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On Kallianpur-Robbins law for fractional Brownian motion

Dedicated to Professor Hiroshi Kunita on the occasion of his 60th birthday

By

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1. Introduction

Let $\{B_t\}_{t\geq 0}$ be a d-dimensional Brownian motion and $V(x)$ be a summable function on $\mathbf{R}^d$ such that

$$\bar{V}:=\int_{\mathbf{R}^d} V(x)\,dx>0.$$

Then the random variables of the form $\int_0^t V(B_u)\,du$ are called the *occupation times* and the following theorem is well known as the Kallianpur-Robbins law.

Theorem A ([6]).

$(d=1)$
$$\lim_{t\to\infty} P\left[\frac{1}{\bar{V}\sqrt{t}}\int_0^t V(B_u)\,du<x\right]=\sqrt{\frac{2}{\pi}}\int_0^x e^{-y^2/2}\,dy,\quad x>0.$$

$(d=2)$
$$\lim_{t\to\infty} P\left[\frac{2\pi}{\bar{V}\log t}\int_0^t V(B_u)\,du<x\right]=1-e^{-x},\quad x>0.$$

This theorem was extended greatly by Darling-Kac ([4]) as follows and has been stimulated the interest of many authors for a long time. (See e.g. [7], [10]. See also H. Kesten [8].) Let $\{X_t\}_{t\geq 0}$ be a temporally homogeneous Markov process with values in a measurable space $(S,\mathscr{B})$ and let $V(x)\geq 0$ be a bounded measurable function on S. Suppose there exists a function $h(s)$, $(s>0)$ which tends to infinity as s goes to 0 such that

$$(1.1)\qquad E_x\left[\int_0^\infty e^{-su}\,V(X_u)\,du\right]\sim h(s),\qquad \text{as } s\to 0$$

uniformly on $\{x \mid V(x)>0\}$. Then,

Theorem B ([4]). (i) *If*

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$$h(s)=(1/s)^{\alpha}L(1/s), \qquad 0\leq\alpha<1,$$

where $L(1/s)$ *is slowly varying as* $s\to 0$, *then*

$$\lim_{t\to\infty}P\Big[\frac{1}{h(1/t)}\int_0^t V(X_u)du\leq x\Big]=G_{\alpha}(x), \qquad x>0,$$

where $G_{\alpha}(x)$ *denotes the distribution function of Mittag-Leffler distribution with index* α: *i. e.*,

$$G_{\alpha}(x)=\frac{1}{\pi\alpha}\int_0^x\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j\,!}\sin\pi\alpha j\;\Gamma(\alpha j+1)y^{j-1}\,dy.$$

(ii) *Conversely, if for some normalizing function* $u(t)>0$,

$$\lim_{t\to\infty}P\Big[\frac{1}{u(t)}\int_0^t V(X_u)du\leq x\Big]=G(x),\ x>0,$$

where $G(x)$ *is a nondegenerate distribution function, then*

$$h(s)=(1/s)^{\alpha}L(1/s)$$

for some $\alpha\in[0,1)$ *and slowly varying* $L(t)$. *Hence* $G(x)=G_{\alpha}(x/b)$ *where* $b>0$ *is an appropriate constant.*

Thus Theorem B extends the Kallianpur-Robbins Theorem from the view point of Markov processes, and under the condition (1.1) limiting distributions of the occupation times should necessarily be Mittag-Leffler distributions with suitable parameters. However, if we do not confine ourselves to Markov processes, other limiting distributions are of course possible. One of the important aspects of Brownian motion is that it is a self-similar Gaussian process, in which sense *fractional Brownian motion* is an extension of Brownian motion. Therefore, it would be of interest to study the occupation-time problem for fractional Brownian motion. Let X^{γ} be a fractional Brownian motion with index γ. That is, X^{γ} is a real-valued centered Gaussian process such that

$$E[X^{\gamma}(t)X^{\gamma}(s)]=\frac{1}{2}\{t^{2\gamma}+s^{2\gamma}-|t-s|^{2\gamma}\}, \qquad s,t\geq 0.$$

A d-dimensional fractional Brownian motion, which we shall denote by $X^{\gamma,d}(t)$ throughout the paper, is defined to be an $\mathbf{R}^d$-valued Gaussian process $(X_1^{\gamma}(t), X_2^{\gamma}(t),\cdots,X_d^{\gamma}(t))$, where X_1^{γ}, X_2^{γ}, $\cdots$are independent copies of X^{γ}. If $0<\gamma d<1$, then the existence of continuous *local time* is known; there exits a jointly continuous random function $l_{\gamma,d}(t,x)$, $t\geq 0$, $x\in\mathbf{R}^d$ such that

$$\int_0^t f(X^{\gamma,d}(u))\,du=\int_{\mathbf{R}^d}f(x)l_{\gamma,d}(t,x)\,dx,$$

for every bounded continuous function $f(x)$. (cf. Berman [1]. See [3] for further references.) Recently, N. Kôno obtained the following result, which is a generalization of Theorem A.

Theorem C ([9]). *Let $V(x)$ be a bounded summable function on $\mathbf{R}^d$ and let $\overline{V}=\int_{\mathbf{R}^d} V(x)\,dx$.*
(i) *If $0<\gamma d<1$, then*

$$\frac{1}{t^{1-\gamma d}}\int_0^t V(X^{\gamma d}(s))ds \xrightarrow{\mathscr{L}} \frac{\overline{V}}{\sqrt{2\pi}^d} l_{\gamma,d}(1,0) \qquad as\ t\to\infty.$$

(ii) *If $\gamma d=1$ and $d\geq 2$, then*

$$\frac{1}{\log t}\int_0^t V(X^{\gamma d}(s))ds \xrightarrow{\mathscr{L}} \frac{\overline{V}}{\sqrt{2\pi}^d} l_1 \qquad as\ t\to\infty,$$

where l_1 is an exponential random variable with mean 1.

In fact, in the cases where $0<\gamma d<1$ the assertion is almost obvious from the self-similarity once we know the existence of the local time but it should be recalled that its explicit law is unknown. So we can now raise a natural question comparing Theorems B and C; whether or not the law of limiting random variable $l_{\gamma,d}(1,0)$ is Mittag-Leffler distribution? As is pointed out in [7], the renewal property of excursions of Markov processes plays an essential role in order to explain why we have Mittag-Leffler distribution as the limiting law of occupation times. From this point of view, we have to conjecture that the answer is no. However, on the other hand , in view of Theorem C (ii), it is also natural to conjecture that the answer is yes. (Recall that the exponential distribution is Mittag-Leffler distribution with index 0.) The aim of this paper is to show that the answer is in fact no. That is, the law of $l_{\gamma,d}$ (1,0) is indeed similar in some sense to G_α $(\alpha=1-\gamma d)$ up to a scaling multiplicative constant, they are distinct unless $\gamma=1/2$ or $\gamma d=1$ (Corollary of Theorem 1). After knowing this we arrive at a new question; why the case $\gamma d=1$ is exceptional? To this question we shall show that, under suitable normalization, the law of $l_{\gamma d}$ (1,0) converges weakly to an exponential distribution when γd approaches 1 (Theorem 2).

2. Results and Proofs

By $C_n(t_1, t_2, \cdots, t_n; \gamma)$ we denote the $n\times n$ matrix with elements

(2.1) $C_n(t_1, t_2, \cdots, t_2; \gamma)_{ij}=\frac{1}{2}\{|t_{i-1}-t_j|^{2\gamma}+|t_i-t_{j-1}|^{2\gamma}-|t_i-t_j|^{2\gamma}-|t_{i-1}-t_{j-1}|^{2\gamma}\}$

where $t_0=0$ throughout the paper. The reader should notice that if $0<t_1<\cdots<t_n$ then $C_n(t_1, t_2, \cdots, t_n; \gamma)$ is the covariance matrix of $\{X^\gamma(t_j)-X^\gamma(t_{j-1})\}_{j=1}^n$.

Lemma 2.1. *For every $n\geq 1$,*

$$E[l_{\gamma,d}(1,0)^n]=\frac{n!}{\sqrt{2\pi}^{nd}}\int\cdots\int_{0<t_1<\cdots<t_n<1}\frac{dt_1\cdots dt_n}{\sqrt{\det C_n(t_1, t_2, \cdots, t_n; \gamma)}^d}. \tag{2.2}$$

Proof. Let V be a bounded continuous function on $\mathbf{R}^d$ such that $\overline{V}(=\int_{\mathbf{R}^d} V(x)dx)=1$, and let

$$(2.3)\qquad A(\varepsilon)=\varepsilon^{-d}\int_{\mathbf{R}^d} V(\varepsilon^{-1}x)\,\ell_{\gamma,d}(1,x)\,dx,\qquad \varepsilon>0.$$

Then it is easy to see that

$$(2.4)\qquad \lim_{\varepsilon\to 0} A(\varepsilon)=l_{\gamma,d}(1,0),\qquad a.\ s.$$

Since (2.3) can be rewritten as

$$A(\varepsilon)=\varepsilon^{-d}\int_0^1 V(\varepsilon^{-1}X^{\gamma,d}(t))\,dt,$$

we have

$$\begin{aligned}E[A(\varepsilon)^n]&=\frac{1}{\varepsilon^{nd}}\int_0^1\cdots\int_0^1 dt_1\cdots dt_n\, E[V(\varepsilon^{-1}X^{\gamma,d}(t_1))\cdots V(\varepsilon^{-1}X^{\gamma,d}(t_n))]\\&=\frac{n!}{\varepsilon^{nd}}\int\cdots\int_{0<t_1<\cdots<t_n<1} dt_1\cdots dt_n\int\cdots\int_{\mathbf{R}^d\times\cdots\times\mathbf{R}^d} g(t_1,\cdots,t_n;x_1,\cdots,x_n)\\&\qquad\times V(\varepsilon^{-1}x_1)\,V(\varepsilon^{-1}(x_1+x_2))\cdots V(\varepsilon^{-1}(x_1+\cdots+x_n))\,dx_1\cdots dx_n\end{aligned}$$

where $g(t_1,\cdots,t_n;x_1,\cdots,x_n)$ is the density of

$$(X^{\gamma,d}(t_1),\,X^{\gamma,d}(t_2)-X^{\gamma,d}(t_1),\,\cdots,\,X^{\gamma,d}(t_n)-X^{\gamma,d}(t_{n-1})),$$

i. e.,

$$\begin{aligned}g(t_1,\cdots,t_n;x_1,\cdots,x_n)&=\frac{1}{\sqrt{2\pi}^{nd}\sqrt{\det C_n(t_1,t_2,\cdots,t_n;\gamma)}^{\,d}}\\&\qquad\times\exp\Big\{-\frac{1}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{n}(C_n(t_1,t_2,\cdots,t_n;\gamma)^{-1})_{ij}\,x_i^k x_j^k\Big\}.\end{aligned}$$

Here, $x_j=(x_j^1,\cdots,x_j^d)\in\mathbf{R}^d$, $j=1,\cdots,n$. Therefore, we have

$$\begin{aligned}E[A(\varepsilon)^n]&=\frac{n\,!}{\sqrt{2\pi}^{nd}}\int\cdots\int_{0<t_1<\cdots<t_n<1}\frac{dt_1\cdots dt_n}{\sqrt{\det C_n(t_1,t_2,\cdots,t_n;\gamma)}^{\,d}}\\&\qquad\times\int\cdots\int_{\mathbf{R}^d\times\cdots\times\mathbf{R}^d} V(x_1)\,V(x_1+x_2)\cdots V(x_1+\cdots+x_n)\\&\qquad\times\exp\Big\{-\frac{\varepsilon^2}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{n}(C_n(t_1,t_2,\cdots,t_n;\gamma)^{-1})_{ij}\,x_i^k x_j^k\Big\}.\end{aligned}$$

Thus, keeping in mind that $\overline{V}=1$, we have

$$\lim_{\varepsilon\to 0} E[A(\varepsilon)^n]=\frac{n!}{\sqrt{2\pi}^{nd}}\int\cdots\int_{0<t_1<\cdots<t_n<1}\frac{dt_1\cdots dt_n}{\sqrt{\det C_n(t_1,t_2,\cdots,t_n;\gamma)}^{\,d}}.$$

Combining this with (2.4) we easily obtain (2.2).

Q.E.D.

Theorem 1. *Suppose* $d \geq 2$, $0 < \gamma d < 1$, *and let* $\alpha = 1 - \gamma d$.

(i)

$$E[l_{\gamma,d}(1,0)^n] \begin{cases} = \dfrac{1}{\sqrt{2\pi}^{nd}} \dfrac{n!\Gamma(\alpha)^n}{\Gamma(\alpha n+1)}, & n=1 \\ > \dfrac{1}{\sqrt{2\pi}^{nd}} \dfrac{n!\Gamma(\alpha)^n}{\Gamma(\alpha n+1)}, & n \geq 2. \end{cases}$$

(ii)

$$E[l_{\gamma,d}(1,0)^n] \leq \frac{1}{\sqrt{\pi}^{nd}} \frac{n!\Gamma(\alpha)^n}{\Gamma(\alpha n+1)}, \qquad n \geq 1.$$

Corollary. *If* $d \geq 2$, *then the law of* $l_{\gamma,d}(1,0)$ *is not Mittag-Leffler distribution with any index: i. e., there do not exist* $0 \leq \beta < 1$ *and* $b > 0$ *such that* $P[l_{\gamma,d}(1,0) \leq x] = G_\beta(x/b)$, $x > 0$.

Proof of Theorem 1. (i) For any positive definite $n \times n$ matrix $A = (a_{ij})$ it holds that $\det A \leq a_{11} \cdots a_{nn}$ the equality holding if and only if A is diagonal. Therefore, we see that

$$\det C_n(t_1, t_2, \cdots, t_n; \gamma) \leq \{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{2\gamma} \quad \text{if} \quad 0 < t_1 < \cdots < t_n,$$

where the equality holds if and only if $n=1$. So Lemma 2.1 yields

$$\begin{aligned} E[l_{\gamma,d}(1,0)^n] &\geq \frac{n!}{\sqrt{2\pi}^{nd}} \int\cdots\int_{0<t_1<\cdots<t_n<1} \{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{\alpha-1} dt_1\cdots dt_n \\ &= \frac{n!}{\sqrt{2\pi}^{nd}} \frac{\Gamma(\alpha)^n}{\Gamma(\alpha n+1)}. \end{aligned}$$

The first equality holds if and only if $n = 1$ and the last equality is well-known as a special case of Dirichlet's integral. (See also the proof of Lemma 2.3 below.)

The latter half (ii) can be proved in a similar way using the next inequality which follows immediately from Lemma 3.3 of Csörgő *et al.* ([3]).

(2.5) $$\det C_n(t_1, t_2, \cdots, t_n; \gamma) \geq (1/2)^n \{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{2\gamma}, \quad 0 < t_1 < \cdots < t_n.$$

Q.E.D.

Proof of Corollary. To begin with recall that Mittag-Leffler distribution G_β can be characterized by its moments;

(2.6) $$\int_0^\infty x^n dG_\beta(x/b) = b^n \frac{n!}{\Gamma(\beta n+1)}, \qquad n = 1, 2, \cdots$$

Suppose $P[l_{\gamma,d}(1,0) \leq x] = G_\beta(x/b)$ for some β and $b > 0$. Then (2.6) and

Stirling's formula yield

$$\lim_{n\to\infty}\frac{1}{n\log n}\log E[l_{\gamma,d}(1,0)^n]=1-\beta.$$

Similarly, we have from Theorem 1 that

$$\lim_{n\to\infty}\frac{1}{n\log n}\log E[l_{\gamma,d}(1,0)^n]=1-\alpha.$$

Thus we have $\beta=\alpha$, but then (2.6) contradicts Theorem 1 (i).

Q.E.D.

Theorem 2. *Let* ≥ 2. *Then*

$$\lim_{\gamma\uparrow 1/d} P[\sqrt{2\pi}^d(1-\gamma d)l_{\gamma,d}(1,0)\leq x]=1-e^{-x},\qquad x>0.$$

For the proof of Theorem 2 we prepare a few lemmas. For every $\beta>0$ and $n=1,2,\cdots$ we put

$$f_n(\beta;a,b)=\int\cdots\int_{D_n(a,b)}(u_1\cdots u_n)^{\beta-1}du_1\cdots du_n,\qquad 0\leq a<b\leq\infty,$$

where

$$D_n(a,b)=\{(u_1,\cdots,u_n)\in\mathbf{R}^n|\, u_1,\cdots,u_n>0,\qquad 0<u_1+\cdots+u_n<1 \text{ and } a<u_{n-1}/u_n<b\}.$$

Lemma 2.3. (i) *For any* a,b $(0<a<b<\infty)$,

$$\lim_{\beta\downarrow 0}\beta^n f_n(\beta;a,b)=0\qquad (n\geq 1).$$

(ii)

$$\lim_{\beta\downarrow 0}\beta^n f_n(\beta;0,\infty)=1\qquad (n\geq 1).$$

Proof. We change the variables as follows:

$$\begin{cases} u_1+u_2+\cdots+u_n=x_1\\ u_2+\cdots+u_n=x_1x_2\\ \cdots\\ u_n=x_1\cdots x_n,\end{cases}$$

or equivalently,

$$\begin{cases} u_1=x_1(1-x_2)\\ u_2=x_1x_2(1-x_3)\\ \cdots\\ u_n=x_1x_2\cdots x_n.\end{cases}$$

Then,

$$\frac{\partial(u_1, \cdots, u_n)}{\partial(x_1, \cdots, x_n)} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0.$$

Therefore, letting $a' = 1/(a+1)$ and $b' = 1/(b+1)$ we have

$$\begin{aligned} f_n(\beta; a, b) &= \int_0^1 x_1^{n\beta-1} dx_1 \int_0^1 x_2^{(n-1)\beta-1} (1-x_2)^{\beta-1}\, dx_2 \cdots \\ &\quad \times \int_0^1 x_{n-1}^{2\beta-1} (1-x_{n-1})^{\beta-1}\, dx_{n-1} \int_{b'}^{a'} x_n^{\beta-1} (1-x_n)^{\beta-1}\, dx_n \\ &= B(n\beta, 1) B((n-1)\beta, \beta) \cdots B(2\beta, \beta) \int_{b'}^{a'} x_n^{\beta-1} (1-x_n)^{\beta-1}\, dx_n \\ &= \frac{\Gamma(\beta)^n}{\Gamma(\beta n+1)} \frac{1}{B(\beta, \beta)} \int_{b'}^{a'} x^{\beta-1} (1-x)^{\beta-1}\, dx \end{aligned}$$

where $B(p, q)$ denotes the usual beta function. Hence, it holds

$$\beta^n f_n(\beta; a, b) = \frac{\Gamma(\beta+1)^n}{\Gamma(\beta n+1)} \frac{\int_{b'}^{a'} x^{\beta-1}(1-x)^{\beta-1}\, dx}{\int_0^1 x^{\beta-1}(1-x)^{\beta-1}\, dx}.$$

Letting $\beta \downarrow 0$, we have the assertion of the lemma.

Q.E.D.

For every $n \geq 1$ and for any a, b $(0 < a < b < \infty)$, let

$$\Lambda_n = \{(t_1, \cdots, t_n) \in \mathbf{R}^n \mid 0 < t_1 < \cdots < t_n < 1\}$$

and

$$\Lambda_n(a, b) = \bigcup_{i \neq j} \{(t_1, \cdots, t_n) \in \Lambda_n \mid \quad a < (t_i - t_{i-1}) / (t_j - t_{j-1}) < b\}.$$

Then,

Lemma 2.4.

(i) $$\lim_{\beta \downarrow 0} \beta^n \int \cdots \int_{\Lambda_n} \{t_1(t_2-t_1) \cdots (t_n - t_{n-1})\}^{\beta-1}\, dt_1 \cdots dt_n = 1.$$

(ii) $$\lim_{\beta \downarrow 0} \beta^n \int \cdots \int_{\Lambda_n(a,b)} \{t_1(t_2-t_1) \cdots (t_n - t_{n-1})\}^{\beta-1}\, dt_1 \cdots dt_n = 0.$$

Proof. Changing the variables, we have (i) from Lemma 2.3 (ii). Keeping in mind the symmetry of $\{u_1, \cdots, u_n\}$ in Lemma 2.3 (i), we also have (ii).

Q.E.D.

Lemma 2.5.

$$\lim_{a \downarrow 0, b \uparrow \infty} \sup_{\Lambda_n \setminus \Lambda_n(a,b)} \left| \frac{\det C_n(t_1, t_2, \cdots, t_n; \gamma)}{\{t_1(t_2-t_1) \cdots (t_n - t_{n-1})\}^{2\gamma}} - 1 \right| = 0,$$

the convergence being uniform for $0 < \gamma < 1/2$ *in wide sense.*

Proof. For $0<t_1<\cdots<t_n$ let $\widehat{C}_n(t_1, t_2, \cdots, t_n; \gamma)$ denote the correlation matrix of $\{X^\gamma(t_j)-X^\gamma(t_{j-1})\}_{j=1}^n$, i. e., $\widehat{C}_n(t_1, t_2, \cdots, t_n; \gamma)$ is the $n\times n$ matrix with elements

$$r_{ij}=\frac{|t_{i-1}-t_j|^{2\gamma}+|t_i-t_{j-1}|^{2\gamma}-|t_i-t_j|^{2\gamma}-|t_{i-1}-t_{j-1}|^{2\gamma}}{2(t_i-t_{i-1})^\gamma(t_j-t_{j-1})^\gamma}$$

Notice that

$$\det \widehat{C}_n(t_1, t_2, \cdots, t_n; \gamma)=\frac{\det C_n(t_1, t_2, \cdots, t_n; \gamma)}{\{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{2\gamma}}. \tag{2.7}$$

As is pointed out by Kôno ([9]) we easily have

$$|r_{ij}|\leq\left(\frac{u_i\wedge u_j}{u_i\vee u_j}\right)^\gamma, \quad i\neq j, \ (u_k=t_k-t_{k-1}).$$

Combining this with the definition of $\Lambda_n\backslash\Lambda_n(a, b)$ we have

$$\lim_{a\downarrow 0, b\uparrow\infty} \sup_{\Lambda_n\backslash\Lambda_n(a,b)} |r_{ij}|=0, \qquad i\neq j. \tag{2.8}$$

Since $r_{11}=r_{22}=\cdots=1$, (2.8) implies

$$\lim_{a\downarrow 0, b\uparrow\infty} \sup_{\Lambda_n\backslash\Lambda_n(a,b)} |\det \widehat{C}_n(t_1, t_2, \cdots, t_n; \gamma)-1|=0. \tag{2.9}$$

The assertion of the lemma follows from (2.7) and (2.9).

Q.E.D.

We are now ready to prove Theorem 2. By Lemma 2.1,

$$\begin{aligned} &E[(\sqrt{2\pi}^d(1-\gamma d)l_{\gamma,d}(1,0))^n] \\ &\quad=n!(1-\gamma d)^n\int\cdots\int_{0<t_1<\cdots<t_n<1}\frac{dt_1\cdots dt_n}{\sqrt{\det C_n(t_1, t_2, \cdots, t_n; \gamma)}^d} \end{aligned} \tag{2.10}$$

We shall first consider the integration on $\Lambda_n(a, b)$: Using (2.5) and Lemma 2.4 (ii), we see

(2.11)

$$\begin{aligned} &\limsup_{\gamma\to 1/d} n!(1-\gamma d)^n\int\cdots\int_{\Lambda_n(a,b)}\frac{dt_1\cdots dt_n}{\sqrt{\det C_n(t_1, t_2, \cdots, t_n; \gamma)}^d} \\ &\leq\limsup_{\gamma\to 1/d} n!2^{\frac{n}{2}}(1-\gamma d)^n\int\cdots\int_{\Lambda_n(a,b)}\{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{(1-\gamma d)-1}dt_1\cdots dt_n \\ &=0 \end{aligned}$$

We next study the integration on $\Lambda_n\backslash\Lambda_n(a, b)$: For any given $\varepsilon>0$, Lemma 2.5 allows us to choose small $a>0$ and large $b<\infty$ so that

$$\frac{1}{\sqrt{\det C_n(t_1, t_2, \cdots, t_n; \gamma)}^d}\leq(1+\varepsilon)\{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{-\gamma d} \quad \text{on } \Lambda_n\backslash\Lambda_n(a, b).$$

Therefore, for such $0<a<b<\infty$,

(2.12)

$$\limsup_{\gamma\to1/d} n!(1-\gamma d)^n\int\cdots\int_{\Lambda_n\setminus\Lambda_n(a,b)}\frac{dt_1\cdots dt_n}{\sqrt{\det C_n(t_1,t_2,\cdots,t_n;\gamma)}^{\,d}}$$

$$\leq\limsup_{\gamma\to1/d} n!(1+\varepsilon)(1-\gamma d)^n\int\cdots\int_{\Lambda_n\setminus\Lambda_n(a,b)}\{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{-\gamma d}dt_1\cdots dt_n$$

$$\leq\limsup_{\gamma\to1/d} n!(1+\varepsilon)(1-\gamma d)^n\int\cdots\int_{\Lambda n}\{t_1(t_2-t_1)\cdots(t_n-t_{n-1})\}^{-\gamma d}dt_1\cdots dt_n$$

$$=(1+\varepsilon)n\,!$$

by Lemma 2.4 (ii). Summing (2.11) and (2.12) we have from (2.10) that

$$\limsup_{\gamma\to1/d} E[(\sqrt{2\pi}^d(1-\gamma d)l_{\gamma,d}(1,0))^n]\leq(1+\varepsilon)n!,\qquad n\geq1.$$

Since $\varepsilon>0$ is arbitrary, this implies

$$\limsup_{\gamma\to1/d} E[(\sqrt{2\pi}^d(1-\gamma d)l_{\gamma,d}(1,0))^n]\leq n!,\qquad n\geq1.$$

The reverse inequality

$$\liminf_{\gamma\to1/d} E[(\sqrt{2\pi}^d(1-\gamma d)l_{\gamma,d}(1,0))^n]\geq n!,\qquad n\geq1$$

follows easily from Theorem1 (i). Therefore we conclude

$$\lim_{\gamma\to1/d} E[(\sqrt{2\pi}^d(1-\gamma d)l_{\gamma,d}(1,0))^n]=n!,\qquad n\geq1,$$

which completes the proof of Theorem 2.

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