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Homological stability of oriented configuration spaces

By

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§1. Introduction

For a connected space M, let $F(M, d)$ be space *ordered configurations* of d distinct points in, M, which is defined by

$$F(M, d) = \{(x_1, \cdots, x_d)\} \in M^d: x_i \neq x_j \text{ if } i \neq j\}.$$

Let Σ_d be the symmetric group of d letters $\{1, 2, \cdots, d\}$. Σ_d acts on $F(M, d)$ freely in the usual manner. The orbit space

$$C_d(M) = F(M, d)/\Sigma_d$$

is called the space of *configuratons* of d distinct points in M. In this paper we shall assume that M is an open manifold, i.e. each component is non-compact and without boundary. Adding a point near one of the ends of M gives (up to homotopy) a stabilization map

$$j_d: C_d(M) \rightarrow C_{d+1}(M).$$

The following is well-known:

Theorem 0 (F. Cohen [6], G. Segal [11]). *If M is an open manifold, then the stabilization map j_d: $C_d(M) \rightarrow C_{d+1}(M)$ is a homology equivalence up to dimension* [d/2].

(We shall call a map f: $X \rightarrow Y$ a *homology equivalence up to dimension m if the induced homomrphism*

$$f_*: H_i(X, \mathbf{Z}) \rightarrow H_i(Y, \mathbf{Z})$$

is bijective when $i < m$ and surjective when $i = m$.)

Remarks. Various special cases of this result were known earlier. For example.
(1) Let $M = R^q$ $(q > 2)$. Then $\lim_{q \to \infty} C_d(\mathbf{R}^q) = K(\Sigma_d, 1)$. The homology stabilization of this space follows from work of Nakaoka ([10]). We can also show this using theorem 0.

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(2) Let $M=\mathbf{R}^2$. Then $C_d(M)=K(Br_d, 1)$. The statement of Theorem 0 in this case was proved by Arnold ([1]).

Let $\widetilde{C}_d(M)=F(M, d)/A_d$, where $A_d\subset\Sigma_d$ is the alternating group of d letters $\{1, \cdots, d\}$. We shall call $\widetilde{C}_d(M)$ the space of *oriented configurations* of d distinct points in M. There is a non-trivial double covering $\widetilde{C}_d(M)\rightarrow C_d(M)$. Adding a point near an end of M gives a stabilization map

$$\tilde{j}_d\colon \widetilde{C}_d(M)\rightarrow C_{d+1}(M).$$

In this note we shall determine the homological stability dimension for the spaces $\widetilde{C}_d(M)$, when M is obtained from a compact Riemann surface by removing finite number of points.

More precisely, we shall prove:

Theorem 1. *Let M be a compact Riemann surface, and let*

$$M'=M\setminus\{n\ points\}$$

where $n\geq 1$. Then the stabilization map

$$\tilde{j}_d\colon \widetilde{C}_d(M')\rightarrow\widetilde{C}_{d+1}(M')$$

is a homology equivalence up to dimension $[(d-1)/3]$. Moreover, this bound is the best possible.

We shall give a proof in the next section, based on the calculations due to Bödingheimer, Cohen, Taylor and Milgram ([2], [3]). First we make some remarks and pose a question:

Remarks. (1) It seems somewhat surprising that the answer is (about) *$d/3$*, not *$d/2$* as in the un-oriented case.
(2) An analogous argument proves a similar result for McDuff's configuration space $C_n^{\pm}(M)$ of "positive and negative particles" ([9]). An application of this will be given in [7].

Question. Is Theorem 1 true for any open manifold?

§2. Proof of Theorem 1

Since $M'=M\setminus\{(n\text{ points})\}\cong\mathbf{C}\setminus\{(n-1)\text{ points}\}$, we shall assume that

$$M'=\mathbf{C}-\{l\text{ points}\}\ (\text{where } l=n-1)$$

and write C_d for $C_d(M')$ and $\widetilde{C}_d$ for $\widetilde{C}_d(M')$. We shall only consider the case $l\geq 1$. The case $l=0$ can be dealt with in a similar way.

We shall show that

$$(*)\qquad H_q(\widetilde{C}_d, \mathbf{F})\rightarrow H_q(\widetilde{C}_{d+1}, \mathbf{F})$$

is bijective for $q<n(d)$ and surjective for $q=n(d)$ if $\mathbf{F}=\mathbf{Z}/p$ (p is any prime)

or $\mathbf{F}=\mathbf{Q}$, where

$$n(d)=\begin{cases}[d/2] & \text{if } \mathbf{F}\neq\mathbf{Z}/3\\ [(d-1)/3] & \text{if } \mathbf{F}=\mathbf{Z}/3\end{cases}$$

Theorem 1 follows from this and the universal coefficient theorem. (The case $\mathbf{F}=\mathbf{Z}/2$ is trivial. Indeed, since $\widetilde{C}_d\to C_d$ is a double covering and the stabilization map $C_d\to C_{d+1}$ is a homology equivalence up to dimension $[d/2]$, the result follows from the Gysin exact sequence.)

We shall make use of the following well known fact ([4]):

Lemma 2. *Let G be a group and $H\subset G$ a subgroup of index 2. Let F be any field of characteristic not equal to 2. Then there is a natural additive isomorphism*

$$H_q(H,\mathbf{F})\cong H_q(G,\mathbf{F})\oplus H_q(G,\mathbf{F}(-1))$$

for any $q\geq 1$, where $\mathbf{F}(-1)$ denotes the field $\mathbf{F}$ with the G-module structure given by

$$g\cdot f=\begin{cases}-f & g\notin H\\ f & g\in H\end{cases}$$

for $f\in\mathbf{F}$ and $g\in G$.

Remark. Although similar result holds for any double coverings, because we do not need it, we omit this here.

Let us take $G=\pi_1(C_d)$ and $H=\pi_1(\widetilde{C}_d)$. Since $\widetilde{C}_d$ is a double covering, H can be identified with a subgroup of G of index 2. It is well known that $F(M', d)$ is a $K(\pi, 1)$-space ([8]). So the spaces $\widetilde{C}_d$ and C_d are also $K(\pi, 1)$-spaces. Hence we can assume $\widetilde{C}_d\simeq K(H, 1)$, $C_d\simeq K(G, 1)$ and we can identify the covering map with the map $K(H, 1)\to K(G, 1)$ induced by the inclusion $H\subset G$. We can thus apply Lemma 2 to obtain:

Lemma 3. *If $\mathbf{F}=\mathbf{Z}/p$ (p any odd prime) or $\mathbf{F}=\mathbf{Q}$, then there is a natural additive isomrphism*

$$H_q(\widetilde{C}_d,\mathbf{F})\cong H_q(C_d,\mathbf{F})\oplus H_q(C_d,\mathbf{F}(-1))$$

for any $q\geq 1$

Now, since $C_d\to C_{d+1}$ is a homology equivalence up to dimension $[d/2]$, Theorem 1 follows directly from the following result:

Lemma 4. *Let q and d be positive integers such that $1\leq q\leq[d/2]$ and $(q, d)\neq(1,2)$.*

(1) *If $\mathbf{F}=\mathbf{Z}/p$ (p prime, $p\geq 7$) or $\mathbf{F}=\mathbf{Q}$, then*

$$H_q(C_d,\mathbf{F}(-1))=0$$

(2) *If $\mathbf{F}=\mathbf{Z}/5$ and $(q, d)\neq(3,6)$, then*

$$H_q(C_d, \mathbf{Z}/5(-1))=0$$

(3) *If* $\mathbf{F}=\mathbf{Z}/3$ *and* $d\geq 3q+2$, *then*

$$H_q(C_d, \mathbf{Z}/3(-1))=0$$

Proof. Let $1\leq q\leq [d/2]$.

By (8.4) of [3], if n is sufficiently large, then

$$H_q(C_d, \mathbf{F}(-1))\cong H_{q+(2n+1)d}(\Omega^2S^{2n+3}\times(\Omega S^{2n+3})^l, \mathbf{F})$$

Note that

$$H_j((\Omega S^{2n+3})^l, \mathbf{F})\cong\begin{cases}\mathbf{F}^{m(\beta)} & \text{if } j=(2n+2)\beta, \quad \beta\geq 0\\ 0 & \text{otherwise}\end{cases}$$

and there is a stable splitting ([5], [12])

$$\Omega^2S^{2n+3}\simeq_s\bigvee_{\alpha\geq 1}\Sigma^{2n}D_\alpha$$

where we take

$$m(\beta)=\binom{\beta+l-1}{l-1} \text{ and } D_\alpha=F(\mathbf{C}, \alpha)_+\wedge_{\Sigma_\alpha}(\wedge^\alpha S^1).$$

Since D_α has the homotopy type of a CW complex of dimension $2\alpha-1$, $H_j(D_\alpha, \mathbf{Z}/p)=0$ for any $j\geq 2\alpha$.

Applying the Künneth formula one can show that

$$(**)\qquad H_q(C_d, \mathbf{F}(-1))\cong\bigoplus_{\alpha=1}^d\tilde{H}_{q+2\alpha-d}(D_\alpha, \mathbf{F})^{m(d-\alpha)}$$

From now on we shall only consider the case $\mathbf{F}=\mathbf{Z}/p$ (p an odd prime). The case $\mathbf{F}=\mathbf{Q}$ can be dealt with analogously.

The following is well known:

Lemma 5. *Let* $p\geq 3$ *be any odd prime.*

(1) *There is a multiplicative isomorphism*

$$\text{(a)}\qquad H_*(\Omega^2S^3, \mathbf{Z}/p)=\mathbf{Z}[x_1, x_2, \cdots]\otimes E[y_0, y_1, y_2, \cdots]$$

where $\deg(x_i)=2p^i-2$ *and* $\deg(y_i)=2p^i-1$.

(2) *There is an additive isomorphism*

$$\text{(b)}\qquad \tilde{H}_*(D_\alpha, \mathbf{Z}/p)=\bigoplus_{J=(\varepsilon_0, m_1, \varepsilon_1, \cdots)\in\mathcal{J}}\mathbf{Z}/p\Big\{\prod_{j\geq 1}x_j^{m_j}\cdot\prod_{j\geq 1}y_j^{\varepsilon_j}\Big\}$$

where we take:

$$\mathcal{J}=\{J=(\varepsilon_0, m_1, \varepsilon_1, \cdots): \varepsilon_j\in\{0,1\}, m_j\geq 0, w(j)=\alpha\}$$

and

$$w(J)=\varepsilon_0+\textstyle\sum_{j\geq 1}p^j(m_j+\varepsilon).$$

From Lemma 5

(c) $$\dim_{\mathbf{Z}/p}\widetilde{H}_{q+2\alpha-d}(D_\alpha, \mathbf{Z}/p)=\mathrm{card}(\mathcal{F})$$

where

$$\mathcal{F}=\{J=(\varepsilon_0, m_1, \varepsilon_1, \cdots)\neq(0,0,\cdots): \varepsilon_j\in\{0,1\}, m_j\geq 0, D(J)=q+2\alpha-d, w(J)=\varepsilon\}$$

and

$$D(J)=\varepsilon_0+\sum_{j\geq 1}\{2(p^j-1)m_j+(2p^j-1)\varepsilon_j\}.$$

Here card(S) denotes the cardinality of a finite set S.

Note that for $J=(\varepsilon_0, \mathrm{m}_1, \varepsilon_1, \cdots)$, if $\mathrm{w}(J)=\alpha$, then

$$D(J)=q+2\alpha-d \Leftrightarrow H(J)=\varepsilon_0+\textstyle\sum_{j\geq 1}(2m_j+\varepsilon_j)=d-q$$

Hence

(d)
$$\mathcal{F}=\{J=(\varepsilon_0, m_1, \varepsilon_1, \cdots)\neq(0,0,\cdots): \varepsilon_j\in\{0,1\}, m_j\geq 0, w(J)=\alpha, H(J)=d-q\}.$$

By (c) and (d) it suffices to show:

CLAIM. *Let* $1\leq q\leq[d/2]$, $1\leq\alpha\leq d$ *and* $(q, d)\neq(1,2)$.

(1) *If* $p\geq 7$ *is an odd prime or* $p=5$ *and* $(q, d)\neq(3,6)$, *then* $\mathcal{F}=\phi$

(2) *If* $p=3$ *and* $d\geq 3q+2$, $\mathcal{F}=\phi$.

Proof of Claim. (1) Assume that $p\geq 5$ is a prime and $J=(\varepsilon_0, m_1, \varepsilon_1, \cdots)\in\mathcal{F}$.

Since $1\leq \mathrm{q}\leq[d/2]\leq d/2$,

$$\varepsilon_0+\sum_{j\geq 1}(2m_j+\varepsilon_j)=H(J)=d-q\geq d/2\geq\alpha/2=\{\varepsilon_0+\sum_{j\geq 1}p^j(m_j+\varepsilon_j)\}/2.$$

Hence

(e) $$\varepsilon_0+\textstyle\sum_{j\geq 1}\{(4-p^j)m_j+(2-p_j)\varepsilon_j\}\geq 0$$

Since $J\neq(0,0,\cdots)$, one can deduce from (e) that

$$J=(\varepsilon_0, m_1, \varepsilon_1, m_2, \varepsilon_2, \cdots)=\begin{cases}(1,0,0,0,0,\cdots) & \text{if } p>7\\ (1,0,0,0,0,\cdots)\text{ or }(1,1,0,0,0,\cdots) & \text{if } p=5\end{cases}$$

Hence

$$(q, d)=\begin{cases}(1,2) & p\geq 7\\ (1,2),\ (3,6) & p=5\end{cases}$$

This is a contradiction.

(2) Assume $d\geq 3q+2$ and $p=3$. Then

$$\begin{aligned}\alpha-\mathrm{d}&=\mathrm{w}(\mathrm{J})-(\mathrm{q}+\mathrm{H}(\mathrm{J}))\\ &=\{\varepsilon_0+\textstyle\sum_{j\geq 1}3^j(m_j+\varepsilon_j)\}-\{\varepsilon_0+\sum_{j\geq 1}(2m_j+\varepsilon_j)\}-q\\ &=\textstyle\sum_{j\geq 1}\{(3^j-2)m_j+(3^j-1)\varepsilon_j\}-\mathrm{q}\\ &\geq\frac{1}{2}\textstyle\sum_{j\geq 1}(2m_j+\varepsilon_j)-q\end{aligned}$$

$$=\frac{1}{2}(d-q-\varepsilon_0)-q \qquad (\text{by } H(J)=d-q)$$
$$=\frac{1}{2}(d-3q-\varepsilon_0)$$
$$\geq\frac{1}{2}\{(3q+2)-3q-1\}=\frac{1}{2}>0$$

Hence $\alpha=w(J)>d$, which is a contradiction.

This completes the proof of Theorem 2.

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