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On H-spaces and exceptional Lie groups

By

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0. Introduction

An H-space is a space which admits a continuous product with unit. F. Borel [1] showed its fundamental group is restricted by the rational cohomology algebra under a certain associativity condition. In particular, if an H-space X satisfies $\mathrm{H}^*(X; \boldsymbol{Q}) \cong \mathrm{H}^*(G: \boldsymbol{Q})$ as an algebra where G is an exceptional Lie group, then $\pi_1(X)$ is a subgroup of the group in the following table.

$G = G_2$	$\pi_1(X) \subset \boldsymbol{Z}/2$
F_4	$\boldsymbol{Z}/8 \times \boldsymbol{Z}/8$
E_6	$\boldsymbol{Z}/8 \times \boldsymbol{Z}/8 \times \boldsymbol{Z}/3 \times \boldsymbol{Z}/5$
E_7	$\boldsymbol{Z}/8 \times \boldsymbol{Z}/8$
E_8	$\boldsymbol{Z}/8 \times \boldsymbol{Z}/8$

As for the mod 2 cohomology, J.Lin showed

Theorem 1 ([4]) *Let X be a 1-connected H-space such that $\mathrm{H}_*(X ; \boldsymbol{F}_2)$ is finite and associative. If $\mathrm{H}^*(X ; \boldsymbol{Q}) \cong \mathrm{H}^*(G ; \boldsymbol{Q})$ as an algebra for an exceptional Lie group G, then $\mathrm{H}^*(X ; \boldsymbol{F}_2) \cong \mathrm{H}^*(G ; \boldsymbol{F}_2)$ as an algebra over the mod 2 Steenrod algebra.*

By adding Serre spectral sequence arguments we can refine these. The purpose of this paper is to prove the following theorem.

Theorem 2 *Let X be a connected homotopy associative H-space such that $\mathrm{H}_*(X ; \boldsymbol{F}_2)$ is finite. Assume that $\mathrm{H}^*(X ; \boldsymbol{Q}) \cong \mathrm{H}^*(G ; \boldsymbol{Q})$ as an algebra, where G is an exceptional Lie group. Then $\pi_1(X)$ and $\mathrm{H}^*(X ; \boldsymbol{F}_2)$* are as follows.

$$G = G_2, F_4, E_8 \qquad \begin{cases} \pi_1(X) = 0, \\ \mathrm{H}^*(X ; \boldsymbol{F}_2) \cong \mathrm{H}^*(G ; \boldsymbol{F}_2) \end{cases}$$

$$G = E_6 \qquad \begin{cases} \pi_1(X) \subset \boldsymbol{Z}/3 \times \boldsymbol{Z}/5, \\ \mathrm{H}^*(X ; \boldsymbol{F}_2) \cong \mathrm{H}^*(E_6 ; \boldsymbol{F}_2) \end{cases}$$

$$G = E_7 \qquad \begin{cases} \pi_1(X) = 0, \\ \mathrm{H}^*(X ; \boldsymbol{F}_2) \cong \mathrm{H}^*(E_7 ; \boldsymbol{F}_2) \end{cases} \text{or} \quad \begin{cases} \pi_1(X) = \boldsymbol{Z}/2, \\ \mathrm{H}^*(X ; \boldsymbol{F}_2) \cong \mathrm{H}^*(\mathrm{Ad}\ (E_7) ; \boldsymbol{F}_2) \end{cases}$$

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The four isomorphisms between the cohomologies preserve the Hopf algebra structure over the mod 2 Steenrod algebra.

For our sake we recall here the projecitve plane and the associated exact sequence :

$$\to \widetilde{H}^i(X) \xrightarrow{\overline{\Psi}} [\widetilde{H}^*(X) \otimes \widetilde{H}^*(X)]^i \xrightarrow{\lambda} \widetilde{H}^{i+2}(P_2X) \xrightarrow{\iota} \widetilde{H}^{i+1}(X) \to .$$

In the sequence P_2X is the projective plane of X and $\overline{\Psi}$ is the reduced coproduct. Our plan to prove Theorem 2 is as follows. §1 is devoted to the proof for the case $G=G_2$ and explains our method. §2 and §3 deal with the cases $G=F_4$ and E_i $(i=6, 7, 8)$, respectively. In these sections we shall make use of the above exact sequence to compute $\pi_1(X)$ as is stated in Theorem 2. Note that in most cases Theorem 1 will be applicable.

Throughout this paper a space is assumed to have the homotopy type of a CW complex *localized at 2*. The symbol X is reserved for the space in Theorem 2. Let H^* denote $H^*(\ ;\boldsymbol{F}_2)$, or on apparent occasions $H^*(X;\boldsymbol{F}_2)$. Last let $\widetilde{X}\to X$ be the universal covering.

Especially for the reason of rather heavy tasks in my previous office I needed to recover. I am very grateful to Professor Akira Kono. Throughout preparation of this paper he advised and encouraged me kindly. Also I would like to thank Dr. Kouichi Inoue, a kind friend who helped me to study.

1. Case $G=G_2$

This easiest case illustrates our method. At first we notice a lemma about the associativity of a covering space.

Lemma 3 *Let $p: E\to B$ be a covering where B is a connected homotopy associative H-space and E is connected. Then E is also a homotopy associative H-space and p is an H-map.*

Proof. An easy application of the lift theorem. Since E is homeomorphic to $\widetilde{B}/M$, where $\widetilde{B}$ is the universal covering space and M is a subgroup is a subgroup of $\pi_1(B)$, it is easy to see E is an H-space and p is an H-map. Thus for a CW-complex K, $[K, \pi_1(B)/M]\to[K,E]\to[K,B]$ is an exact sequence of algebraic loops. Note that the first and the last terms are groups. Each element x lying in $[K,E]$ has a unique right inverse $-x$ ([10]). Let μ denote the multiplication map of E. Because $\pi_1(B)/M$ is discrete, it is immediate to see $\mu(\mu\times 1)-(\mu(1\times\mu))$ is trivial.

Q.E.D.

Suppose $H^*(X;\boldsymbol{Q})\cong H^*(G_2;\boldsymbol{Q})$ as an algebra. If $\pi_1(X)\neq 0$, we can assume $\pi_1(X)=\boldsymbol{Z}/2$ by replacing X with an appropriate covering space if necessary. Lemma 3 ensures X is still homotopy associative after such an exchange. We will show the assumption $\pi_1(X)=\boldsymbol{Z}/2$ leads to a contradiction.

By the hypothesis we have a fibration $\widetilde{X}\to X\to \boldsymbol{R}P^\infty$, where $\widetilde{X}$ is homotopy associative by Lemma 3. Because $\widetilde{X}$ and X are rationally homotopy equivalent, $\widetilde{X}$ fulfils the condition of Theorem 1 and its mod 2 cohomology is isomorphic to that of G_2 ;

$$\mathrm{H}^*(\widetilde{X}) = \frac{\boldsymbol{F}_2[x_3]}{(x_3{}^4)} \otimes \Lambda(x_5),$$

where $\deg x_i=i$, and $Sq^2x_3=x_5$. Set $\mathrm{H}^*\boldsymbol{R}P^\infty=\boldsymbol{F}_2[t]$, where $\deg t=1$. Since X is mod 2 finite, all elements lying in $\mathrm{H}^*(X)$ are of finite height. Thus because of the theorem of Hopf-Borel, the Serre spectral sequence deduces that

$$\mathrm{H}^*(X) = \frac{\boldsymbol{F}_2[t]}{(t^4)} \otimes \Lambda(x_5, \bar{x}_6),$$

where t and x_5 are obvious elements and $\bar{x}_6=Sq^1x_5$. $\mathrm{H}^*(X)$ is primitively generated. In particular there exists a_7 lying in $\mathrm{H}^7(\mathrm{P}_2X)$ such that $\iota(a_7)=\bar{x}_6$. We now have $a_7{}^2=Sq^7a_7=Sq^3Sq^4a_7$, and $\iota(Sq^4a_7)=0$. Hence Sq^4a_7 has an inverse image $\sum a'\otimes a''$ by λ. $Sq^3(\sum a'\otimes a'')$ has $\bar{x}_6\otimes\bar{x}_6$ as a summand, which is inconsistent with the Steenrod action on QH^*. Therefore X is 1-connected. We find that $\mathrm{H}^*(X)$ is isomorphic to $\mathrm{H}^*(G_2)$ as an algebra over the Steenrod algebra by Theorem 1. The Hopf algebra structure is described in [8].

2. Case $G=\boldsymbol{F}_4$

Assume $\pi_1(X)=\boldsymbol{Z}/2$ (, which is equivalent to the assumption $\pi_1(X)\neq 0$). In the present case

$$\mathrm{H}^*(\widetilde{X}) = \frac{\boldsymbol{F}_2[x_3]}{(x_3{}^4)} \otimes \Lambda(x_5, x_{15}, x_{23}).$$

From the Serre spectral sequence for $\widetilde{X}\to X\to\boldsymbol{R}P^\infty$ we have two cases about $\mathrm{H}^*(X)$:

$$\text{(1)}\quad \mathrm{H}^*(X) = \frac{\boldsymbol{F}_2[t]}{(t^4)} \otimes \Lambda(x_5, \bar{x}_6, x_{15}, x_{23}), \qquad \text{or}$$

$$\text{(2)}\quad \mathrm{H}^*(X) = \frac{\boldsymbol{F}_2[t]}{(t^{16})} \otimes \frac{\boldsymbol{F}_2[x_3]}{(x_3{}^4)} \otimes \Lambda(x_5, x_{23}).$$

We find the former case is impossible in a similar way as in §1. Suppose the latter case. To show x_3 is primitive, we need the following useful lemma.

Lemma 4 ([6] *Lemma* 1. 11 *in* §1, *Chap.* 7, [2]) *Suppose that a connected Hopf algebra* A *over a field* $\boldsymbol{k}$ *satisfies* $A=\Delta(y_1, y_2, \cdots, y_n)$ *in dimensions* *less than N, where y_i's are primitive and Δ denotes the simple system of generators. Then $\bar{A}\cdot\bar{A}+\bar{\Psi}^{-1}(PA\otimes PA)=A$ in dimension N, where PA denotes the submodule of the primitive elements and $\bar{A}$ the augmentation ideal. In particular we can choose indecomposable elements of degree N from $\bar{\Psi}^{-1}(PA\otimes PA)$.*

If x_3 is not primitive we can assume $\bar{\Psi}(x_3)=t\otimes t^2$ Thus we get $\bar{\Psi}(Sq^1x_3)=t^2\otimes t^2$. As is easily seen, this cannot happen. Therefore x_3 is primitive. Again by Lemma 4, x_{23} is also primitive. Let a_{24} be an inverse image of x_{23} by ι. Note that $Sq^{24}=Sq^8Sq^{16}+Sq^{23}Sq^1+Sq^{22}Sq^2+Sq^{20}Sq^4$. Since $\mathrm{PH}^{39}=\mathrm{PH}^{24}=\mathrm{PH}^{25}=\mathrm{PH}^{27}=0$, there are a, b, c, d lying in $\mathrm{H}^*\otimes\mathrm{H}^*$ such that $a_{24}{}^2=Sq^8\lambda(a)+Sq^{23}\lambda(b)+Sq^{22}\lambda(c)+Sq^{20}\lambda(d)$. Therefore $Sq^8a+Sq^{23}b+Sq^{22}c+Sq^{20}d=x_{23}\otimes x_{23}$ mod $\mathrm{Im}\bar{\Psi}$. Since $x_{23}\otimes x_{23}$ does not lie in $\mathrm{Im}\bar{\Psi}$, by passing to $\mathrm{QH}^*\otimes\mathrm{QH}^*$, one sees this relation is impossible. Therefore X is 1-connected. The remaining part is verified as in §1.

3. Case $G=E_i$ $(i=6, 7, 8)$

We will first show $\pi_1(X)=0$ in the case $G=E_6$. Suppose $\pi_1(X)=\boldsymbol{Z}/2$. There is a fibration $\tilde{X}\to X\to \boldsymbol{R}P^\infty$, and hence we have two cases:

$$(1)\quad \mathrm{H}^*(X)=\frac{\boldsymbol{F}_2[t]}{(t^4)}\otimes\Lambda(x_5, \bar{x}_6, x_9, x_{15}, x_{17}, x_{23})\quad\text{or}$$

$$(2)\quad \mathrm{H}^*(X)=\frac{\boldsymbol{F}_2[t]}{(t^{16})}\otimes\frac{\boldsymbol{F}_2[x_3]}{(x_3{}^4)}\otimes\Lambda(x_5, x_9, x_{17}, x_{23})$$

As for case (1), by the same reasoning about $\bar{x}_6$ as in §1 one deduces such a cohomology is impossible. When $G=E_7$ or E_8, $x_5{}^2$ obstructs this argument. In case (2), x_3 is primitive as in case (2) of §2, and also is $x_{17}(=Sq^8Sq^4Sq^2x_3)$. The next lemma shows the present case is impossible.

Lemma 5 *Assume* $\mathrm{PH}^{33}=\mathrm{PH}^{34}=\mathrm{QH}^{16}=0$. *If* $\dim\mathrm{PH}^{17}=1$ *and* $\mathrm{PH}^{17}\to\mathrm{QH}^{17}$ *is isomorphic, then* $\mathrm{QH}^{15}\neq 0$.

Proof. Suppose $\mathrm{QH}^{15}=0$. Let x_{17} be the primitive generator of degree 17, then there exists a_{18} lying in $\mathrm{H}^{18}(\mathrm{P}_2X)$ which is mapped to x_{17}. Since $Sq^{18}=Sq^1Sq^{16}Sq^1+Sq^2Sq^{16}$ and $\mathrm{PH}^{33}=\mathrm{PH}^{34}=0$, $a_{18}{}^2=Sq^{18}a_{18}=Sq^1\lambda(\mathrm{a})+Sq^2\lambda(b)$ for some a and b. Then $Sq^1a+Sq^2b=x_{17}\otimes x_{17}$ mod $\mathrm{Im}\bar{\Psi}$. However this is impossible because $\mathrm{QH}^{15}=\mathrm{QH}^{16}=0$

Q. E. D.

Second, we will show $\pi_1(X)=0$ or $\boldsymbol{Z}/2$ for $G=E_7$. For this purpose we shall prove two facts: (a) $\pi_1(X)\neq\boldsymbol{Z}/2\times\boldsymbol{Z}/2$, and (b) $\pi_1(X)\neq\boldsymbol{Z}/4$. In a similar way as in the beginning of §1, fact (a) ensures $\pi_1(X)$ contains at most one factor and then fact (b) implies the required result. We now suppose $\pi_1(X) = \boldsymbol{Z}/2\times\boldsymbol{Z}/2$. We have from the Serre spectral sequence for $\widetilde{X}\to X\to \boldsymbol{R}P^\infty\times\boldsymbol{R}P^\infty$

$$\mathrm{H}^*(X) = \frac{\boldsymbol{F}_2[t_1, t_2]}{(t_1^4, t_2^{16})} \otimes \frac{\boldsymbol{F}_2[x_5, x_9]}{(x_5^4, x_9^4)} \otimes \Lambda(\overline{x}_6, x_{17}, x_{23}, x_{27}),$$

where deg $t_1=$deg $t_2=1$ and other generators are obvious elements. From Lemma 4 we can set $\overline{\Psi}(x_5) = \alpha t_1\otimes t_2^4+\beta t_2\otimes t_2^4\ (\alpha,\ \beta\in\boldsymbol{F}_2)$. Then $x_{17}=Sq^8Sq^4x_5$ is primitive and Lemma 5 is valid in this case. Therefore the cohomology above is not possible and fact (a) follows.

Next we will prove $\pi_1(X)\neq\boldsymbol{Z}/4$. If $\pi_1(X)=\boldsymbol{Z}/4$, the spectral sequence deduces

$$\mathrm{H}^*(X) = \Lambda(t_1, u_2) \otimes \frac{\boldsymbol{F}_2[x_5, x_9]}{(x_5^4, x_9^4)} \otimes \Lambda(\overline{x}_6, x_{15}, x_{17}, x_{23}, x_{27})$$

where the suffices refer to the degrees. (Note that x_3 is not a permanent cycle by Lemma 5.) u_2 has an inverse image a_3 in $\mathrm{H}^3(\mathrm{P}_2X)$. There exists a' lying in $(\mathrm{H}^*\otimes\mathrm{H}^*)^3$ such that $\lambda(a')=Sq^2a_3$. Since $\lambda(Sq^1a')=a_3^2$ we have $Sq^1(a')=t_2\otimes t_2$ mod Im λ. This relation is a contradiction since $Sq^1t_1=t_1^2=0$. We conclude that $\pi_1(X)=0$ or $\boldsymbol{Z}/2$.

Last, we will show $\pi_1(X)\neq\boldsymbol{Z}/2$ when $G=E_8$. If it is not the case, we have

$$\mathrm{H}^*(X) = \frac{\boldsymbol{F}_2[\,t\,]}{(t^4)} \otimes \frac{\boldsymbol{F}_2[x_5, \overline{x}_6, x_9, x_{15}]}{(x_5^8, \overline{x}_6^8, x_9^4, x_{16}^4)} \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29}).$$

(In the spectral sequence x_{15} does not vanish by Lemma 5.) Since $\mathrm{H}^*(X)$ is primitively generated in dimensions less than 15, Lemma 4 deduces

$$\overline{\Psi}(x_{15}) = \alpha x_5\otimes x_5^2+\beta x_9\otimes\overline{x}_6 \qquad (1)$$

We will show such a coproduct cannot occur. The next lemma states the corresponding coproduct in $\mathrm{H}^*(\widetilde{X})$. Here we set

$$\mathrm{H}^*(\widetilde{X}) = \frac{\boldsymbol{F}_2[y_3, y_5, y_9, y_{15}]}{(y_3^{16}, y_5^8, y_9^4, y_{15}^4)} \otimes \Lambda(y_{17}, y_{23}, y_{27}, y_{29}).$$

We may suppose each x_i is mapped to y_i except for $i=3$.

Lemma 6 $\overline{\Psi}(y_{15})=y_5\otimes y_5^2+y_9\otimes y_3^2+y_3\otimes y_3^4.$

Proof. We only sketch the proof. Since the cohomology is primitively generated in dimensions less than 15, we can set $\overline{\Psi}(y_{15})=\alpha y_5\otimes y_5^2+\beta y_9\otimes y_3^2+\gamma y_3\otimes y_3^4\ (\alpha,\ \beta,\ \gamma\in\boldsymbol{F}_2)$. Note that Sq^1y_{15} and Sq^2y_{15} lie in the Hopf subalgebra A generated by primitive elements y_3, y_5, y_9 and y_{17}. Thus $Sq^1\overline{\Psi}(y_{15})=\alpha y_3^2\otimes y_5^2+\beta y_5^2\otimes y_3^2\in\mathrm{Im}\,\overline{\Psi}|_{\overline{A}}$. It is then easy to see that $\alpha=\beta$. In a similar way we deduce $Sq^2\bar{\Psi}(y_{15})=\alpha y_5\otimes y_3{}^4+\gamma y_5\otimes y_3{}^4$ and $\alpha=\gamma$. Thus $\alpha=\beta=\gamma$. We now quote the following theorem.

Theorem 7 ([7]) *Let Z be a 1-connected mod 2 finite H-space satisfying the following conditions, where $n\geqq 3$.*

(a) $H^*(Z)=\boldsymbol{F}_2[x]/(x^4)\otimes R$ *as an algebra, and* $x\in PH^{2^n-1}$.

(b) $QR^{2^n-1}=0$, $\dim QH^{2^{n+1}-3}=\dim QH^{2^{n+1}+1}$.

Then $H^*(Z)$ is not primitively generated.

Assume $\alpha=0$. Because $Sq^1Sq^2Sq^4Sq^8y_{15}=y_{15}{}^2\neq 0$, we have primitive generators Sq^8y_{15}, $Sq^4Sq^8y_{15}$, and $Sq^2Sq^4Sq^8y_{15}$. This is inconsistent with Theorem 7, which completes the proof.

Q. E. D.

It is clear that the coproduct (1) cannot be mapped to the coproduct of y_{15}. Thus $\pi_1(X)\neq\boldsymbol{Z}/2$ and X is simply connected.

Summing up all, we conclude that $\pi_1(X)=0$ when $G=E_i$ $(i=6, 8)$ and that $\pi_1(X)=\boldsymbol{Z}/2$ or 0 when $G=E_7$. Except the case for E_7, $H^*(X)$ is isomorphic to $H^*(G)$ as a Hopf algebra over the Steenrod algebra. For more detailed arguments about the Hopf algebra structure we refer to [8].

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