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Radiation condition for Dirac operators

By

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1. Introduction

In the papers [6] and [7], results from the theory of pseudodifferential operators and spectral analysis of Schrödinger operators were combined to discuss the asymptotic properties of the Dirac operator

$$H=-i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j}+\beta+Q(x). \tag{1.1}$$

Here $i=\sqrt{-1}$, $x=(x_1, x_2, x_3)\in\mathbf{R}^3$ and α_j, β are the Dirac matrices, i.e., 4×4 Hermitian matrices satisfying the anticommutation relation

$$\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}I, \quad (j, k=1, 2, 3, 4) \tag{1.2}$$

with the convention $\alpha_4=\beta$, δ_{jk} being Kronecker's delta and I being the 4×4 identity matrix. The potential $Q(x) = (q_{jk}(x))$ is a 4×4 Hermitian matrix-valued function. In this paper we assume that $Q(x)$ is short-range in the sense that each element q_{jk} satisfies

$$\sup_{x\in\mathbf{R}^3}[(1+|x|)^{1+\varepsilon}|q_{jk}(x)|]<\infty \quad (x\in\mathbf{R}^3, j, k=1, 2, 3, 4), \tag{1.3}$$

where ε is a positive constant. The free Dirac operator H_0 is defined by

$$H_0=-i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j}+\beta. \tag{1.4}$$

The aim of this paper is to show how the Dirac operator and the Schrödinger operator are related to each other and how some properties of the Dirac operator and the solutions of the Dirac equation can be obtained from the corresponding properties of the Schrödinger operator. Since we have from the anticommutation relation (1.2)

$$(H_0)^2=(-\Delta+1)I, \tag{1.5}$$

we can anticipate a close relationship between these two operators. We also

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want to show that some results from the theory of pseudodifferential operators, which were used in [6] and [7], are useful in discussing our problems.

Let $R_0(z)=(H_0-z)^{-1}$ be the resolvent of the free Dirac operator H_0, and let $\Gamma_0(z)$ be the resolvent of the Laplacian $T_0=-\Delta$. Balslev-Helffer [2] gave the formula for the extended resolvents $R_0^{\pm}(\lambda)$ ([2], Lemma 3.1):

$$R_0^{\pm}(\lambda)=\lim_{\eta\downarrow 0}R_0(\lambda\pm i\eta)=\begin{cases}(H_0+\lambda)\Gamma_0^{\pm}(\lambda^2-1) & (\lambda>1),\\ (H_0+\lambda)\Gamma_0^{\mp}(\lambda^2-1) & (\lambda<-1),\end{cases} \tag{1.6}$$

where $\Gamma_0^{\pm}(\lambda)$ are the extended resolvents of T_0 (for the exact definition of the extended resolvent, see § 2). The formula was used to establish the limiting absorption principle for the Dirac operator with a short-range potential ([2], Theorem 3.9).

In this work, we are going to exchange the order of $H_0+\lambda$ and $\Gamma_0^{\pm}(\lambda^2-1)$ in the formula (1.6) to obtain more detailed similarities between Dirac operators and Schrödinger operators (Propositions 2.1 and 2.2). Our strategy is to combine a representation formula for the resolvent $R_0(z)$, which was originated in Yamada [13] and used in [6] and [7] with some known results on Schrödinger operators to study some new properties of the extended resolvent $R^{\pm}(\lambda)$ of the Dirac operator H with a short-range potential Q. Let

$$R^{\pm}(\lambda)f(x)={}^t(v_1^{\pm}(x),v_2^{\pm}(x),v_3^{\pm}(x),v_4^{\pm}(x)), \tag{1.7}$$

where ${}^tA$ is the transposed matrix (or vector) of A, and

$$f\in L_2(\mathbf{R}^3,\ (1+|x|^2)^{\delta}\,dx) \tag{1.8}$$

with a fixed constant δ satisfying $\delta>1/2$. In order to simplify the description, here we assume that $\lambda>1$. After giving a proof of the limiting absorption principle for the Dirac operator (1.1), we are going to prove the following:

(1) Each element $v_j^{\pm}(x)$, $j=1, 2, 3, 4$, satisfies the radiation condition

$$\begin{cases}v_j^{\pm}\in L_2(\mathbf{R}^3,\ (1+|x|^2)^{-\delta}\,dx),\\ (\partial_l\mp i\sqrt{\lambda^2-1}\tilde{x}_l)v_j^{\pm}\in L_2(\mathbf{R}^3,\ (1+|x|^2)^{\delta-1}\,dx),\end{cases} \tag{1.9}$$

where $l=1, 2, 3$, $\partial_l=\partial/\partial x_l$, $\tilde{x}_l=x_l/|x|$, and $1/2<\delta\leq 1$.

(2) $v=R^{\pm}(\lambda)f$ is characterized as a unique solution of the equation $(H-\lambda)v=f$ with the radiation condition (1.9)

(3) Each element $v_j^{\pm}(x)$, $j=1, 2, 3, 4$ satisfies the asymptotic behavior

$$v_j^{\pm}(r\cdot)\sim c(\lambda,f)r^{-1}e^{\pm i\sqrt{\lambda^2-1}r}\quad\text{in } L_2(S^2) \tag{1.10}$$

as $r=|x|\to\infty$, where S^2 is the unit sphere in $\mathbf{R}^3$, and $c(\lambda, f)\in L_2(S^2)$ is determined by λ and f.

We now introduce the notation which will be used in this paper. Let n be a positive integer. For $x=(x_1, x_2, \cdots, x_n)\in\mathbf{R}^n$, $|x|$ denotes the Euclidean norm of x and

$$\langle x\rangle=\sqrt{1+|x|^2}. \tag{1.11}$$

For $s\in\mathbf{R}$ and a positive integer k, we define the weighted Hilbert spaces $L_{2,s}(\mathbf{R}^n)$ and $H_s^k(\mathbf{R}^n)$ by

$$L_{2,s}(\mathbf{R}^n)=\{f:\langle x\rangle^s f\in L_2(\mathbf{R}^n)\}, \tag{1.12}$$

and

$$H_s^k(\mathbf{R}^n)=\{f:\langle x\rangle^s\partial_x^\alpha f\in L_2(\mathbf{R}^n),\ |\alpha|\leq k\}, \tag{1.13}$$

where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, and

$$\partial_x^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}\quad\left(\partial_j=\frac{\partial}{\partial x_j},\quad j=1, 2, \cdots, n\right). \tag{1.14}$$

The inner products and norms in $L_{2,s}(\mathbf{R}^n)$ and $H_s^k(\mathbf{R}^n)$ are given by

$$\begin{cases}(f, g)_s=\displaystyle\int_{\mathbf{R}^n}\langle x\rangle^{2s}f(x)\overline{g(x)}\,dx,\\ \|f\|_s=[(f, f)_s]^{1/2}\end{cases}, \tag{1.15}$$

and

$$\begin{cases}(f, g)_{k,s}=\displaystyle\int_{\mathbf{R}^n}\langle x\rangle^{2s}\sum_{|\alpha|\leq k}\{\partial_x^\alpha f\cdot\overline{\partial_x^\alpha g}\}\,dx\\ \|f\|_{k,s}=[(f, f)_{k,s}]^{1/2},\end{cases} \tag{1.16}$$

respectively. For $n=3$ we set

$$\begin{cases}L_{2,s}(\mathbf{R}^3)=L_{2,s},\\ H_s^k(\mathbf{R}^3)=H_s^k.\end{cases} \tag{1.17}$$

Then the spaces $\mathscr{L}_{2,s}$ and $\mathscr{H}_s^k$ are defined by

$$\begin{cases}\mathscr{L}_{2,s}=[L_{2,s}]^4,\\ \mathscr{H}_s^k=[H_s^k]^4,\end{cases} \tag{1.18}$$

i.e., $\mathscr{L}_{2,s}$ and $\mathscr{H}_s^k$ are direct sums of the Hilbert spaces $L_{2,s}$ and H_s^k, respectively. The inner products and norms in $\mathscr{L}_{2,s}$ and $\mathscr{H}_s^k$ are also denoted by $(\ ,\ )_s$, $\|\ \|_s$ and $(\ ,\ )_{k,s}$, $\|\ \|_{k,s}$, respectively. When $s=0$, we simply write

$$\begin{cases}\mathscr{L}_2=\mathscr{L}_{2,0},\\ \mathscr{H}^k=\mathscr{H}_0^k.\end{cases} \tag{1.19}$$

Let $S(\mathbf{R}^n)$ be the set of all rapidly decreasing functions on $\mathbf{R}^n$. We set

$$\begin{cases} S=S(\mathbf{R}^3), \\ \mathcal{S}=[S]^4. \end{cases} \tag{1.20}$$

For $f={}^t(f_1, f_2, f_3, f_4)\in\mathcal{L}_2$, the Fourier transform $\mathcal{F}f(\xi)=\hat{f}(\xi)$ is defined by

$$\begin{cases} \mathcal{F}f(\xi)=\hat{f}(\xi)={}^t(\hat{f}_1(\xi), \hat{f}_2(\xi), \hat{f}_3(\xi), \hat{f}_4(\xi)), \\ \hat{f}_j(\xi)=\mathrm{l.i.m.}\int_{\mathbf{R}^3} e^{-ix\cdot\xi}f_j(x)\,dx \quad (j=1, 2, 3, 4). \end{cases} \tag{1.21}$$

The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

For a pair of Hilbert spaces X and Y, $\mathbf{B}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y, equipped with the operator norm

$$\|T\|_{(X,Y)}=\sup_{x\in X\setminus\{0\}}\|Tx\|_Y/\|x\|_X, \tag{1.22}$$

where $\|\ \ \|_X$ and $\|\ \ \|_Y$ are the norms in X and Y. We set $\mathbf{B}(X)=\mathbf{B}(X, X)$.

Let us sketch the contents of the paper. In § 2, starting with a representation formula of the resolvent $R_0(z)$ of the free Dirac operator H_0, we shall establish the above results for the free Dirac operator. The general Dirac operator with a short-range potential Q will be discussed in § 3.

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2. The free Dirac operator H_0

Let H be the differential expression given by (1.1) which satisfies (1.2) and (1.3). It is known (e.g., Kato [5], Chapter V, § 5) that H restricted on $[C_0^\infty(\mathbf{R}^3)]^4$ is essentially selfadjoint in $\mathcal{L}_2$. The selfadjoint realization, which has the domain $\mathcal{H}^1$, will be denoted again by H which is the Dirac operator with a short-range potential Q. Similarly, the selfadjoint realization of the expression (1.4) will be denoted again by H_0 which is the free Dirac operator. The operator H_0 has the same domain $\mathcal{H}^1$ as H. Thus, as selfadjoint operators, Hv and H_0v are well-defined only for $v\in\mathcal{H}^1$. When H and H_0 are applied to a locally $\mathcal{H}^1$ function v, they should be interpreted as differential operators rather than selfadjoint operators. The resolvents of H and H_0 will be denoted by $R(z)$ and $R_0(z)$, respectively, i.e.,

$$\begin{cases} R(z)=(H-z)^{-1} \quad (z\in\rho(H)), \\ R_0(z)=(H_0-z)^{-1} \quad (z\in\rho(H_0)), \end{cases} \tag{2.1}$$

where $\rho(H)$ and $\rho(H_0)$ are the resolvent sets of H and H_0, respectively. Let

$$\widehat{L}_0(\xi)=\sum_{j=1}^{3}\alpha_j\xi_j+\beta \quad (\xi=(\xi_1, \xi_2, \xi_3)) \tag{2.2}$$

where α_j, β are as in (1.2). The 4×4 matrix $\widehat{L}_0(\xi)$ is the "Fourier transform" of the operator H_0 in the sense that

$$\mathscr{F}(H_0 f)(\xi)=\widehat{L}_0(\xi)\mathscr{F}f(\xi) \quad (f\in\mathscr{H}^1). \tag{2.3}$$

It follows from the anticommutation relation (1.2) that

$$(\widehat{L}_0(\xi))^2=(|\xi|^2+1)I \tag{2.4}$$

(cf. (1.5)).

Similarly, consider the Schrödinger operator $T=-\Delta+V(x)$ with a short-range potential $V(x)$, and the free Schrödinger operator $T_0=-\Delta$. Here $-\Delta$ is the Laplacian on $\mathbf{R}^n$, and $V(x)$ is a real-valued, measurable function on $\mathbf{R}^n$ such that

$$\sup_{x\in\mathbf{R}^n}[(1+|x|)^{1+\varepsilon}|V(x)|]<\infty, \tag{2.5}$$

where $\varepsilon>0$. The restrictions of T and T_0 on $C_0^\infty(\mathbf{R}^n)$ are essentially selfadjoint, and the selfadjoint realizations in $L_2(\mathbf{R}^n)$ will be denoted again by T and T_0, respectively. The resolvents of T and T_0 will be denoted by

$$\begin{cases} \Gamma(z)=(T-z)^{-1} & (z\in\rho(T)), \\ \Gamma_0(z)=(T_0-z)^{-1} & (z\in\rho(T_0)), \end{cases} \tag{2.6}$$

where $\rho(T)$ and $\rho(T_0)$ are the resolvent sets of T and T_0, respectively.

In this section, we shall start with a representation formula for the resolvent $R_0(z)$ of the free Dirac operator H_0. Let $1<a<b<\infty$. We define $K=K_{a,b}$ by

$$K=\left\{z=\lambda+i\eta\in\mathbf{C}: a\leq|\lambda|\leq b,\ |\eta|\leq\frac{\sqrt{a^2-1}}{2}\right\}. \tag{2.7}$$

Then it is easy to see that

$$z\in K\Rightarrow 0<\frac{3(a^2-1)}{4}\leq \operatorname{Re}(z^2-1)\leq b^2-1. \tag{2.8}$$

Let a, b, K be as above. Then let γ_K be a real-valued function in $C_0^\infty(\mathbf{R}_\xi^3)$ satisfying

$$\gamma_K(\xi)=\begin{cases} 1 & \left(\dfrac{\sqrt{3(a^2-1)}}{4}\leq|\xi|\leq\dfrac{3\sqrt{b^2-1}}{2}\right), \\ 0 & \left(0\leq|\xi|\leq\dfrac{\sqrt{3(a^2-1)}}{6},\ |\xi|\geq 2\sqrt{b^2-1}\right). \end{cases} \tag{2.9}$$

In order to express $R_0(z)$ in terms of the resolvent $\Gamma_0(z)$ of the free Schrödinger operator, we introduce simple pseudodifferential operators: For each $z\in K$, we define

$$\begin{cases} A_{z,K}f=zI+\mathcal{F}^{-1}[\gamma_K(\xi)\hat{L}_0(\xi)]\mathcal{F}f & (f\in\mathcal{S}), \\ B_{z,K}f=\mathcal{F}^{-1}\left[\dfrac{(1-\gamma_K(\xi))\hat{L}_0(\xi)}{|\xi|^2-(z^2-1)}\right]\mathcal{F}f & (f\in\mathcal{S}). \end{cases} \tag{2.10}$$

We note that

$$||\xi|^2-(z^2-1)|\geq\begin{cases} \dfrac{9}{16}(a^2-1) & \text{if } z\in K \text{ and } \xi\in\operatorname{supp}[1-\gamma_K], \\ \dfrac{1}{2}|\xi|^2 & \text{if } z\in K \text{ and } |\xi|\geq\dfrac{3\sqrt{b^2-1}}{2}. \end{cases} \tag{2.11}$$

Proposition 2.1. *Let K be as above. Then for $z\in K$ with* $\operatorname{Im} z\neq 0$

$$R_0(z)f=\Gamma_0(z^2-1)A_{z,K}f+B_{z,K}f \qquad (f\in\mathcal{S}). \tag{2.12}$$

Here, for a vector valued-function $g(x)={}^t(g_1(x), g_2(x), g_3(x), g_4(x))$ on $\mathbf{R}^3$, $\Gamma_0(z)g(x)$ should be interpreted as

$$\Gamma_0(z)g(x)={}^t(\Gamma_0(z)g_1(x), \Gamma_0(z)g_2(x), \Gamma_0(z)g_3(x), \Gamma_0(z)g_4(x)). \tag{2.13}$$

Proof. It follows from (2.3) that

$$R_0(z)f=(2\pi)^{-3}\int_{\mathbf{R}^3}e^{ix\cdot\xi}(\hat{L}_0(\xi)-z)^{-1}\hat{f}(\xi)d\xi. \tag{2.14}$$

Then, using (2.4), we see that

$$R_0(z)f=(2\pi)^{-3}\int_{\mathbf{R}^3}e^{ix\cdot\xi}\left\{\frac{(\hat{L}_0(\xi)+z)}{|\xi|^2-(z^2-1)}\right\}\hat{f}(\xi)d\xi. \tag{2.15}$$

Since we have

$$\hat{L}_0(\xi)+z=[z+\gamma_K(\xi)\hat{L}_0(\xi)]+(1-\gamma_K(\xi))\hat{L}_0(\xi), \tag{2.16}$$

and

$$\Gamma_0(z^2-1)g=(2\pi)^{-3}\int_{\mathbf{R}^3}e^{ix\cdot\xi}\frac{\hat{g}(\xi)}{|\xi|^2-(z^2-1)}d\xi, \tag{2.17}$$

(2.12) follows from (2.15).

On the operators $A_{z,K}$ and $B_{z,K}$ we have

Proposition 2.2. *Let $K=K_{a,b}$ be as in (2.7) and let $A_{z,K}$ and $B_{z,K}$ as in (2.10). Let $s\geq 0$. Then*

(i) For each $z\in K$, $A_{z,K}$ can be uniquely extended to a bounded linear operator on $\mathcal{L}_{2,s}$.

(ii) For each $z \in K$, $B_{z,K}$ can be uniquely extended to a bounded linear operator on $\mathscr{L}_{2,s}$ to $\mathscr{H}^1_s$. Moreover, there exists a constant $C=C_{sK}>0$ such that

$$\|(B_{z_1,K}-B_{z_2,K})f\|_{1,s}\leq C|z_1-z_2|\|f\|_s \quad (f\in\mathscr{S}) \tag{2.18}$$

for all $z_1, z_2\in K$.

Proof. Since each component of $\gamma_K(\xi)\hat{L}_0(\xi)$ belongs to $S^0_{0,0}$, conclusion (i) follows directly from [7, Lemma 5.7]. (For the definition of $S^m_{0,0}$, see [7, section 5]).

Noting (2.11), we see that for any multi-index α there corresponds a constant $C_{\alpha K}>0$ such that

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left\{\frac{1-\gamma_K(\xi)}{|\xi|^2-(z^2-1)}\hat{L}_0(\xi)\right\}\right|\leq C_{\alpha K}\langle\xi\rangle^{-1-|\alpha|} \tag{2.19}$$

for all $z\in K$, where $\langle\xi\rangle=\sqrt{1+|\xi|^2}$. Then by [7, Lemma 5.8], we can deduce that $B_{z,K}$ can be extended to a bounded operator from $\mathscr{L}_{2,s}$ to $\mathscr{H}^1_s$. Moreover, using the identity

$$\frac{1}{|\xi|^2-(z_1^2-1)}-\frac{1}{|\xi|^2-(z_2^2-1)}=\frac{(z_1-z_2)\ (z_1+z_2)}{\{|\xi|^2-(z_1^2-1)\}\ \{|\xi|^2-(z_2^2-1)\}}, \tag{2.20}$$

we have for any α

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left\{\frac{1-\gamma_K(\xi)}{|\xi|^2-(z_1^2-1)}\hat{L}_0(\xi)-\frac{1-\gamma_K(\xi)}{|\xi|^2-(z_2^2-1)}\hat{L}_0(\xi)\right\}\right|$$
$$\leq C_{\alpha K}|z_1-z_2|\langle\xi\rangle^{-1-|\alpha|} \quad (z_1, z_2\in K). \tag{2.21}$$

Hence, appealing to [7, Lemma 5.8], we get (2.18)

In the rest of the paper the extensions of $A_{z,K}$ and $B_{z,K}$ of which existence has been guaranteed by Proposition 2.2, will be denoted again by $A_{z,K}$ and $B_{z,K}$ respectively. Then it is clear that $A_{z,K}$ is a $\mathbf{B}(\mathscr{L}_{2,s})$-valued continuous function on K and $B_{z,K}$ is a $\mathbf{B}(\mathscr{L}_{2,s}, \mathscr{H}^1_s)$-valued continuous function on K.

Now we are in a position to summarize the known results on the limiting absorption principle for the Schrödinger operator with a short-range potential. For $0<a<b<\infty$ and $0<c<\infty$, $L^\pm=L^\pm(a, b, c)$ are defined by

$$L^\pm=\{z=\kappa+i\tau: a\leq\kappa\leq b,\ 0\leq\pm\tau\leq c\}. \tag{2.22}$$

Theorem 2.3 (Agmon [1], Ikebe-Saitō [3], Jäger [4], Saitō [8]-[12]). *Let $n\geq 2$ and $1/2<\delta\leq 1$. Let $T=T_0+V$ be the Schrödinger operator which satisfies (2.5). Let $\Gamma(z)$ be the resolvent of T.*

(i) (Existence of the boundary value of $\Gamma(z)$). Let $\kappa>0$. Then there exist the limits

$$\Gamma^{\pm}(\kappa)=\lim_{\tau\downarrow 0}\Gamma(\kappa\pm i\tau) \quad \text{in } \mathbf{B}(L_{2,\delta}, H^2_{-\delta}). \tag{2.23}$$

(ii) (Continuity of $\Gamma(z)$). Set

$$\Gamma(\kappa)=\Gamma^{+}(\kappa) \quad (\kappa\in[a, b]\subset L^{+}), \tag{2.24}$$

[or

$$\Gamma(\kappa)=\Gamma^{-}(\kappa) \quad (\kappa\in[a, b]\subset L^{-}). \tag{2.25}$$

Then $\Gamma(z)$ is a $\mathbf{B}(L_{2,\delta}, H^2_{-\delta})$-valued, continuous function on L^+ [or L^-].

(iii) (Radiation condition and uniqueness). For $\kappa>0$ and $f\in L_{2,\delta}$, set

$$\begin{cases} u(\kappa+i\tau, f)=u(\cdot, \kappa+i\tau, f)=\Gamma(\kappa+i\tau)f & (\tau\neq 0), \\ u^{\pm}(\kappa, f)=u^{\pm}(\cdot, \kappa, f)=\Gamma^{\pm}(\kappa)f. \end{cases} \tag{2.26}$$

Let $L^{\pm}=L^{\pm}(a, b, c)$ be as in (2.22). Then there exists a positive constant $C=C(a, b, c)$ such that

$$\begin{cases} \|(\partial_j-i\sqrt{\kappa+i\tau}\tilde{x}_j)\, u(\kappa+i\tau, f)\|_{\delta-1}\leq C\|f\|_{\delta} & (\tau\neq 0), \\ \|(\partial_j\mp i\sqrt{\kappa}\tilde{x}_j)\, u^{\pm}(\kappa, f)\|_{\delta-1}\leq C\|f\|_{\delta} \end{cases} \tag{2.27}$$

for any $f\in L_{2,\delta}(\mathbf{R}^n)$ and any $\kappa+i\tau\in L^{+}\cup L^{-}$. Here the branch of $\sqrt{\kappa+i\tau}$ is taken so that $\mathrm{Im}\sqrt{\kappa+i\tau}\geq 0$.

Conversely, $u^{\pm}(\kappa, f)$, $\kappa>0$, is characterized as a unique solution of the equation

$$\begin{cases} (-\Delta+V(x)-\kappa)u=f, \\ u\in L_{2,-\delta}(\mathbf{R}^n)\cap H^2(\mathbf{R}^n)_{\mathrm{loc}}, \\ (\partial_j\mp i\sqrt{\kappa}\tilde{x}_j)u\in L_{2,\delta-1}(\mathbf{R}^n) \quad (j=1, 2, \cdots, n). \end{cases} \tag{2.28}$$

(iv) (Compactness). (a) Let $L^{\pm}=L^{\pm}(a, b, c)$ be as above. Then there exists a positive constant $C=C(a, b, c)$ such that

$$\begin{cases} \|u(\kappa+i\tau, f)\|_{-\delta,E_r}\leq C(1+r)^{-(\delta-1/2)}\|f\|_{\delta} & (\tau\neq 0), \\ \|u^{\pm}(\kappa, f)\|_{-\delta,E_r}\leq C(1+r)^{-(\delta-1/2)}\|f\|_{\delta} \end{cases} \tag{2.29}$$

for any $f\in L_{2,\delta}(\mathbf{R}^n)$ and any $\kappa+i\tau\in L^{+}\cup L^{-}$, where $E_r=\{x\in\mathbf{R}^n: |x|>r\}$, and

$$\|g\|_{-\delta,E_r}=\left[\int_{E_r}\langle x\rangle^{-2\delta}|g(x)|^2dx\right]^{1/2}. \tag{2.30}$$

(b) $\Gamma(\kappa+i\tau)$ $(\kappa>0, \tau\neq 0)$ and $\Gamma^{\pm}(\kappa)$ $(\kappa>0)$ are compact operators from $L_{2,\delta}(\mathbf{R}^n)$ into $L_{2,-\delta}(\mathbf{R}^n)$.

(v) (Asymptotic behavior of $u^{\pm}(\kappa, f)$). Suppose, in addition, that the potential V satisfies

$$V(x)=O(|x|^{-2}) \quad (|x|\to\infty). \tag{2.31}$$

Let $u^{\pm}(\kappa, f)$ be given by (2.26). Then for $\kappa>0$ and $f\in L_{2,1}(\mathbf{R}^n)$, there exists the limit

$$c(\kappa, f)\equiv\lim_{r\to\infty} e^{\mp i\sqrt{\kappa}r}r^{(n-1)/2}u^{\pm}(r\cdot, \kappa, f) \quad in\ L_2(S^{n-1}), \tag{2.32}$$

where S^{n-1} is the unit sphere in $\mathbf{R}^n$, and $u^{\pm}(r\cdot, \kappa, f)$ should be interpreted as the trace of $u^{\pm}(\kappa, f)$ on the sphere S_r with center at the origin and radius r.

These results on the Schrödinger operator are combined with Propositions 2.1 and 2.2 to investigate the resolvent $R_0(z)$ of the free Dirac operator H_0. First we are going to give another proof of the limiting absorption principle.

Theorem 2.4 (Limiting absorption principle for H_0). *Let H_0 be the free Dirac operator satisfying the anticommutation relation* (1.2). *Let δ be such that* $1/2<\delta\leq 1$.

(i) Then, for $\lambda\in(-\infty, -1)\cup(1, \infty)$, there exists the limits

$$R_0^{\pm}(\lambda)=\lim_{\eta\downarrow 0} R_0(\lambda\pm i\eta) \tag{2.33}$$

in $\mathbf{B}(\mathscr{L}_{2,\delta}, \mathscr{H}^1_{-\delta})$, and

$$R^{\pm}{}_0(\lambda)=\begin{cases}\Gamma_0^{\pm}(\lambda^2-1)A_{\lambda,K}+B_{\lambda,K} & (\lambda>1),\\ \Gamma_0^{\mp}(\lambda^2-1)A_{\lambda,K}+B_{\lambda,K} & (\lambda<-1),\end{cases} \tag{2.34}$$

where $K=K_{a,b}$ is taken as in (2.7) such that $1<a\leq|\lambda|\leq b<\infty$.

(ii) The operator $R_0(z)$ is a $\mathbf{B}(\mathscr{L}_{2,\delta}, \mathscr{H}^1_{-\delta})$-valued, continuous function on $J^+=\{z=\lambda+i\eta\colon |\lambda|>1,\ \eta\geq 0\}$ and $J^-=\{z=\lambda+i\eta\colon |\lambda|>1,\ \eta\leq 0\}$. Here the boundary value of $R_0(z)$ on the boundary $(-\infty, -1)\cup(1, \infty)$ is defined by either $R_0^+(\lambda)$ or $R_0^-(\lambda)$ according to (2.34).

Proof. Noting that, for $z=\lambda+i\eta$, $\mathrm{Im}(z^2-1)=2\lambda\eta$, we have (i) and (ii) directly from Propositions 2.1, 2.2 and (i), (ii) of Theorem 2.3.

The following theorem gives a characterization of $R_0^{\pm}(\lambda)$ through the Dirac equation with the radiation condition.

Theorem 2.5 (Radiation condition for H_0). *Let H_0 and δ be as above. Let $\lambda\in(-\infty, -1)\cup(1, \infty)$ and $f\in\mathscr{L}_{2,\delta}$. Set*

$$\begin{cases} v(\lambda+i\eta, f)=R_0(\lambda+i\eta)f={}^t(v_1, v_2, v_3, v_4) & (\eta\neq 0),\\ v^{\pm}(\lambda, f)=R_0^{\pm}(\lambda)f={}^t(v_1^{\pm}, v_2^{\pm}, v_3^{\pm}, v_4^{\pm}).\end{cases} \tag{2.35}$$

(i) Let $K=K_{a,b}$ be as in (2.7). Then there exists a positive constant $C=C(K)$ such that, for $f\in\mathscr{L}_{2,\delta}$ and $z=\lambda+i\eta\in K$,

$$\begin{cases} \|(\partial_l - i\sqrt{z^2-1}\tilde{x}_l)v_j(z,f)\|_{\delta-1} \le C\|f\|_\delta \quad (z=\lambda+i\eta,\ \eta\ne 0), \\ \|(\partial_l \mp i\sqrt{\lambda^2-1}\tilde{x}_l)v_j^\pm(\lambda,f)\|_{\delta-1} \le C\|f\|_\delta \\ \qquad\qquad (j=1,2,3,4,\quad l=1,2,3,\quad 1<a\le\lambda\le b<\infty), \\ \text{or} \\ \|(\partial_l \pm i\sqrt{\lambda^2-1}\tilde{x}_l)v_j^\pm(\lambda,f)\|_{\delta-1} \le C\|f\|_\delta \\ \qquad\qquad (j=1,2,3,4,\quad l=1,2,3,\quad -\infty<-b\le\lambda\le -a<-1). \end{cases} \tag{2.36}$$

(ii) Conversely, $v^\pm(\lambda,f)$ is determined as a unique solution of the equation

$$\begin{cases} (H_0-\lambda)v=f, \\ v\in\mathscr{L}_{2,-\delta}\cap\mathscr{H}^1_{\mathrm{loc}}, \\ (\partial_l \mp i\sqrt{\lambda^2-1}\tilde{x}_l)v_j \in L_{2,\delta-1} \\ \qquad\qquad (\lambda>1,\ j=1,2,3,4 \text{ and } l=1,2,3), \\ \text{or} \\ (\partial_l \pm i\sqrt{\lambda^2-1}\tilde{x}_l)v_j \in L_{2,\delta-1} \\ \qquad\qquad (\lambda<-1,\ j=1,2,3,4 \text{ and } l=1,2,3). \end{cases} \tag{2.37}$$

Proof. (i) directly follows from (i) of Theorem 2.4 and the first half of (iii) of Theorem 2.3 by noting that $B_{z,K_0}f\in\mathscr{H}^1$ by (ii) of Proposition 2.2. Let us prove (ii). We may consider only the case of $v^+(\lambda,f)$ with $\lambda>1$, since other cases can be treated in a quite similar manner. Thus we have only to prove that if v satisfies

$$\begin{cases} (H_0-\lambda)v=0, \\ v\in\mathscr{L}_{2,-\delta}\cap\mathscr{H}^1_{\mathrm{loc}} \\ (\partial_l - i\sqrt{\lambda^2-1}\tilde{x}_l)v_j\in L_{2,\delta-1} \quad (j=1,2,3,4,\quad l=1,2,3), \end{cases} \tag{2.38}$$

then v is identically zero. In fact, from the first relation of (2.38) and (1.5) it follows that

$$(-\Delta+1-\lambda^2)v=0, \tag{2.39}$$

i.e., (each component of) $-\Delta v$ is locally L_2, and hence we have $v\in\mathscr{H}^2_{\mathrm{loc}}$. Thus we see that each component v_j of v satisfies

$$\begin{cases} (-\Delta-(\lambda^2-1))v_j=0, \\ v_j\in L_{2,-\delta}\cap H^2_{\mathrm{loc}}, \\ (\partial_l - i\sqrt{\lambda^2-1}\tilde{x}_l)v_j\in L_{2,\delta-1} \quad (l=1,2,3), \end{cases} \tag{2.40}$$

which is combined with the uniqueness result of the Schrödinger operator ((iii) of Theorem 2.3) to give that each v_j is identically zero, i.e., v is identically zero.

Theorem 2.6 (Asymptotic behavior of $v^{\pm}(\lambda, f)$). *Let H_0, δ and $v^{\pm}(\lambda, f)$ be as above, where $\lambda \in (-\infty, -1) \cup (1, \infty)$ and $f \in \mathcal{L}_{2,1}$. Then there exist the limits*

$$c_j^{\pm}(\lambda, f) \equiv \begin{cases} \lim\limits_{r\to\infty} e^{\mp i\sqrt{\lambda^2-1}r} r\, v_j^{\pm}(r\cdot, \lambda, f) & (\lambda>1), \\ \lim\limits_{r\to\infty} e^{\pm i\sqrt{\lambda^2-1}r} r\, v_j^{\pm}(r\cdot, \lambda, f) & (\lambda<-1). \end{cases} \tag{2.41}$$

To prove the above theorem we need the following lemma.

Lemma 2.7. *Let $g \in H_s^1(\mathbf{R}^n)$ with $s>1/2$. Then we have*

$$\int_{S_r} |g(x)|^2 dS = O(r^{-(2s-1)}) \quad (r\to\infty), \tag{2.42}$$

where S_r is the sphere with center at the origin and radius r, $dS = r^{n-1}d\omega$, and $d\omega$ is the area element on S^{n-1}.

Proof. Let us first assume that $g \in C_0^\infty(\mathbf{R}^n)$. Then, since

$$\begin{aligned} |g(r\omega)|^2 &= \left| -\int_r^\infty \frac{\partial g(t\omega)}{\partial t} dt \right|^2 \\ &\le \int_r^\infty \left| \frac{\partial g(t\omega)}{\partial t} \right|^2 (1+t)^{2s} dt \int_r^\infty (1+t)^{-2s} dt \\ &= (2s-1)^{-1}(1+r)^{-(2s-1)} \int_r^\infty \left| \frac{\partial g(t\omega)}{\partial t} \right|^2 (1+t)^{2s}\, dt, \end{aligned} \tag{2.43}$$

it follows that

$$\begin{aligned} |g(r\omega)|^2 r^{n-1} &\le (2s-1)^{-1}(1+r)^{-(2s-1)} r^{n-1} \int_r^\infty \left| \frac{\partial g(t\omega)}{t} \right|^2 (1+t)^{2s} dt \\ &\le (2s-1)^{-1}(1+r)^{-(2s-1)} \int_r^\infty |(\nabla g)(t\omega)|^2 (1+t)^{2s}\, t^{n-1} dt \end{aligned} \tag{2.44}$$

Integrating the both sides of (2.44) over S^2, we obtain

$$\int_{S_r} |g(x)|^2 dS \le (2s-1)^{-1}(1+r)^{-(2s-1)} \|g\|_{1,s}^2 \quad (r>0). \tag{2.45}$$

Let us next assume that $g \in H_s^1(\mathbf{R}^n)$. Then, since the trace is continuous on $H_s^1(\mathbf{R}^n)$ and $C_0^\infty(\mathbf{R}^n)$ is dense in $H_s^1(\mathbf{R}^n)$, we see that the inequality (2.45) holds for general g, which completes the proof.

Proof of Theorem 2.6. Suppose that $\lambda>1$. Let $K = K_{a,b}$ as in (2.7) such that $1<a<\lambda<b<\infty$. Set

$$\begin{cases} g^{(1)}={}^t(g_1{}^{(1)}, g_2{}^{(1)}, g_3{}^{(1)}, g_4{}^{(1)})=A_{\lambda,K}f, \\ g^{(2)}={}^t(g_1{}^{(2)}, g_2{}^{(2)}, g_3{}^{(2)}, g_4{}^{(2)})=B_{\lambda,K}f. \end{cases} \tag{2.46}$$

Then it follows from (2.34) that

$$R_0^{\pm}(\lambda)f=\Gamma_0^{\pm}(\lambda^2-1)g^{(1)}+g^{(2)}. \tag{2.47}$$

Applying (i) of Proposition 2.2 with $s=1$ to $g^{(1)}$, we see that $g^{(1)}\in\mathscr{L}_{2,1}$, and hence it follows from (v) of Theorem 2.3 that there exists

$$c_j^{\pm}(\lambda, f)\equiv\lim_{r\to\infty} e^{\mp i\sqrt{\lambda^2-1}\,r}\, r\Gamma_0^{\pm}(\lambda^2-1)g_j^{(1)}(r\cdot). \tag{2.48}$$

in $L_2(S^2)$ for $j=1, 2, 3, 4$. On the other hand, from (ii) of Proposition 2.2 we have $g^{(2)}\in\mathscr{H}_1^1$, and hence we can apply Lemma 2.7 to see

$$\lim_{r\to\infty} e^{\pm i\sqrt{\lambda^2-1}\,r}\, rg_j^{(2)}(r\cdot)=0 \quad (\text{in } L_2(S^2)), \tag{2.49}$$

which is combined with (2.48) to give (2.41) for $\lambda>1$. The case that $\lambda<-1$ can be treated in quite a similar way.

Finally we shall prove the compactness of the resolvent $R_0(z)$ and the extended resolvent $R_0^{\pm}(\lambda)$.

Theorem 2.8 (Compactness of $R_0(z)$ and $R_0^{\pm}(\lambda)$). *Let H_0 and δ be as above, and let $K=K_{a,b}$ be as in (2.7).*

(i) Then there exists a positive constant $C=C(K)$ such that, for $f\in\mathscr{L}_{2,\delta}$ and $z=\lambda+i\eta\in K$,

$$\begin{cases} \|v(z, f)\|_{-\delta,E_r}\leq C(1+r)^{-(\delta-1/2)}\|f\|_{\delta} \quad (z=\lambda+i\eta,\ \eta\neq 0), \\ \|v^{\pm}(\lambda, f)\|_{-\delta,E_r}\leq C(1+r)^{-(\delta-1/2)}\|f\|_{\delta}, \end{cases} \tag{2.50}$$

where $v(z, f)$ and $v^{\pm}(\lambda, f)$ are as in (2.35), and E_r and $\|\ \ \|_{-\delta,E_r}$ are as in (iv) of Theorem 2.3 with $n=3$.

(ii) $R_0(\lambda+i\eta)$ ($|\lambda|>1$, $\eta\neq 0$) and $R_0^{\pm}(\lambda)$ ($|\lambda|>1$) are compact operators from $\mathscr{L}_{2,\delta}$ into $\mathscr{L}_{2,-\delta}$.

Proof. We shall show (i) for $z=\lambda+i\eta\in K=K_{a,b}$ with $\eta\neq 0$. The case that $z=\lambda\in[a, b]$ can be treated in the same way. It follows from Propositions 2.1 and 2.2 that

$$R_0(z)f=\Gamma_0(z^2-1)A_{z,K}f+B_{z,K}f\equiv f_1+f_2. \tag{2.51}$$

Note that it follows from Proposition 2.2 that $A_{z,K}$ and $B_{z,K}$ are $\mathbf{B}(\mathscr{L}_{2,\delta})$-valued, continuous function on K. Therefore, it is easy to see that

$$\|f_j\|_{-\delta,E_r}\leq C(1+r)^{-(\delta-1/2)}\|f\|_{\delta} \quad (j=1, 2), \tag{2.52}$$

where we have also used (a) of Theorem 2.3, (iv) to evaluate f_1. This completes the proof of (i).

Let us turn to the proof of (ii). We give the proof only for $R_0(z)$ with $z=\lambda+i\eta$, $\eta\neq 0$. The proof for $R_0^{\pm}(\lambda)$ can be done in exactly the same way. Let $\{f_m\}$ be a bounded sequence in $\mathscr{L}_{2,\delta}$. Then for any $r>0$, we have

$$\int_{|x|\leq r}\sum_{|\alpha|\leq 1}|\partial^{\alpha}R_0(z)f_m(x)|^2dx\leq(1+r^2)^{\delta}\|R_0(z)f_m\|_{1,-\delta}^2.$$

With this inequality, we see that $\{R_0(z)f_m\}$, together with their first derivatives, is $\mathscr{L}_2$-bounded on any compact set in $\mathbf{R}^3$. We now appeal to the Rellich selection theorem to deduce that there exists a subsequence $\{f_{m'}\}$ of $\{f_m\}$ such that $\{R_0(z)f_{m'}\}$ is locally $\mathscr{L}_2$-convergent. Combining this fact with (2.50), we see that $\{R_0(z)f_{m'}\}$ converges in $\mathscr{L}_{2,-\delta}$. This completes the proof of (ii).

3. The Dirac operator H with a short-range potetial

Let H be given at the beginning of § 2, i.e., H is the selfadjoint realization of the Dirac operator with a short-range potential. We have $H=H_0+Q$. On the potential Q we assume the following:

Assumption 3.1. The potential $Q(x)=(q_{jk}(x))$ is a 4×4 Hermitian matrix-valued function satisfying (1.3). Further, each component q_{jk} is a C^1 function on $\mathbf{R}^3$ except at a finite number of singularities, and there exists $R_0>0$ such that

$$\sup_{|x|>R_0}\Big(\sum_{l=1}^{3}|\partial_l q_{jk}(x)|\Big)<\infty \quad (j, k=1, 2, 3, 4). \tag{3.1}$$

Then we have

Proposition 3.2 (Yamada [13], Proposition 2.5). *Let H satisfy Assumption 3.1 and (1.2). Then there are no eigenvalues of H on* $(-\infty, -1)\cup(1, \infty)$.

We set

$$\begin{cases} F(z)=QR_0(z) & (z=\lambda+i\eta,\ |\lambda|>1,\ \eta\neq 0), \\ F^{\pm}(\lambda)=QR_0^{\pm}(\lambda) & (\lambda\in(-\infty, -1)\cup(1, \infty)), \end{cases} \tag{3.2}$$

where $R_0(z)$ and $R_0^{\pm}(\lambda)$ are the resolvent and the extended resolvents of the free Dirac operator H_0, respectively. It follows from (iii) of Theorem 2.8 that $F(z)$ and $F^{\pm}(\lambda)$ are compact operators on $\mathscr{L}_{2,\delta}$, where δ satisfies

$$1/2<\delta\leq\min\left\{1, \frac{1+\varepsilon}{2}\right\}. \tag{3.3}$$

We start with the proof of the invertibility of the operators $I+F(z)$ and $I+$

$F^{\pm}(\lambda)$, where I is the identity operator in $\mathscr{L}_{2,\delta}$.

Proposition 3.3. *Let H satisfy Assumption 3.1 and (1.2), and let δ be as in (3.3). Let $F(z)$ and $F^{\pm}(\lambda)$ be as above.*

(i) Then $F(z)$ and $F^{\pm}(\lambda)$ do not have the eigenvalue -1, and hence we have

$$(I+F(z))^{-1},\ (I+F^{\pm}(\lambda))^{-1}\in\mathbf{B}(\mathscr{L}_{2,\delta}). \tag{3.4}$$

(ii) The operator $(I+F(z))^{-1}$ is a $\mathbf{B}(\mathscr{L}_{2,\delta})$-valued, continuous function on $J^{+}=\{z=\lambda+i\eta\colon |\lambda|>1,\ \eta\geq 0\}$ and $J^{-}=\{z=\lambda+i\eta\colon |\lambda|>1,\ \eta\leq 0\}$. Here the boundary value of $(I+F(z))^{-1}$ on the boundary $(-\infty,-1)\cup(1,\infty)$ is defined by either $(I+F^{+}(\lambda))^{-1}$ or $(I+F^{-}(\lambda))^{-1}$ according as λ is supposed to belong to the boundary of J^{+} or J^{-}.

Proof. Since the proof of (i) for $F(z)$ is trivial, we shall give the proof for $F^{+}(\lambda_0)$ with $\lambda_0>1$. Suppose that $g_0\in\mathscr{L}_{2,\delta},\neq 0$ such that

$$(I+F^{+}(\lambda_0))g_0=0. \tag{3.5}$$

Set $v_0=R_0^{+}(\lambda_0)g_0$ so that we have from (3.5)

$$\begin{cases} v_0\in\mathscr{L}_{2,-\delta}\cap\mathscr{H}^1_{\mathrm{loc}}, \\ (H_0+Q-\lambda_0)v_0=0. \end{cases} \tag{3.6}$$

Then we can follow the arguments in the proof of Theorem 4.1 in Yamada [13] to prove that $v_0\in\mathscr{L}_2,\neq 0$, which contradicts Proposition 3.2. In fact, we have only to set $\lambda_n=\lambda+i/n$, $g_n=g_0$, and $u_n=R_0(\lambda_n)g_0$ in the arguments in Yamada [13], p.570-p.573. This completes the proof of (i). Now we are going to prove the continuity of $(I+F(z))^{-1}$ on J^{+} and, for the sake of simplicity of the notation, set $F(\lambda)=F^{+}(\lambda)$. Since it follows from (ii) of Theorem 2.4 that $F(z)$ is a $\mathbf{B}(\mathscr{L}_{2,\delta})$-valued, continuous function on J^{+}, the continuity follows from the relation

$$(I+F(z))^{-1}-(I+F(z_0))^{-1}=-(I+F(z))^{-1}(F(z)-F(z_0))(I+F(z_0))^{-1}, \tag{3.7}$$

or

$$(I+F(z))^{-1}[I+(F(z)-F(z_0))(I+F(z_0))^{-1}]=(I+F(z_0))^{-1} \tag{3.8}$$

for z, $z_0\in J^{+}$. In fact, if $|z-z_0|$ is sufficiently small, then so is $F(z)-F(z_0)$, and hence the inverse $[I+(F(z)-F(z_0))(I+F(z_0))^{-1}]^{-1}$ exists and is bounded in a neighborhood of z_0. This guarantees the local boundedness of $(I+F(z))^{-1}$ on J^{+}. Once the local boundedness is established, then the continuity follows directly from (3.7) and the continuity of $F(z)$, which completes the proof of (ii).

The main result of this paper follows from the above proposition.

Theorem 3.4 (Limiting absorption principle for H). *Let the Dirac operator H satisfy Assumption 3.1 and (1.2), and let δ be as in (3.3). Let $R(z)$ be the resolvent of H. Let $F(z)$ and $F^{\pm}(\lambda)$ be as in (3.2).*

(i) (Cf. Balslev-Helffer [2].) *Let $z=\lambda+i\eta$ with $|\lambda|>1$ and $\eta\neq 0$. Then we have*

$$R(z)=R_0(z)\,(I+F(z))^{-1}, \tag{3.9}$$

where $R_0(z)$ be the resolvent of the free Dirac operator H_0. Further, there exist the limits

$$R^{\pm}(\lambda)=\lim_{\eta\downarrow 0} R(\lambda\pm i\eta) \quad \text{in } \mathbf{B}(\mathscr{L}_{2,\delta}, \mathscr{H}^{1}_{-\delta}) \tag{3.10}$$

for $|\lambda|>1$, where the extended resolvents $R^{\pm}(\lambda)$ of H are given by

$$R^{\pm}(\lambda)=R_0^{\pm}(\lambda)\,(I+F^{\pm}(\lambda))^{-1} \tag{3.11}$$

with the extended resolvents $R_0^{\pm}(\lambda)$ of the free Dirac operator H_0. The operator $R(z)$ is a $\mathbf{B}(\mathscr{L}_{2,\delta}, \mathscr{H}^{1}_{-\delta})$-valued continuous function on J^{+} and J^{-}, where the boundary value of $R(z)$ on the boundary $(-\infty, -1)\cup(1,\infty)$ is defined by either $R^{+}(\lambda)$ or $R^{-}(\lambda)$ according as λ belongs to the boundary of J^{+} or J^{-}.

(ii) (a) Let $\lambda\in(-\infty,-1)\cup(1,\infty)$ and $f\in\mathscr{L}_{2,\delta}$. Set

$$\begin{cases} v(\lambda+i\eta, f)=R(\lambda+i\eta)f={}^{t}(v_1, v_2, v_3, v_4) & (\eta\neq 0),\\ v^{\pm}(\lambda, f)=R^{\pm}(\lambda)f={}^{t}(v_1^{\pm}, v_2^{\pm}, v_3^{\pm}, v_4^{\pm}). \end{cases} \tag{3.12}$$

Let $K=K_{a,b}$ be as in (2.7). Then there exists a positive constant $C=C(K)$ such that the radiation condition estimates (2.39) hold for $f\in\mathscr{L}_{2,\delta}$ and $z=\lambda+i\eta\in K$.

(b) Conversely, $v^{\pm}(\lambda, f)$ is determined as a unique solution of the equation

$$\begin{cases} (H-\lambda)v=f,\\ v\in\mathscr{L}_{2,-\delta}\cap\mathscr{H}^{1}_{\mathrm{loc}},\\ (\partial_l\mp i\sqrt{\lambda^2-1}\,\tilde{x}_l)v_j\in L_{2,\delta-1}\\ \qquad\qquad (\lambda>1,\ j=1,2,3,4 \text{ and } l=1,2,3),\\ \text{or}\\ (\partial_l\pm i\sqrt{\lambda^2-1}\,\tilde{x}_l)v_j\in L_{2,\delta-1}\\ \qquad\qquad (\lambda<-1,\ j=1,2,3,4 \text{ and } l=1,2,3). \end{cases} \tag{3.13}$$

(iii) Let $v^{\pm}(\lambda, f)$ be as abovve, where $\lambda\in(-\infty,-1)\cup(1,\infty)$ and $f\in\mathscr{L}_{2,1}$. Then there exist the limits

$$c_j^{\pm}(\lambda, f)\equiv\begin{cases} \lim_{r\to\infty} e^{\mp i\sqrt{\lambda^2-1}\,r}\, r v_j^{\pm}(r\cdot, \lambda, f) & (\lambda>1),\\ \lim_{r\to\infty} e^{\pm i\sqrt{\lambda^2-1}\,r}\, r v_j^{\pm}(r\cdot, \lambda, f) & (\lambda<-1). \end{cases} \tag{3.14}$$

(iv) Let $K=K_{a,b}$ be as in (2.7), and let $K^{\pm}$ be as in (2.54). Let $v(z,f)$ and $v^{\pm}(\lambda, f)$ be as above. Then (i) and (ii) of Theorem 2.8 with H_0 replaced by H holds. Thus $R(\lambda+i\eta)$ ($|\lambda|>1$, $\eta\neq 0$) and $R^{\pm}(\lambda)$ ($|\lambda|>1$) are compact operators from $\mathscr{L}_{2,\delta}$ into $\mathscr{L}_{2,-\delta}$.

Proof. Let $z=\lambda+i\eta$ with $|\lambda|>1$ and $\eta\neq 0$. Then, since we have

$$R(z)-R_0(z)=-R(z)QR_0(z)=-R(z)F(z), \tag{3.15}$$

(3.9) follows from (i) of Proposition 3.3. Using the continuity of $F(z)$ ((ii) of Proposition 3.3), we easily see that (3.10) and (3.11) hold. The continuity of $R(z)$ on $J^{\pm}$ follows from the continuity of $F(z)$ and $R_0(z)$ ((ii) of Theorem 2.4), which completes the proof of (i). (ii) - (a), (iii), (iv) can be easily proved from (3.9), (3.11), Proposition 3.3 and Theorems 2.4, 2.5, 2.6 and 2.8.

Let us prove (ii) - (b). We shall prove the case that $v_0={}^t(v_{01}, v_{02}, v_{03}, v_{04})$ satisfies

$$\begin{cases} (H-\lambda)v_0=0, \\ v_0\in\mathscr{L}_{2,-\delta}\cap\mathscr{H}^1_{\mathrm{loc}}, \\ (\partial_l-i\sqrt{\lambda^2-1}\tilde{x}_l)v_{0j}\in L_{2,\delta-1} \\ \qquad (\lambda>1, j=1, 2, 3, 4 \text{ and } l=1, 2, 3) \end{cases} \tag{3.16}$$

with $\lambda>1$. The other cases can be treated similarly. Then, since

$$(H_0-\lambda)v_0=-Qv_0=-(H_0-\lambda)R_0^+(\lambda)Qv_0, \tag{3.17}$$

we have $(H_0-\lambda)(v_0+R_0^+(\lambda)Qv_0)=0$, i.e., $w=v_0+R_0^+(\lambda)Qv_0$ is a solution of the homogeneous equation $(H_0-\lambda)v=0$. Noting that w satisfies the radiation condition, too, we see from (ii) of Theorem 2.5 that $w=0$, i.e.,

$$(I+R_0^+(\lambda)Q)v_0=0. \tag{3.18}$$

Let $f\in\mathscr{L}_{2,\delta}$. Then we have from (3.18)

$$\begin{aligned} 0&=((I+R_0^+(\lambda)Q)v_0, f) \\ &=\lim_{m\to\infty}((I+R_0(\lambda+i/m)Q)v_0, f) \\ &=\lim_{m\to\infty}(v_0, (I+QR_0(\lambda-i/m))f) \\ &=(v_0, (I+F^-(\lambda))f), \end{aligned} \tag{3.19}$$

where (,) is the inner product of $\mathscr{L}_2$, i.e.,

$$(f, q)=\sum_{j=1}^{4}\int_{\mathbf{R}^3}f_j(x)\overline{g_j(x)}\,dx \tag{3.20}$$

with $f(x)={}^t(f_1(x), f_2(x), f_3(x), f_4(x))$ and $g(x)={}^t(g_1(x), g_2(x), g_3(x), g_4(x))$.

It follows from (i) of Proposition 3.3 that $(I+F^{-}(\lambda))$ is onto $\mathscr{L}_{2,\delta}$, and hence (3.19) implies that $v_0=0$. This completes the proof.

Remark 3.5. It follows from (3.11) and (2.37) that, for $f\in\mathscr{L}_{2,\delta}$,

$$R^{\pm}(\lambda)f=\begin{cases}\Gamma_0^{\pm}(\lambda^2-1)A_{\lambda,K}(I+F^{\pm}(\lambda))^{-1}f+B_{\lambda,K}(I+F^{\pm}(\lambda))^{-1}f & (\lambda>1),\\ \Gamma_0^{\mp}(\lambda^2-1)A_{\lambda,K}(I+F^{\pm}(\lambda))^{-1}f+B_{\lambda,K}(I+F^{\pm}(\lambda))^{-1}f & (\lambda<-1),\end{cases} \tag{3.21}$$

where δ is as above, $K=K_{a,b}$ is taken so that $a<|\lambda|<b$. Since the term $B_{\lambda,K}(I+F^{\pm}(\lambda))^{-1}f$ belongs to $\mathscr{H}_\delta^1$, the "main" part of $R^{\pm}(\lambda)f$ is the solution v of the equation

$$(-\Delta-(\lambda^2-1))v=A_{\lambda,K}(I+F^{\pm}(\lambda))^{-1}f \tag{3.22}$$

with an appropriate radiation condition.

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