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Cancellation of lattices and approximation properties of division algebras

By

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§0. Introduction

Let R be a Dedekind domain with the quotient field K. Let Λ be an R-order. In this general setting, it is proved in [3] that Roiter-Jacobinski type Divisibility Theorem holds for Λ-lattices. As a consequence, for a Λ-lattice L, the following two cancellation properties are equivalent.

(c) If L' is a local direct summand of $nL=L\oplus\cdots\oplus L$ for some $n\geq 0$, then $L\oplus L'\simeq M\oplus L'$ implies $L\simeq M$.

(c') If $L\oplus nL\simeq M\oplus nL$ for some $n\geq 0$, then $L\simeq M$.

As was pointed out in [3], putting $\Gamma:=\mathrm{End}_\Lambda L$ and $B:=K\Gamma$, there is an intimate connection between cancellation property and the approximation property of the group of Vaserstein $\widetilde{E}(\widehat{B})$ in the idele topology of $\widehat{B}^\times$, of which precise definitions will be recalled in §1.

Here we only indicate, $\widehat{R}:=\Pi R_p$, the direct product of p-adic completions over all maximal ideals of R, $\widehat{M}:=M\otimes_R\widehat{R}$ for any R-alegbra M, and $\widetilde{E}(C):=<(1+xy)(1+yx)^{-1}|x,y\in C,1+xy\in C^\times>$ for any ring $C\ni 1$. Our first remark is

Proposition 1 *(proof in 1.5). The property (c') for L is equivalent with the following property (c'') of* Γ.

(c'') $\widetilde{E}(\widehat{B})\subset\widehat{\Gamma}^\times B^\times$ *as subsets of* $\widehat{B}^\times$.

0.1. We shall consider, for any finite dimensional K-algebra B, the following three *approximation properties over* R, in the idele topology of $\widehat{B}^\times$.

(a) Strong approximation property:

$\widetilde{E}(B)$ is dense in $\widetilde{E}(\widehat{B})$

(a') $B^\times$-approximation property:

$\widetilde{E}(\widehat{B})$ is contained in the closure of $B^\times$.

(a'') $\widehat{R}^\times B^\times$-approximation property:

$\widetilde{E}(\widehat{B})$ is contained in the closure of $\widehat{R}^\times B^\times$.

There are the obvious implications (a) $\Rightarrow$ (a') $\Rightarrow$ (a''). Our second (rather

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obvious) remark is

Proposition 2 (*proof in 1.2*). *The property* (a$''$) *for B is equivalent with the* (*validity of*) *property* (*c*$''$) *for any Λ-lattice L such that $K\mathrm{End}_{\Lambda}L \simeq B$.*

In the following cases, the property (a) always holds.

(1) B is commutative (since $\widetilde{E}(B)=\widetilde{E}(\widehat{B})=1$, by definitions).

(2) R is semi-local (by the Chinese Remainder Theorem).

(3) $B=M_n(C)$ by some K-algebra $C\,(n\geq 2)$ (cf [3]).

0.2. We shall give the following reduction to division algebras.

Theorem 1 (*proof in 2.3*). *Writing as $B/J(B)=\bigoplus_{i=1}^{m} M_{n_i}(D_i)$, with the Jacobson radical $J(B)$ and the division algebras D_i, in such an ordering that $n_i=1$ $(1\leq i\leq r)$ and $n_i\geq 2\,(r<i\leq n)$, we have*

(*i*) (a$'$) *for* $B \Leftrightarrow$ (a$'$) *for* $D_i\,(1\leq i\leq r)$.

(*ii*) (a) (*resp.* (a$''$) *for* $B\Rightarrow$ (a) (*resp.* (a$''$)) *for* $D_i\,(1\leq i\leq r)$.

Thus the approximation properties of general B can be reduced, more or less, to that of non-commutative division algebras over non-semi-local R, and then under a reasonable restriction, to that of central division ones, by 1.6.

Since PF-fields are the most familiar and important source of non semi-local Dedekind domains, now we restrict our attention to central division algebras over PF-fields and recall some basic facts and known results.

0.3. Assume that K is a PF-field in the sense of Artin [1], Chap.12, and let D be a finite dimensional non-commutative central division algebra over K.

In particular, there is given a set of valuations $\mathfrak{V}$ of K, satisfying the product formula $\prod_{\mathfrak{V}}|x|_v=1$ for any $x\in K^{\times}$. In fact K is either a number field or a function field (of one variable) over the constant field $K_0:=\{x\in K \mid |x|_v\leq 1$ for any $v\in\mathfrak{V}\}$. K is called a global field if either it is a number field or a functin field with $\# K_0<\infty$.

(i) Let P be proper non-empty subset of $\mathfrak{V}$ consisting of non-archimedian valuations. Then $R(P):=\{x\in K \mid |x|_p\leq 1$ for any $p\in P\}$ is a Dedekind domain (with an additional requirement $R(P)\supset K_0$, if K is a function field) having K as its quotient field. Conversely, any such Dedekind domain R in K is obtained as $R=R(P)$ by some P.

Consider the following condition (EC) for D over $R=R(P)$, which is known as Eichler's condition when K is a global field.

(EC) There is at least one $v\in\mathfrak{V}\backslash P$, such that the completion $D_v=D\otimes_K K_v$ is not a division algebra.

(ii) If K is a global field, by Wang-Platonov Theorem (cf. [6]), $[D^{\times},D^{\times}]=\widetilde{E}(D)=$ the kernel of the reduced norm. Hence the well known Eichler-Kneser Strong Approximation Theorem [2], [4] (and its analog due independently to Morita [8] and Swan [9], when K is a function field with $\# K_0<\infty$) implies

(SAT) (a) for D over $R(P) \Leftrightarrow$ (EC) for D over $R(P)$.

0.4. Apart from global fields, we shall prove;

Theorem 2 (proof in 3.4). *For any PF-field K,*
(a″) for D over $R(P) \Rightarrow$ (EC) for D over $R(P)$.

All in all, the most optimistic speculation would be "(a) $\Leftrightarrow$ (a′) $\Leftrightarrow$ (a″) $\Leftrightarrow$ (EC)" for any central division algebras over any PF-fields. In this direction we can extend our previous result [11] as,

Theorem 3 (*proof in 4.4*). *When K is an algebraic function field of one variable over the reals,*
(EC) *for D over $R(P) \Rightarrow$* (a) *for D over $R(P)$.*

1. Idele topology

Let R be a Dedekind domain with the quotient field K. A finitely generated R-module L is called an *R-lattice*, if it is torsion free (or equivalently projective) over R, then $K \otimes_R L$ is a finite dimensionl K-vector space and by the natural embedding $L \to K \otimes L$, one can identify as $K \otimes L = KL$. An R-algebra Λ is called an *R-order* if it is an R-lattice, then $K\Lambda = K \otimes \Lambda$ is a finite dimensional K-algebra. When a finite dimensional K-algebra B is given, we call that Γ is an *R-order of B*, if Γ is an R-order and $B = K\Gamma$.

For a maximal ideal p of R, let R_p *always denote the p-adic completion of R.* Let $\widehat{R} := \prod R_p$, the product over all maximal ideals of R. By the diagonal embedding $R \to \widehat{R}$, $\widehat{R}$ is an R-algebra which is faithfully flat as an R-module. For any R-module M, put

$$M_p := M \otimes_R R_p,\ \widehat{M} := M \otimes_R \widehat{R}.$$

We shall be concerned with only the following two special cases.

1) Γ is an R-order : Then, since Γ is a finitely generated projective R-module, $\widehat{\Gamma} := \Gamma \otimes_R \prod R_p \simeq \prod (\Gamma \otimes_R R_p) = \prod \Gamma_p$.

2) B is a finite dimensional K-algebra : Then $\widehat{B} := B \otimes_R \widehat{R} \simeq B \otimes_K K \otimes_R \widehat{R} \simeq B \otimes_K \widehat{K}$, and since $\widehat{R}$ is faithfully flat over R, one may canonically view as $\widehat{B} \supset \widehat{\Gamma}$, B and $B \cap \widehat{\Gamma} = \Gamma$. Moreover, there is a natural identification $\widehat{B} \simeq \varinjlim \widehat{\Gamma}/r$ $(r \in R \setminus \{0\}) \simeq \prod' B_p$ (w.r.t Γ_p), where the last term denotes the restricted direct product i.e. $\prod' B_p$ (w.r.t. Γ_p) $:= \{x = (x_p) \in \prod B_p | x_p \in \Gamma_p$ for almost all $p\}$. The *adele topology* on B is defined as the unique topology which induces on $\widehat{\Gamma}$ the direct product of p-adic topology $\prod \Gamma_p$, for one (hence any) R-order Γ of B. The name comes from the fact that $\widehat{K}$ with this topology is called the (restricted) adele ring of K.

The *idele topology* in $\widehat{B}^\times$ is defined as the unique topology which induces on $\widehat{\Gamma}^\times$ the direct product of p-adic topologies $\prod \Gamma_p^\times$, for one (hence any) R-order Γ of B. The following explicit description of the idele topology will be useful for us.

1.1. For any R-order Γ of B and non-zero $r\in R$, put

$$(0)\begin{cases} U_p(\Gamma, r) := \Gamma_p^\times \cap (1+r\Gamma_p) = \begin{cases} \Gamma_p^\times & \text{if } r\in R_p^\times \\ 1+r\Gamma_p & \text{if } r\in pR_p. \end{cases} \\ U(\Gamma, r) := \prod_p U_p(\Gamma, r) = \widehat{\Gamma}^\times \cap (1+r\widehat{\Gamma}), \\ \Gamma(r) := R+r\Gamma, \text{ which is an } R\text{-order of } B \text{ again.} \end{cases}$$

By definitions, we have

(1) $\{U(\Gamma, r)|r\in R\backslash\{0\}\}$ is a fundamental system of neighbourhoods of 1 in $\widehat{B}^\times$ in the idele topology (for any one fixed Γ).

(1′) $\{r\widehat{\Gamma}|r\in R\backslash\{0\}\}$ is a fundamental system of neighbourhoods of 0 in $\widehat{B}$ in the adele topology.

Let H be a subgroup of $\widehat{B}^\times$, and $\overline{H}$ will denote the closure of H in $\widehat{B}^\times$.

(2) If $H\cap(1+r\widehat{\Gamma})\subset\widehat{\Gamma}^\times$ for some Γ and $r\in R\backslash\{0\}$, (in particular if $H\cap\widehat{\Gamma}\subset\widehat{\Gamma}^\times$), then the idele topology of $\widehat{B}^\times$ and the adele topology of $\widehat{B}$ induce the same topology on H. Indeed, $H\cap U(\Gamma, rr')=H\cap(1+rr'\widehat{\Gamma})$ or any $r'\in R\backslash\{0\}$.

Since $\Gamma(r)_p^\times=R_p^\times U_p(\Gamma, r)$, we have

(3) $\widehat{R}^\times U(\Gamma, r)=\widehat{\Gamma(r)}^\times$

(4) If $\widehat{R}^\times\subset\overline{H}$, then $HU(\Gamma, r)\supset\widehat{R}^\times$ so that $\overline{H}=\bigcap_{r\neq0}H\widehat{\Gamma(r)}^\times=\bigcap_{r\neq0}\widehat{\Gamma(r)}^\times H$.

1.2. Proof of Proposition 2 §0. For any R-order Γ of B, put $\Lambda=\Gamma^{op}$, the opposite ring of Γ and $L:=\Gamma$, then $\mathrm{End}_\Lambda L=\Gamma$. Hence the condition ((c″) for any L such that $K\ \mathrm{End}_\Lambda\ L\simeq B$) is equivalent with the condition $\widetilde{E}(\widehat{B})\subset\widehat{\Gamma}^\times B^\times$ for any Γ. But we have $\bigcap_\Gamma\widehat{\Gamma}^\times B^\times=\overline{\widehat{R}^\times B^\times}$, since $\widehat{\Gamma}^\times B^\times$ is closed and contains $\widehat{R}^\times B^\times$, so $\overline{\widehat{R}^\times B^\times}\subset\bigcap_\Gamma\widehat{\Gamma}^\times B^\times\subset\bigcap_{r\neq0}\widehat{\Gamma(r)}^\times B^\times$, while we have $\overline{\widehat{R}^\times B^\times}=\bigcap_r\widehat{\Gamma(r)}^\times B^\times$, by (4).

1.3. Results of Vaserstein. Let A be a ring with 1, and $E_n(A)$ be the elementary subgroup of $GL_n(A):=M_n(A)^\times$. By the usual embedding $x\mapsto\begin{pmatrix}x & 0\\ 0 & 1_{n-1}\end{pmatrix}$, we consider as $A^\times=GL_1(A)\subset GL_n(A)$ $(n\geq2)$. Let $\widetilde{E}(A)$ be the group of Vaserstein, i.e. the subgroup of $A^\times$ given by the generators as

$$\widetilde{E}(A) := <(1+xy)(1+yx)^{-1}|x, y\in A, 1+xy\in A^\times>.$$

The commutator subgroup $[A^\times, A^\times]$ is always contained in $\widetilde{E}(A)$. Further, if A is local, $\widetilde{E}(A)=[A^\times, A^\times]$.

If A is semi-local, the well known Lemma of Bass and the fundamental results of Vaserstein ([10], Th.3.6) state:

(5) $GL_n(A)=A^\times E_n(A)$ $(n\geq2)$.

(6) $A^{\times}\cap E_n(A)=\widetilde{E}(A)\quad (n\geq 2)$.

1.4.

Lemma. *Let B be a finite dimensional K-algebra and Γ be an R-order of B. Then the equality* (5) *of Bass* (*resp.* (6) *of Vaserstein*) *holds for $A=\widehat{B}$ or $\widehat{\Gamma}$,* (*where $\widehat{B}$ or $\widehat{\Gamma}$ is not semi-local if R is not semi-local.*)

Proof. In the proof of [10] Th.3.6 (a), where semi-locality of A is assumed, it is in fact proved that

(i) If the ring A satisfies the following condition (5′), then (5) holds.

(5′) For any finitely generated left ideal L and $x\in A$,

$$Ax+L=A\Rightarrow (x+L)\cap A^{\times}\neq\phi.$$

(ii) If A satisfies (5′) and moreover the following (6′), then (6) holds.

(6′) $Ax_1+Ax_2=A\Rightarrow {}^{\forall}y\in A,\ {}^{\exists}v,\ q,\ u\in A$ such that $x_1+vx_2\in A^{\times}, 1-yqv\in A^{\times}$, $x_1+u(x_2+yx_1)\in A^{\times}$, $x_1+u(x_2+yqx_1)\in A^{\times}$.

Now, let $A=\prod' A_p$ (w.r.t C_p) be the restricted direct product of A_p with respect to its subring C_p, over some index p's. If each A_p, C_p satisfies (5′) and (6′), it is easy to see that A itself satisfies (5′) and (6′). This applies for $\widehat{B}$ or $\widehat{\Gamma}$, since B_p and Γ_p are semi-local.

1.5. Proof of Proposition 1 §0. As is well known (cf [3] §2 and §3), the property (c') is equivalent with the following (c''') $\widehat{B}^{\times}\cap GL_n(B)\,GL_n(\widehat{\Gamma})=B^{\times}\widehat{\Gamma}^{\times}$ for $n\geq 2$.

By 1.4, we have

1) $GL_n(B)=B^{\times}E_n(B)$ 2) $GL_n(\widehat{\Gamma})=E_n(\widehat{\Gamma})\widehat{\Gamma}^{\times}$.

Since $E_n(B)$ is dense in $E_n(\widehat{B})$ in the idele topology of $\widehat{B}^{\times}$ (cf [3] 1.2.1)

3) $E_n(B)GL_n(\widehat{\Gamma})=E_n(\widehat{B})GL_n(\widehat{\Gamma})$.

Using 1), 3), 2) in this order, we have : $GL_n(B)GL_n(\widehat{\Gamma})=B^{\times}E_n(B)GL_n(\widehat{\Gamma})$ $=B^{\times}E_n(\widehat{B})GL_n(\widehat{\Gamma})=B^{\times}E_n(\widehat{B})E_n(\widehat{\Gamma})\widehat{\Gamma}^{\times}=B^{\times}E_n(\widehat{B})\widehat{\Gamma}^{\times}$.

Hence, the left hand side of (c‴) $=\widehat{B}^{\times}\cap B^{\times}E_n(\widehat{B})\widehat{\Gamma}^{\times}=B^{\times}(\widehat{B}^{\times}\cap E_n(\widehat{B}))\widehat{\Gamma}^{\times}$ $=B^{\times}\widetilde{E}(\widehat{B})\widehat{\Gamma}^{\times}$, the last equality by 1.4 again. This implies that (c‴) is equivalent with (c″).

1.6. Change of the base field. Let K' be a finite extension field of K contained in the center of B, and let R' be the integral closure of R in K'. Then R' is a Dedekind domain with the quotient K', and B is a finite dimensional K'-algebra. Assume the following condition

(f) R' is a finitely generated R-module.

Then there are canonical isomorphisms $\widehat{R'}\simeq R'\otimes_R\widehat{R}$ and $K'\otimes_{R'}\widehat{R'}\simeq K'\otimes_R\widehat{R}$ (cf. [7] Th.1 and Prop. 4 Chap. II §3), so that $B\otimes_{R'}\widehat{R'}\simeq B\otimes_R\widehat{R}$ including the topology. Hence the approximation property (a) (resp. (a′)) of B over R is equivalent with that of B over R', and (a″) over R implies that over R'.

(i) For a residually separable algebra B (i.e. $B/J(B)$ is separable) the

$B^\times$-approximation problem is reduced, by Theorem 1, to that of a central division algebra.

(ii) If K is a PF-field, the condition (f) always holds (cf. [5] Th.72), so that we get the reduction to a central division algebra even for residually inseparable case.

2. Reduction to a division algebra

Let B be a finite dimensional K-algebra with the Jacobson radical $J=J(B)$, $\varphi: B\to B' := B/J$ be the canonical K-morphism and $\Gamma' := \varphi(\Gamma)$. Then Γ' is an R-order in B', and φ induces the following surjective morphisms : $\varphi_0 : \Gamma\to\Gamma'$, $\hat{\varphi} := \varphi\otimes 1 : \hat{B}=B\otimes\hat{R}\to B'\otimes\hat{R}=\hat{B}'$ and $\hat{\varphi}_0 := \varphi_0\otimes 1 : \hat{\Gamma}=\Gamma\otimes\hat{R}\to\Gamma'\otimes\hat{R}=\hat{\Gamma}'$.

Since $\hat{R}$ is faithfully flat over R,

1) $\mathrm{Ker}\varphi_0=\Gamma\cap J\subset J(\Gamma)$, $\mathrm{Ker}\hat{\varphi}=J\otimes\hat{R}=\hat{J}\subset J(\hat{B})$,

2) Viewing as $\hat{B}\supset\hat{\Gamma}$, B and $\hat{\Gamma}\cap B=\Gamma$, φ_0, $\hat{\varphi}_0$, φ is the restriction of $\hat{\varphi}$ to Γ, $\hat{\Gamma}$, B respectively.

By 1), $1+\hat{J}\subset\hat{B}^\times$ so that $\hat{\varphi}$ induces the exact sequence of groups :

3) $1\to 1+\hat{J}\to\hat{B}^\times\to\hat{B}'^\times\to 1$, and $\hat{\varphi}^{-1}(\hat{B}'^\times)=\hat{B}^\times$.

Consequently, we have

4) $\hat{\varphi}(\tilde{E}(\hat{B}))=\tilde{E}(\hat{B}')$.

By the same reasoning, we have

5) $\hat{\varphi}(\tilde{E}(B))=\tilde{E}(B')$.

Also we have

6) $\hat{\Gamma}^\times=\hat{\varphi}_0^{-1}(\hat{\Gamma}'^\times)$, which in turn implies

7) $\hat{\varphi}(U(\Gamma, r))=U(\Gamma', r)$, in the notation of 1.1.

2.1.

Lemma. *Let H be a subgroup of $\hat{B}^\times$ and $\bar{H}$ be its closure in $\hat{B}^\times$.*

(*i*) $\tilde{E}(\hat{B})\subset\bar{H}\Rightarrow\tilde{E}(\hat{B}')\subset\overline{\hat{\varphi}(H)}$

(*ii*) *If $1+\hat{J}\subset\bar{H}$, then the converse implication* ($\Leftarrow$) *also holds.*

(*iii*) $1+\hat{J}\subset\bar{B}^\times$.

Proof. (i) and (ii) : $(\tilde{E}(\hat{B})\subset\bar{H})\overset{(1)1.1}{\Leftrightarrow}(\tilde{E}(\hat{B})\subset HU(\Gamma, r)$ for any $r\in R\setminus\{0\})$ $\overset{4)\&7)}{\Rightarrow}(\tilde{E}(\hat{B}')\subset\hat{\varphi}(H)U(\Gamma', r))$ for any $r\in R\setminus\{0\})$ $\overset{3)4)\&7)}{\Rightarrow}(\tilde{E}(\hat{B})\subset(1+\hat{J})HU(\Gamma, r)$ $(=HU(\Gamma, r)$ if $\bar{H}\supset 1+\hat{J})$ for any $r\in R\setminus\{0\})$.

(iii) Since any element of $\hat{J}$ is nilpotent, $(1+\hat{J})\cap(1+r\hat{\Gamma})=1+(\hat{J}\cap r\hat{\Gamma})\subset\hat{\Gamma}^\times$, hence by (2) 1.1, the idele topology on $1+\hat{J}$ is induced from the adele topology. Since J is dense in $\hat{J}$ in the adele topology, $1+J$ is dense in $1+\hat{J}$ in the idele topology so that $1+\hat{J}\subset(1+J)U(\Gamma, r)\subset B^\times U(\Gamma, r)$ for any $r\in R\setminus\{0\}$.

2.2.

Lemma. *Let $B=\bigoplus_{i=1}^{m}B_i$ be the ring direct sum of finite dimensional K-algebras. Then we have the following implications.*

(*i*) (a) (*resp.*(a′)) *for* $B\Leftrightarrow$(a) (*resp.* (a′)) *for any* $B_i(1\leq i\leq m)$.

(*ii*) (a″) *for* $B\Rightarrow$(a″) *for any* $B_i(1\leq i\leq m)$.

Proof. Let Γ_i be an R-order of B_i, then $\Gamma:=\oplus\Gamma_i$ is an R-order of B. By the canonical isomorphism $\widehat{B}=B\otimes\widehat{R}\simeq\oplus(B_i\otimes\widehat{R})=\oplus\widehat{B}_i$, $\widehat{B}^\times\simeq\Pi\widehat{B}_i^\times$, $\widehat{\Gamma}\simeq\Pi\widehat{\Gamma}_i^\times$, $U(\Gamma,r)\simeq\Pi U(\Gamma_i,r)$, $\widetilde{E}(B)\simeq\Pi\widetilde{E}(B_i)$ and $\widetilde{E}(\widehat{B})\simeq\Pi\widetilde{E}(\widehat{B}_i)$, the claims are completely obvious.

2.3. Proof of Theorem 1 §0. Put $B_i=M_{n_i}(D_i)$, $n_i=1\,(1\leq i\leq r)$, $n_i\geq 2\,(r<i\leq m)$. Recall that (a) holds for $B_i\,(r<i\leq m)$ ((3) of 0.1) and apply 2.1 and 2.2, then we get the following implications which obviously prove Theorem 1.

(a) for $B\Rightarrow$(a) for $B'\Leftrightarrow$(a) for $D_i(1\leq i\leq r)$

(a′) for B$\Leftrightarrow$(a′) for $B'\Leftrightarrow$(a′) for $D_i(1\leq i\leq r)$

(a″) for $B\Leftrightarrow$(a″) for $B'\Rightarrow$(a″) for $D_i(1\leq i\leq r)$.

3. (a″)$\Rightarrow$(EC) for a PF-field

Let K be a PF-field in the sense of [1], D be a central division K-algebra of dimension n^2, $[D:K]=n^2$. Let $D_v:=D\otimes_K K_v$ be the completion at $v\in\mathfrak{V}$. Let $\mathfrak{N}:D\rightarrow K$ be the reduced norm and $\mathfrak{N}_v:D_v\rightarrow K_v$ be its extension.

If D_v is a division algebra, $D_v\ni x\mapsto|\mathfrak{N}_v x|_v^{1/n}$ defines a norm of D_v as a K_v-vector space. While for any basis $\{e_i|1\leq i\leq n^2\}$ of D over K, writing $x=\sum\xi_i e_i\in D_v$, $x\mapsto\mathrm{Max}|\xi_i|_v$ is also a norm of D_v. Hence there is a constant $c_v>0$ such that

(1) $\mathrm{Max}_i|\xi_i|_v\leq c_v|\mathfrak{N}_v x|_v^{1/n}\,(x=\sum\xi_i e_i)$.

For almost all v, we have : v is non-archimedean; $\{\sum\xi_i e_i|\max_i|\xi_i|_v\leq 1\}$ is a maximal order of D_v ; $|\det\mathrm{Tr}\,(e_ie_j)|_v=1$. Hence for almost all v such that D_v is a division algebra, D_v/K_v is unramified and $|\mathfrak{N}_v x|_v^{1/n}=\mathrm{Max}|\xi_i|_v$. Thus we can choose c_v as

(1′) $c_v=1$ for almost all v such that D_v is a division algebra.

Let R be a Dedekind domain with the quotient field K, so that it has the form $R=R(P):=\{\xi\in K||\xi|_p\leq 1$ for any $p\in P\}$ by some non-empty proper subset P consisting of non-archimedeam valuations of $\mathfrak{V}$. For a fixed R, we can obviously choose a basis $\{e_i|1\leq i\leq n^2\}$ satisfying

(2) $\Gamma=\sum_{i=1}^{n^2}Re_i$ is an R-order of D, and $e_i=1$.

Then $\Gamma(r):=R+r\Gamma$ is also an R-order for any $r(\neq 0)\in R$.

3.1.

Lemma. *Assume that D does not satisfy the Eichler's condition (EC) over*

$R=R(P)$, i.e. the following $\neg(EC)$ is satisfied.

$\neg(EC)$: *D_v is a division algebra for any $v\in\mathfrak{D}\backslash P$.*

(*i*) *Let $\{e_i\}$ be a basis of D satisfying* (2), *then there is a positive constant c depending only on $\{e_i\}$ but not on $r(\neq 0)\in R$ such that*

$$\prod_P |r|_p<c\Rightarrow\Gamma(r)^\times=R^\times.$$

(*ii*) *$\widehat{R}^\times D^\times$ is closed in $\widehat{D}^\times$.*

Proof. (i) It suffices to take $c:=\prod_{\mathfrak{D}\backslash P}c_v^{-1}$ (which is well defined by (1')). Indeed, if $\Gamma(r)^\times\neq R^\times$, there is some $x=\sum\xi_i e_i\in\Gamma(r)^\times$ with $\xi:=\xi_i\neq 0$ for some $i\geq 2$. At $p\in P$,

(3) $|\xi|_p\leq|r|_p=|r|_p|\mathfrak{N}x|_p^{1/n}$.

Using the product formula, (1) at $v\in\mathfrak{D}\backslash P$ and (3) at $p\in P$, the product formula again, in this order, we get

$$1=\prod_{\mathfrak{D}}|\xi|_v=\prod_{\mathfrak{D}\backslash P}|\xi|_v\times\prod_P|\xi|_p\leq\prod_{\mathfrak{D}\backslash P}c_v|\mathfrak{N}x|_v^{1/n}\times\prod_P|r|_p|\mathfrak{N}x|_p^{1/n}$$
$$=\prod_{\mathfrak{D}\backslash P}c_v\times\prod_P|r|_p=c^{-1}\prod_P|r|_p.$$

(ii) Put $R(c):=\{r\in R\backslash\{0\}\mid\prod_P|r|_p<c\}$. If $r\in R(c)$, by (i), we have $\widehat{\Gamma}(r)^\times\cap D^\times=\Gamma(r)^\times=R^\times$. This obviously implies

(4) $\bigcap_{r\in R(c)}(D^\times\widehat{\Gamma(r)}^\times)=D^\times(\bigcap_{r\in R(c)}\widehat{\Gamma(r)}^\times)$.

Then together with (4)1.1, we have

$$\overline{D^\times\widehat{R}^\times}=\bigcap_{r\neq 0}(D^\times\widehat{\Gamma(r)}^\times)\subset\bigcap_{r\in R(c)}(D^\times\widehat{\Gamma(r)}^\times)=D^\times(\bigcap_{r\in R(c)}\widehat{\Gamma(r)}^\times)=D^\times\widehat{R}^\times\subset\overline{D^\times\widehat{R}^\times}.$$

3.2. As usual, we consider $D_p^\times$ as the subgroup of $\widehat{D}^\times$ consisting of the elements $x=(x_p)\in\widehat{D}^\times$ such that $x_q=1$ for $q\in P\backslash\{p\}$, Under this convention, the following is obvious.

(5) $\#P\geq 2\Rightarrow\widehat{R}^\times D^\times\cap D_p^\times\subset K_p^\times$.

If $\#P<\infty$, then R is semi-local and $\overline{D}^\times=\widehat{D}^\times$, hence 3.1 implies

(6) $2\leq\#P<\infty\Rightarrow(EC)$.

Indeed : $\neg(EC)$ implies $\overline{\widehat{R}^\times D^\times}=\widehat{R}^\times D^\times$ so that $D_p^\times\subset\widehat{R}^\times D^\times$ hence $D_p^\times\subset D_p^\times\cap\widehat{R}^\times D^\times\subset K_p^\times$, a contradiction to the assumption that D is non-commutative.

3.3.

Lemma. *Let D be a central division algebra over a PF-field K. Then D_v is not a division algebra for infinitely many $v\in\mathfrak{D}$.*

Proof. If $\mathfrak{D}$ contains at least one archimedean valuation (i.e. if K is a number field), as is well known, much stronger results are known. Assume that $\mathfrak{D}$ consists of non-archimedean valuations. If $\#\{v\in\mathfrak{D}|D_v$ is not a division algebra$\}<\infty$, then obviously we can choose a subset P of $\mathfrak{D}$ such that $2\leq\#P<\infty$ and $\neg$(EC), a contradiction with (6)3.2.

3.4. Proof of Theorem 2 We shall prove:

$\neg(\mathrm{EC}) \Rightarrow [\hat{D}^\times, \hat{D}^\times] \not\subset \overline{\hat{R}^\times D^\times}$.

Suppose not, then $[\hat{D}^\times, \hat{D}^\times] \subset \hat{R}^\times D^\times$ by 3.1, so that $[D_p^\times, D_p^\times] = D_p^\times \cap [\hat{D}^\times, \hat{D}^\times] \subset D_p^\times \cap \hat{R}^\times D^\times \subset K_p^\times$ for any $p \in P$. It is a contradiction, since if x, y do not commute in $D_p^\times$, then one of $[x, y]$ and $[x, 1+y]$ does not belong to $K_p^\times$.

4. (EC)$\Rightarrow$(a) for a real coefficient case

We shall derive our Theorem 3 from our previous result [11], where it is proved only for a special case of $K = \boldsymbol{R}(X)$. For this purpose, we prepare a few lemmas, which are of quite general nature, but regretfully, effectively applicable only for a very restricted situation like in Theorem 3, so that we state them only for PF-fields.

4.1. Let D be a central division algebra over a PF-field K and $R = R(P)$ as in 0.3. For a fixed $p_0 \in P$, as usual, we identify $D_{p_0}^\times$ as the (closed normal) subgroup of $\hat{D}^\times$, consisting of elements $x = (x_p) \in \hat{D}^\times \subset \prod D_p^\times$ with $x_p = 1$ for $p \neq p_0$. Then $\{\tilde{E}(D_p) | p \in P\}$ generates a dense subgroup of $\tilde{E}(\hat{D})$ in $\hat{D}^\times$ (cf. [2] §51). Hence a closed subgroup H of $\hat{D}^\times$ contains $\tilde{E}(\hat{D})$ if and only if it contains $\tilde{E}(D_p) = [D_p^\times, D_p^\times]$ for all $p \in P$. By the Chinese Remainder Theorem, 'all' can be replaced by 'almost all'. In particular we have:

(1) (a) for D over $R \Leftrightarrow \overline{[D_p^\times, D_p^\times] \subset \tilde{E}(D)}$ for almost all p,

and the corresponding (1') (resp. (1'')) for (a') (resp. (a'')).

Let K' be a finite extension field of K, and let P' be the set of all (non-equivalent) valuations of K' lying over P, $P' = \{p' | p' \supset p, p \in P\}$. The integral closure R' of R in K' is given by $R' = \{0\} \cup \{x \in K'^\times | |x|_{p'} \leq 1 \text{ for any } p' \in P'\}$.

Put $D' := D \otimes_K K'$. By 1.6, $\hat{D}' := D' \otimes_{R'} \hat{R}' \simeq D' \otimes_R \hat{R} \supset D \otimes_R \hat{R} = \hat{D}$ as topological rings, and

(2) $\hat{D}'^\times \supset \hat{D}^\times, \hat{D}'^\times \supset \prod_{p' \supset p} D'^\times_{p'} \simeq D'^\times_p \supset D_p^\times$ as topological groups.

In the following $\overline{(\quad)}$ denotes the closure in $\hat{D}'^\times$.

Let consider the following condition (*).

(*) For almost all $p \in P$, $p' \supset p \Rightarrow [D'^\times_{p'}, [D_p^\times, D_p^\times]] = [D'^\times_{p'}, D'^\times_{p'}]$.

Lemma. *Assume that the condition* (*) *holds. Then*
(a'') *for D over $R \Rightarrow$* (a) *for D' over R'.*

Proof. By the Chinese Remainder Theorem, $D'^\times$ is dense in $\prod_{p' \supset p} D'^\times_{p'}$.

Hence, by (2), $[D'^\times_{p'}, [D_p^\times, D_p^\times]] \subset \overline{[D'^\times, [D_p^\times, D_p^\times]]}$, so that the assumption (*) implies

(3) $[D'^\times_{p'}, D'^\times_{p'}] \subset \overline{[D'^\times, [D_p^\times, D_p^\times]]}$ for almost all $p \in P$.

On the other hand we have

(a″) for D over $R \underset{(1'')}{\Leftrightarrow} [D_p^\times, D_p^\times] \subset \overline{\widehat{R}^\times D^\times}$ for almost all $p \in P \Rightarrow [D'^\times, [D_p^\times, D_p^\times]] \subset \overline{[D'^\times, \widehat{R}^\times D^\times]} \subset \overline{[D'^\times, D^\times]} \subset \overline{[D'^\times, D'^\times]} = \widetilde{E}(D')$.

Hence by (3), we have $[D'^\times_{p'}, D'^\times_{p'}] \subset \overline{\widetilde{E}(D')}$ for almost all p, which is equivalent with ((a) for D' over R') by (1).

4.2. Now assume that the constant field $K_0 = \boldsymbol{R}$, i.e. K is an algebraic function field of one variable over the reals.

Recall from [11] that $Br(K) \simeq K^\times / \mathfrak{N}(K(\sqrt{-1})^\times) = K^\times / (K^2+K^2) \cap K^\times$, so that any central division algebra D over K is a quaternion algebra of the form $D \simeq \{-1, f\}$ with $f \in K^\times$. D is trivial if and only if $f \in K^2+K^2$.

We call a valuation $v \in \mathfrak{D}$ real (resp. imaginary) if the residue field is isomorphic to $\boldsymbol{R}$ (resp. $\boldsymbol{C}$). $K(\sqrt{-1})$ is an algebraic function field of one variable over $\boldsymbol{C}$, so the corresponding $\mathfrak{D}'$ is identified with the Riemann surface $\mathfrak{R}$, and $K(\sqrt{-1})$ with the field of all meromorphic functions on $\mathfrak{R}$. Since a real valuation v of K does not decompose on $K(\sqrt{-1})$, the set $RP(K)$ of all real valuations can be embedded in $\mathfrak{R}$ as a finite disjoint union of closed curves. Then we have

$$K = \{\varphi \in K(\sqrt{-1}) \mid \varphi(z) \in \boldsymbol{R} \text{ for } z \in RP(K)\}.$$

Furthermore, as shown in [11],

$$K^2+K^2 = \{f \in K \mid f(z) \geq 0 \text{ for } z \in RP(K)\},$$

so $\{-1, f\}$ is trivial for such f.

Let P be a non-empty proper subset of $\mathfrak{D}$.

Lemma. *If D satisfies* (EC) *over $R(P)$, then D can be written as $D = D_0 \otimes_{R(g)} K$, where $g \in R(P) \setminus \boldsymbol{R}$ and D_0 is a central division $\boldsymbol{R}(g)$-algebra satisfying* (a) *over $\boldsymbol{R}[g]$.*

Proof. (EC) for D means that D_{v_0} is trivial for some $v_0 \in \mathfrak{D} \setminus P$. From Riemann-Roch Theorem, for any $f \in K^\times$ we can find $h \in K^\times$ such that $g := h^2 f$ has the unique pole at v_0. Therefore D can be written as $D = \{-1, g\}$, where $g \in R(P)$ and has the unique pole at v_0.

Since D_{v_0} is trivial, we have either (i) v_0 is imaginary or (ii) v_0 is real and g is positive around v_0. In any case, g is bounded from below on $RP(K)$, since g has no pole other than v_0. So, $g + c \in K^2+K^2$ for some $c \in \boldsymbol{R}$, hence $D = \{-1, g\} = \{-1, g(g+c)\} \simeq D_0 \otimes_{R(g)} K$ where $D_0 = \{-1, g(g+c)\}$ over $\boldsymbol{R}(g)$ which satisfies (EC) over $\boldsymbol{R}[g]$ since $X(X+c)$ is monic and quadratic. From our previous result [11], D_0 satisfies (a) over $\boldsymbol{R}[g]$.

4.3.

Lemma. *If K is an algebraic function field of one variable over $\boldsymbol{R}$, then the condition* $(*)$ *in* 4.1 *is satisfied for any D.*

Proof. Note that D_p is unramified for almost all $p \in P$. If D_p is trivial, then

$D_p^\times = GL(2, K_p)$ and $[D_p^\times, D_p^\times] = SL(2, K_p)$. In this case $[D'^\times_{p'}, [D_p^\times, D_p^\times]] = [GL(2, K'_{p'}), SL(2, K_p)]$ is a normal subgroup of $SL(2, K'_{p'})$ not contained in its center, so it must coincide with $SL(2, K'_{p'})$.

If D_p is an unramified quaternion algebra, then p is real so that $-1 \notin K_p^2$ and $K_p^2 + K_p^2 = K_p^2$. Thus the reduced norm $\mathfrak{N}_p : D_p^\times \to K_p^\times$ maps $D_p^\times$ onto $K_p^{\times 2}$ with the kernel $[D_p^\times, D_p^\times]$. This implies $D_p^\times = K_p^\times [D_p^\times, D_p^\times]$, so that $[D_p^\times, [D_p^\times, D_p^\times]] = [D'^\times_{p'}, D_p^\times] \supset [D_p^\times, D_p^\times]$, hence the left hand side is a normal subgroup of $[D'^\times_{p'}, D'^\times_{p'}]$ containing $i \in [D_p^\times, D_p^\times]$, and as such it coincides with $[D'^\times_{p'}, D'^\times_{p'}]$. (Proof for $D'_{p'} \simeq \{-1, -1\}$ is as follows : Let N be a normal subgroup of $[D'^\times_{p'}, D'^\times_{p'}]$ containing i, then $\{x \in D'_{p'} | x^2 + 1 = 0\} \subset N$ since such x is conjugate with i by Skolem-Noether Theorem. So for any $a \in K'_{p'}$ such that $1\text{-}a^2 \in K'^2_{p'}$, we have $-ai + bj \in N$ (with $a^2 + b^2 = 1$), hence $y : i(-ai + bj) = a + bij \in N$ which satisfies $y^2 - 2ay + 1 = 0$. Thus, again from Skolem-Noether Theorem, every $y \in [D'^\times_{p'}, D'^\times_{p'}]$ belongs to N).

4.4. Proof of Theorem 3 §0. Assume that D satisfies (EC) over $R(P)$. Applying Lemmas 4.1 and 4.3 to the result of Lemma 4.2 (regarding $\boldsymbol{R}(g)$ as K and K as K'), we see that D satisfies (a) over $\boldsymbol{R}[g]_K$, the integral closure of $\boldsymbol{R}[g]$ in K. Since $g \in R(P)$, we have $R(P) \supset \boldsymbol{R}[g]_K$ so that (a) over $\boldsymbol{R}[g]_K$ implies (a) over $R(P)$.

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