J. Math. Kyoto Univ. (JMKYAZ)
36-3 (1996) 619-640

Quasi Sure quadratic variations of two parameter smooth martingales on the Wiener space*†

By

Zong-xia LIANG

1. Introduction

Stimulated by Malliavin calculus, the theory of quasi sure analysis of Wiener functionals has been extensively developed (cf. [4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 19], etc). Recently, J. Ren (cf. [17]) has studied quasi sure properties of quadratic variation of "*smooth martingales*", a notion introduced by P. Malliavin and D. Nualart [8], and his results are concentrated on studying one parameter smooth martingales and two parameter smooth strong martingales. In this paper we generalize his results in more general setting and study the quasi sure properties of two parameter smooth martingales which are not necessary strong martingale in general. This situation is much more difficulty to handle, when Malliavin calculus is involved, because the two parameter stochastic differentiation rules (cf. [1, 21, 24]) and the representation of two parameter square integrable martingales involve "*stochastic integral of the second type*", i.e., $\iint_{\Pi\times\Pi} f(\xi, \eta)dW_\xi dW_\eta$ (cf. [1]). Now let us state our results in more details.

Let N be a two parameter smooth martingale, then by [1] and [21], for each $z \in \Pi = [0, 1]^2$, N can be represented as a sum of stochastic integral of the first type and stochastic integral of the second type,

$$N_z = \int_{R_z} \phi(\eta)dW_\eta + \iint_{R_z\times R_z} \psi(\xi, \eta)dW_\xi dW_\eta$$

where W is a two parameter Wiener process and vanishing on the axes. Let $\bar{N}_z = \int_{R_z} \phi(\eta)dW_\eta$ and $M_z = \iint_{R_z\times R_z} \psi(\xi, \eta)dW_\xi dW_\eta$. It is well known that the quadratic variation processes of $\bar{N}$, M and N are given by $\langle \bar{N}\rangle_z = \int_{R_z} \phi(\eta)^2 d\eta$, $\langle M\rangle_z = \iint_{R_z\times R_z} \psi(\xi, \eta)^2 d\xi d\eta$ and $\langle N\rangle^z = \int_{R_z} \phi(\eta)^2 d\eta + \iint_{R_z\times R_z} \psi(\xi, \eta)^2 d\xi d\eta$ respectively. And by [21], we have for each z that $\langle \bar{N}, M\rangle_z = 0$. We shall prove

* Work supported by NSF and Doctoral Research Foundation of China

† Typeset by latex

Communicated by Prof. K. Ueno, October 11, 1995
Revised February 16, 1996

that the quadratic variation process $\langle N \rangle$ of two parameter smooth martingale N admits an ∞-modification which can be constructed as quasi sure limit of sums of form $\sum_{ij} N(\Delta_{ij})^2$. Our main tool is the quasi sure version of Kolmogrov criterion for the continuity of trajectories of stochastic processes established by J. Ren (cf. [4, 16, 18]).

The organization of this paper is as follows. In section 2 we will briefly recall some basic facts about Malliavin calculus for two parameter Wiener functionals and two parameter processes. In section 3 we will study the quasi sure properties of two parameter smooth martingale $M_z = \iint_{R_z \times R_z} \psi(\xi, \eta) dW_\xi dW_\eta$. In section 4 we will give the main results.

2. Malliavin calculus

The extension of Malliavin Calculus to the case of two parameter Wiener functionals is straightforward. We introduce here those notations and concepts which are necessary for finishing our main results. Let $\Pi = [0, 1]^2$ be our parameter space, $z_1 = (s_1, t_1)$ and $z_2 = (s_2, t_2)$ be two points in Π, we write $z_1 \le z_2$ *iff* $s_1 \le s_2$ and $t_1 \le t_2$, $z_1 < z_2$ *iff* $s_1 < s_2$ and $t_1 < t_2$, and $z_1 \barwedge z_2$ *iff* $s_1 \le s_2$ and $t_1 \ge t_2$. If $z_1 < z_2$, $(z_1, z_2]$ will denote the rectangle $\{z \in \Pi; z_1 < z \le z_2\}$. We put $R_z = [0, z]$, and $z_1 \vee z_2 = (\max(s_1, s_2), \max(t_1, t_2))$. The increment of a function $f: \Pi \to R$ on a rectangle $(z_1, z_2]$ is given by $f((z_1, z_2]) = f(s_1, t_1) - f(s_1, t_2) - f(s_2, t_1) + f(s_2, t_2)$.

Let $(\mathbf{X}, \mathscr{F}, \mu)$ be the canonical probability space associated with the two parameter Wiener process W, that is, $\mathbf{X} = \{\omega: \Pi \to R$, continuous, vanishing on the axes$\}$, μ is the two parameter Wiener measure (cf. [14, 25]), and $\mathscr{F}$ is the completion of the Borel σ-field of $\mathbf{X}$ with respect to μ, $\{\mathscr{F}_z\}$ is the filtration generated by the functions $\{\omega(r), \omega \in \mathbf{X}, r \le z\}$ and the null sects of $\mathscr{F}$. Let $\mathscr{F}_z^1 = \mathscr{F}_{(s,1)}$, $\mathscr{F}_z^2 = \mathscr{F}_{(1,t)}$ for $z = (s, t) \in \Pi$, then $\{\mathscr{F}_z\}_{z \in \Pi}$, $\{\mathscr{F}_z^1\}_{z \in \Pi}$ and $\{\mathscr{F}_z^2\}_{z \in \Pi}$, satisfy the usual conditions of [1]. Let $\mathbf{H} = \left\{\omega \in \mathbf{X}, \text{ there exists } \frac{\partial^2 \omega}{\partial s \partial t} \in L^2(\Pi) \text{ such that } \omega(s, t) = \int_0^s \int_0^t \frac{\partial^2 \omega}{\partial u \partial w} du dv, \text{ for any } z = (s, t) \in \Pi \right\}$ be its Cameron–Martin subspace. $\mathbf{H}$ is a Hilbert space with the inner product $\langle \omega_1, \omega_2 \rangle_{\mathbf{H}} = \int_\Pi \frac{\partial^2 \omega_1}{\partial s \partial t} \cdot \frac{\partial^2 \omega_2}{\partial s \partial t} ds dt$. Then $(\mathbf{X}, \mathbf{H}, \mu)$ forms a classical two parameter Wiener space (cf. [12, 13]).

A smooth functional is a map $F: \mathbf{X} \to R$ such that there exists some $n \ge 1$ and C^∞-function f on R^n with the following properties:

(i) f and all its derivatives have at most polynomial growth order;

(ii) $F(\omega) = f(\omega(z_1), \ldots, \omega(z_n))$ for some $z_1, \ldots, z_n \in \Pi$.

The derivative ∇F of a smooth functional F along any vector $h \in \mathbf{H}$ is given by

$$\langle \nabla F, h \rangle_{\mathbf{H}} = \sum_{k=1}^{n} \frac{\partial g}{\partial x_k}(\omega(z_1), \ldots, \omega(z_n))h(z_k)$$

$$= \int_{\Pi} \xi(\tau)\dot{h}(r)dr \tag{2.1}$$

Where $\xi(r) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\omega(z_1), \ldots, \omega(z_n))I_{R_{z_i}}(r)$, $\dot{h}(z) = \frac{\partial^2 h(s, t)}{\partial s \partial t}$, for $z = (s, t)$.

Similar to [12], we can define the Nth derivative $\nabla^N F$ of F, it determines a square integrable random variable taking values on the Hilbert space $\mathbf{H}^{\otimes N}$ of all continuous N-multilinear forms on $\mathbf{H} \otimes \overset{(N)}{\ldots} \otimes \mathbf{H}$ with the Hilbert–Schmidt norm $\|\cdot\|_{HS}$ (for details cf. [13]). We define Ornstein–Uhlenbeck operator $\mathbf{L}$ on smooth functionals as follows:

$$\mathbf{L}F(\omega) = \sum_{l,k=1}^{n} \frac{\partial^2 f}{\partial x_l \partial x_k}(\omega(z_1), \ldots, \omega(z_n))\Gamma(z_i, z_j)$$

$$- \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\omega(z_1), \ldots, \omega(z_n))\omega(z_k)$$

Where $\Gamma(z_i, z_j) = \min(x_i, x_j) \cdot \min(y_i, y_j)$ if $z_i = (x_i, y_i)$ $i = 1, \ldots, n$. For any integer $r \geq 0$ and any real number $p > 1$, we set

$$\|F\|_{p,2r} = \|(I - \mathbf{L})^r F\| \qquad \textit{and}$$

$$\|F\|'_{p,2r} = \|F\|_p + \|\nabla^{2r} F\|_p$$

$$W_\infty = \bigcap_{p>1, r \geq 0} W^p_{2r}$$

Where W^p_{2r} is the completion of set of smooth functionals with respect the norm $\|\cdot\|_{p,2r}$ (i.e., W^p_{2r} is the Sobolev space of order $2r$ and of power p over $\mathbf{X}$), then we have (cf. [3, 12]) that for any smooth functional F there exists constant c and c' such that

$$c\|F\|_{p,2r} \leq \|F\|'_{p,2r} \leq c'\|F\|_{p,2r} \tag{2.2}$$

and W_∞ is a nice space in the sense that:

(i) W_∞ is an algebra;

(ii) if $F, G \in W_\infty$, then $\mathbf{L}F \in W_\infty$ and $\langle \nabla F, \nabla G \rangle_{\mathbf{H}} \in W_\infty$;

(iii) if $F \in W_\infty$, and let $u: R^d \to R$ be a C^∞-function such that u and all its derivatives have at most polynomial growth order. If we set $F = (F_1, \ldots, F_n)$ then $u \circ F \in W_\infty$ and the following differentiation rules hold:

$$\nabla(u \circ F) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(F) \cdot \nabla F_i \tag{2.3}$$

$$\mathbf{L}(u \circ F) = \left(\sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j} \circ F\right) \cdot \langle \nabla F_i, \nabla F_j \rangle_{\mathbf{H}} + \left(\sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \circ F\right) \cdot \mathbf{L}F_i \tag{2.4}$$

In [1, 21] Cairoli and Walsh, Wong and Zakai introduced the concepts of two parameter martingales, i-martingales ($i = 1, 2$), strong (weak) martingales and stochastic surface integral. With these concepts it is well known that a strong martingale is a martingale, a process is a martingale *iff* it is both an adapted 1-martingale and an adapted 2-martingale, and adapted 1- and 2-martingales are also weak martingales. By [1], for any square integrable martingale N, there exists $\{\mathscr{F}_z^i\}$-predictable increasing process $[N]^i$ ($i = 1, 2$), and $\{\mathscr{F}_z\}$-predictable increasing process $\langle N \rangle$ such that $(N)^2 - [N]^i$ is an $\bar{i}$-martingale ($\bar{i} = 1, 2$ *when* $i = 2, 1$, respectively) and $(N)^2 - \langle N \rangle$ is a weak martingale.

Let $s_i^n = \frac{i}{2^n}$, $t_j^n = \frac{j}{2^n}$, $(i, j = 1, 2, \ldots, 2^n)$, for any $z = (s, t) \in \Pi$. We shall denote the rectangles $\left(\frac{i}{2n} \wedge s, \frac{i+1}{2^n} \wedge s\right] \times \left(\frac{j}{2^n} \wedge t, \frac{j+1}{2^n} \wedge t\right]$, $\left(\frac{i}{2^n} \wedge s, \frac{i+1}{2^n} \wedge s\right] \times \left(0, \frac{j+1}{2^n} \wedge t\right]$ and $\left(0, \frac{j+1}{2^n} \wedge s\right] \times \left(\frac{j}{2^n} \wedge t, \frac{j+1}{2^n} \wedge t\right]$ by $\triangle$, $\triangle_1$ and $\triangle_2$ respectively. By the definition of stochastic integral of the second type and stochastic **Fubini's** theorem (cf. [1, Theorem 2.6]), we have the following.

Proposition 2.1. *For the square integrable martingale* $M_z = \iint_{R_z \times R_z} \phi(\xi, \eta) \cdot dW_\xi dW_\eta$ *we have that*

$$[M]_z^1 = \int_{R_z} \left\{ \int_{R_z} \phi(\xi, \eta) dW_\xi \right\}^2 d\eta \tag{2.5}$$

$$[M]_z^2 = \int_{R_z} \left\{ \int_{R_z} \phi(\xi, \eta) dW_\eta \right\}^2 d\xi \tag{2.6}$$

$$M(\triangle) = \iint_{\triangle_1 \times \triangle_2} \phi(\xi, \eta) dW_\xi dW_\eta \tag{2.7}$$

$$[M]^1(\triangle) = \int_{\triangle_1} \left\{ \int_{\triangle_2} \phi(\xi, \eta) dW_\xi \right\}^2 d\eta \tag{2.8}$$

$$[M]^2(\triangle) = \int_{\triangle_2} \left\{ \int_{\triangle_1} \phi(\xi, \eta) dW_\eta \right\}^2 d\xi \tag{2.9}$$

Moreover we note that the parameter space $T = [0, 1]$ of [20] can be replaced by $T = [0, 1]^2 = \Pi$, then by the definition of stochastic integral and stochastic Fubini's theorem (cf. [1, 24]) we easily deduce from the proposition 5.1 and proposition 5.8 of [20] and (2.2), (2.3) that

$$\mathbf{L}\left(\int_{R_z} f(\eta) dW_\eta \right) = \int_{R_z} (\mathbf{L}f - f)(\eta) dW_\eta \tag{2.10}$$

$$\mathbf{L}\left(\iint_{R_z \times R_z} f(\xi_1, \xi_2) dW_{\xi_1} dW_{\xi_2} \right) = \iint_{R_z \times R_z} (\mathbf{L}f - 2f)(\xi_1, \xi_2) dW_{\xi_1} dW_{\xi_2} \tag{2.11}$$

$$\mathbf{L}\left(\iint_{R_z \times R_z} f(\xi_1, \xi_2) d\xi_1 dW_{\xi_2} \right) = \iint_{R_z \times R_z} (\mathbf{L}f - f)(\xi_1, \xi_2) d\xi_1 dW_{\xi_2} \tag{2.12}$$

We don't give the proofs of (2.10) (2.11) (2.12), for it is entirely similar to the one parameter case (cf. [23]).

Given an open set $\mathbf{O}$ in $\mathbf{X}$, its (r, p)-capacity is defined by (cf. [3, 4, 7, 19])

$$C_{r,p}(\mathbf{O}) \equiv \inf\{\|u\|_{p,2r};\ u \in \mathbf{W}_{2r}^{p},\ \mu \geq 1,\ \mu\text{-}a.e.\ on\ \mathbf{O}\}$$

and for any subset A of $\mathbf{X}$, the capacity is defined to be

$$C_{r,p}(A) \equiv \inf\{C_{r,p}(O);\ O\ is\ open\ and\ O \supset A\}$$

If $C_{r,p}(A) = 0$ for all $p \geq 2$ and for all $r \in \mathbf{N}$, then A is called a slim set (cf. [7]). If some property holds except on a slim set, then we say that it holds quasi surely (abb. q.s.). It is well known (cf. [4]) that for any element f in $\mathbf{W}_\infty = \bigcap_{p>1, r\geq 0} \mathbf{W}_{2r}^{p}$, we can find a function f^* such that (1) $f^* = f$ μ-*a.e.*; (2) for each pair (p, r) and any $\varepsilon > 0$, there exists an open set O with $C_{r,p}(O) < \varepsilon$ such that f^* is continuous on $\mathbf{X}\backslash O$. f^* is referred to as redefinition of f. Obviously, any two redefinitions of a function coincide except on a slim set. Any function with property (2) above is called ∞-quasi continuous. The important tool we will also use is the concept of ∞-modification of a random field.

Definition 2.1 (cf. [4, 18]). *Let $\{X(t), t \in D\}$ be a random field, where D is a domain in R^d. A random field $\{\tilde{X}(t), t \in D\}$ is called an ∞-modification of $\{X(t), t \in D\}$ if*
(1) *$\tilde{X}(t) = X(t)$ a.e. for each $t \in D$;*
(2) *$\tilde{X}(\cdot, \omega)$ are continuous in D q.s.;*
(3) *$\tilde{X}(t, \cdot)$ is ∞-quasi continuous for each $t \in D$.*

Theorem 2.1 (Quasi sure version of Kolmogrov criterion) (cf. [18]). *Suppose that for any pair (p, r) we can find an even number $\beta(p, r)$ and two positive constant $c = c(p, r)$, $\alpha = \alpha(p, r)$ such that*
(1) *$X(t) \in \mathbf{W}_{2r}^{p}$ for each $t \in D$;*
(2) *$(X(t) - X(s))^\beta \in \mathbf{W}_{2r}^{p}$ for each $(t, s) \in D \times D$;*
(3) *$\|(X(t) - X(s))^\beta\|_{p,2r} \leq c\|t - s\|^{\alpha+d}$ for each $(t, s) \in D \times D$,*
where $\|t - s\| = \sum_{j=1}^{d} \|t_j - s_j\|$. Then $\{X(t), t \in D\}$ has an ∞-modification.

In addition, we quote the following theorems which will be used later.

Theorem 2.2 (cf. [4, 18]). *If two processes $X_1(t, \omega)$, $X_2(t, \omega)$ satisfy the conditions of Theorem 2.1 above and if $X_1(t, \omega) \leq X_2(t, \omega)$ a.e. for every t, then $\tilde{X}_1(t, \omega) \leq \tilde{X}_2(t, \omega)$ q.s. for all $t \in D$.*

Theorem 2.3 (Faá di Bruno's inequality) (cf. [4, 16]). *For any fixed $p \geq 2$, $r \in \mathbf{N}$ and $n \geq 2r$, there exists a constant $c = c(n, p, r)$ such that*

$$\|g^n\|_{p,2r} \leq c\|g\|_{4r^2p,2r}^{2r} \max_{0\leq\alpha\leq 2r} [\mathbf{E}\|g\|^{(n-\alpha)2rp}]^{1/2rp}$$

Theorem 2.4. *Suppose that $u \in \mathbf{W}_{2r}^{p}$, u^* is its refinement, then*

$$C_{p,r}(\|u^*\| > \varepsilon) \leq \frac{1}{\varepsilon}\|u\|_{p,2r}$$

for any $\varepsilon > 0$.

Now we define two parameter smooth martingales. As in introduction, for any square integrable martingale N which vanishes on the axes, we have

$$N_z = \phi \cdot W_z + \psi \cdot WW_z \equiv \int_{R_z} \phi(\eta) dW_\eta + \iint_{R_z \times R_z} \psi(\xi, \eta) dW_\xi dW\eta \tag{2.13}$$

where $\phi \in L^2_W \equiv \left\{ f: f \text{ is an } \{\mathscr{F}_z\}\text{-predictable process and } \mathbf{E}\left\{\int_{R_z} |f(\xi)|^2 d\xi\right\} < +\infty, \text{ for any } z \in \Pi \right\}$, $\psi \in L^2_{WW} \equiv \left\{ f: f = \{f(\xi, \eta), \xi, \eta \in \Pi\} \text{ satisfies: (1) } f \text{ is predictable process (cf. [1]), (2) } f(\xi, \eta) = 0 \text{ unless } \xi \wedge \eta, \text{ (3) } \mathbf{E}\left\{\iint_{R_z \times R_z} f(\xi, \eta)^2 d\xi d\eta\right\} < +\infty \text{ for } z \in \Pi \right\}$.

Following P. Malliavin and D. Nualart [8], we say that N represented as (2.13) is smooth if the following conditions are fulfilled:

(C.1) $\phi(z) \in \mathbf{W}_\infty$ and $\psi(\xi, \eta) \in \mathbf{W}_\infty$ for almost $0 \leq z \leq (1, 1)$, and $0 \leq \xi, \eta \leq (1, 1)$.

(C.2) $\int_\Pi \|\phi\|^p_{p,2r} d\eta + \iint_{\Pi \times \Pi} \|\psi(\xi, \eta)\|^p_{p,2r} d\xi d\eta < +\infty$ for all p, r.

3. Quasi sure analysis on two parameter smooth martingale M

Analogously to [8, Theorem 4.2], we have the following.

Theorem 3.1. *Let* $M_z \equiv \psi \cdot WW_z = \iint_{R_z \times R_z} \psi(\xi, \eta) dW_\xi dW_\eta$ *and* ψ *satisfies conditions* (C.1) (C.2), *then*

(i) $M_z \in \mathbf{W}_\infty$ *for all* $z \in \Pi$;

(ii) *There exists a decreasing sequence* $\{O_n, n \geq 1\}$ *of open subsets of* $\mathbf{X}$ *and a function* $\tilde{M}: \left(\bigcup_{n \geq 1} O_n^c\right) \times \Pi \to R$ *such that*

   (a) $\tilde{M}$ *is continuous on* $O_n^c \times \Pi$, *for each* $n \geq 1$;

   (b) $C_{r,p}(O_n) \to 0$, *as* $n \to +\infty$, *for all* p, r;

   (c) $\tilde{M}_z = M_z$ *almost surely, for all* $z \in \Pi$.

Proof. For simplicity, all the constants depending only on M, p, but not on n and the parameter z, will be simply denoted by c. Note that $\langle M \rangle_z = \iint_{R_z \times R_z} \psi(\xi, \eta)^2 d\xi d\eta$. By (2.11), Burkholder's inequality for two parameter martingales (cf. [2, 5, 10, 11]) and Hölder's inequality there exists c such that

$$\sup_z \|M_z\|_{p,2}^p \leq c \iint_{\Pi\times\Pi} \|\psi(\xi,\eta)\|_{p,2}^p d\xi d\eta < +\infty$$

(by (C.1) (C.2))

and in the same way

$$\sup_z \|M_z\|_{p,r}^p \leq c \iint_{\Pi\times\Pi} \|\psi(\xi,\eta)\|_{p,r}^p d\xi d\eta < +\infty$$

for all p, r. In particular, (i) holds.

To prove (ii), for any $z, z' \in \Pi$, we only consider the case $z \leq z'$, and $z \,\overline{\wedge}\, z'$ can be considered in the same way as in previous case. For any $z \leq z'$, we have

$$M_{z'} - M_z = M(D_1) + M(D_2) \tag{3.1}$$

where $D_1 = (0, s'] \times (t, t']$, $D_2 = (s, s'] \times (0, t]$ for $z = (s, t)$, $z' = (s', t') \in \Pi$, and by (2.7) we have

$$M(D_1) = \iint_{D_1\times D_3} \psi(\xi,\eta) dW_\xi dW_\eta$$

$$M(D_2) = \iint_{R_{s't}\times D_2} \psi(\xi,\eta) dW_\xi dW_\eta$$

where $D_3 = (s, s'] \times (0, t']$. Using again Burkholder's inequality and Hölder's inequality we get

$$\begin{aligned}\|M(D_1)\|_{p,2}^p &\leq c\mathbf{E}\left(\iint_{D_1\times D_2} |3\psi - L\psi|^2 d\xi d\eta\right)^{p/2}\\ &\leq cm(D_1\times D_3)^{p/2-1} \iint_{D_1\times D_2} \|\psi\|_{p,2}^p d\xi d\eta\\ &\leq c|t-t'|^{p/2-1} \iint_{\Pi\times\Pi} \|\psi\|_{p,2}^p d\xi d\eta \end{aligned}\tag{3.2}$$

where m stands for Lebesgue measure on $\Pi \times \Pi$.

Similarly

$$\|M(D_2)\| \leq c|s-s'|^{p/2-1} \iint_{\Pi\times\Pi} \|\psi\|_{p,2}^p d\xi d\eta \tag{3.3}$$

A combination of (3.1), (3.2) and (3.3) implies

$$\|M_z - M_{z'}\|_{p,r}^p \leq c[|t-t'|^{p/2-1} + |s-s'|^{p/2-1}]$$

(by (C.1) (C.2))

And we can prove in the same way that

$$\|M_{s,t} - M_{s',t'}\|_{p,2r}^p \le c[|s - s'|^{p/2-1} + |t - t'|^{p/2-1}]$$

for all p, r. Hence by Theorem 2.1 and Theorem 2.3, the proof of (ii) is finished.

Q.E.D

Similar to the proof of Theorem 3.1 of [17] and the proof of (ii) above, we have the following.

Theorem 3.2. $\langle M \rangle = \left\{ \iint_{R_z \times R_z} \psi(\xi, \eta)^2 d\xi d\eta, z \in \Pi \right\}$ *admits an* ∞*-modification.*

We denote by M (resp $\langle M \rangle$) itself its ∞-modification $\tilde{M}$ (resp $\langle \tilde{M} \rangle$). The following theorem is our main result in this section.

Theorem 3.3. *The convergence*

$$\lim_{n\to\infty} \sum_{ij} M(\Delta_{ij}^n(s, t))^2 = \iint_{R_z \times R_z} \psi(\xi, \eta)^2 d\xi d\eta$$

holds uniformly in $z = (s, t) \in \Pi$, *q.s., where* $\Delta_{ij}^n(s, t) = (s_i, s_{i+1}] \times (t_j, t_{j+1}]$, $s_i = \frac{i}{2^n} \wedge s$, $t_j = \frac{j}{2^n} \wedge t$.

Proof. We define a random field parametrized by $[0, 1]^3$ as follows:

$$X(\xi, s, t) = \begin{cases} X(2^{-n}, s, t) + (\xi - 2^{-n})(2^{-n} - 2^{-(n+1)}) & \\ \quad \times (X(2^{-n}, s, t) - X(2^{-(n+1)}, s, t)), & \text{if } \xi \in [2^{-(n+1)}, 2^{-n}], \\ \iint_{R_z \times R_z} \psi(\xi, \eta)^2 d\xi d\eta, & \text{if } \xi = 0. \end{cases}$$

where $X(2^{-n}, s, t) = \sum_{ij} M(\Delta_{ij}^n)^2$, $z = (s, t)$. By Theorem 2.1, 2.2, 2.3, 2.4 and 3.2, it suffices to prove the following facts:

$$\sup_n \|X(2^{-n}, s', t') - X(2^{-n}, s, t)\|_p^p \le c[|s' - s|^{p/2-1} + |t' - t|^{p/2-1}] \tag{3.4}$$

$$\sup_z \|X(2^{-n}, s, t) - X(0, s, t)\|_p^p \le c 2^{-n(p/2-1)} \tag{3.5}$$

$$\sup_{\xi, z} \|X(\xi, z)\|_{p,2r}^p < +\infty \tag{3.6}$$

We divide the proof in three steps.

Proof of (3.4). By the definition of $z \le z'$ and $z \barwedge z'$, we need only to consider two cases $s < s'$, $t < t'$ and $s < s'$, $t > t'$. Since the methods are similar, we only prove (3.4) in the case of $s < s'$ and $t < t'$. Let $\bar{s}_n = \frac{[2^n s]}{2^n}$, $\bar{t}_n = \frac{[2^n t]}{2^n}$, we can reduce the following four cases: $(\bar{s}_n, \bar{t}_n) = (\bar{s}'_n, \bar{t}'_n)$, $(\bar{s}_n, \bar{t}_n) \le (\bar{s}_n, \bar{t}'_n)$, $(\bar{s}_n, \bar{t}_n) \le (\bar{s}'_n, \bar{t}_n)$ and $(\bar{s}_n, \bar{t}_n) < (\bar{s}'_n, \bar{t}'_n)$, so we can only prove (3.4) in two cases of $(\bar{s}_n, \bar{t}_n) = (\bar{s}'_n, \bar{t}'_n)$ and $(\bar{s}_n, \bar{t}_n) < (\bar{s}'_n, \bar{t}'_n)$.

In the first case, we have

$$\begin{aligned}
&\|X(2^{-n}, s, t) - X(2^{-n}, s', t')\|_p^p \\
&= \|M((z_n, z'])^2 - M((z_n, z])^2\|_p^p \\
&\leq \|M((\bar{s}_n, s] \times (t, t']) + M((s, s'] \times (\bar{t}_n, t'])\|_{2p}^p \\
&\leq \|M((\bar{s}_n, s] \times (t, t'])\|_{2p}^p + \|M((s, s'] \times (\bar{t}_n, t'])\|_{2p}^p
\end{aligned} \tag{3.7}$$

where $z_n = (\bar{s}_n, \bar{t}_n)$, $z = (s, t)$.

On the other hand, by (2.7) we have

$$M((\bar{s}_n, s] \times (t, t']) = \iint_{(R_{st'} \backslash R_{st}) \times (R_{st'} \backslash R_{\bar{s}_n t'})} \psi(\xi, \eta) dW_\xi dW_\eta$$

By the Burkholder's inequality for two parameter continuous martingales and proposition 2.4(b) of [1] we get

$$\begin{aligned}
\|M((\bar{s}_n, s] \times (t, t'])\|_{2p}^p &\leq c \left| \mathbf{E}\left(\iint_{(R_{st'} \backslash R_{st}) \times (R_{st'} \backslash R_{\bar{s}_n t'})} \psi(\xi, \eta) d\xi d\eta \right)^p \right|^{1/2} \\
&\leq c m(R_{st'} \backslash R_{st})^{p-1/2} m(R_{st'} \backslash R_{\bar{s}_n t'})^{p-1/2} \left| \iint_{II^2} \mathbf{E} |\psi|^{2p} d\xi d\eta \right|^{1/2} \\
&\leq c |t - t'|^{p/2-1}
\end{aligned} \tag{3.8}$$

Similarly

$$\|M((s, s'] \times (\bar{t}_n, t'])\|_{2p}^p \leq c |s - s'|^{p/2-1} \tag{3.9}$$

Therefore we deduce from (3.7), (3.8) and (3.9) that

$$\|X(2^{-n}, s, t) - X(2^{-n}, s', t')\|_p^p \leq c[|s - s'|^{p/2-1} + |t - t'|^{p/2-1}] \tag{3.10}$$

For the second case, we have

$$\begin{aligned}
&X(2^{-n}, s', t') - X(2^{-n}, s, t) \\
&= M((z_n, z_n'])^2 - M((z_n, z])^2 + \sum_{i=[2^n s]}^{[2^n s']-1} \sum_{j=0}^{[2^n t]-1} M(\Delta_{ij}^n)^2 \\
&\quad + \sum_{i=0}^{[2^n s]-1} \sum_{j=[2^n t]}^{[2^n t']-1} M(\Delta_{ij}^n)^2 + \sum_{i=[2^n s]}^{[2^n s']-1} \sum_{j=[2^n t]}^{[2^n t']-1} M(\Delta_{ij})^2 \\
&\equiv a_1^n + a_2^n + a_3^n + a_4^n
\end{aligned} \tag{3.11}$$

where $\Delta_{ij}^n = \left(\frac{i}{2^n}, \frac{i+1}{2^n}\right] \times \left(\frac{j}{2^n}, \frac{j+1}{2^n}\right]$. Similar to the proof in the first case, we have

$$\|a_1^n\|_p^p \leq c[|s - s'|^{p/2-1} + |t - t'|^{p/2-1}] \tag{3.12}$$

By Burkholder's inequality for two parameter discrete martingales and Hölder's inequality, we have

$$\begin{aligned}
\mathbf{E}|a_2^n|^p &\le c\mathbf{E}|M_{\bar{s}'_n t_n} - M_{\bar{s}_n t_n}|^{2p} \\
&= c\mathbf{E}\left|\iint_{R_{\bar{s}'_n \bar{t}_n}\times(R_{\bar{s}'_n}\backslash R_{\bar{s}_n \bar{t}_n})} \psi(\xi,\eta)dW_\xi dW_\eta\right|^{2p} \\
&\le c\mathbf{E}\left|\iint_{R_{\bar{s}'_n \bar{t}_n}\times(R_{\bar{s}'_n \bar{t}_n}\backslash R_{\bar{s}_n \bar{t}_n})} \psi(\xi,\eta)^2 d\xi d\eta\right|^{p} \\
&\le c|s-s'|^{p-1} \\
&\le c|s-s'|^{p/2-1} \qquad (3.13)
\end{aligned}$$

Similarly

$$\mathbf{E}|a_3^n|^p \le c|t-t'|^{p/2-1} \qquad (3.14)$$

$$\mathbf{E}|a_4^n|^p \le c[|s-s'|^{p/2-1} + |t-t'|^{p/2-1}] \qquad (3.15)$$

A combinaion of (3.11), (3.12), (3.13), (3.14) and (3.14) yields

$$\|X(2^{-n}, s', t') - X(2^{-n}, s, t)\|_p^p \le c[|s-s'|^{p/2-1} + |t-t'|^{p/2-1}]$$

hence the proof of (3.4).

Proof of (3.5). For two parameter martingale $M_z = \iint_{R_z\times R_z} \psi(\xi,\eta)dW_\xi dW_\eta$, by Ito formula (cf. [2]) for two parameter stochastic processes we have the following decomposition

$$\begin{aligned}
&X(2^{-n}, s, t) - X(0, s, t) \\
&= 2\sum_{ij}\iint_{\Delta_{ij}^n(s,t)} M(\Delta_{ij}^n(s\wedge u, t\wedge v))dM_{uv} + 2\sum_{ij}\iint_{\Delta_{ij}^n(s,t)} d\sigma M_{\sigma\tau} d\tau M_{\sigma\tau} \\
&\quad + \sum_{ij}([M]^1(\Delta_{ij}^n(s,t)) - \langle M\rangle(\Delta_{ij}^n(s,t))) \\
&\quad + \sum_{ij}([M]^2(\Delta_{ij}^n(s,t)) - \langle M\rangle(\Delta_{ij}^n(s,t))) \\
&\equiv b_1^n + b_2^n + b_3^n + b_4^n \qquad (3.16)
\end{aligned}$$

Since $\langle M\rangle(u,v) = \int_0^u\int_0^v\int_0^u\int_0^v \psi(\xi,\eta)^2 d\xi d\eta$, noting that $\psi(\xi,\eta)=0$ unless $\xi \wedge \eta$, we have

$$d\langle M\rangle(u,v) = \left\{\int_0^u\int_0^v \psi(\xi_1, v; u, \eta_2)^2 d\xi_1 d\eta_2\right\} dudv \qquad (3.17)$$

Hence we have

$$\mathbf{E}|b_1^n|^p = c(p)\mathbf{E}\left|\sum_{ij}\iint_{\Delta_{ij}^n(s,t)} M(\Delta_{ij}^n(s \wedge u, t \wedge v))dM_{uv}\right|^p$$

(by Burkholder's inequality for two parameter discrete martingales and Hölder's inequality)

$$\leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left|\iint_{\Delta_{ij}^n(s,t)} M(\Delta_{ij}^n(s \wedge u, t \wedge v))dM_{uv}\right|^p$$

(by Burholder's inequality for two parameter continuous martingales)

$$\leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left(\iint_{\Delta_{ij}^n(s,t)} M(\Delta_{ij}^n(s \wedge u, t \wedge v)))^2 d\langle M\rangle(u, v)\right)^{p/2}$$

(by (3.17))

$$\leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left(\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\int_0^u\int_0^v M((s_i, u] \times (t_j, v])^2\psi(\xi_1, v; u, \eta_2)^2 dudvd\xi_1 d\eta_2\right)^{p/2}$$

(by Hölder's inequality)

$$\leq c\sum_{ij}\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\int_0^u\int_0^v |\mathbf{E}|M((s_i, u] \times (t_j, v])|^{2p}|^{1/2} \times |\mathbf{E}|\psi(\xi_1, v, u, \eta_2)|^{2p}|^{1/2} dudvd\xi_1 d\eta_2 \tag{3.18}$$

On the other hand, for any $(u, v) \in \Delta_{ij}^n(s, t)$, we have

$$\mathbf{E}|M((s_i, u] \times (t_j, v])|^{2p}$$

(by Burholder's inequality for two parameter continuous martingales)

$$\leq c\mathbf{E}\left(\iint_{(R_{uv}\backslash R_{ut_j}) \times (R_{uv}\backslash R_{s_i v})} \psi(\xi, \eta)^2 d\xi d\eta\right)^p$$

(by Hölder's inequality)

$$\leq cm(R_{uv}\backslash R_{ut_j})^{p-1} m(R_{uv}\backslash R_{s_i v})^{p-1}\iint_{\Pi^2}\mathbf{E}|\psi(\xi, \eta)^{2p} d\xi d\eta$$

$$\leq c4^{-n(p-1)} \tag{3.19}$$

Substituting (3.19) into (3.18) and taking (C.1) (C.2) into account we get

$$\mathbf{E}|b_1^n|^p \leq c2^{-n(p/2-1)} \tag{3.20}$$

For b_2^n, by using again Burkholder's inequality for two parameter parameter martingales, we get

$$\mathbf{E}|b_2^n|^p \leq c\mathbf{E}\left|\sum_{ij}\left(\iint_{\Delta_{ij}^n(s,t)} d\sigma M_{\sigma\tau} d\tau M_{\sigma\tau}\right)^2\right|^{p/2}$$

(by Hölder's inequality)

$$\leq c4^{n(p/2-11)}\sum_{ij}\mathbf{E}\left|\iint_{\Delta_{ij}^n(s,t)} d\sigma M_{\sigma\tau} d\tau M_{\sigma\tau}\right|^p$$

(by Burkholder's inequality for two parameter continuous martingales)

$$\leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left|\iint_{\Delta_{ij}^n(s,t)} d\sigma[M]^1_{\sigma\tau} d\tau[M]^2_{\sigma\tau}\right|^{p/2}$$

$$\leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left(\sup_{(u,v)\in\Delta_{ij}^n(s,t)}\iint_{\Delta_{ij}^n} d\sigma[M]^1_{\sigma v} d\tau[M]^2_{u\tau}\right)^{p/2}$$

$$\leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left(\sup_{u\in(s_i,s_{i+1}]}([M]^2_{ut_{j+1}}-[M]^2_{ut_j})\right)^p$$

$$+\, c4^{n(p/2-1)}\sum_{ij}\mathbf{E}\left(\sup_{u\in(t_j,t_{j+1}]}([M]^1_{s_{i+1}v}-[M]^1_{s_iv})\right)^p$$

$$\equiv b_{21}^n + b_{22}^n \tag{3.21}$$

Noting that $[M]^2-\langle M\rangle$ is a 1-martingale, by the corollary of Theorem (11.3) of [5] (cf. [5, page 98]) and Burkholder's inequality for two parameter continuous martingales we have

$$b_{21}^n \leq c4^{n(p/2-1)}\sum_{ij}\mathbf{E}(\langle M\rangle(\Delta_{ij}^n(s,t)))^p$$

(by Hölder's inequality)

$$\leq c4^{-np/2}\sum_{ij}\iint_{\Delta_1\times\Delta_2}\mathbf{E}|\psi(\xi,\eta)|^{2p}d\xi d\eta$$

$$= c4^{-np/2}\iint_{R_z\times R_z}\mathbf{E}|\psi(\xi,\eta)|^{2p}d\xi d\eta$$

$$\leq c2^{-np} \tag{3.22}$$

Similarly

$$b_{22}^n \leq c2^{-np} \tag{3.23}$$

Hence we deduce from (3.21), (3.22) and (3.23) that

$$\mathbf{E}|b_2^n|^p \leq c2^{-n(p/2-1)} \tag{3.24}$$

To estimate b_4^n, let $\overline{M}_z \equiv \int_{R_z}\psi(\xi,\eta)dW_\eta$ for $z\in\Pi$, by applying Ito's formula to 1-martingale $\overline{M}_{\cdot,t}=\{\overline{M}_{s,t}, s\in[0,1], \mathscr{F}_z^1\}$ for every fixed $t\in[0,1]$. (In general, $\overline{M}$ is not both martingale and 2-martingale) we have

$$\mathbf{E}|b_4^n|^p = \mathbf{E}\left|2\sum_{ij}\int_{\Delta_1}\int_{s_i\wedge s}^{s_{i+1}\wedge s}(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})d_u\overline{M}_{u,t_{j+1}\wedge t}d\xi\right|^p$$

(by Burkholder's inequality for one parameter discrete martingales and noting that $[M]^2-\langle M\rangle$ is a 1-martingale)

$$\leq c\mathbf{E}\left|\sum_i\left(\sum_j\int_{\Delta_1}\int_{s_i\wedge s}^{s_{i+1}\wedge s}(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})d_u\overline{M}_{u,t_{j+1}\wedge t}d\xi\right)^2\right|^{p/2}$$

(by using Hölder's inequality, three times)

$$\leq c2^{n(p/2-1)}\sum_{ij}\int_{\Delta_1}\mathbf{E}\left|\int_{s_i\wedge t}^{s_{i+1}\wedge s}(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})d_u\overline{M}_{u,t_{j+1}\wedge t}\right|^p d\xi \tag{3.25}$$

Since $\int_0^s(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})d_u\overline{M}_{u,t_{j+1}\wedge t}$ is a martingale w.r.t. $\{\mathscr{F}_z^1\}$, therefore by Burkholder's inequality for one parameter continuous martingales, we get

$$\mathbf{E}\left|\int_{s_i\wedge s}^{s_{i+1}\wedge s}(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})d_u\overline{M}_{u,t_{j+1}\wedge t}\right|^p$$

$$\leq c\mathbf{E}\left(\int_{s_i\wedge s}^{s_{i+1}\wedge s}(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})^2d_u[\overline{M}]^1_{u,t_{j+1}\wedge t}\right)^{p/2}$$

$$= c\mathbf{E}\left(\int_{s_i\wedge s}^{s_{i+1}\wedge s}\int_0^{t_{j+1}\wedge t}(\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t})^2\psi(\xi;u,v)^2dudv\right|^{p/2}$$

(by using Hölder's inequality, twice)

$$\leq c2^{-n(p/2-1)}\int_{s_i\wedge s}^{s_{i+1}\wedge s}\int_0^{t_{j+1}\wedge t}(\mathbf{E}|\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t}|^{2p})^{1/2}$$

$$\times(\mathbf{E}|\psi(\xi,\eta)|^{2p})^{1/2}dudv \tag{3.26}$$

Again by using Burkholder's inequality for one parameter continuous martingales, we get

$$\mathbf{E}|\overline{M}_{u,t_{j+1}\wedge t}-\overline{M}_{s_i\wedge s,t_{j+1}\wedge t}|^{2p}\leq c\mathbf{E}\left|\int_{s_i\wedge s}^{u}\int_0^{t_{j+1}\wedge t}\psi(\xi,\eta)^2d\eta\right|^p$$

(by Hölder's inequality)

$$\leq c2^{-n(p-1)}\int_{s_i}^{s_{i+1}}\int_0^{t_{j+1}}\mathbf{E}|\psi(\xi,\eta)|^{2p}d\eta \tag{3.27}$$

Substituting (3.27) into (3.26) and then (3.26) into (3.25), we get (again by using Hölder's inequality)

$$\mathbf{E}|b_4^n|^n\leq c2^{-n(p-1)/2}\sum_{ij}\left(\iint_{\Delta_1\times\Delta_2}\mathbf{E}|\psi(\xi,\eta)|^{2p}d\eta d\xi\right)$$

$$= c2^{-n(p-1)/2}\iint_{R_z\times R_z}\mathbf{E}|\psi(\xi,\eta)|^{2p}d\xi d\eta$$

$$\leq c2^{-n(p-1)/2} \tag{3.28}$$

For b_3^n, we note that $\hat{M}_z \equiv \int_{R_z} \psi(\xi, \eta) dW_\xi$ is a 2-martingale, $[M]^1 - \langle M \rangle$ is also a 2-martingale. Similar to the estimate of b_4^n we get

$$\mathbf{E}|b_3^n|^p \le c 2^{-n(p-1)/2} \tag{3.29}$$

A combination of (3.16), (3.20), (3.24), (3.28) and (3.29) implies that

$$\sup_z \|X(2^{-n}, s, t) - X(0, s, t)\|_p^p \le c 2^{-n(p/2-1)}$$

hence the proof of (3.5).

Proof of (3.6). To prove (3.6) we need the following propositions which are easily deduced from (2.3), (2.4) and the definition of stochastic integral of the second type.

Proposition 3.1. *For two parametermartingale $M_z = \psi \cdot WW_z$ we have*

⟨1⟩ $$\nabla^n\left(\iint_{R_z\times R_z} \psi(\xi, \eta) dW_\xi dW_\eta\right)(\cdot)$$
$$= \iint_{R_z\times R_z} \nabla^n\psi(\xi, \eta)(\cdot) dW_\xi dW_\eta + n \iint_{R_z\times R_z^\wedge} \nabla^{n-1}\psi(\xi, \eta)(\cdot) dW_\xi d\eta$$
$$+ n \iint_{R_z^\wedge\times R_z} \nabla^{n-1}\psi(\xi, \eta\eta)(\cdot) d\xi dW_\eta$$
$$+ n(n-1) \iint_{R_z^\wedge\times R_z^\wedge} \nabla^{n-2}\psi(\xi, \eta)(\cdot) d\xi d\eta$$

⟨2⟩ $$\nabla^n\left(\iint_{R_z\times R_z} \psi(\xi, \eta) W_\eta d\xi\right)(\cdot)$$
$$= \iint_{R_z\times R_z} \nabla^n\psi(\xi, \eta)(\cdot) dW^n d\xi + n \iint_{R_z\times R_z^\wedge} \nabla^{n-1}\psi(\xi, \eta)(\cdot) d\xi d\eta$$
$$\nabla^n\left(\iint_{R_z\times R_z} \psi(\xi, \eta) d\eta dW_\xi\right)(\cdot)$$
$$= \iint_{R_z\times R_z} \nabla^n\psi(\xi, \eta)(\cdot) d\eta dW_\xi + n \iint_{R_z^\wedge\times R_z} \nabla^{n-1}\psi(\xi, \eta)(\cdot) d\xi d\eta$$

Proposition 3.2. *Let $f, g \in \mathbf{W}_\infty$, then $(f, g) \in \mathbf{W}_\infty$ and*

$$\mathbf{L}(f, g) = (\mathbf{L}f, g) + (f, \mathbf{L}g) + (\nabla f, \nabla g)_{\mathbf{H}}$$

Now we turn to the proof of (3.6). By (C.1) and (C.2), for all p, r, we have $\sup_z \left\| \iint_{R_z\times R_z} \psi(\xi, \eta)^2 d\xi d\eta \right\|_{p,2r}^p < +\infty$. So it suffice to prove that

$$\sup_{n,z} \left\| \sum_{ij} M(\Delta_{ij}^n(s, t))^2 \right\|_{p,2r} < +\infty, \qquad \forall p,\ r. \tag{3.30}$$

By Proposition 3.2 we have the following decomposition:

$$\mathbf{L}\left(\sum_{ij} M(\triangle_{ij}^n(s,t))^2\right) = 2\sum_{ij} M(\triangle_{ij}^n(s,t))\mathbf{L}M(\triangle_{ij}^n(s,t)) + \sum_{ij} \|\nabla M(\triangle_{ij}^n(s,t))\|_{\mathbf{H}}^2$$

$$\equiv 2I_1^n + I_2^n \tag{3.31}$$

In view of (2.11), $\mathbf{L}M$ is a martingale. By (C.1), (C.2) and [2], we get

$$\lim_{n\to+\infty} \sup_z \mathbf{E}|I_1^n(z)|^p = \sup_z \mathbf{E}|\langle M, \mathbf{L}M\rangle_z|^p \le \sup_z (\mathbf{E}\langle M\rangle_z^p)^{1/2} \sup_z (\mathbf{E}\langle \mathbf{L}M\rangle_z^p)^{1/2} < +\infty.$$

Consequently

$$\sup_{n,z} \mathbf{E}|I_1^n(z)|^p < +\infty \tag{3.32}$$

We deduce from Proposition 2.1 and Proposition 3.1 that

$$\begin{aligned}\nabla M(\triangle_{ij}^n(s,t))(u,v) = &\iint_{\triangle_1\times\triangle_2} \nabla\psi(\xi,\eta)(u,v)dW_\xi dW_\eta \\ &+ \iint_{\triangle_2(u,v)\times\triangle_1} \psi(\xi,\eta)dW_\xi d\eta \\ &+ \iint_{\triangle_1(u,v)\times\triangle_2} \psi(\xi,\eta)d\xi W_\eta\end{aligned} \tag{3.33}$$

where $\triangle_1(u,v) = (0, s_{i+1}\wedge u] \times (t_j \wedge v, t_{j+1}\wedge v]$, $\triangle_2(u,v) = (s_i\wedge u, s_{i+1}\wedge u] \times (0, t_{j+1}\wedge v]$. By Proposition 2.1, we have

$$\iint_{\triangle_1\times\triangle_2} \nabla\psi(\xi,\eta)(u,v)W_\xi dW_\eta = ((\nabla\psi)\cdot WW)(\triangle_{ij}^n(s,t))(u,v) \tag{3.34}$$

By the definition of inner product $(\cdot,\cdot)_{\mathbf{H}}$ we have

$$\begin{aligned}&\left\|\iint_{\triangle_2(\cdot,\cdot)\times\triangle_1} \psi(\xi,\eta)dW_\xi)d\eta\right\|_{\mathbf{H}}^2 \\ &= \int_{[0,1]^2} \left|\frac{\partial^2}{\partial u\partial v}\left(\iint_{\triangle_2(u,v)\times\triangle_1} \psi(\xi,\eta)dW_\xi)\,d\eta\right)\right|^2 dudv \\ &= \int_{\triangle_2}\left(\int_{\triangle_1} \psi(\xi,\eta)dW_\xi\right)^2 d\eta \\ &= [M]^1(\triangle_{ij}^n(s,t))\end{aligned} \tag{3.35}$$

Similarly

$$\left\|\iint_{\triangle_1(\cdot,\cdot)\times\triangle_2} \psi(\xi,\eta)dW_\eta d\xi\right\|_{\mathbf{H}}^2 = [M]^2(\triangle_{ij}^n(s,t)) \tag{3.36}$$

From (3.33), (3.34), (3.35) and (3.36) we deduce that

$$\sum_{ij} \|\nabla M(\Delta_{ij}^n(s,t)\|_{\mathbf{H}}^2 \le \sum_{ij} \|(\nabla\psi)\cdot WW(\Delta_{i,j}^n(s,t))\|_{\mathbf{H}}^2 + [M]_z^1 + [M]_z^2$$

$$\equiv I_{21}^n + I_{22}^n + I_{23}^n \tag{3.37}$$

To estimate $I_{21}^n(z)$, we have

$$\mathbf{E}|I_{21}^n(z)|^p = \mathbf{E}\left|\sum_{ij} \|(\nabla\psi)\cdot WW(\Delta_{i,j}^n(s,t))\|_{\mathbf{H}}^2\right|^p$$

(by Burkholder's inequality for Hilbert space-valued martingales with discrete parameter)

$$\le c\mathbf{E}\left[\sup_{ij}\left\|\iint_{R_{(s_{i+1}\wedge s,t_{j+1}\wedge t)}\times R_{(s_{i+1}\wedge s,t_{j+1}\wedge t)}} \nabla\psi(\xi,\eta)dW_\xi dW_\eta\right\|_{\mathbf{H}}\right]^{2p}$$

(by Burkholder's inequality for $\mathbf{H}$-valued martingales with discrete parameter, but in reverse way)

$$\le c\mathbf{E}\left[\iint_{R_z\times R_z} \|\nabla\psi(\xi,\eta)\|_{\mathbf{H}}^2 d\xi d\eta\right]^p$$

(by Hölder's inequality)

$$\le c\iint_{\Pi^2} \mathbf{E}\|\nabla\psi(\xi,\eta)\|_{\mathbf{H}}^{2p} d\xi d\eta < +\infty \qquad (by\ (C.1)\ and\ (C.2)) \tag{3.38}$$

And for I_{22}^n, by [5] we get

$$\mathbf{E}|I_{22}^n(z)|^p \le c\mathbf{E}(\langle M\rangle_z)^p \le c\iint_{\Pi^2} \mathbf{E}|\psi(\xi,\eta)|^{2p} d\xi d\eta < +\infty \tag{3.39}$$

Similarly

$$\mathbf{E}|I_{23}^n(z)|^p \le c\iint_{\Pi^2} \mathbf{E}|\psi(\xi,\eta)|^{2p} d\xi d\eta < +\infty \tag{3.40}$$

From (3.37), (3.38), (3.39) and (3.40) we have

$$\sup_{n,z} \mathbf{E}|I_2^n|^p < +\infty \tag{3.41}$$

A combination of (3.32) and (3.41) implies that

$$\sup_{n,z}\left\|\sum_{ij} M(\Delta_{ij}^n(s,t))^2\right\|_{p,2r} < +\infty \qquad for\ r=1$$

Now we proceed to estimate the fourth order derivatives. By Proposition 3.2 we have

$$\mathbf{L}^2\left(\sum_{ij} M(\triangle_{ij}^n(s,t))^2\right)$$

$$= \sum_{ij} [2M(\triangle_{ij}^n(s,t))\mathbf{L}^2 M(\triangle_{ij}^n(s,t))$$

$$+ 2(\mathbf{L}M(\triangle_{ij}^n(s,t)))^2 + 2(\nabla M(\triangle_{ij}^n(s,t)), \nabla \mathbf{L}M(\triangle_{ij}^n(s,t)))_{\mathbf{H}}$$

$$+ 2(\mathbf{L}\nabla M(\triangle_{ij}^n(s,t)), \nabla M(\triangle_{ij}^n(s,t)))_{\mathbf{H}} + \|\nabla^2 M(\triangle_{ij}^n(s,t))\|_{\mathbf{H}\otimes\mathbf{H}}^2]$$

$$\equiv J_1^n + J_2^n + J_3^n + J_4^n + J_5^n \tag{3.42}$$

The estimation of J_i^n $(i = 1, 2, 3)$ being obtained in a way similar to that of I_1^n and I_2^n, we have that

$$\sup_{n,z} \mathbf{E}(|J_1^n|^p + |J_2^n|^p + |J_3^n|^p) < +\infty \tag{3.43}$$

Since

$$|J_4^n| \leq \sum_{ij} \|\mathbf{L}\nabla M(\triangle_{ij}^n(s,t)\|_{\mathbf{H}}^2 + \sum_{ij} \|\nabla M(\triangle_{ij}^n(s,t)\|_{\mathbf{H}}^2$$

$$\equiv J_{41}^n + J_{42}^n \tag{3.44}$$

Similar to that for I_2^n we get

$$\sup_{n,z} \mathbf{E}|J_{42}^n|^p < +\infty \tag{3.45}$$

To estimate J_{41}^n, we first make the following observation (by (2.11), (2.12) and Proposition 3.1):

$$\mathbf{L}\nabla M(\triangle_{ij}^n(s,t))(u,v) = \iint_{\triangle_1\times\triangle_2} (\mathbf{L}\nabla\psi - 2\nabla\psi)(u,v)dW_\xi dW_\eta$$

$$+ \iint_{\triangle_2(u,v)\times\triangle_1} (\mathbf{L}\psi - \psi)(\xi,\eta)dW_\xi d\eta$$

$$+ \iint_{\triangle_1(u,v)\times\triangle_2} (\mathbf{L}\psi - \psi)(\xi,\eta)dW_\xi d\xi \tag{3.46}$$

If set $\overline{\overline{M}} \equiv (\mathbf{L}\psi - \psi)\cdot WW_z$ and $\overline{\overline{\overline{M}}} \equiv (\mathbf{L}\nabla\psi - 2\psi)\cdot WW_z$ then similar to the proof of I_2^n we get from (3.46) that

$$\sum_{ij} \|\mathbf{L}\nabla M(\triangle_{ij}^n(s,t))\|_{\mathbf{H}}^2 \leq \sum_{ij} \|\overline{\overline{\overline{M}}}(\triangle_{ij}^n(s,t))\|_{\mathbf{H}}^2 + [\overline{\overline{M}}]_z^1 + [\overline{\overline{M}}]_z^2$$

The RHS can be estimated in the same way as for I_2^n and we have

$$\sup_{n,z} \mathbf{E}|J_{41}^n|^p < +\infty \tag{3.47}$$

Therefore

$$\sup_{n,z} \mathbf{E}|J_4^n|^p < +\infty \tag{3.48}$$

To estimate J_5^n, we use Proposition 3.1 to obtain

$$\begin{aligned}
\nabla^2 M(\Delta_{ij}^n(s,t))(u,v) &= \iint_{\Delta_1\times\Delta_2} \nabla^2\psi(\xi,\eta)(u,v)dW_\xi dW_\eta \\
&+ 2\iint_{\Delta_2(u,v)\times\Delta_1} \nabla\psi(\xi,\eta)(u,v)dW_\xi d\eta \\
&+ 2\iint_{\Delta_1(u,v)\times\Delta_2} \nabla\psi(\xi,\eta)(u,v)dW_\eta d\xi \\
&+ 2\iint_{\Delta_1(u,v)\times\Delta_2(u,v)} \psi(\xi,\eta)d\xi d\eta \\
&\equiv c_1 + c_2 + c_3 + c_4 \qquad (3.49)
\end{aligned}$$

Setting

$$\tilde{c}_2 = \tilde{c}_2(u_1, v_1; u_2, u_2) \equiv 2\iint_{\Delta_2(u,v)\times\Delta_1} \nabla\psi(\xi,\eta)(u_2,v_2)dW_\xi d\eta$$

then by the theorem II.10 of M. Reed and B. Simon [15] we have

$$\begin{aligned}
\|c_2\|^2_{\mathbf{H}\otimes\mathbf{H}} &= \iint_{\Pi^2}\left(\frac{\partial^4\tilde{c}}{\partial u_2\partial v_2\partial u_1\partial v_1}\right)^2 du_1dv_1du_2dv_2 \\
&= 4\int_{\Delta_2}\left\|\int_{\Delta_1}\nabla\psi(\xi,\eta)dW_\xi\right\|^2_{\mathbf{H}} d\eta \qquad (3.50)
\end{aligned}$$

Similarly

$$\begin{aligned}
\|c_3\|^2_{\mathbf{H}\times\mathbf{H}} &= 4\int_{\Delta_1}\left\|\int_{\Delta_2}\nabla\psi(\xi,\eta)dW_\eta\right\|^2_{\mathbf{H}} d\xi \\
\|c_4\|^2_{\mathbf{H}\otimes\mathbf{H}} &= 4\iint_{\Delta_1\times\Delta_2}\psi(\xi,\eta)^2 d\xi d\eta \\
&= 4\langle M\rangle(\Delta_{ij}^n(s,t)) \qquad (3.51)
\end{aligned}$$

Hence from (3.49), (3.50) and (3.51) we get

$$\begin{aligned}
J_5^n \le& \sum_{ij}\|(\nabla^2\psi)\cdot WW(\Delta_{ij}^n(s,t)\|^2_{\mathbf{H}\times\mathbf{H}} + 4\sum_{ij}\int_{\Delta_2}\left\|\int_{\Delta_1}\nabla\psi(\xi,\eta)dW_\xi\right\|^2_{\mathbf{H}} d\eta \\
&+ 4\sum_{ij}\int_{\Delta_1}\left\|\int_{\Delta_2}\nabla\psi(\xi,\eta)dW_\eta\right\|^2_{\mathbf{H}} d\xi + 4\iint_{R_z\times R_z}\psi(\xi,\eta)^2 d\xi d\eta \\
\equiv& J_{51}^n + J_{52}^n + J_{53}^n + J_{54}^n \qquad (3.52)
\end{aligned}$$

Since J_{51}^n can be estimated in the same way as for I_2^n, and it is trivial that $\sup_z\left\|\iint_{R_z\times R_z}\psi(\xi,\eta)^2 d\xi d\eta\right\|^p_{p,2r} < +\infty$, so it suffices to estimate J_{52}^n (in view of the symmetric relation of J_{52}^n and J_{53}^n). By Hölder's inequality, we have

$$\mathbf{E}|J_{52}^n|^p \le c2^{n(p-1)} \sum_{ij} \int_{\Delta_2} \mathbf{E} \left\| \int_{\Delta_1} \nabla\psi(\xi, \eta) dW_\xi \right\|_{\mathbf{H}}^{2p} d\eta$$

(by Burkholder's inequality for $\mathbf{H}$-valued martingales w.r.t $\{\mathscr{F}_z^2\}_{z\in\Pi}$)

$$\le c2^{n(p-1)} \sum_{ij} \int_{\Delta_2} \mathbf{E}\left(\int_{\Delta_1} \|\nabla\psi(\xi, \eta)\|_{\mathbf{H}}^2 d\xi \right)^p d\eta$$

(by Hölder's inequality)

$$\le c \sum_{ij} \iint_{\Delta_1 \times \Delta_2} \mathbf{E} \|\nabla\psi(\xi, \eta)\|_{\mathbf{H}}^{2p} d\xi d\eta$$

$$= c \iint_{R_z \times R_z} \mathbf{E} \|\nabla\psi(\xi, \eta)\|_{\mathbf{H}}^{2p} d\xi d\eta$$

$$\le \iint_{\Pi^2} \mathbf{E} \|\nabla\psi(\xi, \eta)\|_{\mathbf{H}}^{2p} d\xi d\eta < +\infty \qquad (by\ (C.1)\ (C.2))$$

Therefore

$$\sup_{n,z} \mathbf{E}|J_5^n|^p < +\infty \tag{3.53}$$

Finally we deduce from (3.42) (3.48) and (3.53) that

$$\sup_{n,z} \left\| \sum_{ij} M(\Delta_{ij}^n(s, t))^2 \right\|_{p,4} < +\infty$$

proving (3.30) for $r = 2$. Doing the same thing for high order derivatives, step by step, we complete the proof of (3.30). Thus finished the proof of Theorem 3.3.

Q.E.D

4. Main results

Theorem 4.1. *Let N be a two parameter smooth martingale represented as* (2.13), *then the convergence*

$$\lim_{n\to+\infty} \sum_{ij} N(\Delta_{ij}^n(s, t))^2 = \int_{R_z} \phi(\eta)^2 d\eta + \iint_{R_z \times R_z} \psi(\xi, \eta)^2 d\xi d\eta$$

holds uniformly in $z = (s, t) \in \Pi$, q.s. where $\Delta_{ij}^n(s, t) = (s_i, s_{i+1}] \times (t_j, t_{j+1}]$ and $s_i = \frac{i}{2^n} \wedge s$, $t_j = \frac{j}{2^n} \wedge t$.

To prove this theorem, in view of Theorem 3.3 above and Theorem 4.3 of [17], we need only to prove the following.

Theorem 4.2. *Let* $\overline{N}_z \equiv \int_{R_z} \phi(\eta) dW_\eta$ *and* $M_z \equiv \iint_{R_z \times R_z} \psi(\xi, \eta) dW_\xi dW_\eta$ *for any* $z \in \Pi$, *where* ϕ, ψ *satisfy conditions* (C.1) *and* (C.2), *then the convergence*

$$\lim_{n \to +\infty} \sum_{ij} M(\Delta_{ij}^n(s, t)) \overline{N}_z(\Delta_{ij}^n(s, t)) = 0$$

holds uniformly in $z = (s, t) \in \Pi$, *q.s.*

Proof. Put $X(2^{-n}, s, t) = \sum_{ij} M(\Delta_{ij}^n(s, t)) \overline{N}(\Delta_{ij}^n(s, t))$. Similar to that of Theorem 3.3 we reduce the proof to proving the following inequalities:

$$\sup_z \|X(2^{-n}, s, t)\|_p^p \leq c 2^{-n(p/2-1)} \tag{4.1}$$

$$\sup_n \|X(2^{-n}, s, t) - X(2^{-n}, s', t')\|_p^p \leq c[|s - s'|^{p/2-1} + |t - t'|^{p/2-1}] \tag{4.2}$$

$$\sup_{n, z} X(2^{-n}, s, t)\|_{p, 2r} < +\infty \tag{4.3}$$

where $z = (s, t)$.

To prove (4.1), by applying Ito formula for two parameter processes to $(M + \overline{N})^2$, $\overline{N}^2$, M^2 and Ito formula for one parameter processes to $\left\{\left(\int_{R_{st_0}} \psi(\xi, \eta) dW_\xi + \phi(\eta)\right)^2, \mathscr{F}_z^1\right\}_{s \in [0,1]}$ and $\left\{\left(\int_{R_{s_0 t}} \psi(\xi, \eta) dW_\eta + \phi(\xi)\right)^2, \mathscr{F}_z^2\right\}_{t \in [0,1]}$ for a fixed $(s_0, t_0) \in \Pi$, we get the following decomposition:

$$\begin{aligned}
\sum_{ij} M(\Delta_{ij}^n) \overline{N}(\Delta_{ij}^n) &= \sum_{ij} \iint_{\Delta_{ij}^n(s,t)} M(\Delta_{ij}^n(s \wedge u, t \wedge v) d\overline{N}_{uv} \\
&\quad + \sum_{ij} \iint_{\Delta_{ij}^n(s,t)} \overline{N}(\Delta_{ij}^n(s \wedge u, t \wedge v)) dM_{uv} \\
&\quad + \sum_{ij} \iint_{\Delta_{ij}^n(s,t)} d\sigma M_{\sigma\tau} d\tau \overline{N}_{\sigma\tau} \\
&\quad + \sum_{ij} \iint_{\Delta_{ij}^n(s,t)} d\sigma \overline{N}_{\sigma\tau} d\tau M_{\sigma\tau} \\
&\equiv d_1^n + d_2^n + d_3^n + d_4^n
\end{aligned} \tag{4.4}$$

where we have used the fact: $\langle \overline{N}, M \rangle = 0$.

Similar to the estimation of b_2^n, we have

$$\begin{aligned}
\mathbf{E}|d_3^n|^p &\leq c 4^{n(p/2-1)} \sum_{ij} \left[\mathbf{E}\left|\sup_{v \in (t_j, t_{j+1}]} ([M]_{s_{i+1}v}^1 - [M]_{s_i v}^1)^p \mathbf{E}\right| \int_0^1 \int_{t_j}^{t_{j+1}} \phi(\xi)^2 d\xi\right|^p\right]^{1/2} \\
&\leq c 4^{n(p/2-1)} \sum_{ij} \left[8^{-n(p-1)} \left(\int_0^{s_{i+1}} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} \int_0^{t_{j+1}} \mathbf{E}|\psi|^{2p} d\xi d\eta\right)\right. \\
&\quad \left. \cdot \int_0^1 \int_{t_j}^{t_{\lambda+1}} \mathbf{E}|\phi|^{2p} d\xi\right]^{1/2}
\end{aligned}$$

$$\leq c2^{-np/2}\left[\sum_{ij}\int_0^{s_{i+1}}\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}}\int_0^{t_{j+1}} \mathbf{E}|\psi|^{2p}d\xi d\eta + \sum_{ij}\int_0^1\int_{t_j}^{t_{\lambda+1}} \mathbf{E}|\phi|^{2p}d\xi\right]$$

$$\leq c2^{-np/2}\left[\iint_{R_z\times R_z} \mathbf{E}|\psi|^{2p}d\xi d\eta + 2^n\int_\Pi \mathbf{E}|\phi|^{2p}d\xi\right]$$

$$\leq c2^{-n(p/2-1)} \tag{4.5}$$

By the symmetric relation of d_3^n and d_4^n, we get

$$\mathbf{E}|d_4^n|^p \leq c2^{-n(p/2-1)} \tag{4.6}$$

The estimation of d_i^n $(i = 1, 2)$ can be done in the same way as for b_1^n, thus we have

$$\sup_z \mathbf{E}(|d_1^n|^p + \mathbf{E}|d_2^n|^p) \leq c2^{-n(p/2-1)} \tag{4.7}$$

A combination of (4.4), (4.5), (4.6) and (4.7) implies the proof of (4.1).

By an argument similar to that of (3.4) and (3.6) we need only to replace $M(\Delta_{ij}^n(s,t))^2$ by $M(\Delta_{ij}^n(s,t))\bar{N}(\Delta_{ij}^n(s,t))$ and use Hölder's inequality $|(\cdot,\cdot)_{\mathbf{H}^{\otimes n}}| \leq c[\|\cdot\|_{\mathbf{H}^{\otimes n}}^2 + \|\cdot\|_{\mathbf{H}^{\otimes n}}^2]$ to estimate $(\mathbf{L}\nabla M(\Delta_{ij}^n(s,t)), \nabla\bar{N}(\Delta_{ij}^n(s,t)))_{\mathbf{H}}$, $(\nabla^2 M(\Delta_{ij}^n(s,t)), \nabla^2\bar{N}(\Delta_{ij}^n(s,t)))_{\mathbf{H}\otimes\mathbf{H}}$, $\ldots$, and so on in the proof of (3.4) and (3.6). We can easily prove (4.2) and(4.3), thus complete the proof. Q.E.D

From the proof of Theorem 4.1 of [17] and that of Theorem 3.1, we also have the following

Theorem 4.3. *Let N be a two parameter smooth martingale represented as* (2.13), *then*

(i) $N_z \in W_\infty$, *for all* $z \in \Pi$;

(ii) *There exists a decreasing sequence* $\{O_n, n \geq 1\}$ *of open subset of* $\mathbf{X}$ *and a function* $\tilde{N}: \left(\bigcup_n O_n^c\right) \times \Pi \to R$ *such that*

- (a) $\tilde{N}$ *is continuous on* $O_n^c \times \Pi$, *for each* $n \geq 1$;
- (b) $C_{r,p}(O_n) \to 0$ *as* $n \to \infty$ *for all* p, r;
- (c) $\tilde{N}_z = N_z$ *almost surely, for all* $z \in \Pi$.

Acknowledgment. The author is very grateful to Prof. Zhiyuan Huang and Prof. Jiagang Ren for encouragement and useful discussions, he also wish to thank the referee for his valuable comments.

Department of Mathematics,
School of Science
Huazhong University of Science and Technology, Wuhan,
Hubei, 430074, P.R. China

References

[1] R. Cairoli and J. B. Walsh, Stochastic integrals in the plane, Acta Math., **134** (1975), 111–183.

[2] L. Chevalier, Martingale continues à deux paramétres, Bull. Sc. Math., **106** (1982), 19–62.

[3] Z. Huang, Foundation of Stochastic Analysis, Wuhan Univ. Press, 1988 (in Chinese).

[4] Z. Huang and J. Ren, Quasi sure stochastic flows, stochastic, **33** (1990), 149–157.

[5] P. Imkeller, Two-Parameter Martingales and Their Quadratic Variation, L.N.M. 1308, 1988.

[6] S. Kusuoka and D. Stroock, Applications of the Malliavin calculus, Part I, Proc. Taniguchi Inter. Symp. on Stochastic Analysis, Katata and Kyoto 1982, 271–306, Kinokuniya, 1984.

[7] P. Malliavin, Implicit functions in finite corank on the Wiener space, Proc. Taniguchi. Intern. Symp. on Stochastic Analysis, Katata and Kyoto 1982, 369–386.

[8] P. Malliavin and D. Nualart, Quasi sure analysis of stochastic flows and Banach space valued smooth functions on the Wiener space, J. Funct. Anal., **112** (1993), 287–317.

[9] P. Malliavin, Differential Analysis in stochastic Analysis, Cours, M.I.T.

[10] D. Nualart, On the quadratic variation of two parameter continuous martingales, Ann. Prob., **12** (1984), 445–457.

[11] D. Nualart, Variation quadratiques et Inéegalités pour les martingales à deux indices, stochastic, **15** (1985), 51–63.

[12] D. Nualart and M. Sanz, Malliavin calculus for two parameter Wiener functionals, Z.W.V.G., **70** (1985), 573–590.

[13] D. Nualart and M. Sanz, Stochastic differential equations in the plane: Smoothness of the solution, J. Multi. Anal., **31** (1989), 1–29.

[14] S. Orey and W. Pruit, Sample functions of the N-parameter Wiener processes, Ann. Prob., **1** (1973), 138–163.

[15] M. Reed and B. Simon, Methods of modern Mathematical Physics, I. Functional Analysis, Academic Press, New York and London, 1972.

[16] J. Ren, Analysis quasi sûre des èquation differentielles stochastiques, Bull. Sc. Math., **(2)114** (1990), 187–213.

[17] J. Ren, Quasi sure quadratic variation of smooth martingales, J. Math. Kyoto. Univ., **34** (1994), 191–206.

[18] J. Ren, On smooth martingales, J. Funct. Anal., **120** (1994), 72–81.

[19] M. Takeda, (r, p)-capacity on the Wiener space and properties of Brownian motion, Z.W.V.G., **68** (1984), 149–162.

[20] A. S. Ustunel and M. Zakai, On the structure of independence on Wiener space, J. Funct. Anal., **90** (1990), 113–137.

[21] E. Wong and M. Zakai, Martingales and stochastic integrals for processes with a multi-dimensional parameter, Z.W.V.G., **29** (1974), 109–122.

[22] S. Watanabe, Stochastic differential equations and Malliavin calculus, Tata Inst. Fund. Res. Bombay, 1984.

[23] D. W. Stroock, The Malliavin calculus: a functional analytic approach, J. Funct. Anal., **44** (1981), 212–257.

[24] E. Wong and M. Zakai, Differentiation formulas for stochastic integral in the plane, Stochastic process. Appl., **6** (1978), 339–349.

[25] J. Yeh, Wiener measure in a space of function of two variables, Trans. Amer. Math. Soc., **95** (1960), 443–450.