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The mod 3 homology of the space of loops on the exceptional Lie groups and the adjoint action

By

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1. Introduction

Let p be a prime number and G be a compact, connected, simply connected and simple Lie group. Let ΩG be the loop space of G. Bott showed $H_*(\Omega G;Z/p)$ is a finitely generated bicommutative Hopf algebra concentrated in even degrees, and determined it for classical groups G ([1]).

Here, let G be an exceptional Lie group, that is, $G=G_2, F_4, E_6, E_7, E_8$. In [2], K. Kozima and A. Kono determined $H_*(\Omega G; Z/2)$ as a Hopf algebra over $\mathscr{A}_2$, where $\mathscr{A}_p$ is the mod p Steenrod Algebra and acts on it dually.

Let Ad : $G\times G\rightarrow G$ and ad : $G\times\Omega G\rightarrow\Omega G$ be the adjoint actions of G on G and ΩG respectively. In [3], the cohomology maps of these adjoint actions are studied and it is shown that $H^*(ad\ ; Z/p)=H^*(p_2; Z/p)$ where p_2 is the second projection if and only if $H^*(G; Z)$ is p-torsion free. For $p=2$, 3 and 5, some exceptional Lie groups have p-torsions on its homology. Moreover in [8, 9] mod p homology map of ad is determined for $(G, p)=(G_2, 2)$, $(F_4, 2)$, $(E_6, 2)$, $(E_7, 2)$ and $(E_8, 5)$. This result is applied to compute the $\mathscr{A}_5$ module structure of $H_*(\Omega E_8; Z/5)$ and $H^*(E_8; Z/5)$ in [9].

For a compact and connected Lie group G, the free loop group of G is denoted by $LG(G)$, i. e. the space of free loops on G equipped with multiplication as

$$\phi\cdot\psi(t)=\phi(t)\cdot\psi(t),$$

and has ΩG as its normal subgroup. Then

$$LG(G)/\Omega G\cong G,$$

and identifying elements of G with constant maps from S^1 to G, $LG(G)$ is equal to the semi-direct product of G and ΩG. This means that the homology of $LG(G)$ is determined by the homology of G and ΩG as module and the algebra structure of $H_*(LG(G); Z/p)$ depends on $H_*(ad\ ; Z/p)$ where

$$ad : G\times\Omega G\rightarrow\Omega G$$

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is the adjoint map. Since the next diagram commutes where λ, λ', and μ are the multiplication maps of ΩG, $LG(G)$ and G respectively and ω is the composition

$$(1_{\Omega G}\times T\times 1_G)\circ(1_{\Omega G\times G}\times ad\times 1_G)\circ(1_{\Omega G}\times \Delta_G\times 1_{\Omega G\times G}),$$

$$\begin{array}{ccccc} \Omega G\times G\times\Omega G\times G & \xrightarrow{\omega} & \Omega G\times\Omega G\times G\times G & \xrightarrow{\lambda\times\mu} & \Omega G\times G \\ \downarrow\cong\times\cong & & & & \downarrow\cong \\ LG(G)\times LG(G) & & \xrightarrow{\lambda'} & & LG(G) \end{array}$$

we can determine directly the algebra structure of $H_*(LG(G);Z/p)$ by the knowledge of the Hopf algebra structure of $H_*(G;Z/p)$, $H_*(\Omega G;Z/p)$ and induced homology map $H_*(ad;Z/p)$. See Theorem 6.12 of [8] for detail.

In this paper we determined the Hopf algebra structure over $\mathcal{A}_3$ of the homology group $H_*(\Omega G;Z/3)$ for $G=F_4$, E_6, E_7 and E_8 by using adjoint action and determine the mod 3 homology map of ad for them. The result is shown in §2.

This paper is organized as follows. We refer to the results of [4, 5, 6] for the structure of $H^*(G)$ and compute $H^*(\Omega G)$ for the lower dimensions and their cohomology operations are partially determined. This is done in §3. In §4 we turn to their homology rings. We determine the algebra structure of $H_*(\Omega G;Z/3)$ and we partly determine the Hopf algebra structure and cohomology operaions on $H_*(\Omega G;Z/3)$. Finally in §5 the homology map of the adjoint action and the rest of the Hopf algebra structure and cohomology operations are determined. The computations are completely algebraic.

2. Results

Let $G(l)$ be the compact, connected, simply connected and simple exceptional Lie group of rank l where $l=4$, 6, 7 or 8. The exponents of $G(l)$ are the integers $n(1)<n(2)<\cdots n(l)$ which are given by the following table:

l	$n(1)$,		$n(2)$,		$\cdots$,		$n(l)$					
4	1		5	7			11					
6	1	4	5	7	8		11					
7	1		5	7		9	11	13	17			
8	1		7				11	13	17	19	23	29

Put $E(l)=\{n(1),\cdots,n(l)\}$ and $\bar{\phi}(t)=\Delta_*(t)-(t\otimes 1+1\otimes t)$ where Δ is the diagonal map. $\mathcal{P}^k_*$ is the dual of the Steenrod operation $\mathcal{P}^k$. Then the results are following:

Theorem 1. *As a Hopf Algebra over* $\mathscr{A}_3$,

$$H_*(\Omega G(l);\, Z/3) \cong \begin{cases} Z/3[t_{2j} \mid j \in E(l) \cup \{3\}]/(t_2^3), & \text{if } l=4, 6, 7 \\ Z/3[t_{2j} \mid j \in E(8) \cup \{3, 9\}]/(t_2^3, t_6^3), & \text{if } l=8 \end{cases}$$

where $|t_{2j}|=2j$.

$$\bar{\phi}(t_{2j}) = \begin{cases} 0, & \text{if } j \neq 3, 9, \\ -t_2^2 \otimes t_2 - t_2 \otimes t_2^2, & \text{if } j=3, \\ t_2^2 t_6^2 \otimes t_2 + t_2 t_6^2 \otimes t_2^2 - t_6^2 \otimes t_6 - t_2^2 t_6 \otimes t_2 t_6 & \\ \quad - t_2 t_6 \otimes t_2^2 t_6 - t_6 \otimes t_6^2 + t_2^2 \otimes t_2 t_6^2 + t_2 \otimes t_2^2 t_6^2, & \text{if } j=9, \end{cases}$$

$$\mathscr{P}_*^{3^r} t_{2j} = 0, \qquad \text{if} \quad r \geq 3,$$

$$\mathscr{P}_*^9 t_{2j} = \begin{cases} t_{22}, & \text{if } j=29, \\ 0, & \text{otherwise} \end{cases}.$$

$\mathscr{P}_*^1 t_{2j}$ *and* $\mathscr{P}_*^3 t_{2j}$ *are given by the following table:*

t_{2j}	t_2	t_6	t_8	t_{10}	t_{14}	t_{16}	t_{18}	t_{22}	t_{26}	t_{34}	t_{38}	t_{46}	t_{58}
$\mathscr{P}_*^1 t_{2j}$	0	t_2	0	0	t_{10}	0	$\varepsilon t_{14} - t_2 t_6^2$	κt_6^3	εt_{22}	$-\varepsilon t_{10}^3$	εt_{34}	εt_{14}^3	t_{18}^3
$\mathscr{P}_*^3 t_{2j}$	0	0	0	0	0	0	t_6	0	t_{14}	t_{22}	$-t_{26}$	t_{34}	0

where ε *and* κ *are* 1 *or* -1.

Remark. In Theorem 1, if t_{2j} does not exist in $H_*(\Omega G(l);\, Z/3)$, we regard t_{2j} as 0 for such j.

Let Ad: $G \times G \to G$ and ad: $G \times \Omega G \to \Omega G$ be the adjoint actions of a Lie group G defined by $\mathrm{Ad}(g, h) = ghg^{-1}$ and $\mathrm{ad}(g, l)(t) = gl(t)g^{-1}$ where $g, h \in G$, $l \in \Omega G$ and $t \in [0, 1]$. These induce the homology maps

$$\mathrm{Ad}_*\colon H_*(G;\, Z/3) \otimes H_*(G;\, Z/3) \to H_*(G;\, Z/3)$$

$$\mathrm{ad}_*\colon H_*(G;\, Z/3) \otimes H_*(\Omega G;\, Z/3) \to H_*(\Omega G;\, Z/3).$$

Theorem 2. *There are generators* y_8 *in* $H_*(G(l);\, Z/3)$ *for* $l=4, 6, 7$ *and* y_8 *and* y_{20} *in* $H_*(E_8; Z/3)$. *We can choose these generators so that* $ad_*(y_i \otimes t_{2j})$ $(i=8, 20)$ *is given by the following table.*

t_{2j}	$ad_*(y_8\otimes t_{2j})$	$ad_*(y_{20}\otimes t_{2j})$	t_{2j}	$ad_*(y_8\otimes t_{2j})$	$ad_*(y_{20}\otimes t_{2j})$
t_2	t_{10}	εt_{22}	t_{22}	$-t_{10}^3$	$-t_{14}^3$
t_6	$t_{14}-t_{10}t_2^2$	$t_{26}-\varepsilon t_{22}t_2^2$	t_{26}	t_{34}	$-t_{46}$
t_8	t_{16}	—	t_{34}	$-t_{14}^3$	εt_{18}^3
t_{10}	κt_6^3	—	t_{38}	$-t_{46}$	t_{58}
t_{14}	t_{22}	t_{34}	t_{46}	$-\varepsilon t_{18}^3$	εt_{22}^3
t_{16}	δt_8^3	—	t_{58}	$-\varepsilon t_{22}^3$	$-t_{26}^3$
t_{18}	$t_{26}+t_{10}t_6^2t_2^2-t_{14}t_6^2$	$t_{38}+\varepsilon t_{22}t_6^2t_2^2-t_{26}t_6^2$			

where $\delta,\ \varepsilon\in Z/3Z$ *and* $\varepsilon\neq 0$. *For other generators* $y_i\in H_*(G(l);Z/3)$, $ad(y_i\otimes t_{2j})=0$ *for all* j.

3. The mod 3 cohomology groups

We recall the results of [4, 5, 6] for the structure of $H^*(G(l);Z/3)$ as the Hopf algebra over $\mathscr{A}_3$.

Theorem 3. *There is an isomorphism:*

$$H^*(G(l);Z/3)\cong\begin{cases}\Lambda(x_{2j+1}\mid j\in E(l)\cup\{3\}-\{11\})\otimes Z/3[x_8]/(x_8^3), & \text{if } l=4,6,7,\\ \Lambda(x_{2j+1}\mid j\in E(8)\cup\{3,9\}-\{11,29\})\otimes Z/3[x_8,x_{20}]/(x_8^3,x_{20}^3), & \text{if } l=8,\end{cases}$$

the coproduct is given by:

x_i	$\bar{\varphi}x_i$
x_{11}	$x_8\otimes x_3$
x_{15}	$x_8\otimes x_7$
x_{17}	$x_8\otimes x_9$
x_{27}	$x_8\otimes x_{19}+x_{20}\otimes x_7$
x_{35}	$x_8\otimes x_{27}-x_8^2\otimes x_{19}+x_{20}\otimes x_{15}+x_8x_{20}\otimes x_7$
x_{39}	$x_{20}\otimes x_{19}$
x_{47}	$-x_8\otimes x_{39}-x_{20}\otimes x_{27}-x_{20}x_8\otimes x_{19}+x_{20}^2\otimes x_7$
others	0

and the cohomology operations are determined by the following table:

x_i	x_3	x_7	x_8	x_9	x_{11}	x_{15}	x_{17}	x_{19}	x_{20}	x_{27}	x_{35}	x_{39}	x_{47}
βx_i	0	x_8	0	0	0	$-x_8^2$	0	x_{20}	0	x_8x_{20}	$-x_8^2x_{20}$	$-x_{20}^2$	$x_8x_{20}^2$
$\mathscr{P}^1x_i$	x_7	0	0	0	x_{15}	εx_{19}	0	0	0	0	εx_{39}	0	0
$\mathscr{P}^3x_i$	0	x_{19}	x_{20}	0	0	x_{27}	0	0	0	$-x_{39}$	x_{47}	0	0

where ε *is 1 or* -1.

If $r>1$ *then* $\mathscr{P}^{3^r}x_i=0$.

Remark. We consider x_i in these tables as 0 when $x_i\notin H^*$.

Recall a Serre fibration:

$$\Omega G(l) \quad \rightarrow \quad * \quad \rightarrow \quad G(l). \quad \text{(A)}$$

First, we compute $H^*(\Omega G(l); Z/3)$ by the Serre spectral sequence associated with the fibration (A). This spectral sequence has a Hopf algebra structure. We can proceed to compute it using degree-reason and Kudo's transgression theorem ([7]) from the previous theorem. For $j \in E(l) - \{9, 11, 29\}$, there are universally transgressive elements $a_{2j} \in H^*(\Omega G(l); Z/3)$, such that $\tau a_{2j} = x_{2j+1}$. Thus we can show that for $j = 9, 11, 15, 21, 27$ and 29, there are a_{2j} such that satisfy

$$\begin{aligned}
&d_7(1 \otimes a_{18}) = x_7 \otimes a_2^6, \quad \text{for } l=4, 6, 7,\\
&d_{11}(1 \otimes a_{30}) = x_{11} \otimes a_{10}^2, \quad \text{for } l=4, 6, 7,\\
&d_{15}(1 \otimes a_{42}) = x_{15} \otimes a_{14}^2, \quad \text{for } l=8,\\
&d_{19}(1 \otimes a_{22}) = x_3 x_8^2 \otimes a_2^2, \quad \text{for } l=4, 6, 7, 8,\\
&d_{19}(1 \otimes a_{54}) = x_{19} \otimes a_2^{18}, \quad \text{for } l=8,\\
&d_{47}(1 \otimes a_{58}) = x_7 x_{20}^2 \otimes a_2^6, \quad \text{for } l=8.
\end{aligned}$$

a_{2j}*'s* are generators of the cohomology group in the low dimensions. The results are the following:

Proposition 4. *For the dimensions less than $2n(l)+2$, the next isomorphism holds:*

$$H^*(\Omega G(l); Z/3) \cong \begin{cases} Z/3[a_{2j} | j \in E(l) \cup \{9\}]/(a_2^9), & \text{if } l=4, 6,\\ Z/3[a_{2j} | j \in E(7) \cup \{15\}]/(a_{10}^3), & \text{if } l=7,\\ Z/3[a_{2j} | j \in E(8) \cup \{21, 27\}]/(a_2^{27}, a_{14}^3), & \text{if } l=8. \end{cases}$$

Now we start to determine the cohomology operations and the coproducts on a_{2j}.

Theorem 5. *For $j \in E(l) - \{9, 11, 29\}$ $a_{2j} \in H^*(\Omega G(l); Z/3)$ is primitive and cohomology operations are determined by*

a_{2j}	a_2	a_8	a_{10}	a_{14}	a_{16}	a_{26}	a_{34}	a_{38}	a_{46}
$\mathcal{P}^1 a_{2j}$	a_2^3	0	a_{14}	εa_2^9	0	0	εa_{38}	0	0
$\mathcal{P}^3 a_{2j}$	0	0	0	a_{26}	0	$-a_{38}$	a_{46}	0	0

If $r>1$ then $\mathcal{P}^{3^r} a_{2j} = 0$.

Proof. For $j \in E(l) - \{9, 11, 29\}$, a_{2j} is transgressive, therefore $\mathcal{P}^i a_{2j} = \mathcal{P}^i \sigma x_{2j+1} = \sigma \mathcal{P}^i x_{2j+1}$. Thus this can be determined by Theorem 3.

For the investigation of a_{2j} which is not transgressive we start from the following theorem. In the next theorem, ϕ means the coproduct of $H^*(\Omega G; Z/3)$ and we set $\bar{\phi}(a)=\phi(a)-(a\otimes 1+1\otimes a)$.

Theorem 6. *For $j=9, 15, 21, 27$, $\bar{\phi}a_{2j}$ is given by the following formula:*

$$\bar{\phi}a a_{2j}=\begin{cases} a_2{}^3\otimes a_2{}^6+a_2{}^6\otimes a_2{}^3, & \text{if } j=9,\\ a_{10}\otimes a_{10}{}^2+a_{10}{}^2\otimes a_{10}, & \text{if } j=15,\\ a_{14}\otimes a_{14}{}^2+a_{14}{}^2\otimes a_{14}, & \text{if } j=21,\\ a_2{}^9\otimes a_2{}^{18}+a_2{}^{18}\otimes a_2{}^9, & \text{if } j=27.\end{cases}$$

Proof. To begin with, we investigate the element a_{18}. Let a_2' be the generator of $H^2(\Omega F_4; Z)$. $H^*(\Omega F_4; Z)$ has no torsion and is a commutative Hopf algebra over Z. Since $a_2{}^9=0$, there is a_{18}' such that $a_2'^9=3a_{18}'$ and $\rho a_{18}'\neq 0$, where ρ is modulo 3 reduction. Then we can choose a_{18} as $\rho a_{18}'$. The coproduct of a_{18}' is computed as follows:

$$\begin{aligned}\phi a_{18}' &= 1/3\phi a_2'^9\\ &=1/3(1\otimes a_2'+a_2'\otimes 1)^9\\ &\equiv a_{18}'\otimes 1+a_2'^3\otimes a_2'^6+a_2'^3+1\otimes a_{18}' \pmod 3.\end{aligned}$$

Thus $\bar{\phi}a_{18}=a_2{}^3\otimes a_2{}^6+a_2{}^6+\otimes a_2{}^3$ is shown.

Consider the inclusion ι: $F_4\to E_7$, we chose $a_{18}\in H^*(\Omega E_7; Z/3)$ so as to satisfy $(\Omega\iota)a^*a_{18}=a_{18}$. Because $(\Omega\iota)^*$ is injective for degrees less than 18, $\bar{\phi}a_{18}=a_2{}^3\otimes a_2{}^6+a_2{}^6\otimes a_2{}^3$ is shown again for this a_{18}. And in the similar way we put $a_{30}=1/3a_{10}{}^3$, $a_{42}=1/3a_{14}{}^3$ and $a_{54}=1/3a_2{}^{27}$ and obtain the coproduct formulas of the statement.

We remark that we can assume that a_{22} and a_{58} are primitive.

Theorem 7. *In Proposition 4 we have that $\mathcal{P}^1 a_{18}=\pm a_{22}$.*

Let $\tilde{G}(l)$ be the 3-connected cover of $G(l)$ and

$$\Omega\tilde{G}(l) \to * \to \tilde{G}(l) \qquad \text{(B)}$$

$$\tilde{G}(l) \xrightarrow{p} G(l) \xrightarrow{i} K(Z, 3) \qquad \text{(C)}$$

$$\Omega\tilde{G}(l) \xrightarrow{\Omega p} \Omega G(l) \xrightarrow{\Omega i} K(Z, 2) \qquad \text{(D)}$$

be Serre fibrations. To prove Theorem 7 we have to compute $H^*(\Omega\tilde{G}; Z/3)$ and $H^*(\tilde{G}; Z/3)$.

Let $\tilde{a}_{2j}$ be $(\Omega p)^*a_{2j}$, for $j\neq 1$. Using the Serre spectral sequence associated with the fibration (D), one can easily show that there are generators $\tilde{a}_{17}\in H^{17}$ for $l=4$, 6, and $\tilde{a}_{53}\in H^{53}$ for $l=8$. We have the following proposition. Let denote $E(l)-\{1\}$ as $\widetilde{E}(l)$.

Proposition 8. *For the dimensions less than $2n(l)+2$, the next isomorphism holds:*

$$H^*(\Omega\widetilde{G}(l);Z/3)\cong\begin{cases} Z/3[\widetilde{a}_{2j}\mid j\in\widetilde{E}(l)\cup\{9\}]\otimes\Lambda(\widetilde{a}_{17}), & \text{if } l=4,6,\\ Z/3[\widetilde{a}_{2j}\mid j\in\widetilde{E}(7)\cup\{15\}]/(\widetilde{a}_{10}{}^3), & \text{if } l=7,\\ Z/3[\widetilde{a}_{2j}\mid j\in\widetilde{E}(8)\cup\{21,27\}]/(\widetilde{a}_{14}{}^3)\otimes\Lambda(\widetilde{a}_{53}), & \text{if } l=8.\end{cases}$$

By computing the Serre spectral sequence associated with (B), it is easy to see $\widetilde{a}_{2j}$, $(j\neq 15, 21)$ is universally transgressive. Let $\widetilde{x}_{i+1}$ be $\tau\widetilde{a}_i$. Then we have the following:

Proposition 9. *For the dimensions less than $2n(l)+2$, the next isomorphism holds:*

$$H^*(\widetilde{G}(l);Z/3)\cong\begin{cases}\Lambda(\widetilde{x}_{2j+1}\mid j\in\widetilde{E}(l)\cup\{9\})\otimes Z/3[\widetilde{x}_{18}], & \text{if } l=4,6,\\ \Lambda(\widetilde{x}_{2j+1}\mid j\in\widetilde{E}(7)), & \text{if } l=7,\\ \Lambda(\widetilde{x}_{2j+1}\mid j\in\widetilde{E}(8)\cup\{27\})\cup Z/3[\widetilde{x}_{54}], & \text{if } l=8.\end{cases}$$

Proof of Therorem 7. It is possible to show that $\mathcal{P}^1a_{18}$ is not zero as follows. Let σ' denotes the cohomology suspension associated to the fibration (C) for $l=4$. It is easy to see $\widetilde{x}_{19}=\sigma'\beta\mathcal{P}^3\mathcal{P}^1u_3$ and $\widetilde{x}_{23}=\sigma'(\beta\mathcal{P}^1u_3)^3$, where u_3 is the generator of $H^3(K(Z,3);Z/3)$. So we get $\mathcal{P}^1\widetilde{x}_{19}=\sigma'\mathcal{P}^1\beta\mathcal{P}^3\mathcal{P}^1u_3=\sigma'\mathcal{P}^4\beta\mathcal{P}^1u_3=\sigma'(\beta\mathcal{P}^1u_3)^3=\widetilde{x}_{23}$, and from this, we have $(\Omega p)^*\mathcal{P}^1a_{18}=\mathcal{P}^1(\Omega p)^*a_{18}=\mathcal{P}^1\widetilde{a}_{18}=\mathcal{P}^1\sigma\widetilde{x}_{19}=\sigma\mathcal{P}^1\widetilde{x}_{19}=\sigma\widetilde{x}_{23}=a_{22}$, where σ is the cohomology suspension associated to (B). Thus $\mathcal{P}^1a_{18}\neq 0$. We fix a_{22} as $\mathcal{P}^1a_{18}$.

4. Homology groups

Theorem 10. *The homology ring of $\Omega G(l)$ is*

$$H_*(\Omega G(l);Z/3)\cong\begin{cases} Z/3[t_{2j}\mid j\in E(l)\cup\{3\}]/(t_2{}^3), & \text{if } l=4,6,7\\ Z/3[t_{2j}\mid j\in E(8)\cup\{3,9\}]/(t_2{}^3,t_6{}^3), & \text{if } l=8.\end{cases}\qquad(1)$$

where $|t_{2j}|=2j$. The coproduct is given by

$$\bar{\phi}(t_{2j})=\begin{cases}0, & \text{if } j\neq 3,9,11,29,\\ -t_2{}^2\otimes t_2-t_2\otimes t_2{}^2, & \text{if } j=3,\\ t_2{}^2t_6{}^2\otimes t_2+t_2t_6{}^2\otimes t_2{}^2-t_6{}^2\otimes t_6-t_2{}^2t_6\otimes t_2t_6 & \\ -t_2t_6\otimes t_2{}^2t_6-t_6\otimes t_6{}^2+t_2{}^2\otimes t_2t_6{}^2+t_2\otimes t_2{}^2t_6{}^2, & \text{if } j=9.\end{cases}$$

Proof. Let t_{2j} be the dual elemet of $a_{2j}\in H_*(\Omega G;\ Z/3)$ as to the monomial basis for $j\in E(l)-\{9\}$ and t_6, t_{18} be the dual element of a_2^3, a_2^9, respectively. It is easy to see $t_2^3=t_6^3=0$ and to show the coproduct formula for t_6 and t_{18}. Thus we can say that statement (1) is true for $*<2n(l)+2$.

Now it is possible to show that there is no truncation in $H_*(\Omega G;\ Z/3)$ other than the parts generated by t_2 and t_6 and that (1) holds for all dimensions. Since $H_*(\Omega G(l)\ ;\ Z/3)$ is the even degree concentrated commutative Hopf algebra, we may suppose

$$H_*(\Omega G(l);\ Z/3)=Z/3[u_i|i\in I]\otimes Z/3)\ [v_j|j\in J]/(v_j^{3^{r_j}}|j\in J).$$

Consider an Eilenberg-Moore spectral sequence:

$$E_2=\mathrm{Ext}_{H_*(\Omega G(l):Z/3)}(Z/3,\ Z/3)\Rightarrow E_\infty=\mathcal{G}r(H^*(G(l);\ Z/3)).$$

Since $E_2=\Lambda(su_i|i\in I)\otimes\Lambda(su_j|j\in J)\otimes Z/3[\theta v_j|j\in J]$, where deg $su_i=(1,\ |u_i|)$, deg $su_j=(1,\ |v_j|)$, and deg $\theta v_j=(2,\ 3^{r_j}|v_j|)$, the essential differentials have the forms: $d_r su_i=(\theta v_j)^{3^{k_j}}$ $(k_j\geq 1)$ or $d_r su_j=(\theta v_{j'})^{3^{l_j}}$ $(l_j\geq 1)$. Because $H^*(G(l)$; $Z/3)$ is a finite dimensional vector space, one can easily show

$$E_\infty=\Lambda(su_i|i\in I')\otimes\Lambda(su_j|j\in J')\otimes Z/3[\theta v_j|j\in J]/((\theta v_j)^{3^{m_j}}|j\in J),\quad (I'\subset I,J'\subset J)$$

and $|I'|+|J'|=|I|$. Here the total dimension of E_∞ is $2^{|I'|+|J'|}3^{\Sigma_{j\in J}m_j}$, $(m_j\geq 1)$ and the total dimension of $H^*(G(l);\ Z/3)$ is $2^{|E(l)|}3^{f(l)}$ where $f(l)=1$ for $l=4,\ 6,\ 7$ and $f(l)=2$ for $l=8$. Thus the indices J of the truncation part satisfy that $|J|\leq f(l)$ and $|I|=|E(l)|$. This means that the truncation parts of $H_*(\Omega G;$ $Z/3)$ is generated by only t_2 and t_6.

Therefore $H_*(\Omega G(l);\ Z/3)$ has the form

$$Z/3[u_i|i\in I]\otimes Z/3[t_2]/(t_2^3)\qquad \text{for } l=4,\ 6,\ 7 \text{ and}$$
$$Z/3[u_i|i\in I]\otimes Z/3[t_2,\ t_6]/(t_2^3,\ t_6^3)\quad \text{for } l=8.$$

Also Theorem 5 means that for $j\in E(l)-\{9\}\ t_{2j}$ is primitive and indecomposable and t_6, t_{18} are indecomposable. Thus

$$\{t_{2j}|j\in\widetilde{E}(l)\}\cup\{t_6\}\subset\{u_i|i\in I\}\quad \text{for } l=4,\ 6,\ 7 \text{ and}$$
$$\{t_{2j}|j\in\widetilde{E}(l)\}\cup\{t_{18}\}\subset\{u_i|i\in I\}\quad \text{for } l=8.$$

Since $|I|=|E(l)|$, the theorem is proved.

Dualizing the result of Theorem 5 and Theorem 7, we obtain the statement of Theorem 1 except for $\mathcal{P}_*^1t_{26}$, $\mathcal{P}_*^1t_{34}$, $\mathcal{P}_*^3t_{34}$, $\mathcal{P}_*^1t_{46}$, $\mathcal{P}_*^1t_{58}$ and $\mathcal{P}_*^9t_{58}$. To determine these operations, we use the adjoint action of $H_*(G(l)\ ;\ Z/3)$ on $H_*(\Omega G(l);\ Z/3)$ which is introduced in the next section.

Remark. The computation of dualizing the result of Theorem 5 and Theorem 7 is not difficult except for $\mathcal{P}_*^1t_{18}$, because $\mathcal{P}_*^nt$ is primitive if t is primitive. Moreover, it is easily shown

$$\bar{\phi}(\mathcal{P}^1_* t_{18}) = \mathcal{P}^1_* \bar{\phi}(t_{18}) = \bar{\phi}(-t_2 t_6^2)$$

and this shows $\mathcal{P}^1_* t_{18} = -t_2 t_6^2$ modulo primitive elements. By Theorem 5 we can see $\mathcal{P}^1 a_{14} = \varepsilon a_2^9$ and this shows that $\mathcal{P}^1_* t_{18} = \varepsilon t_{14} - t_2 t_6^2$.

5. Adjoint action

Put $y * y' = \mathrm{Ad}_*(y \otimes y')$ and $y * t = \mathrm{ad}_*(y \otimes t)$ where $y, y' \in H_*(G; Z/3)$ and $t \in H_*(\Omega G; Z/3)$. The following theorem is the dual result of [3]. Also see [9].

Theorem 11. *For, $y, y', y'' \in H_*(G; Z/3)$ and $t, t' \in H_*(\Omega G; Z/3)$*

(i) $1 * y = y$, $1 * t = t$.

(ii) $y * 1 = 0$, *if* $|y| > 0$, *whether* $1 \in H_*(G; Z/3)$ *or* $1 \in H_*(\Omega G; Z/3)$.

(iii) $(yy') * t = y * (y' * t)$.

(iv) $y * (tt') = \sum (-1)^{|y''||t|} (y' * t)(y'' * t')$ *where* $\Delta_* y = \sum y' \otimes y''$.

(v) $\sigma(y * t) = y * \sigma(t)$ *where σ is the homology suspension.*

(vi) $\mathcal{P}^n_*(y * t) = \sum_i (\mathcal{P}^i_* y) * (\mathcal{P}^{n-i}_* t)$.

$\mathcal{P}^n_*(y * y') = \sum_i (\mathcal{P}^i_* y) * (\mathcal{P}^{n-i}_* y')$.

(vii) $\Delta_*(y * t) = (\Delta_* y) * (\Delta_* t)$

$= \sum (-1)^{|y''||t'|} (y' * t') \otimes (y'' * t'')$

where $\Delta_* y = \sum y' \otimes y''$ and $\Delta_* t = \sum t' \otimes t''$.

And $\bar{\Delta}_*(y * t) = (\Delta_* y) * (\bar{\Delta}_* t)$.

(viii) *If t is primitive then $y * t$ is primitive.*

Also the result of [3] implies the following theorem. See [8].

Theorem 12. *We set a submodule A of $H_*(G; Z/3)$ as*

$$A = Z/3[y_8]/(y_8^3) \qquad \textit{for } G = F_4, E_6, E_7 \textit{ and}$$

$$A = Z/3[y_8, y_{20}]/(y_8^3, y_{20}^3) \qquad \textit{for } G = E_8$$

where y_{2i} is the dual of x_{2i} with respect to the monomial basis. Then there exists a retraction p: $H_*(G; Z/3) \to A$ and the following diagram commutes.

$$\begin{array}{ccc} H_*(G; Z/3) \otimes H_*(\Omega G; Z/3) & \xrightarrow{\boxed{ad_*}} & H_*(\Omega G; Z/3) \\ \downarrow p \otimes 1 & \nearrow \boxed{ad_*} & \\ A \otimes H_*(\Omega G; Z/3) & & \end{array}$$

Remark. By Theorem 3 we can see $\mathcal{P}^3_* y_{20} = y_8$.

Since Ad_* is agreed with the composition $\mu_* \circ (1 \otimes \mu_*) \circ (1 \otimes 1 \otimes \iota_*) \circ (1 \otimes T) \circ (\Delta_* \otimes 1)$ where μ is the multiplication of $G(l)$ and ι is the inverse map, the next theorem follows. See [9].

Theorem 13. *Let $y, y' \in H_*(G)$. If y is primitive,*

$$y * y' = [y, y']$$

where $[y, y'] = yy' - (-1)^{|y||y'|} y'y$.

Now we give the proof of Theorem 2 and finish the proof of Theprem 1. Let y_i be the dual element of $x_i \in H^*(G(l))$ as to the monomial basis. By Theorem 3 and Theorem 13 we see that for $j \in E(l) \cup \{3, 9\} - \{11, 29\}$

$$y_8 * y_{2j+1} = \begin{cases} y_{2j+9} & \text{for } j = 1, 3, 4, 9, 13, \\ -y_{2j+9} & \text{for } j = 19, \\ 0 & \text{others} \end{cases}$$

and

$$y_{20} * y_{2j+1} = \begin{cases} y_{2j+21} & \text{for } j = 3, 7, 9, \\ -y_{2j+21} & \text{for } j = 13, \\ 0 & \text{others.} \end{cases}$$

Since $\sigma t_{2j} = y_{2j+1}$ for $j \in E(l) \cup \{3, 9\} - \{11, 29\}$, Theorem 11 (v) implies

$$\begin{aligned} &\sigma(y_8 * t_{2j}) \neq 0 \quad \text{for } j = 1, 3, 4, 9, 13, 19, \\ &\sigma(y_{20} * t_{2j}) \neq 0 \quad \text{for } j = 3, 7, 9, 13. \end{aligned} \tag{2}$$

Then the equations

$$y_8 * t_2 = t_{10}, \tag{3}$$
$$y_8 * t_8 = t_{16}, \tag{4}$$
$$y_8 * t_{26} = t_{34}, \tag{5}$$
$$y_8 * t_{38} = -t_{46}, \tag{6}$$
$$y_{20} * t_{14} = t_{34}, \tag{7}$$
$$y_{20} * t_{26} = -t_{46} \tag{8}$$

are shown by Theorem 11 (viii). Moreover (2) implies

$$y_8 * t_6 \equiv t_{14}, \tag{9}$$
$$y_8 * t_{18} \equiv t_{26}, \tag{10}$$
$$y_{20} * t_6 \equiv t_{26}, \tag{11}$$
$$y_{20} * t_{18} \equiv t_{38} \tag{12}$$

modulo decomposable elements. Since

$$\begin{aligned} \bar{\phi}(y_8 * t_6) &= -(y_8 * t_2) \otimes t_2{}^2 - (y_8 * t_2{}^2) \otimes t_2 - t_2 \otimes (y_8 * t_2{}^2) - t_2{}^2 \otimes (y_8 * t_2) \\ &= \bar{\phi}(-t_{10} t_2{}^2), \end{aligned}$$

one can see that $y_8 * t_6 \equiv -t_{10}t_2^2$ mod primitive elements. By this and (9), we have

$$y_8 * t_6 = t_{14} - t_{10}t_2^2. \tag{13}$$

The equations

$$y_8 * t_{18} = t_{26} + t_{10}t_2^2t_6^2 - t_{14}t_6^2, \tag{14}$$
$$y_{20} * t_6 = t_{26} - (y_{20} * t_2)t_2^2, \tag{15}$$
$$y_{20} * t_{18} = t_{38} - (y_{20} * t_6)t_6^2 \tag{16}$$

are shown in the similar way.

By the equation (13), we can compute $y_8^3 \otimes t_6$ as

$$\begin{aligned} y_8^3 * t_6 &= y_8^2 * (t_{14} - t_{10}t_2^2) \\ &= y_8^2 * t_{14} + t_{10}^3. \end{aligned}$$

Since $y_8^3 = 0$, $y_8^2 * t_{14} = -t_{10}^3$ and this means $y_8 * t_{14}$ is a non-zero primitive indecomposable element. We redefine t_{22} as

$$t_{22} = y_8 * t_{14}. \tag{17}$$

Then we have

$$y_8 * t_{22} = -t_{10}^3.$$

By Theorem 7 we can set $\mathscr{P}^1_* t_{22} = \kappa t_6^3$ where $\kappa = \pm 1$. Since $\mathscr{P}^1_* t_{22} = \mathscr{P}^1_*(y_8 * t_{14}) = y_8 * t_{10}$, we have

$$y_8 * t_{10} = \kappa t_6^3.$$

By the similar manner, we can compute $y_8^3 * t_{18}$ and obtain $y_8^2 * t_{26} = -t_{14}^3$. Therefore

$$y_8 * t_{34} = y_8^2 * t_{26} = -t_{14}^3. \tag{18}$$

Because t_{16} and t_{46} are primitive, we can set

$$y_8 * t_{16} = \rho_2 t_8^3, \tag{19}$$
$$y_8 * t_{46} = \rho_3 t_{18}^3. \tag{20}$$

Operate $\mathscr{P}^3_*$ to (20) to obtain

$$y_8 * t_{34} = \mathscr{P}^3_*(y_8 * t_{46}) = \rho_3 \mathscr{P}^3_*(t_{18}^3) = \rho_3 \varepsilon t_{14}^3.$$

Thus by (18), we conclude that $\rho_3 = -\varepsilon$. $y_8 * t_{58}$ will be determined after the determination of $y_{20} * t_{58}$.

Here we apply $\mathscr{P}^1_*$ on (5), (6) and (14), $\mathscr{P}^3_*$ on (5) to see

$$\begin{aligned} \mathscr{P}^1_* t_{26} &= \mathscr{P}^1_*(y_8 * t_{18} - t_{10}t_6^2t_2^2 + t_{14}t_6^2) \\ &= \varepsilon y_8 * t_{14} = \varepsilon t_{22}, \\ \mathscr{P}^1_* t_{34} &= \mathscr{P}^1_*(y_8 * t_{26}) = \varepsilon y_8 * t_{22} = -\varepsilon t_{10}^3, \end{aligned}$$

$$\mathcal{P}^1_* t_{46} = -\mathcal{P}^1_*(y_8 * t_{38}) = -\varepsilon y_8 * t_{34} = \varepsilon t_{14}{}^3,$$
$$\mathcal{P}^3_* t_{34} = \mathcal{P}^3_*(y_8 * t_{26}) = y_8 * t_{14} = t_{22}.$$

Next we compute $y_{20} * t_{2i}$. First we apply $\mathcal{P}^1_*$ to (15) to obtain

$$y_{20} * t_2 = \mathcal{P}^1_*(y_{20} * t_6) = \mathcal{P}^1_*(t_{26} - (y_{20} * t_2)t_2{}^2) = \varepsilon t_{22}.$$

From this, (15) and (16) imply that

$$y_{20} * t_2 = \varepsilon t_{22},$$
$$y_{20} * t_6 = t_{26} - \varepsilon t_{22} t_2{}^2,$$
$$y_{20} * t_{18} = t_{38} + \varepsilon t_{22} t_6{}^2 t_2{}^2 - t_{26} t_6{}^2.$$

$y_{20}{}^3 * t_6$ is computed as

$$\begin{aligned} 0 = y_{20}{}^3 * t_6 &= y_{20}{}^2 * (y_{20} * t_6) \\ &= y_{20}{}^2 * (t_{26} - \varepsilon t_{22} t_2{}^2) \\ &= y_{20}{}^2 * t_{26} + \varepsilon t_{22}{}^3. \end{aligned}$$

Thus $y_{20} * t_{46} = -y_{20}{}^2 * t_{26} = \varepsilon t_{22}{}^3$.

The similar computation of $y_{20}{}^3 * t_{18}$ implies

$$y_{20}{}^2 * t_{38} = -t_{26}{}^3.$$

Thus $y_{20} * t_{38}$ is a non zero primitive indecomposable element and we redefine t_{58} as $y_{20} * t_{38}$. Hence

$$y_{20} * t_{38} = t_{58}, \qquad (21)$$
$$y_{20} * t_{58} = -t_{26}{}^3. \qquad (22)$$

By applying $\mathcal{P}^3_*$ to (22), we have

$$y_8 * t_{58} = \mathcal{P}^3_*(y_{20} * t_{58}) = -\mathcal{P}^3_*(t_{26}{}^3) = -\varepsilon t_{22}{}^3.$$

We obtain also

$$y_{20} * t_{22} = \varepsilon \mathcal{P}^1_*(y_{20} * t_{26}) = -\mathcal{P}^1_* t_{46} = -t_{14}{}^3$$

by applying $\mathcal{P}^1_*$ to (8).

Since t_{34} is primitive, we can set $y_{20} * t_{34} = \rho_4 t_{18}{}^3$ $(\rho_4 \in Z/3)$. Operating $\mathcal{P}^3_*$ to the both sides of this equation, $\rho_4 \varepsilon t_{14}{}^3$ is computed as follows:

$$\begin{aligned} \rho_4 \varepsilon t_{14}{}^3 &= \rho_4 \mathcal{P}^3_*(t_{18}{}^3) \\ &= \mathcal{P}^3_*(y_{20} * t_{34}) \\ &= y_8 * t_{34} + y_{20} * t_{22} \\ &= t_{14}{}^3. \end{aligned}$$

So $y_{20} * t_{34} = \varepsilon t_{18}{}^3$ is shown. Now ad_* is determined except for $y_8 * t_{16}$.

Finally we operate $\mathcal{P}^1_*$ to (21) and $\mathcal{P}^9_*$ to (22) and see

$$\mathcal{P}^1_* t_{58} = \mathcal{P}^1_*(y_{20} * t_{38}) = y_{20} * (\mathcal{P}^1{}_* t_{38}) = \varepsilon y_{20} * t_{34} = t_{18}{}^3,$$

$$y_{20} * (\mathcal{P}_*^9 t_{58}) = \mathcal{P}_*^9 (y_{20} * t_{58}) = -\mathcal{P}_*^9 (t_{26}{}^3) = -t_{14}{}^3.$$

These equations imply that

$$\mathcal{P}_*^1 t_{58} = t_{18}{}^3, \ \mathcal{P}_*^9 t_{58} = t_{22}.$$

This completes the proof of Theorem 1.

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