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Projective elements in K-theory and self maps of ΣCP^∞

By

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1. Introduction and statements of results

In this paper, we will work in the homotopy category of based spaces and based maps. Given a space X, we denote the reduced K-theory by $K(X)$ and the homology group of integral coefficients by $H_*(X)$. Let CP^∞ be the infinite dimensional complex projective space. Let η be the canonical line bundle over CP^∞ and i: $CP^\infty \to BU$ be the classifying map of the virtual bundle $\eta - 1$. Since BU has a loop space structure which is derived from the Whitney sum of complex vector bundles, there exists a unique extension of i to the loop map j: $\Omega\Sigma CP^\infty \to BU$.

In this paper we investigate the following problems:

Given an element $\alpha \in K(X)$, when does there exist a lift $\widehat{\alpha} \in [X, \Omega\Sigma CP^\infty]$ such that $j_*(\widehat{\alpha}) = \alpha$? If α has a lift, how we can construct the lift $\widehat{\alpha}$?

Define

$$PK(X) = \{\alpha \in K(X) \mid \exists\ \widehat{\alpha} \in [X, \Omega\Sigma CP^\infty] \text{ such that } j_*(\widehat{\alpha}) = \alpha\}.$$

If an element $\alpha \in K(X)$ belongs to $PK(X)$, we call that α is projective.

The significance of the above problem is as follows:

The James splitting theorem [2] implies that there exists a loop map θ: $BU \to \Omega^\infty\Sigma^\infty CP^\infty$ such that the following diagram commutes:

$$\begin{array}{ccc} \Omega\Sigma CP^\infty & \xrightarrow{E^\infty} & \Omega^\infty\Sigma^\infty CP^\infty \\ \Big\downarrow j & \nearrow \theta & \\ BU. & & \end{array}$$

Therefore, given an element $\alpha \in K(X)$, we have the stable map, $adj.(\theta(\alpha))$: $\Sigma^\infty X \to \Sigma^\infty CP^\infty$. Using the information of $K(X)$, we can calculate the induced homomorphism [3], [4] of $adj.(\theta(\alpha))_*$: $H_*(X) \to H_*(CP^\infty)$. If α has a lift $\widehat{\alpha}$, then this implies that the stable map $adj.(\theta(\alpha))$ and its induced

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homomorphism come from the unstable map $adj.(\widehat{\alpha}): \Sigma X \to \Sigma CP^\infty$. These imply that the determination of $PK(X)$ gives complete information of the image of the homomorphism:

$$[\Sigma X, \Sigma CP^\infty] \to \mathrm{Hom}(H_*(X), H_*(CP^\infty)).$$

However, since the above homomorphism factors through $\mathrm{Hom}(H_*(X), H_*(\Omega\Sigma CP^\infty))$, it is desiable to obtain the image of

$$[X, \Omega\Sigma CP^\infty] \to \mathrm{Hom}(H_*(X), H_*(\Omega\Sigma CP^\infty)).$$

So, if possible, we want to have the information of not $adj.(\widehat{\alpha})_*$ but $\widehat{\alpha}_*: H_*(X) \to H_*(\Omega\Sigma CP^\infty)$. Thus we need the geometry of the lift $\widehat{\alpha}$.

Now we shall state our main results.

Since CP^∞ is an H-space, we have a map

$$CP^\infty \wedge CP^\infty \to \Omega\Sigma CP^\infty,$$

which is the adjoint of the Hopf construction: We will show that

Theorem 1.1. *The adjoint of the Hopf construction of CP^∞ has an extension*

$$\# : \Omega\Sigma CP^\infty \wedge \Omega\Sigma CP^\infty \to \Omega\Sigma CP^\infty,$$

such that the following diagram commutes:

$$\begin{array}{ccc} \Omega\Sigma CP^\infty \wedge \Omega\Sigma CP^\infty & \xrightarrow{\#} & \Omega\Sigma CP^\infty \\ \downarrow{\scriptstyle j\wedge j} & & \downarrow{\scriptstyle j} \\ BU \wedge BU & \xrightarrow{\otimes} & BU, \end{array} \qquad (1.1)$$

where $\otimes: BU \wedge BU \to BU$ is the map which represents the external tensor product $K(X) \otimes K(Y) \to K(X \wedge Y)$.

As the properties of $PK(X)$, we have

Theorem 1.2. *$PK(X)$ has the following properties.*

(1) *$PK(X)$ is an additive subgroup of $K(X)$,*

(2) *if $\alpha \in PK(X)$ and $\beta \in PK(Y)$, then $\alpha \otimes \beta \in PK(X \wedge Y)$,*

(3) *if $\alpha \in PK(X)$, then $\varphi^k(\alpha) \in PK(X)$ for all $k \in Z$, where φ^k is the Adams operation,*

(4) *if X is a finite complex, then for any $\alpha \in K(X)$, there exists a number N such that $N\alpha \in PK(X)$,*

(5) *if X is a finite complex, then for large N, $PK(\Sigma^N X) = K(\Sigma^N X)$,*

(6) *if $\alpha \in K(X)$ is a linear combination of line bundles of virtual dimension 0, then $\alpha \in PK(X)$.*

As an application, we consider the group $[CP^\infty, \Omega\Sigma CP^\infty]$.

Recall that $K(CP^\infty_+) \cong \mathbf{Z}[[x]]$, where $x = \eta - 1$. Since CP^∞ is the classifying space of complex line bundles, it is easily seen that $PK(CP^\infty) =$

$K(CP^{\infty})$. However, we would like to find the canonical lift of $x^n \in K(CP^{\infty})$.

Let f_1: $CP^{\infty} \to \Omega\Sigma CP^{\infty}$ and ζ_1: $S^2 \to \Omega\Sigma CP^{\infty}$ be the inclusions and inductively define

$$f_{n+1}\colon CP^{\infty} \xrightarrow{\bar{\Delta}} CP^{\infty} \wedge CP^{\infty} \xrightarrow{f_1 \wedge f_n} \Omega\Sigma CP^{\infty} \wedge \Omega\Sigma CP^{\infty} \xrightarrow{\#} \Omega\Sigma CP^{\infty}, \quad (1.2)$$

$$\zeta_{n+1}\colon S^{2n+2} = S^2 \wedge S^{2n} \xrightarrow{\zeta_1 \wedge \zeta_n} \Omega\Sigma CP^{\infty} \wedge \Omega\Sigma CP^{\infty} \xrightarrow{\#} \Omega\Sigma CP^{\infty}, \quad (1.3)$$

where $\bar{\Delta}$ is the reduced diagonal map.

Remark 1.3. The above definition of ζ_n coincides with the one in [5].

Let CP_n^{∞} be the stunted projective space CP^{∞}/CP^{n-1} and p: $CP^{\infty} \to CP_n^{\infty}$ be the projection.

Theorem 1.4. *$\{f_n\}$ and $\{\zeta_n\}$ have the following properties.*

(1) $j_*(f_n) = x^n$ *in* $K(CP^{\infty})$,

(2) f_n: $CP^{\infty} \to \Omega\Sigma CP^{\infty}$ *factors as* $CP^{\infty} \xrightarrow{p} CP_n^{\infty} \xrightarrow{g_n} \Omega\Sigma CP^{\infty}$, *such that the restriction to the bottom sphere of the map* g_n *coincides with the map* ζ_n.

(3) $j_*(\zeta_n)$ *is the generator of* $\pi_{2n}(BU) \cong \mathbf{Z}$,

(4) *Let* $C(f_n, f_m)$ *be the commutator in the group* $[CP^{\infty}, \Omega\Sigma CP^{\infty}]$ *of* f_n *and* f_m. *Then*

$$i^* C(f_n, f_m) = q^* \langle \zeta_n, \zeta_m \rangle$$

where i: $CP^{n+m} \to CP^{\infty}$ *is the inclusion,* q: $CP^{n+m} \to S^{2n+2m}$ *is the projection and* $\langle \zeta_n, \zeta_m \rangle$ *is the Samelson product in* $\pi_*(\Omega\Sigma CP^{\infty})$.

Let h: $\pi_*(\Omega\Sigma CP^{\infty}) \to H_*(\Omega\Sigma CP^{\infty})$ be the Hurewicz homomorphism. Recall that $\tilde{H}_*(CP^{\infty}) \cong \mathbf{Z}\{\beta_1, \beta_2, \cdots\}$, where $\beta_n \in H_{2n}(CP^{\infty})$ is the standard generator. Therefore $H_*(\Omega\Sigma CP^{\infty})$ is the tensor algebra generated by $\{\beta_1, \beta_2, \cdots\}$. Let χ: $\Omega\Sigma CP^{\infty} \to \Omega\Sigma CP^{\infty}$ be the map of loop inverse. Then

Theorem 1.5.

$$h(\zeta_n) = \begin{cases} \beta_1 & \text{if } n=1, \\ (n-1)!\sum_{i=1}^{n} \chi_*(\beta_{n-i})(i\beta_i - \beta_1\beta_{i-1}) & \text{if } n \geq 2, \end{cases} \quad (1.4)$$

where the product in the above equation is the one in the tensor algebra and β_0 *means* $1 \in H_0(\Omega\Sigma CP^{\infty})$.

Corollary 1.6. *If* $n \geq 3$, *then the group* $[\Sigma CP^n, \Sigma CP^n]$ *is not commutative.*

Let $\tilde{f}_i \in [\Sigma CP^{\infty}, \Sigma CP^{\infty}]$ be the adjoint of f_i. Then about the composition structures of $\tilde{f}_i$ we get

Theorem 1.7. *The composition* $\tilde{f}_i \circ \tilde{f}_j$ *can be written as a linear* *combination of* $\widetilde{f}_n$*'s for* $n \leq ij$.

Throughout this paper, we use the symbol $+$ as the product in $[\Sigma X, \Sigma CP^\infty]$ or $[X, \Omega\Sigma CP^\infty]$, although these groups are not in general abelian.

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2. The extension of $X\wedge Y\to Z$ to $\Omega\Sigma X\wedge\Omega\Sigma Y$

Given a map f: $X\wedge Y\to\Omega Z$, there exist extensions of f to $\Omega\Sigma X\wedge\Omega\Sigma Y$. We fix a choice of extensions as follows: Take $x\in X$. Define f_x: $Y\to\Omega Z$ by $f_x(y)=f(x, y)$. Extend f_x to a loop map $\bar{f}_x$: $\Omega\Sigma Y\to\Omega Z$. Note that such an extension is unique. Therefore we have a map g: $X\wedge\Omega\Sigma Y\to\Omega Z$ which is an extension of f. Similarly, from g we have a map h: $\Omega\Sigma X\wedge\Omega\Sigma Y\to\Omega Z$. Note that $h(\alpha_1+\alpha_2, \beta)=h(\alpha_1, \beta)+h(\alpha_2, \beta)$, $h(\iota(x), \beta_1+\beta_2)=h(\iota(x), \beta_1)+h(\iota(x), \beta_2)$, and $h(\iota(x), \iota(y))=f(x, y)$, where ι: $X\to\Omega\Sigma X$ is the canonical inclusion. From now on we denote h by $L(f)$.

Our extension $L(f)$ of f can be described through the following commutative diagram:

$$\begin{array}{ccc} X^m\times Y^n & \xrightarrow{\ d\ } & (X\times Y)^{mn} \\ \downarrow & & \downarrow \\ X^m\wedge Y^n & \longrightarrow & (X\wedge Y)^{mn} \\ \downarrow{\scriptstyle \alpha_m\wedge\alpha_n} & & \downarrow{\scriptstyle \alpha_{mn}} \\ \Omega\Sigma X\wedge\Omega\Sigma Y & \xrightarrow[L(\iota)]{} & \Omega\Sigma(X\wedge Y)\xrightarrow[\bar{f}]{}\Omega Z \end{array} \tag{2.1}$$

where

$$\begin{aligned} d(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n) = {} & (x_1, y_1),\ (x_1, y_2),\ \cdots,\ (x_1, y_n), \\ & (x_2, y_1),\ (x_2, y_2),\ \cdots,\ (x_2, y_n), \\ & \cdots \\ & (x_m, y_1),\ (x_m, y_2),\ \cdots,\ (x_m, y_n), \end{aligned}$$

ι: $X\wedge Y\to\Omega\Sigma(X\wedge Y)$ is the inclusion, α_m is the composite: $X^m\xrightarrow{\iota^m}(\Omega\Sigma X)^m\xrightarrow{+}\Omega\Sigma X$, $\bar{f}$ is the canonical extension of f to the loop map and other maps are standard ones. It is clear that $L(f)=\bar{f}\circ L(\iota)$.

Proof of Theorem 1.1. Let p_i:$CP^\infty\times CP^\infty\to CP^\infty$ be the i-th projection ($i=1$, 2). Since $CP^\infty\cong K(\mathbf{Z}, 2)$, it has the unique multiplication μ: $CP^\infty\times CP^\infty\to CP^\infty$. Let $\widetilde{H(\mu)}$: $CP^\infty\wedge CP^\infty\to\Omega\Sigma CP^\infty$ be the adjoint of the Hopf construction $H(\mu)$: $\Sigma CP^\infty\wedge CP^\infty\to\Sigma CP^\infty$. Then the following equation characterizes $\widetilde{H(\mu)}$ [1].

$$\widetilde{H(\mu)}\circ\pi=-\iota\circ p_2-\iota\circ p_1+\iota\circ\mu. \tag{2.2}$$

where π: $CP^\infty \times CP^\infty \to CP^\infty \wedge CP^\infty$ is the canonical projection.

Put

$$\# = L(\widetilde{H(\mu)})): \Omega\Sigma CP^\infty \wedge \Omega\Sigma CP^\infty \to \Omega\Sigma CP^\infty.$$

Lemma 2.1. *The following diagram commutes:*

$$\begin{array}{ccc} CP^\infty \wedge CP^\infty & \xrightarrow{\widetilde{H(\mu)}} & \Omega\Sigma CP^\infty \\ {\scriptstyle i\wedge i}\downarrow & & \downarrow{\scriptstyle j} \\ BU\wedge BU & \xrightarrow[\otimes]{} & BU. \end{array}$$

Proof.

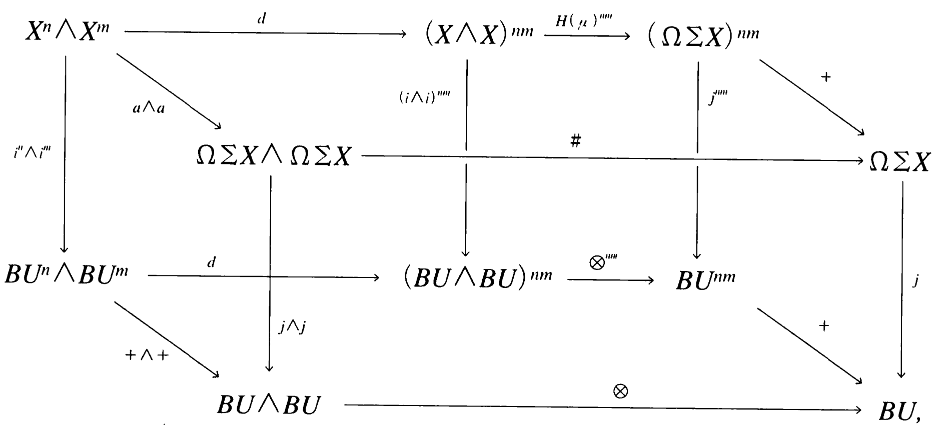
$$\begin{aligned} j\circ\widetilde{H(\mu)}\circ\pi &= j\circ(-\tau\circ p_2 - \iota\circ p_1 + \iota\circ\mu) && \text{by } (2.2) \\ &= -i\circ p_2 - i\circ p_1 + i\circ\mu && \text{since } j \text{ is a loop map} \\ &= -1\otimes\eta + 1\otimes 1 - \eta\otimes 1 + 1\otimes 1 + \eta\otimes\eta - 1\otimes 1 \\ &= (\eta-1)\otimes(\eta-1) \\ &= \otimes\circ(i\wedge i)\circ\pi. \end{aligned}$$

Since π^* is mononorphic, we have the desired result.

Now consider the diagram (1.1). For convenience, put $X = CP^\infty$ and consider the James model X_∞ instead of $\Omega\Sigma X$. Let X_n be the n-th James filtration of X_∞. Consider the Milnor exact sequence:

$$0 \to \varprojlim{}^1 [\Sigma X_n \wedge X_m, BU] \to [X_\infty \wedge X_\infty, BU] \to \varprojlim [X_n \wedge X_m, BU] \to 0.$$

Since $[\Sigma X_n \wedge X_m, BU] = 0$, in order to show the commutativity of the diagram (1.1), it is enough to show that the restriction to $X_n \wedge X_m$ of the diagram (1.1) commutes. To see this, consider the following diagram



where α: $X^n \to \Omega\Sigma X$ is the composite $X^n \xrightarrow{i^n} (\Omega\Sigma X)^n \xrightarrow{+} \Omega\Sigma X$, $+$ means the loop sum. Note that the image of α is X_n. From the bundle theory and the previous lemma, we see that the above diagram commutes (up to homotopy). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2.

(1) is clear, since j: $\Omega\Sigma CP^\infty \to BU$ is a loop map.

(2) follows from Theorem 1.1.

(3) Recall that $[CP^\infty, CP^\infty] \cong H^2(CP^\infty) \cong \mathbf{Z}$. Take any integer $k \in \mathbf{Z}$ and consider the following commutative diagram:

$$\begin{array}{ccc} CP^\infty & \xrightarrow{[k]} & CP^\infty \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ BU & \xrightarrow{\varphi^k} & BU. \end{array}$$

Since φ^k is additive, the above diagram can be extended uniquely to the following commutative diagram:

$$\begin{array}{ccc} \Omega\Sigma CP^\infty & \xrightarrow{\Omega\Sigma[k]} & \Omega\Sigma CP^\infty \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ BU & \xrightarrow{\varphi^k} & BU. \end{array}$$

Thus, (3) follows.

(4) Let $p_1, p_2, \cdots$ be all primes. Put $r_k = (p_1 p_2 \cdots p_k)^k$. Let $Y = \Omega\Sigma CP^\infty$ or BU. For any integer $n \in \mathbf{Z}$, consider the n-fold loop multiplication map n: $Y \to Y$. Then clearly, the following diagram commutes:

$$\begin{array}{ccc} \Omega\Sigma CP^\infty & \xrightarrow{n} & \Omega\Sigma CP^\infty \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ BU & \xrightarrow{n} & BU. \end{array}$$

Consider the telescope of the following sequence:

$$Y \xrightarrow{r_1} Y \xrightarrow{r_2} Y \xrightarrow{r_3} \cdots,$$

then this telescope gives the rational localization of Y [9]. Recall that $(\Omega\Sigma CP^\infty)_{\mathbf{Q}} \cong \Pi_k K(\pi_k(\Omega\Sigma CP^\infty) \otimes \mathbf{Q}, k)$ and $BU_{\mathbf{Q}} \cong K(\pi_k(BU) \otimes \mathbf{Q}, k)$, where $K(\pi, k)$ is the Eilenberg MacLane space. Note that j_*: $(\Omega\Sigma CP^\infty) \otimes \mathbf{Q} \to \pi_*(BU) \otimes \mathbf{Q}$ is split-epi. Thus we get the splitting map between the product of the Eilenberg MacLane spaces and so we get the splitting map of $j_{\mathbf{Q}}$: $(\Omega\Sigma CP^\infty)_{\mathbf{Q}} \to BU_{\mathbf{Q}}$. Since X is a finite complex, $[X, Tel(Y)] = \lim_{r_k}[X, Y]$. This implies that $\lim_{r_k} j_*$: $\lim_{r_k}[X, \Omega\Sigma CP^\infty] \to \lim_{r_k}[X, BU]$ is onto as sets.

Therefore, for any element $\alpha \in K(X) = [X, BU]$, there exists an element $\beta \in [X, \Omega\Sigma CP^\infty]$ such that $(\lim j_*)([\beta]) = [\alpha]$, where $[\alpha]$ means the equivalence class of α in the direct limit. This implies that α is equivalent to $j_*(\beta)$ in the direct limit. Now, the proof of (4) easily follows.

(5) First, from Theorem 1.1, we have the following commutative diagram:

$$\begin{array}{ccccccc}
S^2\wedge\Omega\Sigma CP^\infty & \longrightarrow & CP^\infty\wedge\Omega\Sigma CP^\infty & \longrightarrow & \Omega\Sigma CP^\infty\wedge\Omega\Sigma CP^\infty & \xrightarrow{\#} & \Omega\Sigma CP^\infty \\
\downarrow{\scriptstyle 1\wedge j} & & \downarrow{\scriptstyle i\wedge j} & & \downarrow{\scriptstyle j\wedge j} & & \downarrow{\scriptstyle j} \\
S^2\wedge BU & \longrightarrow & BU\wedge BU & \xrightarrow{=} & BU\wedge BU & \xrightarrow{\otimes} & BU.
\end{array}$$

Taking the adjoint we have

$$\begin{array}{ccc}
\Omega\Sigma CP^\infty & \xrightarrow{f} & \Omega^3\Sigma CP^\infty \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Omega^2 j} \\
BU & \underset{\cong}{\xrightarrow{\beta}} & \Omega^2 BU,
\end{array} \qquad (3.1)$$

where β is the map which represents the Bott periodicity $K(X) \underset{\cong}{\xrightarrow{\beta}} K(\Sigma^2 X)$. By iterating the above diagram we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
CP^\infty & \longrightarrow & \Omega\Sigma CP^\infty & \xrightarrow{f} & \Omega^3\Sigma CP^\infty & \xrightarrow{\Omega^2 f} & \cdots & \xrightarrow{\Omega^{2N-2} f} & \Omega^{2N+1}\Sigma CP^\infty \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \Omega^2 j} & & & & \downarrow{\scriptstyle \Omega^{2N} j} \\
BU & \xrightarrow{=} & BU & \underset{\cong}{\xrightarrow{\beta}} & \Omega^2 BU & \underset{\cong}{\xrightarrow{\Omega^2\beta}} & \cdots & \underset{\cong}{\xrightarrow{\Omega^{2N-2}\beta}} & \Omega^{2N} BU.
\end{array}$$

Thus taking the adjoint we have the next commutative diagram:

$$\begin{array}{ccc}
\Sigma^{2N} CP^\infty & \longrightarrow & \Omega\Sigma CP^\infty \\
{\scriptstyle \beta^N i}\downarrow & \swarrow{\scriptstyle j} & \\
BU. & &
\end{array} \qquad (3.2)$$

Given any element $\alpha \in K(X) \cong [X, BU]$, by Segal-Beker theorem [7], since X is compact, for large N there exists a map $\hat{\alpha}: X \to \Omega^{2N}\Sigma^{2N} CP^\infty$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \Omega^{2N}\Sigma^{2N}CP^{\infty} \\ & \overset{\hat{\alpha}}{\nearrow} & \downarrow j_N \\ X & \xrightarrow{\alpha} & BU, \end{array}$$

where j_N is the canonical extension of $i{:}CP^{\infty}\to BU\xrightarrow{\cong}\Omega^{2N}BU$. Note that $\beta^N\circ j_N=\Omega^{2N}\beta^N i$. Therefore, taking the adjoint we have the commutative diagram:

$$\begin{array}{ccc} & & \Sigma^{2N}CP^{\infty} \\ & \overset{adj.\,\hat{\alpha}}{\nearrow} & \downarrow \beta^N i \\ \Sigma^{2N}X & \xrightarrow{\beta^N\alpha} & BU, \end{array} \tag{3.3}$$

By the diagrams (3.2) and (3.3), we have completed the proof of (5).

(6) Suppose that $\alpha=\sum_i a_i(\alpha_i-1)$, where $a_i\in\mathbf{Z}$ and α_i's are line bundles. Let g_i: $X\to CP^{\infty}$ be the classifying map of the line bundle α_i. Consider the element $\widehat{\alpha}\in[X,\ \Omega\Sigma CP^{\infty}]$ which is defined by $\widehat{\alpha}=\sum_i a_i(\iota\circ g_i)$, where ι: $CP^{\infty}\to\Omega\Sigma CP^{\infty}$ is the inclusion. Then it is clear that $j\circ\widehat{\alpha}=\alpha$.

This completes the proof of Theorem 1.2.

The following proposition gives examples of $PK(X)$.

Proposition 3.1. (1) *for* $n\geq 1$, $K(CP^n)=PK(CP^n)$.

(2) *Let* HP^n *be the quaternionic projective space,* ξ *be the canonical quaternionic line bundle over* HP^n *and* $y=c'(\xi)-2$, *where* c' *is the complexification. If* $n\geq 2$, *then* $y\notin PK(HP^n)$.

Proof. (1) is clear from (6) in Theorem 1.2. We shall prove (2). We use (3) of Theorem 1.4 which is proved in the next section. Recall that $HP^2=S^4\cup_{\nu_4}e^8$, where ν_4: $S^7\to S^4$ is the Hopf bundle. Since the restriction to S^4 of $y\in K(HP^2)$ is the generator of $K(S^4)$, and since $\pi_4(\Omega\Sigma CP^{\infty})\cong\mathbf{Z}$ generated by ζ_2 (See (1.3)), we see that

$$y\in PK(HP^2)\qquad\text{if and only if}\qquad \zeta_2\circ\nu_4=0.$$

On the other hand, using the quasi-fibration [8]

$$CP^{\infty}\to\Sigma CP^{\infty}\wedge CP^{\infty}\xrightarrow{H(\mu)}\Sigma CP^{\infty},$$

we see that

$$\zeta_2\circ\nu_4=0\qquad\text{if and only if}\qquad i\circ\nu_5=0,$$

where i: $S^5\to\Sigma CP^{\infty}\wedge CP^{\infty}$ is the inclusion. Assume that $i\circ\nu_5=0$. Then there exists a map g: $\Sigma HP^2 \to \Sigma CP^\infty \wedge CP^\infty$ such that the following diagram commutes:

$$\begin{array}{ccc} S^5 & = & S^5 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \Sigma HP^2 & \xrightarrow{g} & \Sigma CP^\infty \wedge CP^\infty \end{array}$$

Consider the cohomology group of $\mathbf{Z}/2$ coefficient. Recall that

$$H^*(CP^\infty; \mathbf{Z}/2) = \mathbf{Z}/2[x],$$
$$H^*(HP^2; \mathbf{Z}/2) = \mathbf{Z}/2[u]/(u^3),$$

where $x \in H^2(CP^\infty, \mathbf{Z}/2)$ and $u \in H^4(HP^\infty; \mathbf{Z}/2)$ are the generators. From the above diagram we see that $g^*(xy) = u$. Now consider the Steenrod operation. Recall that $Sq^4(xy) = x^2y^2 = Sq^2(x^2y)$ and $Sq^4(u) = u^2$. By dimensional reason, we get

$$0 = Sq^2(g^*(x^2y)) = g^*(Sq^2(x^2y)) = g^*(x^2y^2) = g^*(Sq^4(xy))$$
$$= Sq^4(g^*(xy)) = Sq^4(u) = u^2 \neq 0.$$

This is a contradiction. This implies that $i \circ \nu_5 \neq 0$.

Remark 3.2. It is known that $\pi_7(\Omega\Sigma CP^\infty) \cong \mathbf{Z}/2$ [6] whose generator is $\xi \circ \eta_6$, where $\xi \in \pi_6(\Omega\Sigma CP^\infty) \cong \mathbf{Z} \oplus \mathbf{Z}$ is characterized by the Hurewicz homomorphism: $h(\xi) = \beta_1\beta_2 - \beta_2\beta_1$ in $H^*(\Omega\Sigma CP^\infty)$. And it holds that $2\xi = \langle \zeta_1, \zeta_2 \rangle$ (Cf. § 5). The above proposition implies that there is a relation $\xi \circ \eta_6 = \zeta_2 \circ \nu_4$ in $\pi_7(\Omega\Sigma CP^\infty)$.

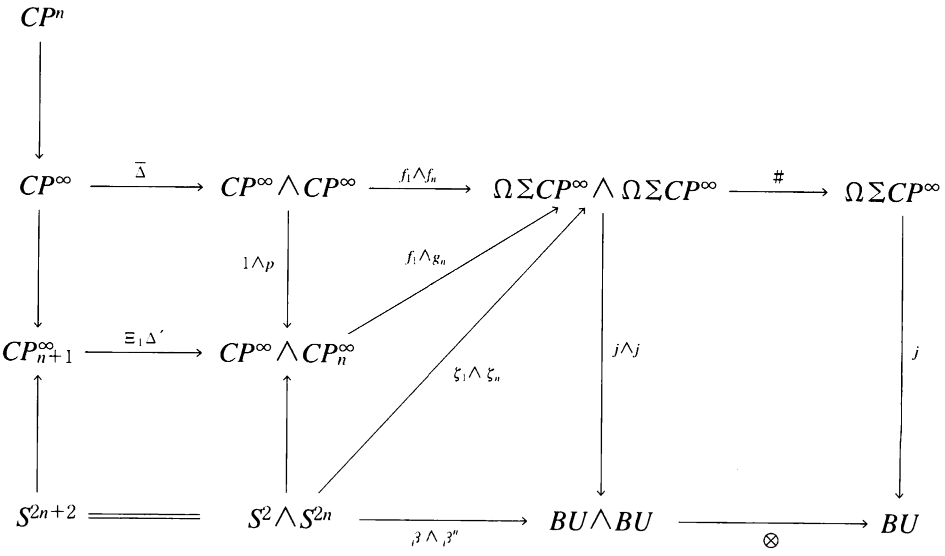
Remark 3.3. Consider the S^2-bundle; $S^2 \to CP^{2n+1} \to HP^n$. There is a stable map t: $\Sigma^\infty HP^n \to \Sigma^\infty CP^{2n+1}$ called the transfer map. The above proposition implies that there exists no unstable map τ: $\Sigma HP^n \to \Sigma CP^{2n+1}$ for $n \geq 2$ such that $\Sigma^\infty \tau = t$. Because, if $\Sigma^\infty \tau =$ t, then it follows that the restriction to the bottom sphere of the adjoint of τ is $\pm \zeta_2$. This implies that $\zeta_2 \circ \nu_4 = 0$.

4. Proof of Theorem 1.4

First we prove (1). By induction on n, the following commutative diagram gives the proof of (1):

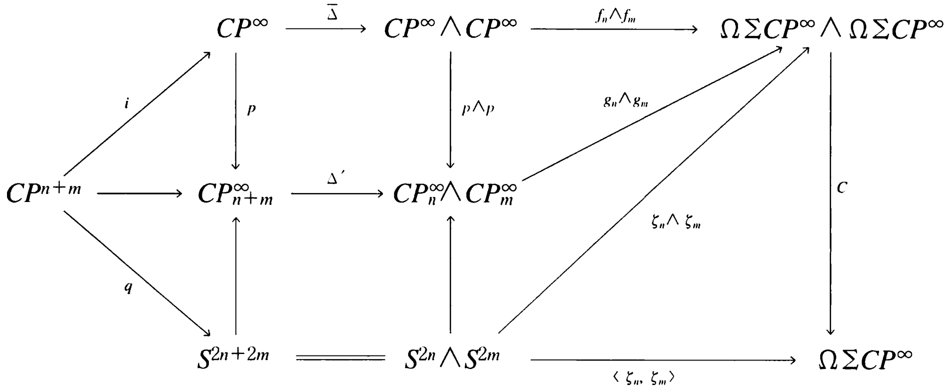
$$\begin{array}{ccccccc} CP^\infty & \xrightarrow{\bar{\Delta}} & CP^\infty \wedge CP^\infty & \xrightarrow{f_1 \wedge f_n} & \Omega\Sigma CP^\infty \wedge \Omega\Sigma CP^\infty & \xrightarrow{\#} & \Omega\Sigma CP^\infty \\ & & & \searrow^{x \wedge x^n} & \downarrow{\scriptstyle j \wedge j} & & \downarrow{\scriptstyle j} \\ & & & & BU \wedge BU & \xrightarrow{\otimes} & BU \end{array}$$

Next we prove (2) and (3). Consider the following diagram:



The commutativity of th above diagram is clear from the inductive assumption. This completes the proof of (2) and (3).

Let $C:\Omega\Sigma CP^{\infty}\wedge\Omega\Sigma CP^{\infty}\to\Omega\Sigma CP^{\infty}$ be the commutator map with respect to the loop sum. The proof of (4) follows by chasing the following commutative diagram:



This completes the proof of Theorem 1.4.

5. Proof of Theorem 1.5

Lemma 5.1. *Let* $\mu:CP^{\infty}\times CP^{\infty}\to CP^{\infty}$ *be the product of* CP^{∞} *and* $\widetilde{H(\mu)}$: $CP^{\infty}\wedge CP^{\infty}\to\Omega\Sigma CP^{\infty}$ *be the adjoint of the Hopf construction* $H(\mu)$: $\Sigma CP^{\infty}\wedge CP^{\infty}\to\Sigma CP^{\infty}$. *Then*

$$\widetilde{H(\mu)}_*(\beta_m\otimes\beta_n)=\sum_{\substack{0\leq i\leq m\\ 0\leq j\leq n}}\binom{i+j}{i}\chi_*(\beta_{n-j})\chi_*(\beta_{m-i})\beta_{i+j} \tag{5.1}$$

where χ: $\Omega\Sigma X\to\Omega\Sigma X$ *is the loop inverse and the products in the right hand side mean the one in the tensor algebra.*

Proof. (2.2) implies that the following diagram commutes:

$$\begin{array}{ccccccc}
X\times X & \xrightarrow{\Delta^3\times\Delta^3} & X^3\times X^3 & \xrightarrow{T} & (X\times X)^3 & \xrightarrow{p_2\times p_1\times\mu} & X^3 \\
 & & & & & & \downarrow{\scriptstyle\iota^3} \\
 & & & & & & (\Omega\Sigma X)^3 \\
\downarrow{\scriptstyle\pi} & & & & & & \downarrow{\scriptstyle\chi\times\chi\times 1} \\
 & & & & & & (\Omega\Sigma X)^3 \\
 & & & & & & \downarrow{\scriptstyle +} \\
X\wedge X & & & \xrightarrow{\widetilde{H(\mu)}} & & & \Omega\Sigma X,
\end{array}$$

where $X=CP^\infty$ and T is the map of changing the order of factors. Now recall the following formula:

$$\Delta_*(\beta_n) = \sum_{0\leq i\leq n}\beta_{n-i}\otimes\beta_i \tag{5.2}$$

$$\mu_*(\beta_i\otimes\beta_j) = \binom{i+j}{i}\beta_{i+j},$$

$$p_{k*}(\beta_i\otimes\beta_j) = \begin{cases}\beta_i & \text{if } k=1 \text{ and } j=0,\\ \beta_j & \text{if } k=2 \text{ and } i=0,\\ 0 & \text{otherwise.}\end{cases}$$

Also recall that $+$ induces just the multiplication in the tensor algebra $\otimes\,{}^*\widetilde{H}_*(CP^\infty)$. Now the proof of Lemma follows easily from the above diagram and (5.2).

Now we shall prove Theorem 1.5. Since $\chi_*(\beta_1)=-\beta_1$, using Lemma 5.1 it is easy to see

$$\begin{aligned}\#_*(\beta_1\otimes\beta_{n-1}) &= \widetilde{H(\mu)}_*(\beta_1\otimes\beta_{n-1})\\ &=\sum_{i=1}^{n}\chi_*(\beta_{n-i})\,(i\beta_i-\beta_1\beta_{i-1})\\ &=n\beta_n+\textit{decomposables}.\end{aligned}$$

On the other hand, since, β_1 is primitive, using (2.1) we see

$$\#_*(\beta_1 \otimes \textit{decomposables}) = 0.$$

Therefore, by induction,

$$\begin{aligned} h(\zeta_n) &= \#_*(\beta_1 \otimes h(\zeta_{n-1})) \\ &= \#_*(\beta_1 \otimes ((n-1)!\beta_{n-1} + \textit{decomposables}) \\ &= (n-1)!\,\#_*(\beta_1 \otimes \beta_{n-1}) \\ &= (n-1)!\sum_{i=1}^{n} \chi_*(\beta_{n-i})\,(i\beta_i - \beta_1\beta_{i-1}). \end{aligned}$$

This completes the proof of Theorem 1.5.

Corollary 5.2. *In $H_*(\Omega\Sigma CP^\infty)$, for $n \geq 2$, the following elements are spherical.*

$$(n-1)!\sum_{i=1}^{n} i\chi_*(\beta_{n-i})\,\beta_i$$

$$(n-1)!\sum_{i=1}^{n} \chi_*(\beta_{n-i})\,\beta_1\beta_{i-1}$$

Proof. Let $\zeta_1' = \zeta_1$. Define $\zeta_n' \in \pi_{2n}(\Omega\Sigma CP^\infty)$ inductively by

$$\zeta_{n+1}' \colon S^{2n+2} = S^{2n} \wedge S^2 \xrightarrow{\zeta_n' \wedge \zeta_1'} \Omega\Sigma CP^\infty \wedge \Omega\Sigma CP^\infty \xrightarrow{\#} \Omega\Sigma CP^\infty. \tag{5.3}$$

Then, by similar arguments, we get

$$h(\zeta_n') = \begin{cases} \beta_1 & \text{if } n=1, \\ (n-1)!\sum_{i=1}^{n} i\chi_*(\beta_{n-i})\,\beta_i & \text{if } n \geq 2 \end{cases} \tag{5.4}$$

Therefore from Theorem 1.5, te result follows.

Proof of Corollary 1.6. By (4) of Theorem 1.4,

$$i^*C(f_1, f_2) = q^* < \zeta_1, \zeta_2 > \qquad \text{in } [CP^3, \Omega\Sigma CP^\infty].$$

On the other hand,

$$\begin{aligned} h(<\zeta_1, \zeta_2>) &= h(\zeta_1)h(\zeta_2) - h(\zeta_2)h(\zeta_1) \\ &= \beta_1(2\beta_2 - \beta_1^2) - (2\beta_2 - \beta_1^2)\beta_1 \qquad \text{by Theorem 1.5} \\ &= 2(\beta_1\beta_2 - \beta_2\beta_1) \neq 0. \end{aligned}$$

This means that $\tilde{f}_1$ and $\bar{f}_2$ does not commutes in $[\Sigma CP^3, \Sigma CP^\infty]$.

Since i^*: $[\Sigma CP^n, \Sigma CP^\infty] \to [\Sigma CP^3, \Sigma CP^\infty]$ is homomorphism of groups and since $[\Sigma CP^n, \Sigma CP^\infty] \cong [\Sigma CP^n, \Sigma CP^n]$, the result follows.

6. Composition of $\{f_i\}$

In this section, we consider not $[CP^\infty, \Omega\Sigma CP^\infty]$ but $[\Sigma CP^\infty, \Sigma CP^\infty]$ to study the composition structures. For convenience, we use the following notation: Let $f, g \in [\Sigma X, \Sigma CP^\infty]$. Define

$$\natural(f, g) \text{ as the adjoint of } \# \circ (\tilde{f} \wedge \tilde{g}) \circ \bar{\Delta},$$

where $\tilde{f}$ and $\tilde{g}$ is the adjoint of f and g, respectively. Then from the construction of $\#$, it holds (Cf. §2)

$$\natural(f, g+h) = \natural(f, g) + \natural(f, h) \qquad \text{for } f = \Sigma f',\ g, h \in [\Sigma X, \Sigma CP^\infty]. \tag{6.1}$$

$$\natural(f+g, h) = \natural(f, g) + \natural(f, h) \qquad \text{for any } f, g, h \in [\Sigma X, \Sigma CP^\infty].$$

Throughout this section we write the adjoint of f_i by the same letter. Under this convention, we restate Theorem 1.7.

Theorem 6.1. *The composition $f_i \circ f_j$ can be written as a linear combination f_n's for $n \leq ij$.*

The proof is divided into three parts.

(1) f_n can be written as a linear combination of $\Sigma[i]$'s for $i \leq n$,

(2) $\Sigma[n]$ can be written as a linear combination of f_i's for $i \leq n$,

(3) the composite of linear combinations of $\Sigma[i]$'s can be also written as ones of $\Sigma[i]$'s,

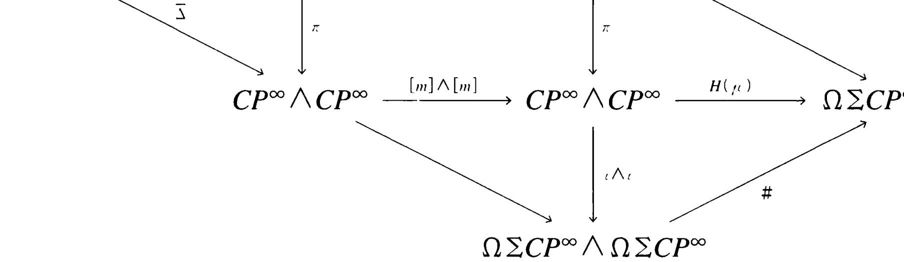
where $[k]: CP^\infty \to CP^\infty$ is the corresponding map to $k \in \mathbf{Z} \cong H^2(CP^\infty)$.

(3) is standard [10]. Before giving the proof of (1) and (2), we need

Lemma 6.2.

$$\natural(\Sigma[m], \Sigma[n]) = -\Sigma[n] - \Sigma[m] + \Sigma[m+n], \tag{6.2}$$

Proof. The following diagram commutes from (2.2) and the definition of $\#$.



Thus, taking the adjoint we have the desired result.

We prove (1) and (2) by induction. Suppose that

$$f_n=\left\{\sum a_i\Sigma[k_i]\right\}+\Sigma[n],$$

where $a_i\in\mathbf{Z}$ and $k_i\leq n-1$. Then, by definition,

$$\begin{aligned}
f_{n+1}&=\natural\,\Sigma([1],f_n)\\
&=\natural\,(\Sigma[1],\left\{\sum a_i\Sigma[k_i]\right\}+\Sigma[n])\\
&=\left\{\sum a_i\,\natural\,(\Sigma[1],\Sigma[k_i])\right\}+\natural\,(\Sigma[1],\Sigma[n]) && \text{by (6.1)}\\
&=\left\{\sum a_i(-\Sigma[k_i]-\Sigma[1]+\Sigma[k_i+1])\right\}\\
&\qquad-\Sigma[n]-\Sigma[1]+\Sigma[n+1] && \text{by lemma 6.2}\\
&=\left\{\sum b_j\Sigma[l_j]\right\}+\Sigma[n+1].
\end{aligned}$$

This proves (1). The proof of (2) follows easily.

This completes the proof of Theorem 6.1.

Example 6.3.

$$\begin{aligned}
f_1&=\Sigma[1]\\
f_2&=-2\Sigma[1]+\Sigma[2]\\
f_3&=-2(-2\Sigma[1]+\Sigma[2])-\Sigma[2]-\Sigma[1]+\Sigma[3]\\
\Sigma[2]&=2f_1+f_2\\
\Sigma[3]&=3f_1+3f_2+f_3
\end{aligned}$$

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