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Relative entropy and mixing properties of interacting particle systems

By

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1. Introduction

Consider a Markov semigroup S_t on a compact metric space X that has the Feller property. If we start the process with some initial distribution μ, then it is not always true that for large t the distribution $\mu_t = \mu S_t$ is close to an invariant distribution for the process. While any limit point as $T \to \infty$ of the time average $\frac{1}{T}\int_0^t \mu_t dt$ is always an invariant measure, the same can not be claimed for limit points of μ_t itself. The simplest examples are provided by deterministic flows. However for any Markov chain on a finite state space, continuous time rules out periodic behavior and μS_t has a limit as $t \to \infty$ and this is always an invariant measure.

The natural question that arises is to determine if under some suitable conditions on the Markov semigroup S_t one can still claim that all possible limit points of μS_t are invariant measures. Such a result in conjunction with a uniqueness theorem for invariant measures will establish the convergence of μS_t to the unique invariant measure giving us a mixing result.

It has been conjectured that in the context of interacting particle systems the answer is in the affirmative under some very mild restrictions. Let $X = F^{\mathbf{Z}^d}$, where F is a finite set. The state η of the system is described by its values $\eta(x)$ for $x \in \mathbf{Z}^d$. The infinitesimal generator of the particle system is given by

$$\Omega f(\eta) = \sum_{T \subset \mathbf{Z}^d} \int_{F^T} c_T\ (d\xi, \eta)\ (f(\eta^\xi) - f(\eta)) \tag{1}$$

where the summation runs over all finite subsets of $\mathbf{Z}^d$. Here $c_T(d\xi, \eta)$ describes the rates for Poisson events that change the current configuration η to a new configuration η^ξ that has been altered on the finite index set $T \subset \mathbf{Z}^d$ from η to ξ. A whole family of such Poisson events are taking place simultaneously and the infinitesimal generator reflects that. Of course a whole lot of these $c_T(\cdot,\cdot)$ may be 0. We say that a particle system has bounded flip rates if there is a bound on the sum of all the Poisson rates that could affect a loca-

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tion $x\in\mathbf{Z}^d$, i. e.,

$$\sup_{x\in Z^d}\left\{\sum_{T\ni x}\sup_{\eta\in X}\int_{F^T}c_T(d\xi,\eta)\right\}<\infty \tag{2}$$

And we will say that a particle system has finite range R if the following two conditions are satisfied:
a) $c_T(\cdot,\cdot)\equiv 0$ whenever the diameter of T exceeds R, i. e. $\sup_{x,y\in T}|x-y|\geq R$.
b) For every $A\subset F^T$ the function $c_T(A,\eta)$ does not depend on η (u) if dist $(T,u)\geq R$.
Under these conditions the operator Ω is well defined for $f\in\mathfrak{D}$, the space of 'tame functions' i.e functions that depend on $\{\eta(x)\}$ only for x in some finite set.

The operator Ω on $\mathfrak{D}$ has a natural extension that defines a Markov generator (see theorem 3.9 of Liggett [1]).

The following theorem was recently proved by Mountford [2] when the dimension $d=1$.

Theorem 1. *Let η_t be an infinite particle system on $X=F^{\mathbf{Z}}$ with bounded flip rates and of finite range. Let μ be some probability measure on X such that*

$$\lim_{n\to\infty}\mu S_{t_n}=\nu$$

exists, where $t_n\nearrow\infty$. Then ν is invariant.

We provide a different proof in this note. The proof of Mountford uses ABL coupling. We replace it by a relative entropy argument. Our proof is shorter and we believe that it can bemodified and applied to other situations.

idea of proof. In order to illustrate the idea behind the proof we will show that if S_t is a Markov semigroup with a bounded generator

$$\Omega f(x)=\int_X[f(y)-f(x)]\pi(x,dy) \tag{3}$$

then for any μ and any fixed $0<\gamma<\infty$, $\|\mu S_{\tau+t}-\mu S_t\|\to 0$ as $t\to\infty$, where $\|\alpha\|$ denotes the total variation norm of the signed measure α. If we take some $\lambda>0$ and consider the generator $(1+\lambda)\Omega$ instead of Ω, by a Girsanov type formula it is seen that the new process is absolutely continuous with respect to the old one and one can get a bound on the relative entropy of the new process with respect to the original one during any finite time interval $[0,t]$. A simple estimate, for small λ, gives the bound $H\leq Ct\lambda^2$ for the relative entropy where C is a bound for the operator Ω. By making the choice of $\lambda=\frac{\tau}{t}$ we get $H=o(1)$ as $t\to\infty$ for fixed τ. Since the marginal at time t of the speededup process with generator $\left(1+\left(\frac{\tau}{t}\right)\right)\Omega$ is $\mu S_{\tau+t}$, and the relative entropy controls the variational distance the proof is complete.

Our proof of Mountford's theorem is basically a modification of the above idea. In order to have a bounded Ω we need to put things in a finite box and then try to control the influence of effects coming from outside the box. This is done through the finite range condition. There is a fight between the smallness of the entropy bound and the growth of the domain that influences what goes on in a finite box. When $d=1$ the entropy bound wing, giving us the result. The details are carried out in the next two sections.

it is enough to show that for any arbitrary finite M, local function f depending on sites $[-M, M]$, and any $\tau>0$,

$$\lim_{t\to\infty}\left|\int S_t f d\mu - \int S_{t+\tau} f d\mu\right| = 0 \tag{4}$$

To do this we define a new process governed by the time dependent infinitesimal generator

$$\Omega^{t,\tau}(s) = \lambda^{t,\tau}(s)\,\Omega \tag{5}$$

where $\lambda^{t,\tau}(s)$: $[0, \infty) \to [1, \infty)$ is chosen so that

$$\int_0^t \lambda^{t,\tau}(s)\ ds = t+\tau \tag{6}$$

The new proces will then have transition operators $S_{0,t}^{\lambda(\cdot)} = S_{t+\tau}$. Because Ω is an unbounded operator, processes with generators $\Omega(s) = \lambda(s)\,\Omega$ are not absolutely continuous with respect to the original one. But this situation can be remedied by truncating the processes.

Suppose we have a function $h(s) \geq M$, defined for $0 \leq s \leq t$, we truncate the processes to $[-h(s), h(s)]$ at time $s \in [0, t]$. We define the generators

$$\Omega^h(s) f(\eta) = \sum_{T\subset[-[h(s)],[h,(s)]]} \int_{F^T} c_T(d\xi, \zeta)\,(f(\zeta^\xi) - f(\zeta))$$

$$\Omega^{\lambda,h}(s) f(\zeta) = \lambda^{t,\tau}(s) \sum_{T\subset[-[h(s)],[h,(s)]]} \int_{F^T} c_T(d\xi, \zeta)\,(f(\zeta^\xi) - f(\zeta))$$

We define four processes P, P^λ, P^h and $P^{\lambda,h}$ corresponding to the generators Ω, $\Omega^\lambda(s)$, $\Omega^h(s)$ and $\Omega^{\lambda,h}(s)$ respectively all having the same initial distribution μ. The processes P and P^λ have $\mu_t = \mu S_t$ and $\mu_{t+\tau} = \mu S_{t+\tau}$ for marginal distributions at time t. P^h and $P^{\lambda,h}$ are governed by truncated generators and the relative entropy of $P^{\lambda,h}$ with respect to P^h can be estimated by means of a Girsanov formula and will be carried out in section 3. In section 2 we will estimate the effect of the truncation. If we denote by μ_t, μ_t^λ, μ_t^h and $\mu_t^{\lambda,h}$ the marginals at time t of the four processes P, P^λ, P^h and $P^{\lambda,h}$, the relative entropy estimate in section 3, will tell us how close $\mu_t^{\lambda,h}$ is to λ_t^h. In section 2 we will estimate the total variation on the σ-field corresponding to $[-M, M]$ of the differences

$\mu_t - \mu_t^h$ and $\mu_t^\lambda - \mu_t^{\lambda,h}$.

2. Construction of truncated processes

A theorem of Liggett (theorem 3.9 (c) [1]) provides the basis for estimating the difference between the truncated and untruncated processes. First some notation: for any bounded function f on $F^{\mathbf{Z}}$ let us define

$$\Delta_f(x) = \sup \{[f(\eta) - f(\eta']; \eta, \eta': \eta(y) = \eta'(y) \quad \forall y \neq x\} \tag{7}$$

and

$$D(X) = \{f: \sum_{x \in \mathbf{Z}} \Delta_f(x) < \infty\} \tag{8}$$

Theorem 2. *Let S_t be the semigroup of a finite range, bounded flip rate particle system on $F^{\mathbf{Z}}$. Then for $f \in D(X)$*

$$\Delta_{S_t f}(x) \leq e^{t\Gamma} \Delta_f(x) \tag{9}$$

Here Γ is an operator defined on $l_1(\mathbf{Z})$, the Banach space of all functions β on $\mathbf{Z}$ such that $\|\beta\|_{l_1(\mathbf{Z})} ;= \sum_x |\beta(x)| < \infty$, by

$$\Gamma\beta(y) = \sum_x \beta(x) \gamma(x, y)$$

where $c_T(y) = \sup_{\eta_1 \eta_2 : \eta_1(x) = \eta_2(x), \forall x \neq y} \{\|c_T(\eta_1, d\xi) - c_T(\eta_2, d\xi)\|_T\}$ and $\gamma(x, y) = \sum_{T \ni x} c_T(y)$

We assume with out loss of generality that $\sup_{x \in \mathbf{Z}} \{\sum_{T \ni x} \sup_{\eta \in \mathbf{Z}} c_T(F^T, \eta)\} \leq 1$ in what follows.

Corollary 1. *Let Ω be the generator of a particle system with bounded flip rates and dinite range R. Then ior each local function f depending on sites on $[-M, M]$ we have*

$$\Delta_{S_t f}(x) \leq 2 \|f\|_\infty \exp\left[-y \log \frac{y}{ebt}\right] \tag{10}$$

whenver $|x| \geq M + 2Rbt$. Here $b = 2(4R+1)$, $y = \dfrac{|x| - M}{2R}$, and $\|f\|_\infty = \sup_{\eta \in F^{\mathbf{Z}}} |f(\eta)|$.

Proof. Note that

$$e^{t\Gamma}(x) = \sum_{n=0}^{\infty} \frac{(t\Gamma)^n}{n!} \Delta_f(x)$$

And since $\gamma(x, y) = 0$ whenever $|x - y| > 2R$, we have

$$\Gamma^n \Delta_f(x) \leq 2^n (4R+1)^n \sup_{y : |y - x| \leq 2Rn} \Delta_f(y).$$

But f is a function depending only on sites on $[-M, M]$, so that

$$\sup_{y:|y-x|\leq 2Rn} \Delta_f(y) \leq 2\|f\|_\infty \chi(M+2nR-x),$$

where $\chi(y)$ is the indicator function of the interval $[0, \infty)$. Therefore

$$\Delta_{S_t f}(x) \leq 2\|f\|_\infty \sum_{n\geq[(|x|-M)/2R]} \frac{(2t(4R+1))^n}{n\,!}$$

from which (10) follows.

Theorem 3. *Let ζ be a finite range, bounded flip rate particle system with state space $X=F_{\mathbf{Z}}$. Let ζ^h be the truncated process, with the same initial distribution, corresponding to the function $h(s)$ with the property that $h(s)\geq Rc(t-s)+R+Rk+M$ for some k that we may choose later to depend on t. The difference in the variational norm between μ_t and μ_t^h, the distributions at time t of the two processes ζ and ζ^h on the σ-field corresponding to the sites $[-M, M]$ is bounded by*

$$\|\mu_t^h - \mu_t\| \leq \frac{2}{Rb} e^{-kR}$$

provided $c=2ebe^R$.

Proof. Let $u(s, \zeta)=S_{t-s}f(\zeta)$, where f is any local function depending on sites on $[-M, M]$. Then since

$$u(s, \zeta_s^h) - u(0, \zeta_0^h) - \int_0^s \left(\frac{\partial u(\sigma, \zeta_\sigma^h)}{\partial \sigma} + (\Omega_\sigma^h u)(\sigma, \zeta_\sigma^h)\right) dv$$

is a martingale and $\frac{\partial u}{\partial s}+\Omega_s^h u=(\Omega_s^h-\Omega)u$, we have

$$(S_{0,t}^h f)(\zeta) - (S_t f)(\zeta) = \int_0^t (S_s^h(\Omega_s^h - \Omega) S_{t-s} f)(\zeta ds) \tag{11}$$

Now for any $g\in d(X)$ we have

$$|(\Omega_s^h - \Omega) g(\zeta)| \leq \sum_{|x|\geq h(s)-R} \Delta_g(x)$$

and since $S_{t-s}f\in D(X)$, we have

$$\sup_{\zeta\in X} |(S_{0,t}^h f)(\zeta) - (S_t f)(\zeta)| \leq \int_0^t \sum_{|x|\geq h(s)-R} \Delta_{S_{t-s}f}(x)\, ds. \tag{12}$$

But from corollary 1 and the fact that $c=2ebe^R=4e(4R+1)e^R$,

$$\int_0^t \sum_{|x|\geq h(s)-R} \Delta_{S_{t-s}f}(x)\, ds \leq \|f\|_\infty \int_0^t \sum_{n\geq Rcs+kR} e^{-n}\, ds$$

$$\leq \frac{2\|f\|_\infty}{bRe^R(e-1)}e^{-kR}$$

The proof is now complete.

Remarks. Exactly the same type of estimates are valid for comparing the speeded up process P^λ with its truncated version. One slight difference of no consequence is that the flip rates may not be bounded by 1. Since our speedups are going to be very minor we assume that $0 \leq \lambda(s) \leq 2$. If we had assumed intially that all flip rates are bounded by $\frac{1}{2}$ then Theorem 3 would apply for the speeded up version as well. So we have

$$\|\mu_t^{\lambda,h} - \mu^\lambda\| \leq \frac{2}{bRe^R(e-1)}e^{-kR} \tag{13}$$

3. Relative entropy of truncated processes

The processes P^h and $P^{\lambda,h}$ have the same initial distribution μ and time dependent gnerators $\Omega^h(s)$ and $\Omega^{\lambda,h}(s) = \lambda(s)\Omega^h(s)$ repectively. Since $\Omega^h(s)$ is a bounded operator on $[0, t]$, Girsanov formula can be used to calculate the relative entropy of $P^{\lambda,h}$ with respect to P^λ on the σ-field $\mathscr{F}_t^0$ generated by $\{\eta(s): 0 \leq s \leq t\}$. Since

$$H(\mu_t^{\lambda,h}|\mu_t^h) \leq H(P^{\lambda,h}|P^h)$$

if we can estimate $H(P^{\lambda,h}|P^h)$, and show that it tends to zero for some reasonable choices of k, λ, and h, we will be done.

Lemma 1.

$$H(P^{\lambda,h}|P^h) \leq 2\int_0^t h(s)\,(\lambda^{t,\tau}(s)-1)^2\,ds \tag{14}$$

Proof. Denote by N(s) the total number of jumps ocurring up to time s. Then the Radon-Nikodym derivative of $P^{\lambda,h}$ respect to P^h on $\mathscr{F}_t^0$ is given by the following formula

$$\frac{dP^{\lambda,h}}{dP^h} = \exp\left\{\int_0^t \log\lambda^{t,\tau}(s)\,dN(s) - \int_0^t \sum_{T\subset[-h(s),h(s)]} c_T(F^T, \zeta_s)\,(\lambda^{t,\tau}(s)-1)\,ds\right\}$$

and a direct computation yields

$$H(P^{\lambda,h}|P^h) = \int_0^t E_{P^{\lambda,h}}\Big(\sum_{T\subset[-h(s),h(s)]} c_T(F^T, \zeta_s)\Big)(\lambda^{t,\tau}(s)\,\log\lambda^{t,\tau}(s) - (\lambda^{t,\tau}(s)-1))\,ds$$

Equation (14) now follows from the assumption of bounded flip rates and the inequality $\ln x \leq x-1$ for $x \leq 1$.

The following lemma is elementary.

Lemma 2. *Let* $\mathscr{A}_{t,\tau}=\{\lambda^{t,\tau}(\cdot)\in L^1([0, t]):\ \int_0^t(\lambda^{t,\tau}(s)-1)\,ds=\tau\}$ *and* $h(s)=Rc(t-s)+R+Rk+M$. *Then*

$$\inf_{\lambda\in\mathscr{A}_{t,\tau}}\int_0^t h(s)\,\lambda^{t,\tau}(s)-1)^2\,ds=Rc\gamma^2\left\{\log\frac{Rct+Rk+M+R}{Rk+M+R}\right\}^{-1} \tag{15}$$

and the infimum is attained at $\lambda^{t,\tau}(s)=1+\dfrac{\alpha}{h(s)}$ *with*

$$\alpha=Rc\tau\left\{\log\frac{Rct+Rk+M+R}{Rk+M+R}\right\}^{-1}.$$

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