\overline{\Gamma}))$ and $R^1p_{Q\times C*}r^*\mathcal{E}[-C_0]$ are locally free sheaves of rank c_2. We denote these sheaves by $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively. Applying $R^\cdot p_{Q\times C*}$ to an exact sequence

$$0 \to r^*\mathcal{E}[-C_0] \otimes \mathcal{O}(-\overline{\Gamma}) \to r^*\mathcal{E}[-C_0] \to r^*\mathcal{E}[-C_0]_{\overline{\Gamma}} \to 0\,,$$

we obtain an exact sequence

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to R^1p_{Q\times C*}r^*\mathcal{E}[-C_0]_{\overline{\Gamma}} \to 0\,.$$

For a point y of Q,

$$\begin{aligned}\operatorname{Supp} R^1p_{C*}(r^*\mathcal{E}_y(-C_0)_{|\Gamma_\pi}) &= \{P \in C \mid \mathcal{E}_{y|\pi^{-1}(P)} \not\cong \mathcal{O}_{\pi^{-1}(P)}^{\oplus r}\} \\ &= \{P \in C \mid \pi^{-1}(P) \text{ is a jumping line of } \mathcal{E}_y\}\,.\end{aligned}$$

The homomorphism $\mathcal{O}_{Q\times C} \to (\det \mathcal{V}_2) \otimes (\det \mathcal{V}_1)^{-1}$ induced by $\mathcal{V}_1 \to \mathcal{V}_2$ defines a family of $PGL(N)$-invariant effective Cartier divisors on C parametrized by Q. We denote this relative Cartier divisor by $\mathcal{J}$. By virtuue of the Grothendieck-Riemann-Roch theorem, the relative degree of $\mathcal{J}$ is c_2. Hence it defines a morphism $Q \to \mathcal{H}ilb_C^{c_2} = S^{c_2}C$, which is $PGL(N)$-invariant. Therefore, we obtain a morphism

$$\lambda_{\overline{M}(r,c_1,c_2)}: \overline{M}(r, c_1, c_2) \to S^{c_2}C\,. \tag{2.1}$$

For simplicity, we sometimes denote $\lambda_{\overline{M}(r,c_1,c_2)}$ by λ.

Let det: $\overline{M}(r, c_1, c_2) \to Pic^0(X)$ be the determinant map: $E \mapsto \det(E) \otimes \mathcal{O}_X(-df)$, where $d = \deg c_1$. Then $\lambda \times \det$ defines a morphism $\overline{M}(r, c_1, c_2) \to S^{c_2}C \times Pic^0(X)$.

Remark 2.1. For a point P of C, let $i_P: S^{c_2-1}C \to S^{c_2}C$ be the inclusion sending D to $D + P$. Then $\mathcal{J}_{|Q\times\{P\}}$ is the pull-back of the divisor $S^{c_2-1}C$.

2.2. We set $\Delta_{i,j} := \{(P_1, P_2, \dots, P_n) \in X^n \mid P_i = P_j\}$, $\Delta := \bigcup_{i<j}\Delta_{i,j}$, $\Delta_0 := \Delta_{smooth}$, and $\Delta_1 := \Delta_{singular}$. Let $\varpi: \tilde{X}^n \to X^n \setminus \Delta_1$ be the blowing up of $X^n \setminus \Delta_1$ along Δ_0.

Then the symmetric group $\mathfrak{S}_n$ acts on $\tilde{X}^n$ and the quotient $\tilde{X}^n/\mathfrak{S}_n$ is an open subscheme of $\mathcal{H}ilb_X^n$. We denote the quotient map by ς. We set $E_{i,j} := \varpi^{-1}(\Delta_{i,j})$ and $E := \sum_{i<j} E_{i,j}$. Then the following proposition holds.

Proposition 2.1. *ς^*: $Pic(\mathcal{H}ilb_X^n) \to Pic(\tilde{X}^n)$ is injective.*

Proof. Since the codimension of the complement of $\tilde{X}^n/\mathfrak{S}_n$ in $\mathcal{H}ilb_X^n$ is 2, we have $Pic(\mathcal{H}ilb_X^n) \cong Pic(\tilde{X}^n/\mathfrak{S}_n)$. Let L be a line bundle on $\mathcal{H}ilb_X^n$ such that $\varsigma^* L \cong \mathcal{O}_{\tilde{X}^n}$. Then it induces an action of $\mathfrak{S}_n$ on $\mathcal{O}_{\tilde{X}^n}$. Since $\mathfrak{S}_n$ is a finite group and $H^0(\tilde{X}^n, \mathcal{O}_{\tilde{X}^n}) \cong H^0(X^n \backslash \Delta, \mathcal{O}_{X^n \backslash \Delta}) = \mathbf{C}$, this action is represented by an element ϱ of the character group $\mathrm{Char}(\mathfrak{S}_n) \cong \mathbf{Z}/2\mathbf{Z}$. Let $s_{1,2}$ be the element of $\mathfrak{S}_n$ which permutes 1 and 2. Since the action of $s_{1,2}$ on $E_{1,2}$ is trivial, the action of $s_{1,2}$ on $\mathcal{O}_{\tilde{X}^n}$ is trivial. Hence we obtain $\varrho = 0$, which implies that $L \cong \mathcal{O}_{\mathcal{H}ilb_X^n}$. Therefore ς^* is injective.

Corollary 2.2. *$H^2(\mathcal{H}ilb_X^n, \mathbf{Z})$ is torsion free.*

Proof. Let $P_1, P_2, \ldots, P_{n-1}$ be $n-1$ distinct points of X and $\mathfrak{p}: X \to X^n$ the morphism sending $P \in X$ to $P \times P_1 \times \cdots \times P_{n-1} \in X^n$. Then we see that the restriction of $\mathfrak{p}^*$ to $Pic^0(X^n)^{\mathfrak{S}_n}$: $Pic^0(X^n)^{\mathfrak{S}_n} \to Pic^0(X)$ is an isomorphism, where $Pic^0(X^n)^{\mathfrak{S}_n}$ is the $\mathfrak{S}_n$-invariant subgroup of $Pic^0(X^n)$. We can easily show that $\mathfrak{q}: X \xrightarrow{\mathfrak{p}} X^n \to S^n X \to S^n C$ induces an isomorphism $\mathfrak{q}^*: Pic^0(S^n C) \to Pic^0(X)$. Since $\mathcal{H}ilb_X^n$ is a subscheme of $M(1, 0, n)$ and $\lambda_{|\mathcal{H}ilb_X^n}$ is the natural morphism $\mathcal{H}ilb_X^n \to S^n X \to S^n C$, we get that $\varsigma^*(Pic^0(\mathcal{H}ilb_X^n)) = Pic^0(\tilde{X}^n)^{\mathfrak{S}_n} = Pic^0(X^n)^{\mathfrak{S}_n}$. We shall first prove that $\varsigma^*: NS(\mathcal{H}ilb_X^n) \to H^2(\tilde{X}^n, \mathbf{Z})^{\mathfrak{S}_n}$ is injective. Since $Pic(\tilde{X}^n) \cong Pic(X^n) \oplus \bigoplus_{i<j} \mathbf{Z} E_{i,j}$ and $H^2(\tilde{X}^n, \mathbf{Z}) \cong H^2(X^n, \mathbf{Z}) \oplus \bigoplus_{i<j} \mathbf{Z} E_{i,j}$, we obtain that $Pic(\tilde{X}^n)^{\mathfrak{S}_n} \cong Pic(X^n)^{\mathfrak{S}_n} \oplus \mathbf{Z}E$ and $H^2(\tilde{X}^n, \mathbf{Z})^{\mathfrak{S}_n} \cong H^2(X^n, \mathbf{Z})^{\mathfrak{S}_n} \oplus \mathbf{Z}E$. Since the kernel of $c_1: Pic(X^n)^{\mathfrak{S}_n} \to H^2(X^n, \mathbf{Z})^{\mathfrak{S}_n}$ is $Pic^0(X^n)^{\mathfrak{S}_n} = \varsigma^*(Pic^0(\mathcal{H}ilb_X^n))$, Proposition 2.1 implies that the kernel of the composition $c_1 \circ \varsigma^*: Pic(\mathcal{H}ilb_X^n) \to H^2(\tilde{X}^n, \mathbf{Z})^{\mathfrak{S}_n}$ is $Pic^0(\mathcal{H}ilb_X^n)$. Hence $\varsigma^*: NS(\mathcal{H}ilb_X^n) \to H^2(\tilde{X}^n, \mathbf{Z})^{\mathfrak{S}_n}$ is injective. The torsion-freeness of $H^1(X, \mathbf{Z})$ and the Künneth formula imply that

$$H^2(X^n, \mathbf{Z}) \cong H^2(X, \mathbf{Z})^{\oplus n} \oplus (H^1(X, \mathbf{Z}) \otimes H^1(X, \mathbf{Z}))^{\oplus \binom{n}{2}}.$$

Since $H^2(X, \mathbf{Z})$ is also torsion free, $H^2(X^n, \mathbf{Z})$ and $H^2(\tilde{X}^n, \mathbf{Z})$ are torsion free. Hence $NS(\mathcal{H}ilb_X^n)$ is torsion free. Therefore $H^2(\mathcal{H}ilb_X^n, \mathbf{Z})$ is also torsion free.

By using this corollary, we shall compare the cohomologies of $S^{c_2}C$ and $\mathcal{H}ilb_X^{c_2}$.

Lemma 2.3. *$H^*(S^{c_2}C, \mathbf{Z})$ is a direct summand of $H^*(\mathcal{H}ilb_X^{c_2}, \mathbf{Z})$.*

Proof. Let $Z \subset S^{c_2}C \times C$ be the universal subscheme. Then $(1_{S^{c_2}C} \times \pi)^* Z_{|S^{c_2}C \times C_0} \subset S^{c_2}C \times X$ defines a flat family of subschemes of length c_2 on X. It defines a morphism $\sigma\colon S^{c_2}C \to \mathscr{H}ilb_X^{c_2}$ such that $(\sigma \times 1_X)^*(\mathscr{Z}) = (1_{S^{c_2}C} \times \pi)^* Z_{|S^{c_2}C \times C_0}$, where $\mathscr{Z}$ is the universal subscheme on $\mathscr{H}ilb_X^{c_2} \times X$. From the construction of λ, we obtain that $\lambda \circ \sigma = 1_{S^{c_2}C}$. Therefore $H^*(S^{c_2}C, \mathbf{Z})$ is a direct summand of $H^*(\mathscr{H}ilb_X^{c_2}, \mathbf{Z})$.

Lemma 2.4. *$H^1(\mathscr{H}ilb_X^{c_2}, \mathbf{Z}) \cong H^1(S^{c_2}C, \mathbf{Z})$ and $H^2(\mathscr{H}ilb_X^{c_2}, \mathbf{Z}) \cong H^2(S^{c_2}C, \mathbf{Z}) \oplus \mathbf{Z}^{\oplus 2}$ for $c_2 \geq 2$.*

Proof. From Corollary 2.2, we obtain that $H^2(\mathscr{H}ilb_X^{c_2}, \mathbf{Z})$ is torsion free. Combining the result of Göttsche [Gö] with that of Macdonald [Mc], we get $b_1(\mathscr{H}ilb_X^{c_2}) = b_1(S^{c_2}C)$ and $b_2(\mathscr{H}ilb_X^{c_2}) = b_2(S^{c_2}C) + 2$. By Corollary 2.2 and Lemma 2.3, we get $H^1(\mathscr{H}ilb_X^{c_2}, \mathbf{Z}) \cong H^1(S^{c_2}C, \mathbf{Z})$ and $H^2(\mathscr{H}ilb_X^{c_2}, \mathbf{Z}) \cong H^2(S^{c_2}C, \mathbf{Z}) \oplus \mathbf{Z}^{\oplus 2}$.

2.3. In this subsection, we assume that $c_2 \geq 2$. Let $P_1, P_2, \ldots, P_{c_2-1}$ be $c_2 - 1$ distinct points of X, and we denote the ideal sheaf of $\{P_1, \ldots, P_{c_2-1}\}$ (resp. the ideal sheaf of $\{P_1, \ldots, P_{c_2-2}\}$) by I (resp. I'). Let $\tilde{X} \to X$ be the blowing up of X at P_{c_2-1}. C_1 denotes the exceptional divisor. We set $\tilde{X}_1 = \tilde{X} \backslash \{P_1, \ldots, P_{c_2-2}\}$. Γ_b denotes the graph of $b\colon \tilde{X}_1 \to X$. Let $p_2\colon \tilde{X}_1 \times X \to X$ be the projection. On $\tilde{X}_1 \times X$, there is a surjective homomorphism $p_2^* I \to \mathcal{O}_{\Gamma_b}(-C_1) \otimes p_2^* I' \to 0$, where we also denote the pull-back of C_1 to $\tilde{X}_1 \times X$ by C_1. Let I_Z be the kernel of this homomorphism. Then I_Z defines a flat family of ideals of $\mathcal{O}_X$ parametrized by $\tilde{X}_1$. Thus it defines a morphism $\gamma\colon \tilde{X}_1 \to \mathscr{H}ilb_X^{c_2}$. This induces a homomorphism $\gamma^*\colon Pic\,(\mathscr{H}ilb_X^{c_2}) \to Pic\,(\tilde{X}_1)$ sending L to $\gamma^*(L)$.

Lemma 2.5. *$\gamma^*\colon Pic\,(\mathscr{H}ilb_X^{c_2}) \to Pic\,(\tilde{X}_1)$ is a surjective homomorphism.*

Proof. For a line bundle $\mathcal{O}_X(D)$ on X, $\det(p_{\mathscr{H}ilb_X^{c_2}!} I_{\mathscr{Z}}[D])$ defines a line bundle on $\mathscr{H}ilb_X^{c_2}$. In the Grothendieck group $K(\tilde{X}_1)$,

$$p_{\tilde{X}_1!}(I_Z[D]) = p_{\tilde{X}_1!}(p_2^* I(D)) - p_{\tilde{X}_1!}(\mathcal{O}_{\Gamma_b}(-C_1) \otimes p_2^* I'(D)).$$

Since $p_{\tilde{X}_1!}(\mathcal{O}_{\Gamma_b}(-C_1) \otimes_{\mathcal{O}_{\tilde{X}_1 \times X}} p_2^* I'(D)) = \mathcal{O}_{\tilde{X}_1}(-C_1 + D) \otimes_{\mathcal{O}_X} I'$ and $\gamma^*(p_{\mathscr{H}ilb_X^{c_2}!}(I_{\mathscr{Z}}[D])) = p_{\tilde{X}_1!}(I_Z[D])$, we get $\gamma^*(\det(p_{\mathscr{H}ilb_X^{c_2}!} I_{\mathscr{Z}}(D))) \cong \mathcal{O}_{\tilde{X}_1}(C_1 - D)$. Therefore γ^* is surjective.

Let $\xi\colon C \to S^{c_2}C$ be the morphism such that $\xi(P) = P + \sum_{i=1}^{c_2-1} \pi(P_i)$. Then $\lambda \circ \gamma = \xi \circ \pi \circ b$. This implies the image of $H^2(S^{c_2}C, \mathbf{Z})$ in $H^2(\tilde{X}_1, \mathbf{Z})$ is generated by f, (by Remark 2.1, the image of $S^{c_2-1}C$ is f.) Since $\mathbf{Z}^{\oplus 2} \cong H^2(\mathscr{H}ilb_X^{c_2}, \mathbf{Z})/H^2(S^{c_2}C, \mathbf{Z}) \to H^2(\tilde{X}_1, \mathbf{Z})/\mathbf{Z}f \cong \mathbf{Z}^{\oplus 2}$ is surjective, it is an isomorphism. Therefore, $H^2(\mathscr{H}ilb_X^{c_2}, \mathbf{Z})/H^2(S^{c_2}C, \mathbf{Z})$ is generated by images of $\det p_{\mathscr{H}ilb_X^{c_2}!} I_{\mathscr{Z}}[D]$, where $\mathcal{O}_X(D)$ belongs to $Pic\,(X)$. In particular, $Pic\,(\mathscr{H}ilb_X^{c_2})/Pic\,(S^{c_2}C) \to H^2(\mathscr{H}ilb_X^{c_2}, \mathbf{Z})/H^2(S^{c_2}C, \mathbf{Z})$ is surjective. Since μ is a section of λ, there is the following exact and commutative diagram.

(2.2)

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & Pic\,(\mathcal{H}ilb_X^{c_2})/Pic\,(S^{c_2}C) & \longrightarrow & \mathbf{Z}^{\oplus 2} & & \\
 & & \uparrow & & \uparrow & & \\
H^1(\mathcal{H}ilb_X^{c_2}, \mathcal{O}) & \longrightarrow & Pic\,(\mathcal{H}ilb_X^{c_2}) & \longrightarrow & H^2(\mathcal{H}ilb_X^{c_2}, \mathbf{Z}) & \longrightarrow & H^2(\mathcal{H}ilb_X^{c_2}, \mathcal{O}) \\
\uparrow{\scriptstyle a} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle d} \\
H^1(S^{c_2}C, \mathcal{O}) & \longrightarrow & Pic\,(S^{c_2}C) & \longrightarrow & H^2(S^{c_2}C, \mathbf{Z}) & \longrightarrow & H^2(S^{c_2}C, \mathcal{O}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

Since $H^1(\mathcal{H}ilb_X^{c_2}, \mathbf{Z}) \cong H^1(S^{c_2}C, \mathbf{Z})$, a is an isomorphism. Thus $Pic^0\,(\mathcal{H}ilb_X^{c_2}) \cong Pic^0(S^{c_2}C)$. By diagram chasing, we get

$$
\begin{aligned}
Pic\,(\mathcal{H}ilb_X^{c_2})/Pic\,(S^{c_2}C) &\cong H^2(\mathcal{H}ilb_X^{c_2}, \mathbf{Z})/H^2(S^{c_2}C, \mathbf{Z}) \\
&\cong H^2(\tilde{X}_1, \mathbf{Z})/\mathbf{Z}f.
\end{aligned}
\tag{2.3}
$$

Fix a point x of X. Let $\mathscr{P}$ be a Poincaré line bundle of degree $d := \deg c_1$ on $Pic^0\,(X) \times X$ such that $\mathscr{P}_{|\{L\} \times X} \cong L \otimes \mathcal{O}_X(df)$ and $\mathscr{P}_{|Pic^0(X) \times \{x\}} \cong \mathcal{O}_{Pic^0(X)}$. Since $Pic^0\,(Pic^0\,(X)) = Alb\,(X)$, it defines a morphism $X \to Alb\,(X)$ sending $y \in X$ to $\mathscr{P}_{|Pic^0(X) \times \{y\}}$. Let $M(1, c_1, c_2) \cong \mathcal{H}ilb_X^{c_2} \times Pic^0\,(X)$ be a decomposition and $\mathscr{I} = I_{\mathscr{Z}} \otimes \mathscr{P}$ a decomposition of a universal family, where $I_{\mathscr{Z}}$ is the universal ideal sheaf of colength c_2 on $\mathcal{H}ilb_X^{c_2} \times X$ and we identify the pull-backs of $I_{\mathscr{Z}}$ and $\mathscr{P}$ to $M(1, c_1, c_2) \times X$ with $I_{\mathscr{Z}}$ and $\mathscr{P}$ respectively. In the same way, we can show that $Pic^0(M(1, c_1, c_2)) \cong Pic^0\,(S^{c_2} \times Pic^0\,(X))$ and

(2.4)

$$
\begin{aligned}
Pic\,(M(1, c_1, c_2))/Pic\,(S^{c_2}C \times Pic^0\,(X)) &\cong H^2(M(1, c_1, c_2), \mathbf{Z})/H^2(S^{c_2}C \times Pic^0\,(X), \mathbf{Z}) \\
&\cong H^2(\tilde{X}_1, \mathbf{Z})/\mathbf{Z}f.
\end{aligned}
$$

Remark 2.2. If $c_2 = 1$, then $\mathcal{H}ilb_X^{c_2} \cong X$. Hence we get that

$$
\begin{aligned}
Pic\,(M(1, c_1, c_2))/Pic\,(S^{c_2}C \times Pic^0\,(X)) &\cong H^2(M(1, c_1, c_2), \mathbf{Z})/H^2(S^{c_2}C \times Pic^0\,(X), \mathbf{Z}) \\
&\cong H^2(X, \mathbf{Z})/\mathbf{Z}f
\end{aligned}
$$

for $c_2 = 1$.

3. Structure of *Pic* $(\overline{M}(r, c_1, c_2))$

3.1. We shall use the notation in **2.1**. By choosing sufficiently large m, we may assume that

$$H^i(X, E(-l + mH_n)) = 0\,, \tag{3.1}$$

for $i > 0$, $E \in M(r, c_1, c_2)$ and all fibres l of π.

Lemma 3.1. *Let Q_0 be the open subscheme of Q which parametrizes quotients $\mathcal{O}_X(-m)^{\oplus N} \to E$ such that quotient sheaves E are stable and $R^1\pi_* E = 0$. Then* codim $(Q \backslash Q_0) \geq 2$ *for* $r \geq 2$.

Proof. By Lemma 1.4, $\operatorname{codim}(Q\backslash Q^s) \geq 2$. Hence we shall show that the codimension of $Q^s \backslash Q_0$ is at least 2. Let l be a fibre of π and set $D_l := \{y \in Q^s | H^1(l, \mathcal{E}_{y|l}) \neq 0\}$. Then $Q^s\backslash Q_0 = \bigcup_l D_l$, where l runs all fibres of π. Hence it is sufficient to show that $\operatorname{codim}(D_l) \geq 3$. We set $B_l := \{y \in Q^s | \dim_{\mathbf{C}} H^1(l, \mathcal{E}_{y|l}(-1)) > 1\}$. Then it is easy to see that $D_l \subset B_l$. Hence it is sufficient to show that $\operatorname{codim}(B_l) \geq 3$. For a point y of Q, $\mathcal{E}_y(-l) \to \mathcal{E}_y$ is injective, and hence $\mathcal{E}_{|l}$ is flat over Q^s. Then we obtain a family of quotients $\mathcal{O}_{\mathcal{Q} \times l}[-m]^{\oplus N} \to \mathcal{E}_{|l}$. Thus we get a morphism res: $Q^s \to \mathcal{Q}uot_{\mathcal{O}_l(-m)^{\oplus N}/l}$. Let $\mathcal{Q}$ be the union of connected components which contain im (res). We denote the universal quotient on $\mathcal{Q} \times l$ by $\mathcal{O}_{\mathcal{Q}\times l}[-m]^{\oplus N} \to \mathcal{E}^l$. For a point y of Q, we set $K := \ker(\mathcal{O}_X(-m)^{\oplus N} \to \mathcal{E}_y)$. Then there is an exact sequence $\operatorname{Hom}(K, \mathcal{E}_y) \to \operatorname{Hom}(K_{|l}, \mathcal{E}_{|l}) \to \operatorname{Ext}^1(K, \mathcal{E}_y(-l))$. By (3.1), $\operatorname{Ext}^1(K, \mathcal{E}_y(-l)) \cong \operatorname{Ext}^2(\mathcal{E}_y, \mathcal{E}_y(-l))$. By the choice of H_n, $(K_X + l, H_n) < 0$. Since $\mathcal{E}_y$ is stable, the Serre duality implies that $\operatorname{Ext}^1(K, \mathcal{E}_y(-l)) = 0$. Thus res induces a surjective homomorphism between tangent spaces. Let $\mathcal{Q}_1 := \{y \in \mathcal{Q} | H^1(l, \mathcal{E}^l_y(m)) = 0\}$ be the open subscheme of $\mathcal{Q}$. Then $\mathcal{Q}_1$ is smooth and contain im (res). In order to compute the codimension of B_l, it is sufficient to compute the codimension of B'_l, where $B'_l := \{y \in \mathcal{Q}_1 | \dim_{\mathbf{C}} H^1(l, \mathcal{E}^l_y(-1)) > 1\}$.

We shall use the same method as in the proof of Proposition 1.3. We shall only treat the case that $\mathcal{E}^l_y$ is not locally free. Another case is similar. Let $\mathcal{H}^l$ be the set of sequences of polynomials $h = (h_1, h_2, \ldots, h_s)$ such that h_1 is the Hilbert polynomial of the torsion submodule $(\mathcal{E}^l_y)_T$ of $\mathcal{E}^l_y$, $y \in \mathcal{Q}_1$ and $h_2 - h_1, \ldots, h_s - h_1$ are the Hilbert polynomials of filters of the Harder-Narasimhan filtration of $(\mathcal{E}^l_y)_F := \mathcal{E}^l_y/(\mathcal{E}^l_y)_T$, $y \in Q_1$. Since $(\mathcal{E}^l_y)_F$ is a torsion free quotient of $\mathcal{O}_X(-m)^{\oplus N}$ and $\deg(\mathcal{E}^l_y)_F \leq \deg \mathcal{E}^l_y$, the boundedness theorem of Grothendieck implies that $\mathcal{H}^l$ is a finite set. We shall consider the flag scheme f_h: $\mathcal{F}lag^h_{\mathcal{E}^l/\mathcal{Q}_1 \times X} \to \mathcal{Q}_1$ and the open subscheme $HN(h)$ whose point F corresponds to a filtration $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \mathcal{E}^l_y$, $y = f_h(F)$ such that $\chi(F_i(m)) = h_i(m)$, F_1 is the torsion submodule of $\mathcal{E}^l_y$ and $F_2/F_1 \subset \cdots \subset F_s/F_1$ is the Harder-Narasimhan filtration of $\mathcal{E}^l_y/F_1$. In the same way as in the proof of Proposition 1.3, we see that $HN(h) \to \mathcal{Q}_1$ is an immersion and $\operatorname{codim}(HN(h)) = -\sum_{i<j} \chi(gr_i(\mathcal{E}^l_y), gr_j(\mathcal{E}^l_y))$. In order to prove our claim, it is sufficient to classify $\mathcal{E}^l_y$ such that $\operatorname{codim}(HN(h)) = 2$. Since $-\chi(gr_1(\mathcal{E}^l_y), gr_j(\mathcal{E}^l_y)) = \operatorname{rk}(gr_j(\mathcal{E}^l_y)) \dim_{\mathbf{C}}(gr_1(\mathcal{E}^l_y))$, we get that $r = 2$, $\dim_{\mathbf{C}} gr_1(\mathcal{E}^l_y) = 1$ and $-\sum_{1 \leq i < j} \chi(gr_i(\mathcal{E}^l_y), gr_j(\mathcal{E}^l_y)) = 0$. Thus $\mathcal{E}^l_y \cong \mathbf{C}_P \oplus \mathcal{O}_l \oplus \mathcal{O}_l(-1)$, where $P \in l$. Therefore $\operatorname{codim}(B'_l) \geq 3$.

We denote $Q_0/PGL(N)$ by $M(r, c_1, c_2)_0$. Replacing E by $E(nf)$, we may assume that $R^1 p_{Q_0 *}\mathscr{E}[-l] = 0$ for all fibres l, $p_{Q_0 *}\mathscr{E}$ is locally free and $q^* p_{Q_0 *}\mathscr{E} \to \pi_{Q_0 *}\mathscr{E}$ is surjective, where $q: Q_0 \times C \to Q_0$ is the projection. Let $g: G = Gr(p_{Q_0 *}\mathscr{E}, r-1) \to Q_0$ be the Grassmannian bundle over Q_0 parametrizing rank $r-1$ subbundles of $p_{Q_0 *}\mathscr{E}$ and $\mathscr{U}$ the universal subbundle of rank $r-1$. We set

$$G' := \{x \in G \mid \mathscr{U}_x \otimes \mathcal{O}_C \to \pi_* \mathscr{E}_{g(x)} \text{ is injective as a bundle homomorphism}\}.$$

Since $p_{Q_0 *}\mathscr{E}$ is $GL(N)$-linearized, $PGL(N)$ acts on G and G'. Let $\mathbf{G}$ be the quotient of G' by $PGL(N)$. Then we can apply the same argument as in [D-N, 7.3.2 and 7.3.3] to the family $\pi_{Q_0 *}\mathscr{E}$. Since $\pi_* \mathscr{E}_{g(x)}$ is a vector bundle of degree $a := \deg c_1 - c_2$, we obtain the following exact sequence:

$$0 \to Pic\,(M(r, c_1, c_2)_0) \to Pic\,(\mathbf{G}) \to \mathbf{Z}\Big/\left(\frac{a}{n}\right)\mathbf{Z} \to 0, \tag{3.2}$$

where $n = \gcd(r, a, c_2)$. Let $t: T = \mathbf{P}(\mathscr{H}om\,(\mathcal{O}_{G'}^{\oplus(r-1)}, \mathscr{U})^\vee) \to G'$ be the projective bundle and N the tautological line bundle on T. On T, there is a homomorphism $\tau: \mathcal{O}_T^{\oplus(r-1)} \to t^*\mathscr{U} \otimes N$. Let $T' = \{x \in T \mid \tau_x \text{ is an isomorphism}\}$ be an open set of T. Setting $\tilde{\mathscr{E}} = (g \circ t \times 1_X)^*\mathscr{E}$ and $\tilde{\mathscr{U}} = p_T^* t^* \mathscr{U}$, we obtain an injective homomorphism on $T' \times X$: $\mathcal{O}_{T' \times X}^{\oplus(r-1)} \to \tilde{\mathscr{U}} \otimes p_{T'}^* N \to \tilde{\mathscr{E}} \otimes p_{T'}^* N$. Let T'' be the open subscheme of T' whose point y satisfies the $\tilde{\mathscr{E}}_y/\mathcal{O}_X^{\oplus(r-1)}$ is torsion free.

Lemma 3.2. *$T' \backslash T''$ is at least of codimension 2.*

Proof. We shall prove that $R := \{y \in G' \mid \mathscr{E}_{g(y)}/\mathcal{O}_X^{\oplus(r-1)} \text{ is not torsion free}\}$ is at least of codimension 2 in G'. We simply denote $\mathscr{E}_{g(y)}$ by E. We note that if $E/\mathcal{O}_X^{\oplus(r-1)}$ is locally free in codimension 1, then $E/\mathcal{O}_X^{\oplus(r-1)}$ is torsion free. Hence if $\mathcal{O}_l^{\oplus(r-1)} \to E_{|l}$ is injective for all fibres l, then $E/\mathcal{O}_X^{\oplus(r-1)}$ is torsion free. In the proof of Lemma 3.1, we proved that the codimension of B_l in Q is at least 3. Hence we may assume that $E_{|l}$ is isomorphic to $\mathcal{O}_l^{\oplus r}$, $\mathcal{O}_l(1) \oplus \mathcal{O}_l(-1) \oplus \mathcal{O}_l^{\oplus(r-2)}$, or $\mathbf{C}_P \oplus \mathcal{O}_l(-1) \oplus \mathcal{O}_l^{\oplus(r-1)}$. Assume that E is locally free and let l be a fibre with $E_{|l} \cong \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1) \oplus \mathcal{O}_l^{\oplus(r-2)}$. Let $r: H^0(X, E) \to H^0(l, E_{|l})$ be the restriction map. In $G' \cap Gr(H^0(X, E), r-1)$, the locus of $V \subset H^0(X, E)$ with $\dim(r(V) \cap H^0(l, \mathcal{O}_l(1))) \geq 2$ is at least of codimension 2. Hence in a neighborhood of y, $\operatorname{codim} R \geq 2$. If E is not locally free, then it is easy to see that $\operatorname{codim}(R \cap Gr(H^0(X, E), r-1)) \geq 1$ in $Gr(H^0(X, E), r-1)$. We set $U := \{z \in Q_0 \mid \mathscr{E}_z \text{ is not locally free}\}$. Then $\operatorname{codim} U \geq 1$ in Q_0, and hence $\operatorname{codim} R \geq 2$.

The quotient $\tilde{\mathscr{E}} \otimes p_{T''}^* N/\mathcal{O}_{T'' \times X}^{\oplus(r-1)}$ is a flat family of torsion free sheaves of rank 1 with Chern classes c_1, c_2. Therefore, $\tilde{\mathscr{E}} \otimes p_{T''}^* N/\mathcal{O}_{T'' \times X}^{\oplus(r-1)}$ can be written as $I_Z \otimes \det(\tilde{\mathscr{E}} \otimes p_{T''}^* N)$ where I_Z is a flat family of ideals of colength c_2. Thus we obtain an extension

$$0 \to \mathcal{O}_{T'' \times X}^{\oplus(r-1)} \to \tilde{\mathscr{E}} \otimes p_{T''}^* N \to I_Z \otimes \det(\tilde{\mathscr{E}} \otimes p_{T''}^* N) \to 0. \tag{3.3}$$

We set $\mathbf{T} = T''/PGL(N)$. Then in the same way as in [D-N, 7.3.4], we obtain the following exact sequence:

$$0 \to Pic\,(\mathbf{G}) \to Pic\,(\mathbf{T}) \to \mathbf{Z}/(r-1)\mathbf{Z} \to 0\,. \tag{3.4}$$

Combining (3.2) with (3.4), we obtain

$$\#(Pic\,(\mathbf{T})/Pic\,(M(r, c_1, c_2)_0)) = \frac{(r-1)a}{n}\,. \tag{3.5}$$

3.2. For simplicity, we denote $M(1, c_1, c_2)$ by M, where c_1 and c_2 are the same as in **3.1**. Let $\mathscr{V} := \mathrm{Ext}^1_{p_M}(\mathscr{I}, \mathcal{O}_{M\times X}^{\oplus(r-1)})$ be the relative extension sheaf on M, where $\mathscr{I}$ is the universal family in **2.3**. The base change theorem implies that $\mathscr{V}$ is locally free. Let $\mu\colon \mathbf{P} = \mathbf{P}(\mathscr{V}^\vee) \to M$ be the projective bundle and v a divisor which defines the tautological line bundle on $\mathbf{P}$. For simplicity, we also denote $(\mu \times 1_X)^*\mathscr{I}$ by $\mu^*\mathscr{I}$ and $p_{\mathbf{P}}^*(v) = v \times X$ by v. Since $\mathrm{Hom}_{p_{\mathbf{P}}}(\mu^*\mathscr{I}, \mathcal{O}_{\mathbf{P}\times X}^{\oplus(r-1)}) = 0$, we get

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}\times X}}(\mu^*\mathscr{I} \otimes \mathcal{O}_{\mathbf{P}}(-v), \mathcal{O}_{\mathbf{P}\times X}^{\oplus(r-1)}) &\cong H^0(\mathbf{P}, \mathrm{Ext}^1_{p_{\mathbf{P}}}(\mu^*\mathscr{I} \otimes \mathcal{O}_{\mathbf{P}}(-v), \mathcal{O}_{\mathbf{P}\times X}^{\oplus(r-1)}))\\ &\cong H^0(\mathbf{P}, \mu^*\mathscr{V} \otimes \mathcal{O}_{\mathbf{P}}(v))\\ &\cong \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}}}(\mu^*\mathscr{V}^\vee, \mathcal{O}_{\mathbf{P}}(v))\,.\end{aligned} \tag{3.6}$$

Therefore the natural surjective homomorphism $\mu^*\mathscr{V}^\vee \to \mathcal{O}_{\mathbf{P}}(v)$ defines a universal family of extensions:

$$0 \to \mathcal{O}_{\mathbf{P}\times X}^{\oplus(r-1)} \to \mathscr{F} \to \mu^*\mathscr{I} \otimes \mathcal{O}_{\mathbf{P}}(-v) \to 0\,. \tag{3.7}$$

For simplicity, we denote $\mathscr{H}\!ilb_X^{c_2}$ by H. We set $\tilde{D} := \{I_Z \in H \mid R^1\pi_* I_Z \neq 0\}$. We assume that $c_2 \geq 2$. Then $\tilde{D}$ is not empty. Let $\Gamma_\pi \subset X \times C$ be the graph of the projection π and $\bar{\Gamma}$ the pull-back of Γ_π to $H \times X \times C$. Let $r\colon H \times X \times C \to H \times X$ be the projection. By using the base change theorem, $R^1 p_{H\times C*}(r^* I_{\mathscr{Z}} \otimes \mathcal{O}(-\bar{\Gamma}))$ and $R^1 p_{H\times C*} r^* I_{\mathscr{Z}}$ are locally free sheaves of rank $c_2 + g$ and $c_2 + g - 1$ respectively. We denote these sheaves by $\mathscr{V}_3$ and $\mathscr{V}_4$ respectively. Applying $R^{\cdot} p_{H\times C*}$ to an exact sequence

$$0 \to r^* I_{\mathscr{Z}} \otimes \mathcal{O}(-\bar{\Gamma}) \to r^* I_{\mathscr{Z}} \to r^* I_{\mathscr{Z}|\bar{\Gamma}} \to 0\,,$$

we obtain an exact sequence

$$0 \to p_{H\times C*} r^* I_{\mathscr{Z}|\bar{\Gamma}} \to \mathscr{V}_3 \to \mathscr{V}_4 \to R^1 p_{H\times C*} r^* I_{\mathscr{Z}|\bar{\Gamma}} \to 0\,.$$

By using the determinantal subscheme defined by the homomorphism $\mathscr{V}_3 \to \mathscr{V}_4$, we can define the multiplicity of $\tilde{D}$. By the Porteous' formula, we get

$$c_1(\mathcal{O}_{\tilde{D}}) = q_{1*} c_2(\mathscr{V}_3 - \mathscr{V}_4)\,, \tag{3.8}$$

where $q_1\colon H \times C \to H$ is the projection.

Lemma 3.3. *$\tilde{D}$ is irreducible.*

Proof. Since $\tilde{D}$ is a divisor on H, it is sufficient to show that $\tilde{D}_1 := \tilde{D}_{|\tilde{X}^{c_2}/\mathfrak{S}_{c_2}}$ is irreducible, where $\tilde{X}^{c_2}/\mathfrak{S}_{c_2}$ is the open subscheme defined in **2.2**. Since $\tilde{D}_1$ does not contain the exceptional divisor $\varsigma(E)$ of $\omega\colon \tilde{X}^{c_2}/\mathfrak{S}_{c_2} \to (X^{c_2} \backslash \Delta_1)/\mathfrak{S}_{c_2}$, we show that $\omega(\tilde{D}_1)$ is irreducible. Since $(X \times_C X \times X^{c_2-2})\backslash\Delta_1$ is irreducible and its image to $(X^{c_2}\backslash\Delta_1)/\mathfrak{S}_{c_2}$ is $\omega(\tilde{D}_1)$, $\omega(\tilde{D}_1)$ is irreducible. Thus $\tilde{D}_1$ is an irreducible divisor.

In **3.3**, we shall show that $\tilde{D}$ is a reduced divisor.

We set $H_0 := H\backslash\tilde{D}$ and $M_0 = H_0 \times Pic^0(X)$ ($M_0 = M$ for $c_1 = 1$). Let $\mathbf{P}_0 := \mathbf{P} \times_M M_0$ be an open set of $\mathbf{P}$. We set $\mathbf{P}^s = \{y \in \mathbf{P} | \mathscr{F}_y \text{ is a stable sheaf}\}$ and $\mathbf{P}_0^s = \mathbf{P}^s \cap \mathbf{P}_0$. We shall prove the following.

(3.9) $$\mathbf{T} \cong \mathbf{P}_0^s .$$

Proof. On $\mathbf{P}_0^s$, $p_{\mathbf{P}_0^s *}(\mathscr{F}[m])$ is locally free and $p_{\mathbf{P}_0^s}^* p_{\mathbf{P}_0^s *}(\mathscr{F}[m]) \to \mathscr{F}[m]$ is surjective. Let $\{U_i\}$ be an open covering of $\mathbf{P}_0^s$ such that $p_{U_i *}(\mathscr{F}[m]) \cong \mathcal{O}_{U_i}^{\oplus N}$, and then $\rho_i: \mathcal{O}_{U_i \times X}[-m]^{\oplus N} \to \mathscr{F}_{|U_i}$ defines a morphism $h_i: U_i \to Q_0$ such that ρ_i is the pull-back of $\mathcal{O}_{Q_0 \times X}[-m]^{\oplus N} \to \mathscr{E}$. $\mathcal{O}_{U_i}^{\oplus(r-1)} = p_{U_i *}(\mathcal{O}_{U_i \times X}^{\oplus(r-1)}) \subset p_{U_i *}\mathscr{F}_{|U_i}$ induces a lifting of h_i to G', moreover the isomorphism of $\mathcal{O}_{U_i}^{\oplus(r-1)}$ to the pull-back of the universal subbundle induces a lifting to T''. Thus, we obtain a morphism $\bar{h}_i: U_i \to \mathbf{T} = T''/PGL(N)$, which satisfy $\bar{h}_i = \bar{h}_j$ on $U_i \cap U_j$. Thus we obtain $\bar{h}: \mathbf{P}_0^s \to \mathbf{T}$. Conversely the extension (3.3) gives a morphism $k': T'' \to \mathbf{P}_0^s$ such that the pull-back of (3.7) is (3.3). Since k' is $PGL(N)$-invariant, it induces a morphism $k: \mathbf{T} \to \mathbf{P}_0^s$. It is easy see $k \circ h = 1_{\mathbf{P}_0^s}$ and $h \circ k = 1_{\mathbf{T}}$. Thus $\mathbf{T} \cong \mathbf{P}_0^s$.

Lemma 3.4. *If* $d := \deg c_1$ *is sufficiently large, then* codim $(\mathbf{P}_0\backslash\mathbf{P}_0^s) \geq 2$ *except for the case that* $g = 1$, $r | c_1$ *and* $c_2 = 1$.

Proof. We set

$$\mathbf{P}_0' := \{y \in \mathbf{P}_0 | \pi_* \mathscr{F}_y \text{ is generated by global sections and } h^1(C, \pi_* \mathscr{F}_y) = 0\} .$$

In order to prove this lemma, it is sufficient to show that codim $(\mathbf{P}_0\backslash\mathbf{P}_0') \geq 2$ for sufficiently large d, and codim $(\mathbf{P}_0'\backslash(\mathbf{P}_0')^s) \geq 2$ except for the case that $g = 1$, $r | c_1$ and $c_2 = 1$, where $(\mathbf{P}_0')^s$ is the open subscheme of $\mathbf{P}_0'$ parametrizing stable sheaves.

Since $R^1\pi_* \mathscr{F}_y = 0$, $y \in \mathbf{P}_0$ and $\deg c_1$ is sufficiently large, (3.7) induces an exact sequence

$$\mathbf{C}^{\oplus g(r-1)} \cong H^1(C, \mathcal{O}_C^{\oplus(r-1)}) \to H^1(C, \pi_* \mathscr{F}_y) \to 0 .$$

Thus we get

(3.10) $$h^1(C, \pi_* \mathscr{F}_y) \leq (r-1)g \qquad \text{for } y \in \mathbf{P}_0 .$$

By using sufficient large m (which depends on d), we may assume that $p_{\mathbf{P}}^* p_{\mathbf{P}*}(\mathscr{F}[m]) \to \mathscr{F}[m]$ is surjective and $R^i p_{\mathbf{P}*}(\mathscr{F}[m]) = 0$, $i > 0$. Let Q_1 be the connected component of $\mathscr{Q}uot_{\mathcal{O}_X(-m)^{\oplus N}/X}$ which contains Q and $\mathcal{O}_{Q_1 \times X}[-m]^{\oplus N} \to \mathscr{E}$ denotes the universal quotient sheaf on $Q_1 \times X$. Let $Q_2(i)$ be the locally closed subset of Q_1 whose point y satisfies that $R^1\pi_* \mathscr{E}_y = 0$ and $h^1(C, \pi_*(\mathscr{E}_y)) = i$, $0 \leq i \leq (r-1)g$. In the same way as in **3.1**, we can construct $Q_2(i)$-schemes $t_i: T(i) \to Q_2(i)$ whose point z corresponds to an injective homomorphism $\mathcal{O}_X^{\oplus(r-1)} \to \mathscr{E}_{t_i(z)}$ with torsion free cokernel, up to multiplication by constants. Let $T(i)^0$ be an open set of $T(i)$ such that for a point z of $T(i)^0$, the exact sequence $0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E}_{t_i(z)} \to \mathcal{E}_{t_i(z)}/\mathcal{O}_X^{\oplus(r-1)} \to 0$ does not split. Then there are morphisms $k_i: T(i)^0 \to \mathbf{P}_0$. By (3.10), we get $\bigcup_{i=0}^{g(r-1)} \mathrm{im}\,(k_i) = \mathbf{P}_0$ and $\bigcup_{i=1}^{g(r-1)} \mathrm{im}\,(k_i) = \mathbf{P}_0 \backslash \mathbf{P}_0'$. In Lemma 3.5, we shall show that the action of $PGL(N)$ on $T(i)^0$ is set-theoretically free. Hence, in order to estimate $\mathrm{codim}\,(\mathbf{P}_0 \backslash \mathbf{P}_0')$ and $\mathrm{codim}\,(\mathbf{P}_0' \backslash (\mathbf{P}_0')^s)$, it is sufficient to compute $\dim T(0) - \dim T(i)$, $i > 0$ and $\mathrm{codim}\,(T(0) \backslash T(0)^s)$.

Let $\mathcal{O}_X(-m)^{\oplus N} \to E$ be a point of $Q_2(0)$ which has the Harder-Narasimhan filtration $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$. Let $h_i(x)$ be the Hilbert polynomial of F_i, $1 \le i \le s$. By Lemma 1.3, the locus of quotients whose Harder-Narasimhan filtrations have the Hilbert polynomials $(h_1(x), h_2(x), \dots, h_s(x))$ is at least of codimension $\dim \mathrm{Ext}^1_{F,+}(E, E)$. We shall use the same notation in Lemma 1.3. Then we obtain that

$$\begin{aligned} \dim \mathrm{Ext}^1_{F,+}(E, E) &= -\sum_{i<j} \chi(E_i, E_j) \\ &= -\sum_{i<j} \{r_i d_j - r_j d_i + r_i r_j (1-g) - r_i e_j - r_j e_i\} > 0 \,. \end{aligned}$$

Moreover we see that $\dim \mathrm{Ext}^1_{F,+}(E, E) = 1$ if and only if $s = 2$, $r_2 d_1 - r_1 d_2 = 0$, $g - 1 = 0$ and $r_1 e_2 + r_2 e_1 = 1$. Since F is the Harder-Narasimhan filtration, $e_1/r_1 < e_2/r_2$. This implies $e_1 = 0$ and $r_1 e_2 = 1$. Thus $r_1 = e_2 = 1$. By Lemma 1.4, the locus of properly semi-stable sheaves is at least of codimension 2. Therefore we obtain that $\mathrm{codim}\,(T(0) \backslash T(0)^s) = \mathrm{codim}\,(Q_2(0) \backslash Q_2(0)^s) \ge 2$ except for the case that $g = 1$, $r | c_1$ and $c_2 = 1$, where $Q_2(0)^s = \{z \in Q_2(0) | \mathcal{E}_z \text{ is a stable sheaf}\}$ and $T(0)^s = t_0^{-1}(Q_2(0)^s)$.

For a point y of $Q_2(i)$, $\dim t_i^{-1}(y) = (h^0(X, \mathcal{E}_y) - (r-1))(r-1) + (r-1)^2 - 1 = h^0(X, \mathcal{E}_y)(r-1) - 1$. Since $i \le (r-1)g$, $\dim T(0) - \dim T(i) \ge \dim Q_2(0) - \dim Q_2(i) - (r-1)^2 g$. Therefore it is sufficient to show that $\dim Q_2(0) - \dim Q_2(i) \ge (r-1)^2 g + 2$, $i > 0$ for sufficiently large d. Let $\mathcal{B}$ be the set of torsion free sheaves E which satisfy

(i) $\mathrm{rk}\,(E) = r$, $c_1(E) = c_1$ and $c_2(E) = c_2$,

(ii) $R^1\pi_* E = 0$,

(iii) for the Harder-Narasimhan filtration $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$, $\max\{\mathrm{rk}\,(E/F_i)\deg(F_i) - \mathrm{rk}\,(F_i)\deg(E/F_i)\} < (r-1)^2 g + 2$.

Since $\mathcal{B}$ is a bounded set, there is an integer a such that for any member E of $\mathcal{B}$, $\pi_* E(af)$ is generated by global sections and $h^1(C, \pi_* E(af)) = 0$. Replacing c_1 by $c_1 + raf$, we may assume that $a = 0$. If $h^1(X, E) \ne 0$, then E does not belong to $\mathcal{B}$, and hence $\dim \mathrm{Ext}^1_{F,+}(E, E) \ge (r-1)^2 g + 2$, where F is the Harder-Narasimhan filtration of E. Hence $\mathrm{codim}\,(Q_2(i)) \ge (r-1)^2 g + 2$ for $i \ge 1$. Therefore $\dim T(0) - \dim T(i) \ge 2$ for $i \ge 1$. Thus $\mathrm{codim}\,(\mathrm{im}\,(\bar{k}_i)) \ge 2$ for $i \ge 1$. Since $\mathrm{codim}\,(Q \backslash Q^s) \ge 2$, we see that $\mathrm{codim}\,(\mathbf{P} \backslash \mathbf{P}^s) \ge 2$, except for the case that $g = 1$, $r | c_1$ and $c_2 = 1$.

Remark 3.1. $\mathbf{P}_0^s \to M(r, c_1, c_2)_0$ extends to a morphism $\mathbf{P}^s \to M(r, c_1, c_2)$. We shall show that $\mathbf{P}^s \backslash \mathbf{P}_0^s$ is an open dense subset of $\mathbf{P} \backslash \mathbf{P}_0$. Since $M(r, c_1, 1) = M(r, c_1, 1)_0$, we may assume that $c_2 \ge 2$. By Remark 1.5 and Lemma 3.1, there is a μ-stable vector bundle E which belongs to $M(r, c_1, c_2 - 1)_0$. Let l be a jumping line of E: $E_{|l} \cong \mathcal{O}_l^{\oplus(r-2)} \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1)$. Let F be the kernel of a surjection $E \to \mathcal{O}_l(-1) \to k_P$, where P is a point on l. It is easy to see that $F_{|l} \cong \mathcal{O}_l^{\oplus(r-2)} \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l(-2) \oplus k_P$ and $\pi^*\pi_* E (\subset E)$ is a subsheaf of F. Hence there is an exact sequence $0 \to \mathcal{O}_X^{\oplus(r-1)} \to F \to I_Z \otimes \det(F) \to 0$. Thus $\mathbf{P}^s \backslash \mathbf{P}_0^s$ is not empty for $c_2 \geq 2$. Since the complement of $\mathbf{P}_0$ is the pull-back of the divisor $\tilde{D} \times Pic^0(X)$, it is irreducible. Therefore we get our claim. Combining Lemma 3.4 with this, we get that $Pic(\mathbf{P}) = Pic(\mathbf{P}^s)$ and $Pic(M(r, c_1, c_2)) \to Pic(\mathbf{P}_0^s)$ lifts to a homomorphism $Pic(M(r, c_1, c_2)) \to Pic(\mathbf{P})$ for $c_2 \geq 2$.

Lemma 3.5. *The action of $PGL(N)$ on $T(i)^0$ is set-theoretically free.*

Proof. For a quotient $\mathcal{O}_X(-m)^{\oplus N} \to E$, the stabilizer of the action of $PGL(N)$ on $Q_2(i)$ is the automorphism group $\mathrm{Aut}(E)$. If $\varphi \in \mathrm{Aut}(E)$ fixes $\mathcal{O}_X^{\oplus(r-1)} \subset E$, then $\varphi - 1$ induces a homomorphism $I := E/\mathcal{O}_X^{\oplus(r-1)} \to E$. If the composition $I \to E \to I$ is not zero, then $E \cong \mathcal{O}_X^{\oplus(r-1)} \oplus I$. Hence $\varphi - 1$ induces $I \to \mathcal{O}_X^{\oplus(r-1)}$ which is 0. Therefore $\varphi = 1$. Thus the action of $PGL(N)$ on $T(i)^0$ is set-theoretically free.

If $c_2 = 1$, then we shall identify $\mathcal{H}ilb_X^1$ with X. Hence $M = X \times Pic^0(X)$. We note that $\mathbf{P} = \mathbf{P}_0$ and $M = M_0$.

Lemma 3.6. *If $g = 1$, $\deg c_1 = rk$ and $c_2 = 1$, then for the codimension 1 component D of $\mathbf{P} \backslash \mathbf{P}^s$, $\mathcal{O}_{\mathbf{P}}(D)$ is isomorphic to $\mathcal{M} := \mathcal{O}_{\mathbf{P}}((r-1)(rk-1)v) \otimes (\iota \circ \mu)^*(\mathcal{P} \otimes (p_{Pic^0(X)}^* \det p_{Pic^0(X)!}\mathcal{P})^\vee)^{\otimes(r-1)}$, where $\iota: X \times Pic^0(X) \to Pic^0(X) \times X$ is the morphism sending (x, L) to (L, x).*

Proof. By the proof of Lemma 3.4, for a general point y of D, the Harder-Narasimhan filtration of $\mathcal{F}_y$ is $0 \subset L \subset \mathcal{F}_y$ where L is a line bundle of degree k. We note that $\mathcal{F}_y/L$ belongs to $M(r-1, (r-1)k, 1)$. By using the irreducibility of $Pic^0(X)$ and $M(r-1, (r-1)k, 1)$, we can easily prove that D is irreducible. We shall fix a point p of C. We set

$$W := \{(y, q) \in \mathbf{P} \times C \mid H^0(X, \mathcal{F}_y(-kf) \otimes \pi^*\mathcal{O}_C(q-p)) \neq 0\}.$$

Since $\deg \mathcal{F}_y(-kf) = 0$, $y \in \mathbf{P}$, the Riemann-Roch theorem implies that $W = \{(y, q) \in \mathbf{P} \times C \mid h^1(X, \mathcal{F}_y(-kf) \otimes \pi^*\mathcal{O}_C(q-p)) > 1\}$. Since $\mathrm{codim}(D) = 1$ and the Harder-Narasimhan filtration is unique, we obtain $\mathrm{codim}(W) = 2$. Let Γ_π be the graph of $\pi: X \to C$, and $\mathcal{Q} = \mathcal{O}(\Gamma_\pi - \pi^{-1}(p) \times C)$ a line bundle on $X \times C$, which is a universal line bundle with $c_1 = 0$ on X. Let $r: \mathbf{P} \times X \times C \to X \times C$ and $s: \mathbf{P} \times X \times C \to \mathbf{P} \times X$ be projections. For simplicity, we denote $s^*(\mu^*\mathcal{I}(-v)[-kf]) \otimes r^*\mathcal{Q}$ by $\mathcal{J}$. By the exact sequence (3.7), there is an exact sequence of sheaves on $\mathbf{P} \times X \times C$:

$$0 \to r^*\mathcal{Q}[-kf]^{\oplus(r-1)} \to s^*\mathcal{F} \otimes r^*\mathcal{Q}[-kf] \to \mathcal{J} \to 0.$$

Since $\mathcal{A} := p_{\mathbf{P} \times C *}\mathcal{J}$ is a locally free sheaf of rank $(r-1)k - 1$ and $\mathcal{B} := R^1 p_{\mathbf{P} \times C *}(r^*\mathcal{Q}[-kf])^{\oplus(r-1)}$ is a locally free sheaf of rank $(r-1)k$, the Porteous' formula implies that $\tilde{c}_2(\mathscr{B} - \mathscr{A})$ is an integer multiple of the class of W in $A^*(\mathbf{P} \times C)$. Let $u: \mathbf{P} \times C \to \mathbf{P}$ be the projection. We denote the projections $X \times Pic^0(X) \times C \to X$, $X \times Pic^0(X) \times C \to Pic^0(X)$ and $X \times Pic^0(X) \times C \to C$ by w_1, w_2 and w_3 respectively. We also denote the projections $X \times Pic^0(X) \times C \to X \times Pic^0(X)$, $X \times Pic^0(X) \times C \to Pic^0(X) \times C$ and $X \times Pic^0(X) \times C \to X \times C$, by w_{12}, w_{23} and w_{13} respectively. Let Δ_C (resp. Δ_X) be the diagonal of $C \times C$ (resp. $X \times X$). Then $\mathscr{I} = I_{\Delta_X} \otimes \mathscr{P}$, where we identify the pull-backs of I_{Δ_X} and $\mathscr{P}$ to $X \times Pic^0(X) \times X$ with I_{Δ_X} and $\mathscr{P}$ respectively.

By using the exact sequence

$$0 \to I_{\Delta_X} \to \mathcal{O}_{X \times X} \to \mathcal{O}_{\Delta_X} \to 0 \tag{3.11}$$

we see that

$$\begin{aligned} & p_{\mathbf{P} \times C!}(s^*(\mu^*(I_{\Delta_X} \otimes \mathscr{P}) \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf] \otimes r^*\mathscr{Q}) \\ &\quad = p_{\mathbf{P} \times C!}(s^*(\mu^*\mathscr{P} \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf]) \otimes r^*\mathscr{Q}) \\ &\qquad - p_{\mathbf{P} \times C!}(s^*(\mathcal{O}_{\tilde{\Delta}_X} \otimes \mathscr{P} \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf]) \otimes r^*\mathscr{Q}) \\ &\quad = p_{\mathbf{P} \times C!}(s^*(\mu^*\mathscr{P} \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf]) \otimes r^*\mathscr{Q}) \\ &\qquad - (\mu \times 1_C)^*(w_{12}^*\iota^*\mathscr{P} \otimes w_1^*\mathcal{O}_X(-kf) \otimes w_{13}^*\mathscr{Q}) \otimes u^*\mathcal{O}_{\mathbf{P}}(-v), \end{aligned}$$

where $\tilde{\Delta}_X$ is the pull-back of Δ_X to $\mathbf{P} \times X$. By using the exact sequence

$$0 \to \mathcal{O}_{C \times C} \to \mathcal{O}_{C \times C}(\Delta_C) \to \mathcal{O}_{\Delta_C} \to 0, \tag{3.12}$$

we see that

$$\begin{aligned} & p_{\mathbf{P} \times C!}(s^*(\mathscr{P} \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf]) \otimes r^*\mathscr{Q}) \\ &\quad = p_{\mathbf{P} \times C!}(s^*(\mathscr{P} \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf]) \otimes r^*(\pi \times 1_C)^*\mathcal{O}_{C \times C}(-p \times C)) \\ &\qquad + p_{\mathbf{P} \times C!}(s^*(\mathscr{P} \otimes \mathcal{O}_{\mathbf{P}}(-v)[-kf]) \otimes r^*(\pi \times 1_C)^*\mathcal{O}_{\Delta_C}(-p \times C)) \\ &\quad = (\mu \times 1_C)^*w_2^*(p_{Pic^0(X)!}(\mathscr{P}[-kf - \pi^{-1}(p)])) \otimes u^*\mathcal{O}_{\mathbf{P}}(-v) \\ &\qquad + (\mu \times 1_C)^*(w_{23}^*(1_{Pic^0(X)} \times \pi)_!(\mathscr{P}(-kf)) \otimes w_3^*\mathcal{O}_C(-p)) \otimes u^*\mathcal{O}_{\mathbf{P}}(-v). \end{aligned}$$

Hence we get

$$\begin{aligned} \mathscr{A} &= (\mu \times 1_C)^*w_2^*(p_{Pic^0(X)!}(\mathscr{P}[-kf - \pi^{-1}(p)])) \otimes u^*\mathcal{O}_{\mathbf{P}}(-v) \\ &\quad + (\mu \times 1_C)^*(w_{23}^*(1_{Pic^0(X)} \times \pi)_!(\mathscr{P}(-kf)) \otimes w_3^*\mathcal{O}_C(-p)) \otimes u^*\mathcal{O}_{\mathbf{P}}(-v) \\ &\quad - (\mu \times 1_C)^*(w_{12}^*\iota^*\mathscr{P} \otimes w_1^*\mathcal{O}_X(-kf) \otimes w_{13}^*\mathscr{Q}) \otimes u^*\mathcal{O}_{\mathbf{P}}(-v). \end{aligned} \tag{3.13}$$

By using (3.12), we also see that

$$\begin{aligned} R^1 p_{\mathbf{P} \times C*}(r^*\mathscr{Q}[-kf]) &= (\mu \times 1_C)^*w_3^*R^1 p_{C*}\mathscr{Q}[-kf] \\ &= (\mu \times 1_C)^*w_3^*(\mathcal{O}_C^{\oplus(r+1)} - \mathcal{O}_C(-kp' - p)), \end{aligned}$$

where $p' = \pi(f)$. Hence we get

(3.14) $$\mathscr{B} = (\mu \times 1_C)^* w_3^* (\mathscr{O}_C^{\oplus(k+1)} - \mathscr{O}_C(-kp' - p))^{\oplus(r-1)}.$$

In order to compute $u_* \tilde{c}_2(\mathscr{B} - \mathscr{A})$, we may ignore $\tilde{c}_2(E)$, where E is the pull-back of a sheaf on $\mathbf{P}$. In particular, we do not need the second Chern class of

$$(\mu \times 1_C)^* w_2^* (p_{Pic^0(X)!}(\mathscr{P}[-kf - \pi^{-1}(p)])) \otimes u^* \mathscr{O}_{\mathbf{P}}(-v).$$

Since the relative degree of $\mathscr{P}[-kf - \pi^{-1}(p)]$ is $(r-1)k - 1$,

(3.15) $$\begin{aligned}&\tilde{c}_1\{(\mu \times 1_C)^* w_2^* (p_{Pic^0(X)!}(\mathscr{P}[-kf - \pi^{-1}(p)])) \otimes u^* \mathscr{O}_{\mathbf{P}}(-v)\} \\ &\quad = \tilde{c}_1\{(\mu \times 1_C)^* w_2^* (\det p_{Pic^0(X)!}(\mathscr{P}[-kf - \pi^{-1}(p)]))\} - ((r-1)k - 1) u^*(v).\end{aligned}$$

A simple calculation shows that

(3.16) $$\begin{aligned}&w_{12*}\{\tilde{c}_1(w_{12}^* \iota^* \mathscr{P}[-kf - \pi^{-1}(p)]) \cdot \tilde{c}_1(w_{23}^*(1_{Pic^0(X)} \times \pi)_! \mathscr{P}[-kf - \pi^{-1}(p)])\} \\ &\quad = ((r-1)k - 1)\tilde{c}_1(\iota^* \mathscr{P}[-kf - \pi^{-1}(p)]).\end{aligned}$$

By using (3.13), (3.14), (3.15) and (3.16), we can show that $u_* \tilde{c}_2(\mathscr{B} - \mathscr{A}) \equiv \tilde{c}_1(\mathscr{M})$ mod $Pic\,(Pic^0\,(X))$. Let $\mathscr{M}'$ be the line bundle constructed by using $p_{Pic^0(X)}^* L \otimes \mathscr{P}$, $L \in Pic\,(Pic^0\,(X))$. Since v is replaced by $v + (p_{Pic^0(X)} \circ \iota \circ \mu)^* L$, we get that $\mathscr{M} \cong \mathscr{M}'$. We shall identify $Pic^0\,(X)$ with C and assume that $(1_C \times \pi)_! \mathscr{P} = \mathscr{O}_{C \times C}(\Delta_C + (rk - 1)(C \times p'))$. Then we see that $\det p_{C!} \mathscr{O}_{C \times C}(\Delta_C + C \times d) = \mathscr{O}_C(d)$, where d is a divisor on C and p_C is the first projection. Then we can show that $u_* \tilde{c}_2(\mathscr{B} - \mathscr{A}) = \tilde{c}_1(\mathscr{M})$. Hence we get $\mathscr{M} \cong \mathscr{O}_{\mathbf{P}}(u_*(mW)) = \mathscr{O}_{\mathbf{P}}(mD)$, $m > 0$. In the proof of Proposition 3.14, we shall prove that $m = 1$.

3.3. We assume that $c_2 \geq 2$ in this subsection. By the construction of the family on $\tilde{X}_1$ (cf. **2.3**) and (3.8), a direct computation shows that the pull-back of $\tilde{D}$ to $\tilde{X}_1$ is $\sum_{i=1}^{c_2-2} \pi^{-1}(\pi(P_i)) + (\pi^{-1}(\pi(P_{c_2-1})) - C_1)$. Hence we get that

(3.17) $\tilde{D}$ is a reduced divisor and the image of $\tilde{D}$ in $NS(\tilde{X}_1)$ is $(c_2 - 1)f - C_1$.

Combining this with (2.4), we see that $Pic\,(S^{c_2}C \times Pic^0\,(X)) \to Pic\,(M_0)$ is injective and

(3.18) $$Pic\,(M_0)/Pic\,(S^{c_2}C \times Pic^0\,(X)) \cong NS(\tilde{X}_1)/(\mathbf{Z}f \oplus \mathbf{Z}C_1) \cong \mathbf{Z}C_0.$$

Hence we get

(3.19) $$Pic\,(\mathbf{P}^s)/Pic\,(S^{c_2}C \times Pic^0\,(X)) \cong \mathbf{Z}C_0 \oplus \mathbf{Z}v.$$

By the construction of morphisms λ, det and h, we obtain the following commutative diagram.

(3.20) $$\begin{array}{ccc} \mathbf{P}_0^s & \xrightarrow{h} & \mathbf{T} \\ \downarrow & & \downarrow \\ M_0 & & M(r, c_1, c_2)_0 \\ \downarrow{\scriptstyle \lambda_{\overline{M}(1,c_1,c_2)} \times \det} & & \downarrow{\scriptstyle \lambda_{\overline{M}(r,c_1,c_2)} \times \det} \\ S^{c_2}C \times Pic^0\,(X) & = & S^{c_2}C \times Pic^0\,(X) \end{array}$$

Therefore, the homomorphism $Pic\,(S^{c_2}C \times Pic^0\,(X)) \to Pic\,(M(r, c_1, c_2)_0)$ is injective. Hence we regard $Pic\,(S^{c_2}C \times Pic^0\,(X))$ as a subgroup of $Pic\,(M(r, c_1, c_2)_0)$. Then there is an inclusion

$$
\begin{aligned}
(3.21) \quad Pic\,(M(r, c_1, c_2)_0)/Pic\,(S^{c_2}C \times Pic^0\,(X)) &\hookrightarrow Pic\,(\mathbf{T})/Pic\,(S^{c_2}C \times Pic^0\,(X)) \\
&\cong Pic\,(\mathbf{P}_0^s)/Pic\,(S^{c_2}C \times Pic^0\,(X)) \\
&\cong \mathbf{Z}C_0 \oplus \mathbf{Z}\nu\,.
\end{aligned}
$$

Remark 3.2. If $c_2 = 1$, then $\tilde{D} = \emptyset$. By using Remark 2.2, we see that (3.20) and (3.21) also hold, unless $g = 1$, $r|c_1$ and $c_2 = 1$. If $g = 1$, and $r|c_1$ and $c_2 = 1$, then Lemma 3.6 implies that

$$
(3.22) \quad Pic\,(M(r, c_1, c_2)_0)/Pic\,(S^{c_2}C \times Pic^0\,(X)) \hookrightarrow \mathbf{Z}C_0 \oplus \mathbf{Z}\nu/((r-1)a)\mathbf{Z}\nu\,,
$$

where $a = \deg c_1 - c_2$.

3.4. We shall recall Drezet's construction of line bundles on $M(r, c_1, c_2)$ ([D1], [D2], [D-N]). Let $K(X)$ be the Grothendieck group of X. Let $K^0(X)$ be the subgroup of $K(X)$ which is generated by $\mathcal{O}_X - \mathcal{O}_X(-D)$ and $\mathcal{O}_{C_0} - \mathcal{O}_{C_0}(-D')$, $D, D' \in Pic^0\,(X)$. Then $K^0(X) \cong Pic^0\,(X) \oplus Alb\,(X)$. We shall represent the class in $K(X)$ of $\mathcal{O}_X$, $\mathcal{O}_X(-f)$, $\mathcal{O}_X(-C_0)$ and $\mathcal{O}_X(-C_0 - f)$ by e_1, e_2, e_3 and e_4 respectively. Then $K(X) \cong K^0(X) \oplus L$, where L is the free $\mathbf{Z}$-module of rank 4 generated by e_i, $1 \le i \le 4$. Let ε be the class in $K(X)$ of a torsion free sheaf of rank r with Chern classes c_1, c_2 and set

$$
(3.23) \qquad K(r, c_1, c_2) = \{x \in K(X) | \chi(\varepsilon \otimes x) = 0\}\,.
$$

We set $a := \deg c_1 - c_2$. Since $\chi(\varepsilon \otimes e_1) = r(1-g) + a$, $\chi(\varepsilon \otimes e_2) = r(1-g) + a - r$ and $\chi(\varepsilon \otimes e_3) = \chi(\varepsilon \otimes e_4) = -c_2$, $K(r, c_1, c_2) = K^0(X) \oplus K$ where $K = \{\sum_{i=1}^4 x_i e_i \in L | x_1(r(1-g) + a) + x_2(r(1-g) + a - r) - x_3 c_2 - x_4 c_2 = 0\}$. For an element x in $K(r, c_1, c_2)$, $\mathcal{L} := \det p_{Q!}(\mathcal{E} \otimes [x])$ defines a $GL(N)$-linearized line bundle on Q, where $[x]$ is the image of x to $K(Q \times X)$. Since the action of the center is the multiplication by $\chi(\varepsilon \otimes x)$-th power of constants, it is the trivial action. Therefore it defines a line bundle on $M(r, c_1, c_2)$. Thus we obtain a homomorphism

$$
(3.24) \qquad \kappa\colon K(r, c_1, c_2) \to Pic\,(M(r, c_1, c_2))\,.
$$

Moreover let S be a smooth variety and $\mathcal{G}$ a flat family of torsion free sheaves of rank r with Chern classes c_1, c_2 parametrized by S. Then $x \mapsto \det p_{S!}(\mathcal{G} \otimes [x])$ defines a homomorphism

$$
(3.25) \qquad \kappa_{\mathcal{G}}\colon K(r, c_1, c_2) \to Pic\,(S)\,.
$$

Lemma 3.7. *Assume that $\mathscr{G}$ is a flat family of stable sheaves and let $\sigma: S \to M(r, c_1, c_2)$ be the morphism defined by $\mathscr{G}$. Let σ^* be the homomorphism induced by σ, then $\kappa_{\mathscr{G}} = \sigma^* \circ \kappa$.*

Proof. $S' = S \times_{M(r,c_1,c_2)} Q \to S$ is a principal $PGL(N)$-bundle and hence $Pic(S) = Pic^{PGL(N)}(S')$. We denote the pull-back of $\mathscr{G}$ and $\mathscr{E}$ to $S' \times X$ by $\mathscr{G}'$ and $\mathscr{E}'$ respectively. Setting $R = \mathrm{Hom}_{p_{S'}}(\mathscr{E}', \mathscr{G}')$, we get $\mathscr{E}' \otimes p_{S'}^* R \cong \mathscr{G}'$. From this, we obtain $\det p_{S'!}(\mathscr{G}' \otimes [x]) \cong \det p_{S'!}(\mathscr{E}' \otimes p_{S'}^* R \otimes [x]) \cong \det p_{S'!}(\mathscr{E}' \otimes [x]) \otimes R^{\otimes \chi(\varepsilon \otimes x)} = \det p_{S'!}(\mathscr{E}' \otimes [x])$. Since $H^0(PGL(N), \mathcal{O}_{PGL(N)}^\times) = \mathbf{C}^\times$, $Pic^{PGL(N)}(S') \to Pic(S')$ is injective. Therefore we get $\kappa_{\mathscr{G}}(x) \cong \sigma^*(\kappa(x))$.

3.5. In this subsection, we shall treat the case where $c_2 \geq 2$. In particular, we shall prove Theorem 0.1.

Lemma 3.8. *$K(r, c_1, c_2) \to Pic(M(r, c_1, c_2)_0)/Pic(S^{c_2}C \times Pic^0(X))$ is surjective.*

Proof. Let I_Z be the family of ideal sheaves defined in **2.3**. The family $I_Z[df]$ of torsion free sheaves of rank 1 defines a morphism $\tilde{X}_1 \to M$. We denote $\tilde{X}_1 \times_M \mathbf{P}_0$ by $\tilde{P}$. Let F be the pull-back of $\mathscr{F}$ to $\tilde{P} \otimes X$. We also denote the pull-back of $I_Z[df]$ to $\tilde{P} \times X$ by $I_Z[df]$. Then there is an exact sequence

$$0 \to \mathcal{O}_{\tilde{P} \times X}^{\oplus(r-1)} \to F \to I_Z[df] \otimes \mathcal{O}_{\tilde{P}}(-v) \to 0,$$

where we also denote $p_{\tilde{P}}^*(v)$ by v. We shall define $A_i \in Pic(\tilde{P})/Pic^0(X)$ $(1 \leq i \leq 4)$ as follows:

$$(3.26)\quad \begin{cases} A_1 := \det p_{\tilde{P}!} F = \mathcal{O}_{\tilde{P}}(C_1 - c_1) \otimes \mathcal{O}_{\tilde{P}}(-(1 - g + a)v), \\ A_2 := \det p_{\tilde{P}!}(F[-f]) = \mathcal{O}_{\tilde{P}}(C_1 - c_1 + f) \otimes \mathcal{O}_{\tilde{P}}(-(1 - g + a - 1)v), \\ A_3 := \det p_{\tilde{P}!}(F[-C_0]) = \mathcal{O}_{\tilde{P}}(C_1 - c_1 + C_0) \otimes \mathcal{O}_{\tilde{P}}(c_2 v), \\ A_4 := \det p_{\tilde{P}!}(F[-C_0 - f]) = \mathcal{O}_{\tilde{P}}(C_1 - c_1 + C_0 + f) \otimes \mathcal{O}_{\tilde{P}}(c_2 v), \end{cases}$$

where $a = \deg c_1 - c_2$. Let $\phi: L \to Pic(\tilde{P})/Pic^0(X) \cong (\mathbf{Z}C_0 \oplus \mathbf{Z}f \oplus \mathbf{Z}C_1 \oplus \mathbf{Z}v)/(\mathbf{Z}((c_2 - 1)f - C_1))$ be the homomorphism such that $\phi(e_i) = A_i$ (cf. (3.17)). Then the restriction of ϕ to K is the morphism induced by $\kappa_{F|K}$. Then $e_3 - e_4$ belongs to K and $\phi(e_3 - e_4) = f$. It induces the homomorphism $\phi': L/\mathbf{Z}(e_3 - e_4) \to N := \mathbf{Z}C_0 \oplus \mathbf{Z}v \cong Pic(\tilde{P})/(Pic^0(X) + \mathbf{Z}f)$. $L' := \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}e_3$ is isomorphic to $L/\mathbf{Z}(e_3 - e_4)$ and under this isomorphism, $K \cap L'$ is isomorphic to $K/\mathbf{Z}(e_3 - e_4)$. Let K' be the kernel of the homomorphism $\psi: L' \to \mathbf{Z}$ such that $\psi(e_1) = r(1 - g) + a$, $\psi(e_2) = r(1 - g) + a - r$ and $\psi(e_3) = -c_2$. Then ψ is the restriction of the homomorphism $\chi(\varepsilon \otimes \_): L \to \mathbf{Z}$ to L', and hence $K' = K \cap L'$. We set $n := \gcd(r, a, c_2)$. Then $\operatorname{im} \psi = n\mathbf{Z}$. It is easy to see that $\ker \phi'$ is generated by $((1 - g) + a - 1)e_1 - ((1 - g) + a)e_2$. Hence $\psi(\ker \phi') = (r - 1)a\mathbf{Z}$. Then there is the following exact and commutative diagram.

$$
(3.27)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker\phi' & = & \ker\phi' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & L' & \xrightarrow{\psi} & n\mathbf{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\phi'} & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & N & \longrightarrow & N/K' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Therefore $N/K' \cong n\mathbf{Z}/\ker\phi' \cong \mathbf{Z}\Big/\frac{(r-1)a}{n}\mathbf{Z}$. Thus we get that

$$(3.28)\qquad \bar{\kappa}_{\mathscr{F}|K'}\colon K' \xrightarrow{\kappa_{\mathscr{F}|K'}} Pic\,(\mathbf{P}_0) \to Pic\,(\mathbf{P}_0)/Pic\,(S^{c_2}C \times Pic^0\,(X))$$

is injective and

$$(3.29)\qquad \operatorname{coker}\,(\bar{\kappa}_{\mathscr{F}|K'}) \cong \mathbf{Z}\Big/\frac{(r-1)a}{n}\mathbf{Z}$$

(cf. **3.3**). Combining this with (3.5), (3.21), and Lemma 3.7, we obtain our lemma.

Corollary 3.9. *$Pic\,(M(r, c_1, c_2)_0) \cong Pic\,(S^{c_2}C \times Pic^0\,(X)) \oplus \mathbf{Z}^{\oplus 2}$ for $c_2 \geq 2$.*

Proof. We note that $Pic\,(S^{c_2}C \times Pic^0(X))$ is a subgroup of $\mathrm{Pic}\,(M(r, c_1, c_2)_0)$. By the proof of Lemma 3.8, we get $K' \cong Pic\,(M(r, c_1, c_2)_0)/Pic\,(S^{c_2}C \times Pic^0\,(X))$. Hence we obtain this corollary.

Lemma 3.10. *The restriction of $\kappa\colon K(r, c_1, c_2) \to Pic\,(M(r, c_1, c_2))$ to $K^0(X)$ is injective and its image is $(\lambda \times \det)^*(Pic^0\,(S^{c_2}C \times Pic^0\,(X)))$.*

Proof. We shall first consider the case that $r = 1$. We denote $\tilde{X}_1 \times Pic^0\,(X)$ by Y and let $\beta\colon Y \to \mathscr{Hilb}_X^{c_2} \times Pic^0\,(X)$ be the morphism induced by γ. We shall show that $\beta^* \circ \kappa$ is injective. Let $r\colon Y \to Pic^0\,(X)$ be the projection. For simplicity, we also denote pull-backs of I_Z in **2.3** and $\mathscr{P}$ to $Y \times X$ by I_Z and $\mathscr{P}$ respectively. By the definition of I_Z, we see that

$$(3.30)\qquad \det\,(p_{Y!}(I_Z \otimes \mathscr{P}[D])) = r^* \det\,(p_{Pic^0(X)!}(\mathscr{P}[D])) \otimes \Big(\Big(\bigotimes_{i=1}^{c_2-1} r^*\mathscr{P}_{P_i}\Big) \otimes (\bar{b}^*(\mathscr{P}[D - C_1]))\Big)^{\vee},$$

where $\mathscr{O}_X(D) \in Pic^0\,(X)$, $\mathscr{P}_{P_i} := \mathscr{P}_{|Pic^0(X) \times \{P_i\}}$ and $\bar{b}\colon \tilde{X}_1 \times Pic^0\,(X) \to Pic^0\,(X) \times X$ is the morphism sending (x, L) to $(L, b(x))$ (cf. **2.3**). Thus we get

$$\beta^* \circ \kappa(\mathcal{O}_X - \mathcal{O}_X(-D)) = r^* \det (p_{Pic^0(X)!}(\mathscr{P} - \mathscr{P}[-D])) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D),$$

$$\beta^* \circ \kappa(\mathcal{O}_X(-C_0) - \mathcal{O}_X(-C_0 - D)) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D).$$

If $D \cdot C_0 = \sum_i a_i Q_i$, $a_i \in \mathbf{Z}$, then $\det (p_{Pic^0(X)!}(\mathscr{P} - \mathscr{P}[-D])) \cong \bigotimes_i (\mathscr{P}_{Q_i})^{\otimes a_i}$. Therefore, for an element D of $Pic^0(X)$, we see that

$$\beta^* \circ \kappa(\mathcal{O}_X - \mathcal{O}_X(-D)) = \bigotimes_i (r^* \mathscr{P}_{Q_i})^{\otimes a_i} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D),$$

$$\beta^* \circ \kappa(\mathcal{O}_{C_0} - \mathcal{O}_{C_0}(-D)) = \bigotimes_i (r^* \mathscr{P}_{Q_i})^{\otimes a_i}.$$

Thus $\beta^* \circ \kappa$ is injective and its image is $Pic^0(Y) \cong Pic^0(\tilde{X}_1) \times Alb(X)$. Since λ^* and $\gamma^* \circ \lambda^*$ are isomorphisms (cf. **2.3**), we obtain that $\kappa(K^0(X)) = (\lambda \times \det)^*(Pic^0(S^{c_2}C \times Pic^0(X)))$.

For general cases, by using (3.7), we get $\det (p_{\mathbf{P}!}\mathscr{F}[-D]) = \det (p_{\mathbf{P}!} \cdot (\mu^* \mathscr{I}(-v)[-D]))$. To avoid confusion, we denote the homomorphism $K^0(X) \to Pic(r, c_1, c_2))$ by κ_r. For the morphism $t: \mathbf{P}^s \to M(r, c_1, c_2)$, we get $t^* \circ \kappa_r(\mathcal{O}_X - \mathcal{O}_X(-D)) = \det (p_{\mathbf{P}^s!}(\mu^* \mathscr{I} - \mu^* \mathscr{I}[-D])) = \mu^* \circ \kappa_1(\mathcal{O}_X - \mathcal{O}_X(-D))$ and $t^* \circ \kappa_r(\mathcal{O}_{C_0} - \mathcal{O}_{C_0}(-D)) = \det (p_{\mathbf{P}^s!}((\mu^* \mathscr{I} - \mu^* \mathscr{I}[-D]) - (\mu^* \mathscr{I}[-C_0] - \mu^* \mathscr{I}[-C_0 - D]))) = \mu^* \circ \kappa_1(\mathcal{O}_{C_0} - \mathcal{O}_{C_0}(-D))$. The assertion follows from these.

Remark 3.3. Replacing the morphism $\tilde{X}_1 \to \mathscr{H}ilb_X^{c_2}$ by $X \to \mathscr{H}ilb_X^1$, we see that Lemma 3.8 and Lemma 3.10 also hold for $c_2 = 1$ (since C_1 does not appear in the case, we can ignore that part).

Lemma 3.11. *For a point Q of C, let $i_Q: S^{n-1}C \to S^nC$ be an inclusion sending D to $D + Q$. Then we obtain the following. For $n \geq 2$.*

$$Pic(S^nC) \cong Pic(J^n) \oplus \mathbf{Z}\mathcal{O}(i_Q(S^{n-1}C)),$$

$$Pic(S^nC \times Pic^0(X)) \cong Pic(J^n \times Pic^0(X)) \oplus \mathbf{Z}\mathcal{O}(i_Q(S^{n-1}C) \times Pic^0(X)),$$

where J^n is the divisor class group of degree n.

Proof. Let $\mathscr{D} \subset S^nC \times C$ be the universal family of divisors such that $\mathscr{D}_{|\{D\} \times C} = D$. The line bundle $\mathcal{O}(\mathscr{D})$ defines a morphism $j: S^nC \to J^n$. Then $\mathscr{D}_Q := \mathscr{D}_{|S^nC \times \{Q\}}$ defines an effective divisor on S^nC and $\mathscr{D}_Q = i_Q(S^{n-1}C)$. Let $\mathscr{P}^n$ be a Poincaré line bundle of degree n. If $n \geq 2g$, then $S^nC \cong \mathbf{P}(E^\vee)$, where $E := q_{J^n *}(\mathscr{P}^n)$ is locally free sheaf on J^n, and hence $H^2(S^nC, \mathbf{Z}) \cong H^2(J^n, \mathbf{Z}) \oplus \mathbf{Z}\mathcal{O}_{\mathbf{P}(E^\vee)}(1)$. For a line bundle $L \in J^n$, we see that $\mathscr{D}_{Q|j^{-1}(L)} = \mathbf{P}(H^0(C, L(-Q))^\vee) \subset \mathbf{P}(H^0(C, L)^\vee)$. Therefore $H^2(S^nC, \mathbf{Z}) \cong H^2(J^n, \mathbf{Z}) \oplus \mathbf{Z}\mathcal{O}(\mathscr{D}_Q)$. By [Mc, 12.2 and 4.2], $i_P^*: H^l(S^{k+1}C, \mathbf{Z}) \to H^l(S^kC, \mathbf{Z})$ is an isomorphism for $k \geq l$. Hence $Pic(S^{k+1}C) \cong Pic(S^kC)$ for $k \geq 2$. Therefore, $Pic(J^n) \to Pic(S^nC)$ is injective and $Pic(S^nC) \cong Pic(J^n) \oplus \mathbf{Z}\mathcal{O}(\mathscr{D}_Q)$ for $n \geq 2$. We also obtain the second relation.

Let $\alpha_1: \mathscr{H}ilb_X^n \to Alb(X)$ be the morphism induced by the Albanese map $\mathfrak{a}: X \to Alb(X)$ in **2.3** (i.e. let Z be a 0-dimensional subscheme of X with $c_2(I_Z) = n$ and $\sum_{i=1}^n P_i$ the associated cycle, then $\alpha_1(z) = \sum_{i=1}^n \mathfrak{a}(P_i)$) and $\alpha_2: \mathscr{H}ilb_X^n \to S^nC \to J^n$ the composition of $\lambda': \mathscr{H}ilb_X^n \hookrightarrow M(1, 0, n) \xrightarrow{\lambda} S^nC$ and j. We shall choose an isomorphism $\zeta\colon J^n \to Alb(X)$ such that $\alpha_1 = \zeta \circ \alpha_2$. We set $\alpha = (\zeta \circ j \circ \lambda_{\overline{M}(r,a_1,c_2)}) \times$ det.

Proposition 3.12. *α^* is injective and $Pic(\overline{M}(r, c_1, c_2)) \cong Pic(Alb(X) \times Pic^0(X)) \oplus K$ for $c_2 \geq 2$.*

Proof. Since $\overline{M}(r, c_1, c_2)$ is locally factorial (Corollary 1.5) and the complement of $M(r, c_1, c_2)_0$ is at least of codimension 2 (Lemma 3.1), we shall compute $Pic(M(r, c_1, c_2)_0)$. In the proof of Corollary 3.9, we saw that $Pic(M(r, c_1, c_2)_0) \cong Pic(S^{c_2}C \times Pic^0(X)) \oplus K'$. Since $Pic(S^{c_2}C \times Pic^0(X)) \cong Pic(J^{c_2} \times Pic^0(X)) \oplus \mathbf{Z}\mathcal{O}(S^{c_2-1}C \times Pic^0(X))$ and $(\lambda \times \det)^*(\mathcal{O}(S^{c_2-1}C \times Pic^0(X))) \equiv \kappa(e_3 - e_4) \bmod Pic(Alb(X) \times Pic^0(X))$ (see Remark 2.1 and Lemma 3.10), we get $Pic(M(r, c_1, c_2)_0) \cong Pic(J^{c_2} \times Pic^0(X)) \oplus K$. Therefore, we obtain our proposition.

Proof of Theorem 0.1.
(i) and (iv) follow from Proposition 3.12. By Lemma 3.10 and Proposition 3.12, we get the following exact and commutative diagram.

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & Pic^0(X) \times Alb(X) & \longrightarrow & K(r, c_1, c_2) & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \kappa} & & \| & & \\ 0 & \longrightarrow & Pic(Alb(X) \times Pic^0(X)) & \xrightarrow{\alpha^*} & Pic(\overline{M}(r, c_1, c_2)) & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

Hence κ is injective and $\operatorname{im}(\kappa) \cap \operatorname{im}(\alpha^*) \cong Pic^0(X) \times Alb(X)$. Thus (ii) and (iii) hold.

Let ω be the dualizing sheaf of $\overline{M}(r, c_1, c_2)$. Then we get the following theorem, whose proof is the same as that in [D-N, Theorem E].

Theorem 3.13. *Let E be a vector bundle of rank r with first Chern class c_1. Then $\omega = \kappa(E^\vee - E^\vee \otimes K_X) \otimes (\lambda \times \det)^* \mathcal{L}^\vee$, where $\mathcal{L} := \det(p_{Pic^0(X)!}(\mathcal{P} \otimes [\det E^\vee - \det E^\vee \otimes K_X]))$ and $[x]$ is the image of $x \in K(X)$ in $K(Pic^0(X) \times X)$ (cf.* 3.4).

Proof. It is sufficient to compute $\det p_{\mathbf{P}^s!}(\mathcal{F}, \mathcal{F})$. From the exact sequence (3.7), we obtain

$$p_{\mathbf{P}!}(\mathcal{F}, \mathcal{F}) = p_{\mathbf{P}!}(\mu^*\mathcal{I}, \mu^*\mathcal{I}) + (r-1)\{p_{\mathbf{P}!}(\mu^*\mathcal{I}(-v)) + p_{\mathbf{P}!}(u^*\mathcal{I}(-v), \mathcal{O}_{\mathbf{P}\times X})\}$$
$$+ (r-1)^2 p_{\mathbf{P}!}(\mathcal{O}_{\mathbf{P}\times X}).$$

Thus we see that

$$\det p_{\mathbf{P}!}(\mathcal{F}, \mathcal{F}) = \mu^*(\det p_{M!}(\mathcal{I}, \mathcal{I}) \otimes (\det p_{M!}(\mathcal{I}) \otimes \det p_{M!}(\mathcal{I}, \mathcal{O}_{M\times X}))^{\otimes(r-1)})$$
$$\otimes\, \mathcal{O}_{\mathbf{P}}(-2d(r-1)v),$$

where $d := \deg c_1$. By the relative duality, we get $\det p_{M!}(\mathscr{I}, \mathcal{O}_{M\times X}) \cong \det p_{M!}(\mathscr{I}[K_X])^\vee$. Thus

(3.31) $$\det p_{\mathbf{P}!}(\mathscr{F}, \mathscr{F}) = \mu^*(\det p_{M!}(\mathscr{I}, \mathscr{I}) \otimes (\det p_{M!}(\mathscr{I} \otimes [\mathcal{O}_X - K_X]))^{\otimes(r-1)}) \otimes \mathcal{O}_{\mathbf{P}}(-2d(r-1)v).$$

In the same way as in **2.3**, we shall identify the pull-backs of $I_{\mathscr{Z}}$ and $\mathscr{P}$ to $M \times X$ with $I_{\mathscr{Z}}$ and $\mathscr{P}$ respectively. Since $\operatorname{codim} \operatorname{Supp}(\mathcal{O}_{M\times X}/I_{\mathscr{Z}}) = 2$, $\tilde{c}_1(p_{M!}(\mathcal{O}_{M\times X}/I_{\mathscr{Z}}, \mathcal{O}_{M\times X}/I_{\mathscr{Z}})) = 0$. Hence, by using the relative duality, we see that $-\tilde{c}_1(p_{M!}(I_{\mathscr{Z}}, I_{\mathscr{Z}})) = \tilde{c}_1(p_{M!}(\mathcal{O}_{M\times X}/I_{\mathscr{Z}} \otimes [\mathcal{O}_X - K_X]))$. Since $(\mathcal{O}_X - K_X) \otimes E^\vee = (\mathcal{O}_X - K_X) \otimes (\mathcal{O}_X^{\oplus(r-1)} \oplus \det E^\vee)$, we obtain the following.

(3.32)
$$\det(p_{M!}(I_{\mathscr{Z}} \otimes \mathscr{P} \otimes [E^\vee - E^\vee \otimes K_X])) \otimes [\det(p_{M!}(\mathscr{P} \otimes [\det E^\vee - \det E^\vee \otimes K_X]))]^\vee$$
$$= \det(p_{M!}(I_{\mathscr{Z}}, I_{\mathscr{Z}})) \otimes [\det(p_{M!}(I_{\mathscr{Z}} \otimes \mathscr{P} \otimes [\mathcal{O}_X - K_X]))]^{\otimes(r-1)}.$$

If I is an element of M, then $\chi(I \otimes E^\vee) - \chi(I \otimes E^\vee \otimes K_X) = 2d$. Therefore

(3.33) $$\kappa_{\mathbf{P}^s}(E^\vee - E^\vee \otimes K_X) \cong \mu^* \det(p_{M!}(I_{\mathscr{Z}} \otimes \mathscr{P} \otimes [E^\vee - E^\vee \otimes K_X])) \otimes \mathcal{O}_{\mathbf{P}^s}(-2d(r-1)v).$$

By using (3.31), (3.32) and (3.33), we get our theorem.

3.6. We shall treat the case where $c_2 = 1$.

Proposition 3.14. *If* $c_2 = 1$, *then*

$$Pic(\overline{M}(r, c_1, c_2)) \cong \begin{cases} Pic(C \times Pic^0(X)) \oplus \mathbf{Z} & \text{if } g = 1 \text{ and } r|c_1 \\ Pic(C \times Pic^0(X)) \oplus \mathbf{Z}^{\oplus 2} & \text{otherwise}. \end{cases}$$

Proof. We note that (3.28) and (3.29) hold for $c_2 = 1$ (see Remark 3.3). Unless $g = 1$ and $r|c_1$, Lemma 3.4 implies that Lemma 3.8 and its proof also hold, and we can argue as in Corollary 3.9. Thus we get $Pic(\overline{M}(r, c_1, c_2)) \cong Pic(C \times Pic^0(X)) \oplus \mathbf{Z}^{\oplus 2}$. Hence we assume that $g = 1$ and $r|c_1$ and compute $Pic(\overline{M}(r, c_1, c_2))$. We also complete the proof of Lemma 3.6 (i.e. we shall show that $m = 1$). Since $c_2 = 1$, r and $a = \deg c_1 - c_2$ are relatively prime. Hence $n = \gcd(r, a, c_2) = 1$. We set $R := Pic(C \times Pic^0(X))$. Let D be the divisor defined in Lemma 3.6. By using (3.26) (see Remark 3.3), we see that $\kappa_{\mathscr{F}}((a-r)e_1 - ae_2) \equiv \mathscr{M} = \mathcal{O}(mD) \bmod R$, in particular, $(a-r)e_1 - ae_2$ belongs to $\ker(\bar{\kappa}_{|K'})$, where $\bar{\kappa}_{|K'}\colon K' \xrightarrow{\kappa_{|K'}} Pic(M(r, c_1, 1)) \to Pic(M(r, c_1, 1))/R$ (we use Lemma 3.7). By using (3.5), (3.21), and Lemma 3.7, we get the following exact and commutative diagram:

(3.34)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker(\bar{\kappa}_{|K'}) & & \mathbf{Z}D & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K' & \xrightarrow{\bar{\kappa}_{\mathcal{F}|K'}} & Pic\,(\mathbf{P})/R & \longrightarrow & \operatorname{coker}(\bar{\kappa}_{\mathcal{F}|K'}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\kappa}_{|K'}} & & \downarrow & & \downarrow{\scriptstyle q} & & \\
0 & \longrightarrow & Pic\,(M(r, c_1, 1))/R & \longrightarrow & Pic\,(\mathbf{P}^s)/R & \longrightarrow & U & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where U is a finite abelian group with $\# U = (r-1)a$ (cf. (3.5)). Since $\operatorname{coker}(\bar{\kappa}_{\mathcal{F}|K'}) \cong \mathbf{Z}/(r-1)a\mathbf{Z}$ ((3.29)), q is an isomorphism. By using the snake lemma, we get that $\ker(\bar{\kappa}_{|K'}) \cong \mathbf{Z}D$ and $\bar{\kappa}_{|K'}$is surjective. Since a and r are relatively prime, $(a-r)e_1 - ae_2$ is a primitive element of K'. Hence we get that $\ker(\bar{\kappa}_{|K'}) = \mathbf{Z}((a-r)e_1 - ae_2)$. Therefore $Pic\,(\overline{M}(r, c_1, c_2)) \cong Pic\,(C \times Pic^0\,(X)) \oplus \mathbf{Z}$ and m must be 1. Thus we completed the proof of Lemma 3.6.

Remark 3.4. Let L be a line bundle of degree d. Let $M(2, L, c_2)^0$ be the open subset of $M(2, L, c_2)$ consisting of stable vector bundles and set $D := M(2, L, c_2) \backslash M(2, L, c_2)^0$. Then D is an irreducible divisor. In the same way as in [S], we can show that $\mathcal{O}_{M(2,L,c_2)}(D) \cong \kappa[K_X + \mathcal{O}_X((2g-2+c_2)f) + (2-2g-d-c_2)(\mathcal{O}_X(-C_0) - \mathcal{O}_X(-C_0-f))]$. A simple calculation shows that $K_X + \mathcal{O}_X((2g-2+c_2)f) + (2-2g-d-c_2)(\mathcal{O}_X(-C_0) - \mathcal{O}_X(-C_0-f)) \equiv c_2(e_1+e_2) + (c_2+d+e+2-2g)(e_4-e_3) - 2e_3 \bmod K^0(X)$. We assume that $c_2 \geq 2$. Then we obtain that

$$
Pic\,(M(2, L, c_2)^0) \cong \begin{cases} Pic\,(Alb\,(X)) \oplus \mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } d+e \equiv c_2 \equiv 0 \bmod 2 \\ Pic\,(Alb\,(X)) \oplus \mathbf{Z}^{\oplus 2} & \text{otherwise}\,. \end{cases}
$$

4. *Pic* $(\overline{M}(r, c_1, c_2))$ in the case where $g = 0$

4.1. We shall next treat the case that $g = 0$. We shall first prove analogous statements as in Proposition 1.3 and Lemma 1.4.

Proposition 4.1. (1) *For an integer d with $0 < d < r$, there is a μ-semi-stable sheaf E of rank r with $c_1(E) = df$ and $c_2(E) = c_2$ if and only if $c_2 \geq \max\{r-d, d\}$.*

(2) *Under the condition in (1), there is a stable sheaf E of rank r with $c_1(E) = df$ and $c_2(E) = c_2$ unless $d/r = c_2/r = 1/2$.*

Proof. (1) For torsion free sheaves F_1, F_2 such that $F_{i|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r_i}$, $i = 1, 2$, the Serre duality implies that $\mathrm{Ext}^2(F_1, F_2) = \mathrm{Hom}(F_2, F_1(K_X))^{\vee} = 0$. Let E be a μ-semi-stable sheaf of rank r with $c_1(E) = df$ and $c_2(E) = c_2$. Then $\mathrm{Hom}(\mathcal{O}_X(f), E) = \mathrm{Hom}(E, \mathcal{O}_X) = 0$. Hence $\chi(\mathcal{O}_X(f), E) = d - c_2 \le 0$ and $\chi(E, \mathcal{O}_X) = r - c_2 - d \le 0$. Therefore $c_2 \ge \max\{d, r-d\}$, in particular $\Delta(E) = c_2/r \ge 1/2$.

Conversely we assume that $c_2 \ge \max\{d, r-d\}$. We shall use the quot-scheme Q^x and the notation in Proposition 1.3. Then it is sufficient to prove that $\dim \mathrm{Ext}^1_{F,+}(E, E) > 0$, where E is a torsion free sheaf of rank r with $c_1(E) = df$ and $c_2(E) = c_2$ which is not μ-semi-stable, and $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ is the Harder-Narasimhan filtration of E. Since (1.6) holds for any g, we obtain that $\dim \mathrm{Ext}^1_{F,+}(E, E) = -\sum_{i<j}\chi(E_i, E_j)$, where $E_i = gr_i(E)$. Thus we shall show that $-\sum_{i<j}\chi(E_i, E_j) > 0$. By the proof of (1.6), $-\chi(E_i, E_j) = \dim \mathrm{Ext}^1(E_i, E_j) \ge 0$. Therefore it is sufficient to show that $\chi(E_i, E_j) \ne 0$ for some i and j. We assume that $-\chi(E_1, E_s) = r_1 r_s(d_1/r_1 - d_s/r_s + e_1/r_1 + e_s/r_s - 1 = 0$. Then $e_1/r_1 + e_s/r_s < 1$. Thus $e_1/r_1 < 1/2$ or $e_s/r_s < 1/2$. We shall treat the case that $e_1/r_1 < 1/2$. The other case is similar. We note that

$$
\begin{aligned}
(4.1) \qquad r_1^2(1 - 2e_1/r_1) &= \chi(E_1, E_1) \\
&= \chi(E_1, E) - \sum_{j>1}\chi(E_1, E_j) \\
&= r_1 r(d/r - c_2/r - d_1/r_1 - e_1/r_1 + 1) - \sum_{j>1}\chi(E_1, E_j)\,.
\end{aligned}
$$

If $d/r - c_2/r - d_1/r_1 - e_1/r_1 + 1 \le 0$, then we get $0 < r_1^2(1 - 2e_1/r_1) \le -\sum_{j>1}\chi(E_1, E_j)$. So it is sufficient to show that $d/r - c_2/r - d_1/r_1 - e_1/r_1 + 1 \le 0$. By our assumption, $d/r - c_2/r \le 0$. Thus we shall prove that $-d_1/r_1 - e_1/r_1 + 1 \le 0$. Since F is the Harder-Narasimhan filtration of E with respect to μ-semi-stability, $d_1 > r_1 d/r > 0$. If $d_1 \ge r_1$, then $-d_1/r_1 - e_1/r_1 + 1 \le 0$ follows from $e_1 \ge 0$ (the Bogomolov-Gieseker inequality). If $0 < d_1 < r_1$, then the necessary condition for μ-semi-stable sheaves (which was showed in the first paragraph of this proof) implies that $e_1 \ge \max\{r_1 - d_1, d_1\}$. Therefore $-d_1/r_1 - e_1/r_1 + 1 \le 0$ for $d > 0$.

(2) Let E be a μ-semi-stable sheaf. Let $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ be the Harder-Narasimhan filtration or a Jordan-Hölder filtration of E. Then $-\chi(E_i, E_j) = r_i r_j(e_i/r_i + e_j/r_j - 1)$. Since $d_i/r_i = d/r \ne 0$, (1) implies that $e_i/r_i \ge 1/2$. We shall first assume that $d/r \ne 1/2$. Since $e_i/r_i > 1/2$, we get $e_i/r_i - 1/2 \ge 1/2r_i$. Thus $-\chi(E_i, E_j) \ge (r_i + r_j)/2$. Since $d/r \ne 0$, $1/2$, we see that $r \ge 3$ and hence $-\sum_{i<j}\chi(E_i, E_j) \ge r/2 \ge 3/2$. We next assume that $d/r = 1/2$. Then $r_i = 2d_i$ for $1 \le i \le s$. Since $c_2/r > 1/2$, there is an integer i such that $e_i/r_i > 1/2$. Since r_i is even, $e_i/r_i \ge 1/2 + 1/r_i$. Hence $-\chi(E_i, E_j) = -\chi(E_j, E_i) \ge r_j \ge 2$. Therefore, there is a stable sheaf.

Remark 4.1. Let Q be the scheme in Lemma 1.4. Then the above proof also implies that $\mathrm{codim}(Q \backslash Q^s) \ge 2$.

Remark 4.2. By using the same method as in Proposition 4.1(1), we get that $\dim \mathrm{Ext}^1_{F,+}(E, E) = -\sum_{i<j}\chi(E_i, E_j) > 0$ for the case that $c_1(E) = 0$ and

$c_2(E) \geq 0$, where F is the Harder-Narasimhan filtration of E with respect to μ-semi-stability.

Lemma 4.2. *Let Q_1 and Q^s be open subschemes of the quot-scheme Q^x in the proof of Proposition* 1.3. *Assume that $0 < d < r$ and $c_2/r > 1/2$. Then* codim $(Q_1 \backslash Q^s) = 1$ *if and only if $c_2 = r - d$ or $c_2 = d$.*

Proof. By the proof of Proposition 4.1, it is sufficient to consider $-\sum_{i<j}\chi(E_i, E_j)$ associated with the Harder-Narasimhan filtration of E with respect to μ-semi-stability. (i) If e_1/r_1, $e_s/r_s > 1/2$, then it is easy to see that $-\chi(E_1, E_s) > r_s d_1 - r_1 d_s > 0$. Thus $-\sum_{i<j}\chi(E_i, E_j) \geq 2$.

(ii) Assume that $e_1/r_1 \leq 1/2$ or $e_s/r_s \leq 1/2$. We shall treat the case that $e_1/r_1 \leq 1/2$. The other case is similar. We first treat the case that $e_1/r_1 = 1/2$. By (4.1), $-\sum_{j>1}\chi(E_1, E_j) = r_1 r(c_2/r - d/r + d_1/r_1 + e_1/r_1 - 1)$. We shall show that $r_1 r(c_2/r - d/r + d_1/r_1 + e_1/r_1 - 1) \geq 2$. If $d_1 \geq r_1$, then $r_1 r(d_1/r_1 + e_1/r_1 - 1) \geq re_1 \geq 2$. Since $c_2/r - d/r \geq 0$ (Proposition 4.1(1)), we obtain that $r_1 r(c_2/r - d/r + d_1/r_1 + e_1/r_1 - 1) \geq 2$. If $0 < d_1 < r_1$, then $1/2 = e_1/r_1 \geq \max\{1 - d_1/r_1, d_1/r_1\}$. Hence $d_1/r_1 = 1/2$, in particular $r_1 > 1$. Since $1/2 = d_1/r_1 > d/r$ and $c_2/r \geq 1 - d/r$, we get $c_2/r > d/r$, and hence $r_1 r(c_2/r - d/r) = r_1(c_2 - d) \geq 2$. Thus $r_1 r(c_2/r - d/r + d_1/r_1 + e_1/r_1 - 1) \geq 2$. Therefore we obtain that $\sum_{j>1}\chi(E_1, E_j) \geq 2$.

We next treat the case that $e_1/r_1 < 1/2$. Assume that $-\sum_{i<j}\chi(E_i, E_j) = 1$. By the proof of Proposition 4.1, we get $0 < r_1^2(1 - 2e_1/r_1) \leq -\sum_{j>1}\chi(E_1, E_j) \leq -\sum_{j>i}\chi(E_i, E_j) = 1$. Hence $r_1^2(1 - 2e_1/r_1) = 1$, $r_1 r(d/r - c_2/r - d_1/r_1 - e_1/r_1 + 1) = 0$ and $-\sum_{1<i<j}\chi(E_i, E_j) = 0$, which imply that $r_1 = 1$, $e_1 = 0$, $d_1 = 1$ and $c_2 = d$. Since $F_2/F_1 \subset F_3/F_1 \subset \cdots \subset F_s/F_1 = E/F_1$ is the Harder-Narasimhan filtration of E/F_1, if $s > 2$, then by the proof of Proposition 4.1 and Remark 4.2, we see that $-\sum_{1<i<j}\chi(E_i, E_j) > 0$. Hence $s = 2$. Thus the Harder-Narasimhan filtration of E is

$$0 \subset \mathcal{O}_X(f) \subset E\,. \tag{4.2}$$

Conversely, if r, c_1 and c_2 satisfy the above condition, then we can easily show that codim $(Q^x \backslash Q^s) = 1$ (in the case where $c_2 = d$, a general member of codimension 1 components is a quotient $\mathcal{O}_X(-m)^{\oplus N} \to E$ such that the Harder-Narasimhan filtration of E is (4.2).)

Proposition 4.3. *There is a stable sheaf E of rank $r \geq 2$ with $c_1(E) = 0$ and $c_2(E) = c_2$ if and only if $c_2 \geq r$.*

Proof. Let E be a stable sheaf of rank r with $c_1(E) = 0$ and $c_2(E) = c_2$. Since $H^0(X, E) = 0$, we get $0 \geq \chi(E) = r - c_2$. Conversely, we assume that $c_2 \geq r$. We shall use the same method as in the proof of Proposition 4.1. Let $F\colon 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ be the Harder-Narasimhan filtration of a torsion free sheaf E of rank r with $c_1(E) = 0$ and $c_2(E) = c_2$ with respect to μ-semi-stability. If $e_1/r_1 \geq 1/2$ and $e_s/r_s \geq 1/2$, then $-\chi(E_1, E_s) = r_1 r_s(d_1/r_1 - d_s/r_s + e_1/r_1 + e_s/r_s -$

$1) \geq 2$. Hence we assume that $e_1/r_1 < 1/2$ or $e_s/r_s < 1/2$. We may assume that $e_1/r_1 < 1/2$ (the other case is similar). Then we see that $0 < r_1^2(1 - 2e_1/r_1) = r_1 r(d/r - d_1/r_1 - e/r - e_1/r_1 + 1) - \sum_{j>1} \chi(E_1, E_j)$. Since $r_1 r(d/r - d_1/r_1) = -rd_1 \leq -2$ and $e/r \geq 1$, we obtain that $-\sum_{j>1} \chi(E_1, E_j) \geq 2$. Therefore, there is a μ-semi-stable sheaf E. Let $F: 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ be the Harder-Narasimhan filtration or a Jordan-Hölder filtration of E with respect to semi-stability or stability respectively. Then $r_1^2 \leq -\sum_{j>1} \chi(E_1, E_j)$. Therefore there is a stable sheaf.

Remark 4.3. By the proof of this proposition, we can easily show that $-\sum_{i<j} \chi(E_i, E_j) = 1$ if and only if (i) $s = 2$, $r_1 = 1$, $e_1 = 0$ and $e = r$, or (ii) $r = e = 2$ and $r_1 = e_1 = 1$. Thus the Harder-Narasimhan filtration of E is $0 \subset \mathcal{O}_X \subset E$.

Definition 4.1. For a pair $(r, \mu) \in H^0(X, \mathbf{Q}) \times H^2(X, \mathbf{Q})$ with $0 \leq \deg \mu < 1$, we set

$$\delta(r, \mu) := |\deg \mu - 1/2| + 1/2 .$$

This definition is similar to the definition of $\delta(r, \mu)$ in [D-L]. Then we obtain similar result as in [D1].

Theorem 4.4. *We assume that* $0 \leq \deg c_1/r < 1$ *and* $c_2/r > 1/2$. *Then*

$$Pic\,(\overline{M}(r, c_1, c_2)) = \begin{cases} \mathbf{Z}^{\oplus 2} & \text{for } c_2/r = \delta(r, c_1/r)\,, \\ \mathbf{Z}^{\oplus 3} & \text{for } c_2/r > \delta(r, c_1/r)\,. \end{cases}$$

Proof. We first assume that $c_2/r > \delta(r, c_1/r)$. Then, in the same way as in the proof of Lemma 3.4, we obtain that $\operatorname{codim}(\mathbf{P}_0 \backslash \mathbf{P}_0^s) \geq 2$ for $c_1' = c_1 + rnf$, $n \gg 0$. Hence the proof is the same as that in Proposition 3.12. We next assume that $c_2/r = \delta(r, c_1/r)$. We shall use the quot scheme Q_1 and the notation in Lemma 3.4 for $c_1' = c_1 + rnf$, $n \gg 0$. By virtue of the above propositions, $Pic^{PGL(N)}(Q_1^{ss}) = Pic\,(\overline{M}(r, c_1', c_2)) = Pic\,(\overline{M}(r, c_1, c_2))$ even if $\operatorname{codim}(Q_1^{ss} \backslash Q_1^s) = 1$, (we use [D-N, Theorem 2.3] and the proof of [D-N, Prposition 4.1]). For simplicity, we set $Q_3 := Q_2(0)$ and $T_3 := T(0)^0$. By our choice of n, $Q_3^{ss} = \{y \in Q_1^{ss} | R^1\pi_*\mathcal{E} = 0\}$. Lemma 3.1 implies that $Pic^{PGL(N)}(Q_3^{ss}) = Pic^{PGL(N)}(Q_1^{ss}) = Pic\,(\overline{M}(r, c_1, c_2))$. We get the following commutative diagram.

$$(4.3) \qquad \begin{array}{ccc} Pic^{PGL(N)}(Q_3^{ss}) & \longrightarrow & Pic^{PGL(N)}(T_3^{ss}) \\ \| & & \uparrow \\ Pic\,(\overline{M}(r, c_1, c_2)) & \longrightarrow & Pic\,(\mathbf{P}_0^{ss}) \end{array}$$

Since $Pic^{PGL(N)}(Q_3^{ss}) \to Pic^{PGL(N)}(T_3^{ss})$ is injective, $Pic\,(\overline{M}(r, c_1, c_2)) \to Pic\,(\mathbf{P}_0^{ss})$ is also injective. Then in the same way as in Proposition 3.14, we can prove that $Pic\,(\overline{M}(r, c_1, c_2))$ is generated by the image of $\kappa: K(r, c_1, c_2) \to Pic\,(\overline{M}(r, c_1, c_2))$. For simplicity, we set $\mathcal{G} := \mathcal{E}[-nf]_{|Q_3 \times X}$. We denote the codimension 1 component of $Q_3 \backslash Q_3^{ss}$ by D. If $r/2 < d < r$, then $\kappa_{\mathcal{G}}(-\mathcal{O}_X(-f)) = (\det p_{Q_3!}\mathcal{G}[-f])^\vee$ is the divisor $\mathcal{O}_{Q_3}(D)$. In fact, by (4.2) and the Riemann-Roch theorem, we get that $D \subset \{y \in Q_3 | H^0(X, \mathcal{G}_y(-f)) \neq 0\} = \{y \in Q_3 | H^1(X, \mathcal{G}_y(-f)) \neq 0\}$. Since $H^2(X, \mathcal{G}_y(-f)) = 0$, the base change theorem implies that $\det(R^1 p_{Q_3 *}\mathcal{G}[-f])$ is a multiple of $\mathcal{O}_{Q_3}(D)$. Since $p_{Q_3 *}\mathcal{G}[-f] = 0$ and $-\mathcal{O}_X(-f)$ is a primitive element of $K(r, c_1, c_2)$, we get $\mathcal{O}_{Q_3}(D) \cong \kappa_{\mathcal{G}}(-\mathcal{O}_X(-f))$. Therefore, we obtain that $Pic(\overline{M}(r, c_1, c_2)) \cong K(r, c_1, c_2)/\mathbf{Z}\mathcal{O}_X(-f) \cong \mathbf{Z}^{\oplus 2}$. For an integer d with $0 < d \leq r/2$, we see that $\mathcal{O}_{Q_3}(D) = \det(-p_{Q_3 !}(\mathcal{G}, \mathcal{O}_{Q_3 \times X}))$. By the relative duality, we get $\det(-p_{Q_3 !}(\mathcal{G}, \mathcal{O}_{Q_3 \times X})) \cong \det(p_{Q_3 !}(\mathcal{G}[K_X]))$. Hence $Pic(\overline{M}(r, c_1, c_2)) \cong K(r, c_1, c_2)/\mathbf{Z}K_X \cong \mathbf{Z}^{\oplus 2}$. If $d = 0$, then in the same way as above, we see that $\mathcal{O}_{Q_3}(D) = \kappa_{\mathcal{G}}(-\mathcal{O}_X)$ and $Pic(\overline{M}(r, c_1, c_2)) \cong \mathbf{Z}^{\oplus 2}$.

Remark 4.4. If $c_2/r = 1/2$, then the following holds.

$$M(r, r/2f, r/2) = \begin{cases} \mathbf{P}^1, & r = 2, \\ \emptyset, & \text{otherwise}. \end{cases}$$

In fact $M(2, 1/2f, 1/2)$ is a smooth projective unirational curve. Hence $M(2, 1/2f, 1/2) = \mathbf{P}^1$. In the notation of Lemma 1.4, we see that Q is irreducible (cf. Remark 1.6) and $\operatorname{codim}(Q \backslash Q^s) = 0$, and hence we get $M(r, r/2f, r/2) = \emptyset$.

5. Appendix

5.1. We shall slightly generalize Theorem 0.1.

Proposition 5.1. *Let H be an ample divisor such that $(K_X + f, H) < 0$. Then $Pic(M_H(r, c_1, c_2)) \cong Pic(M(r, c_1, c_2))$.*

Proof. (1) We shall first show that $W_1 := M_H(r, c_1, c_2) \backslash M(r, c_1, c_2)$ is at least of codimension 2. Let l be a fibre of π and E an element of W_1. Since the locus of E such that E is not locally free on a neighborhood of l is at least of codimension 2, we may assume that E is locally free on a neighborhood of l. Since $(K_X + l, H) < 0$, we see that $\operatorname{Ext}^2(E, E(-l)) \cong \operatorname{Hom}(E, E(l + K_X))^\vee = 0$. Hence the restriction map: $\operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1(E_{|l}, E_{|l})$ is surjective. By Remark 1.2, if $E_{|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r}$, then the Harder-Narasimhan filtration, or a Jordan-Hölder filtration of E with respect to $C_0 + nf$, $n \gg c_2$: $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ satisfies that $(c_1(F_i), f) = 0$, $1 \leq i \leq s$. Then we get that $(\mu(F_1), C_0) \geq (\mu(E), C_0)$ and if $(\mu(F_1), C_0) = (\mu(E), C_0)$, then $\chi(F_1)/\operatorname{rk}(F_1) \geq \chi(E)/\operatorname{rk}(E)$. Then it is easy to see that E is not stable with respect to H. Therefore $E_{|\pi^{-1}(\eta)} \not\cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus r}$. We set $W_1^0 := \{E \in W_1 | E_l \cong \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1) \oplus \mathcal{O}_l^{\oplus(r-2)}\}$. Applying deformation theory to $E_{|l}$, we get that $W_1 \backslash W_1^0$ is at least of codimension 2 in $M_H(r, c_1, c_2)$. Hence we shall compute $\operatorname{codim} W_1^0$. For an element E of W_1^0, there is a filtration $F: 0 \subset F_1 \subset F_2 \subset F_3 = E$ such that (i) $E_i := F_i/F_{i-1}$ are torsion free for $1 \leq i \leq 3$, and (ii) $E_{1|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}(1)$, $E_{2|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}^{\oplus(r-2)}$ and $E_{3|\pi^{-1}(\eta)} \cong \mathcal{O}_{\pi^{-1}(\eta)}(-1)$. We call this filtration the Harder-Narasimhan filtration of E with respect to f. By (ii), we get $\operatorname{Ext}^2(E_j, E_i) = 0$ for $j > i$, and hence $\operatorname{Ext}^2_{F,-}(E, E) = 0$. By using [D-L, Proposition 1.3, 1.5, and 1.7], we see that

$$\begin{aligned}\operatorname{codim} W_1^0 &= \max_E \dim \operatorname{Ext}^1_{F,+}(E, E)\\ &\geq \max_E \left\{-\sum_{i=0}^{2}(-1)^i \dim \operatorname{Ext}^i_{F,+}(E, E)\right\}\\ &= \max_E \left\{-\sum_{i<j} \chi(E_i, E_j)\right\},\end{aligned}$$

where E runs over all elements of W_1^0 and F is the Harder-Narasimhan filtration of E with respect to f. By using (ii), we see that $\operatorname{Ext}^k(E_i, E_{i+1}) = 0$ for $k \neq 1$. Hence $-\chi(E_i, E_{i+1}) \geq 0$. We shall show that $-\chi(E_1, E_3) \geq 2$. We set $c_1(E_3) - c_1(E_1) = -2C_0 + af$ and $H = mC_0 + nf$, where a, m, $n \in \mathbf{Z}$. Since E is stable with respect to H, $(-2C_0 + af, H) \geq 0$. Combining this and $(K_X + f, H) < 0$, we obtain that $e + 1 - (2 - 2g) < 2n/m \leq 2e + a$. Thus $a + e \geq 2g$. By using the Riemann-Roch theorem, we get that $-\chi(E_1, E_3) \geq a + e + 1 - g \geq g + 1 \geq 2$. Therefore $\operatorname{codim} W_1^0 \geq 2$.

(2) We shall next show that the codimension of $W_2 := M(r, c_1, c_2) \backslash M_H(r, c_1, c_2)$ is at least 2. Let E be an element of W_2. Then $\operatorname{Ext}^2(E, E(-C_0)) \cong \operatorname{Hom}(E, E(C_0 + K_X))^\vee = 0$. In the same way as in the proof of (1), we may assume that E is locally free on a neighborhood of C_0. Then the restriction map: $\operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1(E_{|C_0}, E_{|C_0})$ is surjective. Let $F: 0 \subset F_1 \subset \cdots \subset F_{s-1} \subset F_s = E$ be the Harder-Narasimhan filtration or a Jordan-Hölder filtration of E with respect to H. Then $(F_i, f) \leq 0$ for $1 \leq i \leq s$ and $(F_j, f) < 0$ for some j. $(K_X + f, H) < 0$ implies that $n > 0$ and that if $e \geq 1$, then $n > m$. Since $(\mu(F_j), H) \geq (\mu(E), H) = (\mu(E), mC_0)$, we get $(\mu(F_j), mC_0) > (\mu(E), mC_0)$. Thus $\deg(\mu(F_{j|C_0})) > \deg(\mu(E_{|C_0}))$, moreover if $e \geq 1$ then $\deg(\mu(F_{j|C_0})) > \deg(\mu(E_{|C_0})) + 1/\operatorname{rk}(F_j)$. Applying deformation theory to $E_{|C_0}$, we see that

$$\begin{aligned}\operatorname{codim}(W_2) &\geq \dim \operatorname{Ext}^1_{F,+}(E_{|C_0}, E_{|C_0})\\ &\geq \operatorname{rk}(F_j)\operatorname{rk}(E)\{\deg(\mu(F_{j|C_0})) - \deg(\mu(E_{|C_0}))\} + g - 1.\end{aligned}$$

Therefore if $\operatorname{codim}(W_2) = 1$, then $g = 1$, $e \leq 0$, $s = 2$ and $\operatorname{rk}(E)\deg(F_{j|C_0}) - \operatorname{rk}(F_j)\deg(E_{|C_0}) = 1$. We set $gr_i(E) = E_i$, $i = 1, 2$. Then $-\chi(E_1, E_2) = \operatorname{rk}(E_1)\operatorname{rk}(E_2)(\Delta(E_1) + \Delta(E_2) - (\mu(E_1) - \mu(E_2))^2/2 - (K_X, \mu(E_1) - \mu(E_2))/2)$. By the Bogomolov-Gieseker inequality, we get $\Delta(E_i) \geq 0$. Since $K_X = -2C_0 - ef$ in $NS(X)$ and $e \leq 0$, $(K_X, \mu(E_2) - \mu(E_1)) \geq (-2C_0, \mu(E_2) - \mu(E_1)) \geq 2/\operatorname{rk}(E_1)\operatorname{rk}(E_2)$. It is easy to see that $((\mu(E_1) - \mu(E_2))^2) < 0$. Therefore $-\chi(E_1, E_2) \geq 2$. Hence we obtain that $\operatorname{codim} W_2 \geq 2$.

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