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A duality theorem in Hopf algebras and its application to Morava K-theory of $B\mathbf{Z}/p^r$

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0. Introduction

Let $E^*(\ )$ be a complex orientable theory. Then choosing an orientation class $x \in E^2(BS^1)$, we have an isomorphism $E^*(BS^1) \cong E_*[[x]]$, where E_* is the coefficient ring. Let N be a natural number and let $[N]x = x +_F \cdots +_F x$ (N times) be the N-sequence, where $x +_F y = F(x, y)$ is the formal group law of the theory E^*. Note that $[N]x$ is the Euler class of the standard S^1-bundle

$$S^1 \to B\mathbf{Z}/N \to BS^1$$

Therefore if $[N]x$ is not a zero-divisor, from the Gysin sequence it follows that

$$E^*(B\mathbf{Z}/N) \cong E_*[[x]]/([N]x).$$

Suppose that $E^*(B\mathbf{Z}/N)$ is a finitely generated free E_*-module. Then

$$E^*(B\mathbf{Z}/N \times B\mathbf{Z}/N) \cong E^*(B\mathbf{Z}/N) \otimes_{E_*} E^*(B\mathbf{Z}/N)$$

and the product map $m : B\mathbf{Z}/N \times B\mathbf{Z}/N \to B\mathbf{Z}/N$ imduces a ring homomorphism

$$m^* : E^*(B\mathbf{Z}/N) \to E^*(B\mathbf{Z}/N) \otimes_{E_*} E^*(B\mathbf{Z}/N).$$

Thus $E^*(B\mathbf{Z}/N)$ is a bicommutative Hopf algebra over E_* and so is its dual

$$\hom_{E_*}(E^*(B\mathbf{Z}/N), E_*).$$

In this paper we shall study a duality between the algebraic groups of such Hopf algebras and their duals. For typical application we consider the p-adic Morava $K(n)$-theory. Let $\overline{K(n)}^*(\ )$ be the p-adic Morava $K(n)$-theory of period 2 so that the coefficient ring is

$$\overline{K(n)}_* = \mathbf{Z}_p[v_n, v_n^{-1}, t, t^{-1}]/(t^{p^n-1} - v_n)$$

where $\deg t = 2$ and $\mathbf{Z}_p$ is the ring of p-adic integers. For a $\mathbf{Z}_p$-algebra R we define

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$$\overline{K(n)}^*_R(\ ) = \overline{K(n)}^*(\ ) \otimes R.$$

If R is a finitely generated free $\mathbf{Z}_p$-module, then $\overline{K(n)}^*_R(\ )$ will be a complex orientable cohomology theory. Throughout this paper for any local field L we denote by O_L its ring of integers. Suppose K_n is such an extension of the p-adic number field $\mathbf{Q}_p$ that the residue field $O_{K_n}/\mathrm{m} \cong \mathbf{F}_{p^n}$. The algebraic group associated with the Hopf algebra

$$\overline{K(n)}^0_{O_{K_n}}(B\mathbf{Z}/p^r) = O_{K_n}[[x]]/[p^r]x,$$

where $[p]x$ is the p-series of the Lubin-Tate formal group law, can be described by the local class field theory (see [3]). Using our duality theorem we get that the group of homotopy classes of ring spectra maps

$$\boldsymbol{B}\mathbf{Z}/p^r{}_+ \to \overline{K(n)}_{O_L}$$

is isomorphic to $(\mathbf{Z}/p^r)^n$. Here L is certain finite free extension of O_{K_n}, $\boldsymbol{B}\mathbf{Z}/p^r{}_+$ is the suspension spectrum of $B\mathbf{Z}/p^r{}_+$ and $\overline{K(n)}_L$ is the ring spectrum of cohomology theory $\overline{K(n)}^*_L(\ )$. This is a generalization of the result of the first named author [4].

1. A duality theorem

In this section by an R-Hopf algebra we mean a bicommutative Hopf algebra A over a domain R such that A is finitely generated free R-module. If A is such a Hopf algebra, then for the dual

$$A^* = \hom_{R\text{-module}}(A, R)$$

we have an isomorphism $A^* \otimes_R A^* \cong (A \otimes_R A)^*$ and A^* is an R-Hopf algebra in our sense. We fix the notations:

$m : A \otimes_R A \to A$ multiplication,
$u : R \to A$ unit,
$\Delta : A \to A \otimes_R A$ comultiplication,
$\varepsilon : A \to R$ counit (augmentation),
$s : A \to A$ coinverse (antipode).

Definition 1. 1. For an R-Hopf algebra A and an R-algebra S, let

$$f, g \in \hom_{R\text{-module}}(A, S).$$

We define a convolution of f and g by

$$f * g = m' \circ (f \otimes g) \circ \Delta \in \hom_{R\text{-module}}(A, S),$$

where m′ is the multiplication in S.

Lemma 1. 2. *Let A be an R-Hopf algebra and let S be a commutative R-algebra. Then* $\hom_{R\text{-}algebra}(A, S)$ *becomes a group with respect to convolution.*

Proof. Since S is commutative $m' \circ (f \circ g) \circ \Delta \in \hom_{R\text{-}algebra}(A, S)$. Let u' be unit of R-algebra S. It is easy to check that $u' \circ \varepsilon$ is the unit of convolution and $f \circ s$ is the inverse of f.

Definition 1. 3. For any Hopf algebra, $b \in A$ is called grouplike element of A, if $\Delta b = b \otimes b$.

Lemma 1. 4. *Let A be an R-Hopf algebra and let S be a commutative R-algebra. Then the set {grouplike elements in $A \otimes_R S$} is a group with respect to multiplication in $A \otimes_R S$.*

Proof. It is easy to see that the product of grouplike elements is again grouplike. Let $b \in A \otimes_R S$ be a grouplike element. Then $\varepsilon(b) = 1$ and by definition $b \cdot s(b) = \varepsilon(b) = 1$. So $s(b) = b^{-1}$.

Definition 1. 5. For an R-Hopf algebra A and a commutative R-algebra S, the groups $\hom_{R\text{-}algebra}(A, S)$ and {grouplike elements in $A \otimes_R S$} are denoted by $G_R(A)(S)$ and $G_R^*(A)(S)$, respectively.

Lemma 1. 6. *Let A be an R-Hopt algebra. The grouplike elements in A are linearly independent.*

Proof. Let K be the quotient field of R. Then $A \otimes_R K$ is a Hopf algebra over K. We consider that $A \subset A \otimes_R K$ and it is easy to see that grouplike elements in A have the same property in $A \otimes_R K$. So it is enough to show that grouplike elements in $A \otimes_R K$ are linearly independent. Let b, $b_i \in A \otimes_R K$ are grouplike elements with $b = \sum \lambda_i b_i$. We may assume that b_i are independent. For any grouplike element b we have $\varepsilon(b) = 1$. Then $1 = \varepsilon(b) = \sum \lambda_i \varepsilon(b_i) = \sum \lambda_i$. But $\Delta b = b \otimes b = \sum \lambda_i \lambda_j b_i \otimes b_j$ and $\Delta b = \sum \lambda_i \Delta b_i = \sum \lambda_i b_i \otimes b_i$. The $b_i \otimes b_j$ are linearly independent, so by comparing coefficients we get $\lambda_{ij} = 0$ for $i \neq j$ and $\lambda_i^2 = \lambda_i$. As $\sum \lambda_i = 1$, follows that $\sum \lambda_i b_i$ equals some b_j.

Lemma 1. 7. *Let A and S be as in the previous lemma. Then*

$$G_R(A)(S) \cong G_R^*(A^*)(S).$$

Proof. Consider an S-algebra map $\psi : A \otimes_R S \to S$. This map corresponds to an element $b \in (A \otimes_R S)^* = A^* \otimes_R S$. In other words for any $f \in A \otimes_R S$, $\psi(f) = \langle f, b \rangle$. We denote such ψ by ψ_b. By definition we have:

$$\begin{aligned} \langle f \otimes g, \delta b \rangle &= \langle f \cdot g, b \rangle = (f \cdot g) = \psi_b(f)\,\psi_b(g) \\ &= \langle f, b \rangle \cdot \langle g, b \rangle = \langle f \otimes g, b \otimes b \rangle. \end{aligned}$$

Here f and g are any elements of $A \otimes_R S$. So $f \otimes g$ span $(A \otimes_R S) \otimes_S (A \otimes_R S)$

and we get that $\Delta b = b \otimes b$. Now group operation in $G_R(A)(S)$ is defined exactly as multiplication in dual Hopf algebra $(A \otimes_R S)^* = A^* \otimes_R S$.

Corollary 1. 8. *For an R-Hopf algebra A and a commutative R-algebra S,* $G_R(A)(S)$ *and* $G_R^*(A)(S)$ *are finite abelian groups.*

Proposition 1. 9. *Let A and B are finite dimensional Hopf algebras over an algebraically closed field K. Suppose* $f : A \to B$ *is Hopf algebra monomorphism, then induced group homomorphism*

$$G(f) : G_K(B)(K) \to G_K(A)(K)$$

is an epimorphism.

Proof. See [1], page 180.

Theorem 1.10. *Let A be a free bicommutative finite dimensional Hopf algebra over an algebraically closed field K. Then* $G_K(A)(K) \cong G_K(A^*)(K)$.

Proof. In this proof we shell write simply $G(\ )$ and $G^*(\ )$ instead of $G_K(\ )(K)$ and $G^*{}_K(\ )(K)$. Consider the inclusion $G^*(A) \to A$. Since elements in $G^*(A)$ are linearly independent, we can extend to the group algebra $K[G^*(A)]$ and we get a Hopf algebra monomorphism

$$i : K[G^*(A)] \to A.$$

Then by Proposition 1. 9 induced homomorphism

$$\xi_A = G(i) : G(A) \to G(K[G^*(A)])$$

is an epimorphism. Note that

$$G(K[G^*(A)]) = \hom_{K\text{-}algebra}(K[G^*(A)], K) \cong \hom(G^*(A), K^*) = G^*(A)^\wedge.$$

$G^*(A)^\wedge$ is the character group of $G^*(A)$. Note that the adjoint of $\xi_A : G(A) \to G^*(A)^\wedge$ is the natural pairing $\varphi : G(A) \times G^*(A) \to K^*$ given by $\varphi(\alpha, x) = \alpha(x)$. The same argument for A^* gives an epimorphism $\xi_{A^*} : G(A^*) \to G^*(A^*)^\wedge$ and taking the character dual we have a monomorphism

$$(G^*(A^*)^\wedge)^\wedge \to G(A^*)^\wedge.$$

Since all groups are finite abelian we have $(G^*(A^*)^\wedge)^\wedge \cong G^*(A^*)$. By Lemma 1. 7 $G^*(A^*) \cong G(A)$ and $G(A^*) \cong G^*(A)$. It is easy to check that the resulting map $G(A) \to G^*(A)^\wedge$ is nothing but ξ_A. Hence ξ_A is isomorphism.

Let K be a discrete valuation field with maximal ideal of O_K generated by π. Let A be an O_K-Hopf algebra. In $A \otimes_{O_K} K$ we have induced K-Hopf algebra structure.

Lemma 1. 11. *If* $u \in A \otimes_{O_K} K$ *is grouplike element, then* $u \in A$.

Proof. Let $u \in A \otimes_{O_K} K$ be a grouplike element. Assume that $u \notin A$. We consider additive bases of A, $e_1, e_2, \cdots, e_n$. Then $u = \sum b_i e_i$, for some $b_i \in K$. Without losing generality we assume that b_1 have minimal valuation. Let $v(b_1) = -x < 0$. Then $\pi^x u \in A$ and

$$\Delta(\pi^x u) = \pi^x u \otimes u = \sum_{i,j} \pi^x b_i b_j e_i \otimes e_j$$

must be in $A \otimes A$. But $v(\pi^x b_1^2) = -x < 0$. So $\Delta(\pi^x u) \notin A \otimes A$ and we get the contradiction.

Definition 1. 12. Let S be a quotient field of domain R and let $\bar{S}$ be an algebraic closure of S. We define a splitting field of R-Hopf algebra A as a subfield of $\bar{S}$ containing splitting fields of both S-algebras $A \otimes_R S$ and $A^* \otimes_R S$.

Proposition 1. 13. *Let K be a discrete valuation field and let A be a free bicommutative finite dimensional O_K-Hopf algebra. If L is a splitting field of A then $G_{O_K}(A)(O_L) \cong G_{O_K}(A^*)(O_L)$.*

Proof. For the simplicity of the notation we write G instead of G_{O_K}. From Theorem 1.10 it is clear that

$$G(A)(L) \cong G(A^*)(L).$$

Here L is finite over K and thus the extension of the valuation of K to L is discrete. Now according to Lemma. 1. 11 we have

$$G(A)(L) = G(A)(O_L)$$

and

$$G(A^*)(L) = G(A^*)(O_L)$$

Hence $G(A)(O_L) \cong G(A^*)(O_L)$.

2. Topological Application

Now we consider ring spectra maps

$$\boldsymbol{BZ}/p^r{}_+ \to \overline{K(n)}_R.$$

Group operation among such ring spectra maps f and g is defined by following composition

$$\boldsymbol{BZ}/p^r{}_+ \xrightarrow{d} \boldsymbol{BZ}/p^r{}_+ \wedge \boldsymbol{BZ}/p^r{}_+ \xrightarrow{f \wedge g} \overline{\boldsymbol{K}(n)}_R \wedge \overline{\boldsymbol{K}(n)}_R \xrightarrow{\mu} \overline{\boldsymbol{K}(n)}_R.$$

Let us denote Hopf algebra $\overline{K(n)}^0_R(B\mathbf{Z}/p^r)$ by $A_R(n, r)$,

$$A_R(n, r) = \overline{K(n)}^0_R(B\mathbf{Z}/p^r) = R[[x]]/[p^r]x.$$

In this case $[p]x$ is p-series of Lubin-Tate formal group law $F(x, y)$. Thus we can choose such an orientation x, that

$$[p]x = px - x^{p^n}$$

So $A_R(n, r)$ is free of rank p^n over R. We write its algebraic dual as $A_R^*(n, r)$,

$$A_R^*(n, r) = \hom_{R\text{-}module}(A_R(n, r), R).$$

Let f be a spectra map of degree 0

$$f : \boldsymbol{BZ}/p^r{}_+ \to \overline{\boldsymbol{K}(n)}_R.$$

We can consider f as an element of cohomology ring

$$f = f(x) \in \overline{K(n)}_R^0\ (B\mathbf{Z}/p^r) = A_R(n, r)$$

and thus we can consider f as a homomorphism

$$f : A_R^*(n, r) \to R.$$

Lemma 2. 1. *The following conditions are equivalent* :

a) $f \in \hom_{R\text{-}algebra}(A_R^*(n. r), \mathrm{R})$;

b) *f is a ring spectra map* $\boldsymbol{BZ}/p^r{}_+ \to \overline{\boldsymbol{K}(n)}_R$

c) $f(F(x, y)) = f(x)f(y)$.

Proof. a) means that following diagram is commutative

$$\begin{array}{ccc} \mathrm{A}_R^*(n, r) \otimes_R A_R^*(n, r) & \to & A_R^*(n, r) \\ \downarrow f\otimes f & & \downarrow f \\ R \otimes_R R & = & R. \end{array}$$

Considering the dual diagram we get

$$\begin{array}{ccc} A_R(n, r) \otimes_R A_R(n, r) & \overset{\Psi}{\leftarrow} & A_R(n, r) \\ \uparrow f^* \otimes f^* & & \uparrow f^* \\ R \otimes_R R & = & R \end{array}$$

where $f^*(1) = f(x)$ and $\Psi(x) = F(1 \otimes x, x \otimes 1)$. Thus commutativity of this diagram implies that a) $\Leftrightarrow$ c).

b) means that we have commutative diagram

$$\begin{array}{ccc} \boldsymbol{BZ}/p^r{}_+ \wedge \boldsymbol{BZ}/p^r{}_+ & \overset{f\wedge f}{\to} & \overline{\boldsymbol{K}(n)}_R \wedge \overline{\boldsymbol{K}(n)}_R \\ \downarrow m & & \downarrow \mu \\ \boldsymbol{BZ}/p^r{}_+ & \overset{f}{\to} & \overline{\boldsymbol{K}(n)}_R. \end{array}$$

We can consider that $f\circ m$, $\mu\circ(f\wedge f)\in\overline{K(n)}^{0}_{R}(B\mathbf{Z}/p^r\times B\mathbf{Z}/p^r)$. By definition we have that

$$f\circ m=m^{*}(f(x))=f(F(x,y))$$

and

$$\mu\circ(f\wedge f)=f(x)f(y).$$

So b) $\Leftrightarrow$ c).

Let $\mathbf{C}_p$ be the completion field of the algebraic closure of p-adic numbers $\mathbf{Q}_p$. Instead of $O_{\mathbf{C}_p}$ we shall simply write O.

Theorem 2. 2. $G_O(A_O(n,r))(O)\cong(\mathbf{Z}/p^r)^n$.

Proof. See [3], Lemma 4. 7. (ii) and Lemma 4.8. (ii)

Theorem 2. 3. *Let L be a splitting field of $A_{O_{K_n}}(n,r)$. Then the group G of homotopy classes of ring spectra maps*

$$B\mathbf{Z}/p^r{}_{+}\rightarrow\overline{K(n)}_{O_L}$$

is isomorphic to $(\mathbf{Z}/p^r)^n$.

Proof. According to the Lemma 2.1 we see that G is equal to $G_{O_L}(A^{*}_{O_L}(n,r))(O_L)$ as a set. Considering definitions of group operations in G and $G_{O_L}(A^{*}_{O_L}(n,r))(O_L)$ we find out that in fact there is a group isomorphism

$$G\cong G_{O_L}(A^{*}_{O_L}(n,r))(O_L).$$

$A^{*}_{O_L}(n,r)$ and L satisfy the condition of the Proposition 1. 13. Thus

$$G_{O_L}(A^{*}_{O_L}(n,r))(O_L)\cong G_{O_L}(A^{*}_{O_L}(n,r))(O_L).$$

But it is clear that

$$G_{O_L}(A_{O_L}(n,r))(O_L)\cong G_O(A_O(n,r))(O)$$

and from Theorem 2. 2 follows that $G\cong(\mathbf{Z}/p^r)^n$.

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