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Bogomolov conjecture for curves of genus 2 over function fields

By

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1. Introduction

Let k be an algebraically closed field, X a smooth projective surface over k, Y a smooth projective curve over k, and $f: X \to Y$ a generically smooth semi-stable curve of genus $g \geq 2$ over Y. Let K be the function field of Y, $\overline{K}$ the algebraic closure of K, and C the generic fiber of f. Let $j: C(\overline{K}) \to \mathrm{Jac}(C)(\overline{K})$ be a morphism given by $j(x) = (2g-2)x - \omega_c$ and $\| \|_{NT}$ the semi-norm arising from the Néron-Tate height pairing on $\mathrm{Jac}(C)(\overline{K})$. We set

$$B_C(P; r) = \{x \in C(\overline{K}) \mid \|j(x) - P\|_{NT} \leq r\}$$

for $P \in \mathrm{Jac}(C)(\overline{K})$ and $r \geq 0$, and

$$r_C(P) = \begin{cases} -\infty & \text{if } \#(B_C(P; 0)) = \infty \\ \sup\{r \geq 0 \mid \#(B_C(P; r)) < \infty\} & \text{otherwise.} \end{cases}$$

Bogomolov conjecture claims that, if f is non-isotrivial, then $r_C(P)$ is positive for all $P \in \mathrm{Jac}(C)(\overline{K})$. Even to say that $r_C(P) \geq 0$ for all $P \in \mathrm{Jac}(C)(\overline{K})$ is non-trivial because it contains Manin-Mumford conjecture, which was proved by Raynaud. Further, it is well known that the above conjecture is equivalent to say the following.

Conjecture 1.1 (Bogomolov conjecture). If f is non-isotrivial, then

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) > 0.$$

Moreover, we can think the following effective version of Conjecture 1.1.

Conjecture 1.2 (Effective Bogomolov conjecture). In Conjecture 1.1, there is an effectively calculated positive number r_0 with

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) \geq r_0.$$

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Let x be a node of a singular fiber $f^{-1}(y)$ over $y \in Y$, and i an integer defined in the following way. Let $h: Z \to f^{-1}(y)$ be the partial normalization of $f^{-1}(y)$ at x. If Z is connected, then $i = 0$. Otherwise, i is the minimum of arithmetic genera of two connected components of Z. We say the node x of the singular fiber $f^{-1}(y)$ is of type i. We denote by $\delta_{i,y}$ (resp. δ_i) the number of nodes of type i in $f^{-1}(y)$ (resp. all singular fibers). In [2], we proved that, if f is non-isotrivial and the stable model of $f: X \to Y$ has only irreducible fibers, then Conjecture 1.2 holds. More precisely,

$$\inf_{P \in \mathrm{Jac}(C)(\bar{K})} r_C(P) \geq \begin{cases} \sqrt{12(g-1)} & \text{if } f \text{ is smooth,} \\ \sqrt{\dfrac{(g-1)^3}{3g(2g+1)}\delta_0} & \text{otherwise.} \end{cases}$$

In this note, we would like to show the following result.

Theorem 1.3. *If f is non-isotrivial and $g = 2$, then f is not smooth and*

$$\inf_{P \in \mathrm{Jac}(C)(\bar{K})} r_C(P) \geq \sqrt{\frac{2}{135}\delta_0 + \frac{2}{5}\delta_1}.$$

2. Notations and ideas

In this section, we use the same notations as in §1. Let $\omega^a_{X/Y}$ be the dualizing sheaf in the sense of admissible pairing. (For details concerning admissible pairing, see [5] or [2].) First we note the following theorem (cf. [5, Theorem 5.6] or [2, Corollary 2.3]).

Theorem 2.1. *If $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a > 0$, then*

$$\inf_{P \in \mathrm{Jac}(C)(\bar{K})} r_C(P) \geq \sqrt{(g-1)(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a},$$

where $(\cdot)_a$ is the admissible pairing.

From now on, we assume $g = 2$. By the above theorem, in order to get Theorem 1.3, we need to estimate $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a$. First of all, we can set

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a = (\omega_{X/Y} \cdot \omega_{X/Y}) - \sum_{y \in Y} e_y, \tag{2.2}$$

where e_y is the number coming from the Green function of $f^{-1}(y)$. This number depends on the configuration of $f^{-1}(y)$. So, let us consider the classification of semistable cruves of genus 2. Let E be a semistable curve of genus 2 over k and E' the stable model of E, that is, E' is a curve obtained by contracting all (-2)-curves in E. It is well known that there are 7-types of stable curves of genus 2. Thus, we have the classification of semistable curves of genus 2 according to type of E' as in Table 1. (In Table 1, the symbol A_n for a node means that the dual graph of (-2)-curves over the node is same as A_n type graph.) The exact value of e_y can be found in Table 2 and will be calculated in §3.

Next we need to think an estimation of $(\omega_{X/Y} \cdot \omega_{X/Y})$ in terms of type of $f^{-1}(y)$. According to Ueno [4], there is the canonical section s of

$$H^0(Y, \det(f_*(\omega_{X/Y}))^{10})$$

such that $d_y = \mathrm{ord}_y(s)$ for $y \in Y$ can be exactly calculated under the assumption that $\mathrm{char}(k) \neq 2, 3, 5$. The result can be found in Table 2. Prof. Liu points out that by works of T. Saito [3] and Q. Liu [1], the value d_y in Table 2 still holds even if $\mathrm{char}(k) \leq 5$.

Let δ_y be the number of singularities in $f^{-1}(y)$, i.e., $\delta_y = \delta_{0,y} + \delta_{1,y}$. Then, by Noether formula,

$$\deg(\det(f_*(\omega_{X/Y}))) = \frac{(\omega_{X/Y} \cdot \omega_{X/Y}) + \sum_{y \in Y} \delta_y}{12}.$$

On the other hand, by the definition of d_y,

$$\sum_{y \in Y} d_y = 10 \deg(\det(f_*(\omega_{X/Y}))).$$

Thus, we have

$$(\omega_{X/Y} \cdot \omega_{X/Y}) = \sum_{y \in Y} \left(\frac{6}{5} d_y - \delta_y\right). \tag{2.3}$$

Hence, by (2.2) and (2.3),

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a = \sum_{y \in Y} \left(\frac{6}{5} d_y - \delta_y - e_y\right). \tag{2.4}$$

According to Table 2, we can see that

$$d_y = \delta_{0,y} + 2\delta_{1,y}.$$

Moreover, using Table 2 and an inequality:

$$\frac{abc}{ab + bc + ca} \leq \frac{a + b + c}{9},$$

we can show

$$e_y \leq \frac{5}{27} \delta_{0,y} + \delta_{1,y}.$$

Therefore, by (2.4), we have the following theorem.

Theorem 2.5. *If f is non-isotrivial, then f is not smooth and*

$$(\omega^{a}_{X/Y}\cdot\omega^{a}_{X/Y})_a \geq \frac{2}{135}\delta_0+\frac{2}{5}\delta_1.$$

Note that non-smoothness of f can be easily derived from the fact that the moduli space $\mathcal{M}_2$ of smooth curves of genus 2 is an affine variety.

3. Calculation of e_y

Let us start calculations of e_y. If the stable model of a fiber is irreducible, e_y is calculated in [2]. Thus it is sufficient to calculate e_y for II (a), IV (a,b), VI (a,b,c) and VII (a,b,c). In these cases, the stable model has two irreducible components. Let $f^{-1}(y) = C_1+\cdots+C_n$ be the irreducible decomposition of $f^{-1}(y)$. We set

$$D_y=\sum_{i=1}^{n}(\omega_{X/Y}\cdot C_i)v_i,$$

where v_i is the vertex in G_y corresponding to C_i. Especially, we denote by P and Q corresponding vertexes to stable components. Then, $D_y = P+Q$. Let μ and g_μ be the measure and the Green function defined by D_y. In the same way as in the Proof of Theorem 5.1 in [2],

$$e_y=-g_\mu(D_y,D_y)+4c(G_y,D_y),$$

where $c(G_y, D_y)$ is the constant coming from g_μ. By the definition of $c(G_y, D_y)$,

$$c(G_y,D_y)=g_\mu(P,P)+g_\mu(P,D_\mu).$$

Therefore, we have

$$e_y=7g_\mu(P,P)-g_\mu(Q,Q)+2g_\mu(P,Q).$$

Here claim:

Lemma 3.1. *$g_\mu(P,P)=g_\mu(Q,Q)$. In particular,*

$$e_y=6g_\mu(P,P)+2g_\mu(P,Q).$$

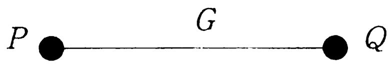
Proof. By the definition of $c(G_y, D_y)$,

$$c(G_y,D_y)=g_\mu(P,P)+g_\mu(P,P+Q)=g_\mu(Q,Q)+g_\mu(Q,P+Q).$$

Thus, we can see $g_\mu(P,P)=g_\mu(Q,Q)$.

In the following, we will calculate e_y for each type II (a), IV (a,b), VI (a,b,c) and VII (a,b,c). First we present the dual graph of each type and then show its calculation.

Type II(a).



length $(G) = a$

In this case, $\mu = \frac{\delta_P}{2} + \frac{\delta_Q}{2}$ by [5, Lemma 3.7]. We fix a coordinate $s : G \to [0, a]$ with $s(P) = 0$ and $s(Q) = a$. If we set

$$g(x) = -\frac{s(x)}{2} + \frac{a}{4},$$

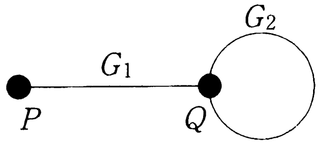
then, $\Delta(g) = \delta_P - \mu$ and $\int_G g\mu = 0$. Thus, $g(x) = g_\mu(P, x)$. Hence

$$g_\mu(P, P) = \frac{a}{4} \quad \text{and} \quad g_\mu(P, Q) = -\frac{a}{4}.$$

Thus

$$e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = a.$$

Type IV(a, b).



We fix coordinates $s : G_1 \to [0, a]$ and $t : G_2 \to [0, b)$ with $s(P) = 0$, $s(Q) = a$ and $t(Q) = 0$. In this case, $\mu = \frac{\delta_P}{2} + \frac{dt}{2b}$ by [5, Lemma 3.7]. We set

$$g(x) = \begin{cases} -\frac{s(x)}{2} + \frac{b + 12a}{48} & \text{if } x \in G_1, \\ \frac{1}{2}\left(\frac{t(x)^2}{2b} - \frac{t(x)}{2}\right) + \frac{b - 12a}{48} & \text{if } x \in G_2 \end{cases}$$

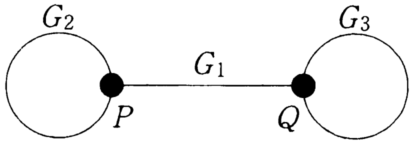
Then, g is continuous, $\Delta(g|_{G_1}) = \frac{\delta_P}{2} - \frac{\delta_Q}{2}$, and $\Delta(g|_{G_2}) = \frac{\delta_Q}{2} - \frac{dt}{2b}$. Thus, $\Delta(g) = \delta_P - \mu$. Moreover, $\int_G g\mu = 0$. Therefore, $g(x) = g_\mu(P, x)$. Hence

$$g_\mu(P, P) = \frac{b + 12a}{48} \quad \text{and} \quad g_\mu(P, Q) = \frac{b - 12a}{48}.$$

Thus

$$e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = a + \frac{b}{6}.$$

Type VI(a, b, c).



We fix coordinates $s: G_1 \to [0, a]$, $t: G_2 \to [0, b)$ and $u: G_3 \to [0, c)$ with $s(P) = 0$, $s(Q) = a$, $t(P) = 0$ and $u(Q) = 0$. In this case, $\mu = \frac{dt}{2b} + \frac{du}{2c}$ by [5, Lemma 3.7]. We set

$$g(x) = \begin{cases} \frac{1}{2}\left(\frac{t(x)^2}{2b} - \frac{t(x)}{2}\right) + \frac{b+c+12a}{48} & \text{if } x \in G_2, \\ -\frac{s(x)}{2} + \frac{b+c+12a}{48} & \text{if } x \in G_1, \\ \frac{1}{2}\left(\frac{u(x)^2}{2c} - \frac{u(x)}{2}\right) + \frac{b+c-12a}{48} & \text{if } x \in G_3. \end{cases}$$

Then, g is continuous, $\Delta(g|_{G_1}) = \frac{\delta_P}{2} - \frac{\delta_Q}{2}$, $\Delta(g|_{G_2}) = \frac{\delta_P}{2} - \frac{dt}{2b}$, and $\Delta(g|_{G_3}) = \frac{\delta_Q}{2} - \frac{du}{2c}$.

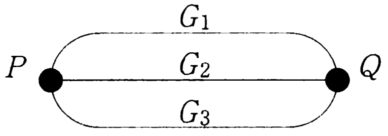
Thus, $\Delta(g) = \delta_P - \mu$. Moreover, $\int_G g\mu = 0$. Therefore, $g(x) = g_\mu(P, x)$. Hence

$$g_\mu(P, P) = \frac{b+c+12a}{48} \quad \text{and} \quad g_\mu(P, Q) = \frac{b+c-12a}{48}$$

Thus

$$e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = a + \frac{b+c}{6}.$$

Type VII(a,b,c).



length $(G_1) = a$
length $(G_2) = b$
length $(G_3) = c$

We fix coordinates $s: G_1 \to [0, a]$, $t: G_2 \to [0, b]$ and $u: G_3 \to [0, c]$ with $s(P) = 0$, $s(Q) = a$, $t(P) = 0$, $t(Q) = b$, $u(P) = 0$ and $u(Q) = c$. In this case, $\mu = \frac{ds}{3a} + \frac{dt}{3b} + \frac{du}{3c}$ by [5, Lemma 3.7]. We set

$$g(x) = \begin{cases} \frac{s(x)^2}{6a} - \left(\frac{1}{6} + \frac{1}{2}\frac{bc}{ab+bc+ca}\right)s(x) + \frac{a+b+c}{108} + \frac{1}{4}\frac{abc}{ab+bc+ca} & \text{if } x \in G_1, \\ \frac{t(x)^2}{6b} - \left(\frac{1}{6} + \frac{1}{2}\frac{ac}{ab+bc+ca}\right)t(x) + \frac{a+b+c}{108} + \frac{1}{4}\frac{abc}{ab+bc+ca} & \text{if } x \in G_2, \\ \frac{u(x)^2}{6c} - \left(\frac{1}{6} + \frac{1}{2}\frac{ab}{ab+bc+ca}\right)u(x) + \frac{a+b+c}{108} + \frac{1}{4}\frac{abc}{ab+bc+ca} & \text{if } x \in G_3 \end{cases}$$

Then, g is continuous and $\Delta(g) = \Delta(g|_{G_1}) + \Delta(g|_{G_2}) + \Delta(g|_{G_3}) = \delta_P - \mu$. Moreover, $\int_G g\mu = 0$. Therefore, $g(x) = g_\mu(P, x)$. Hence

$$e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = \frac{2}{27}(a + b + c) + \frac{abc}{ab + bc + ca}.$$

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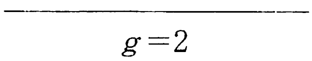
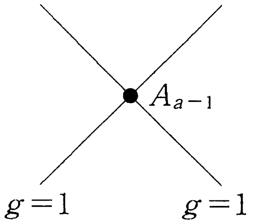
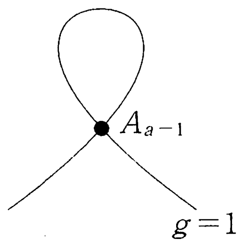
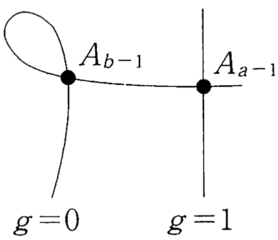
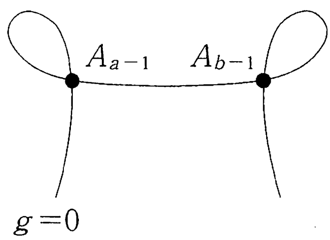
References

[1] Q. Liu, Conducteur et discriminant minimal de courbes de genre 2, Comp. Math., **94** (1994), 51-79.
[2] A. Moriwaki, Bogomolov conjecture over function fields for stable curves with only irreducible fibers, to appear in Comp. Math.
[3] T. Saito, The discriminants of curves of genus 2, Comp. Math., **69** (1989), 229-240.
[4] K. Ueno, Discriminants of curves of genus 2 and arithmetic surfaces, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi NAGATA, (1987), 749-770.
[5] S. Zhang, Admissible pairing on a curve, Invent. Math., **112** (1993), 171-193.

TABLE 1. Classification of semistable curve E of genus 2

Type of E	Description of the stable model E' of E	Figure of E' and types of singularities by contracting (-2)-curves in E
I	a smooth curve of genus 2	
II (a)	two elliptic curves meeting at one point transversally	
III (a)	an elliptic curve with one node	
IV (a,b)	a smooth elliptic curve and a rational curve with one node, which meet at one point transversally	
V (a, b)	a rational curve with two nodes	

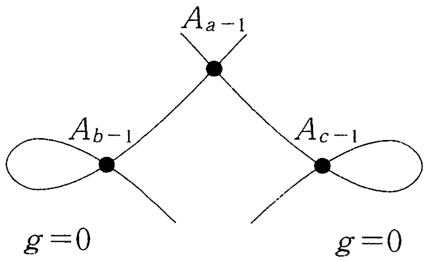
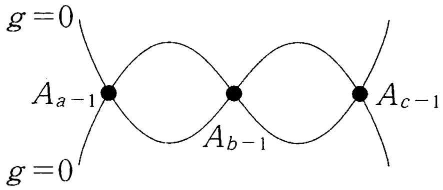
VI (a,b,c)	two rational curves with one node, which meet at one point transversally	
VII (a,b,c)	two smooth rational curves, which meet at three points transversally	

TABLE 2. δ_y, d_y and e_y

Type	δ_y	d_y	e_y
I	0	0	0
II (a)	a	$2a$	a
III (a)	a	a	$\frac{a}{6}$
IV (a,b)	$a+b$	$2a+b$	$a+\frac{b}{6}$
V (a,b)	$a+b$	$a+b$	$\frac{a+b}{6}$
VI (a,b,c)	$a+b+c$	$2a+b+c$	$a+\frac{b+c}{6}$
VII (a,b,c)	$a+b+c$	$a+b+c$	$\frac{2}{27}(a+b+c)+\frac{abc}{ab+bc+ca}$