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On explicit constructions of rational elliptic surfaces with multiple fibers

By

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This note is a supplement to our previous paper [F], where I studied the basic property of rational elliptic surfaces with multiple fibers S through the logarithmic transformations. It is also a nine-points-blowing-up of $\mathbf{P}^2$, but any (-1)-curve is a multi-section of its elliptic fibering over $\mathbf{P}^1$. If the nine points P_i $(1 \le i \le 9)$ on $\mathbf{P}^2$, which are the center of blowing-ups, are mutually distinct and the multiple fiber is of type ${}_mI_0$, it is obtained from the pencil generated by m-fold cubic which passes through p_i's and an irreducible curve of degree $3m$ which has an ordinary singularity of multiplicity m at each p_i and is non-singular outside them. And the anti-pluricanonical map $\Phi_{|-mK_S|} : S \to \mathbf{P}^1$ gives the unique structure of an elliptic fibration. Such a pencil (called Halphen pencil) already appeared in [Nag], §4, Theorem (1), case (へ), when Nagata constructed a rational surface with infinitely many (-1)-curves. Also Hironaka and Matsumura [H-M] applied it to construct examples of a curve C in a smooth projective surface F, where C satisfies $G1$ conditions in F, but not $G2$ conditions. On the other hand, when part of the nine points p_i's on $\mathbf{P}^2$ are infinitely near, the Halphen pencil degenerates into a more complicated one. Any (-1)-curve e on S is an m-sheeted covering of the base curve $\mathbf{P}^1$, branching over the point where the multiple fiber lie with the ramification index m. Hence, it is not at all easy to find nine (-1)-curves on S, see how they intersect the irreducible components of each singular fiber and repeat blowing-downs to $\mathbf{P}^2$.

Here, we shall describe an *explicit* construction of rational elliptic surfaces with multiple fibers through the 'Halphen transform' in the sense of [H-L], which is some kind of *birational transformations.*

We recall the following result.

Theorem (A) ([F], [H-L]). *Let C be a non-singular cubic (resp. a nodal cubic) in $\mathbf{P}^2$ with the fixed inflexion point Q on C such that C should be given the natural group structure with Q as the identity. Take nine points p_i $(1 \le i \le 9)$ on C (which may be infinitely near) and let S be the surface obtained by blowing up $\mathbf{P}^2$ at p_i's $(1 \le i \le 9)$. Then S has the structure of an elliptic surface with one multiple fiber of multiplicity m if and only if $\sum_{i=1}^{9} p_i$ is of order m in the elliptic curve (resp.*

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multiplicative group $\mathbf{C}^*$*), where* $\pm$ *means the additive (resp. multiplicative) group law in* C.

Now, we shall apply this theorem to the following situation. Let $f : S \to \mathbf{P}^1$ be a rational elliptic surface with ${}_mI_0$ (resp. ${}_mI_1$)-type multiple fiber mE, where $m \geq 1$ and E is a non-singular elliptic curve (resp. a rational curve with one node). By Kodaira, $D := N_{E/S} \in \mathrm{Pic}^0(S)$ is of finite order m. Suppose that there exist mutually distinct (-1)-curves $e_1, \dots, e_{2t}$ $(1 \leq t \leq 4)$, such that $\Delta := \sum_{i=1}^{t} p_i - \sum_{j=t+1}^{2t} p_j \in \mathrm{Pic}^0(E) \simeq E$ (resp. $\mathbf{C}^*$) is of finite order, where $p_i := e_i \cap E_i$. Then $D + \Delta \in \mathrm{Pic}^0(E) \simeq E$ (resp. $\mathbf{C}^*$) is of finite order l (≥ 1).

Proposition. *Under the above assumption, if we blow down* (-1)*-curves* $e_{t+1}, \dots, e_{2t}$ *on* S *and blow up at* t *points* $p_1, \dots, p_t$, *we obtain a new surface* X. *Then* X *has the structure of an elliptic surface* $g : X \to \mathbf{P}^1$ *with* ${}_lI_0$*-type multiple fiber* lE', *where* E' *is the strict transform of* E.

Remark (1). This birational transformation is some kind of Halphen transform in the sense of [H–L]. Under the above transformation, the general fiber of f is mapped to the g-horizontal curve, the strict transform $\bar{e}_i$ $(1 \leq i \leq t)$ of e_i are contained in the fibers of g and the types of singular fibers of X are quite different from that of the original S.

Proof. Let $h : S \to S_1 \to \cdots \to S_t$ be a succession of blowing-downs of (-1)-curves $e_{t+1}, \dots, e_{2t}$ on S. By [F], Proposition (1.1), the relatively minimal model of S is isomorphic to either $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ or $\sum_2$. Since $C_1(S_t)^2 = t \leq 4$, S_t is not relatively minimal. By the same method as in the proof of [F] (ibid), we can further contract $(9-t)$ (-1)-curves $\Delta_1, \dots, \Delta_{9-t}$ on S_t to get $\mathbf{P}^2$. Let $\mu : S_t \to \mathbf{P}^2$ be a contraction morphism and put $Q_j := \mu(\Delta_j)$ $(1 \leq j \leq 9-t)$, $P_i' := \mu \circ h(P_i)$ $(1 \leq i \leq 9)$, $\bar{E} := \mu \circ h(E)$. From the construction, X (resp. S) can be obtained by blowing up $\mathbf{P}^2$ at nine points $Q_1, \dots, Q_{9-t}$ and $P_{t+1}', \dots, P_{2t}'$ (resp. $P_1', \dots, P_t'$). Under the canonical identification $E \simeq \bar{E} \simeq E'$, we have $[Q_1 + \cdots + Q_{9-t} + P_{t+1}' + \cdots + P_{2t}' - 9o] \simeq D + \Delta$ in $\mathrm{Pic}^0(\bar{E}) \simeq \bar{E}$ (resp. $\mathbf{C}^*$), where o is an inflexion point of $\bar{E}$. Hence by Theorem (A), $g := \Phi_{|lE'|} : X \to \mathbf{P}^1$ gives the unique structure of an elliptic surface with ${}_lI_0$ (resp. ${}_lI_1$)-type multiple fiber lE'.

In this note, we are mainly concerned with the simplest case where S is a rational elliptic surface with many *torsion sections*. To be more precise, we treat the case where S is a rational elliptic surface with sections, $m = 1$, $t = 1$ and $e_1 - e_2$ is a torsion section of order l in the above situation. Then X has the structure of an elliptic surface with ${}_lI_0$ (resp. ${}_lI_1$)-type multiple fiber lE'.

Note that two torsion sections never intersect.

Remark (2). With the method of torsion sections, one cannot obtain any multiple fiber ${}_mI_0$ $(m \geq 7)$ by Miranda and Persson's results (Persson: Math.Z.205, 1–47 (1990) and Miranda: Math.Z.205, 191–211 (1990), by which we know the list of configuration of singular fibers as well as torsions of the Mordell-Weil groups.

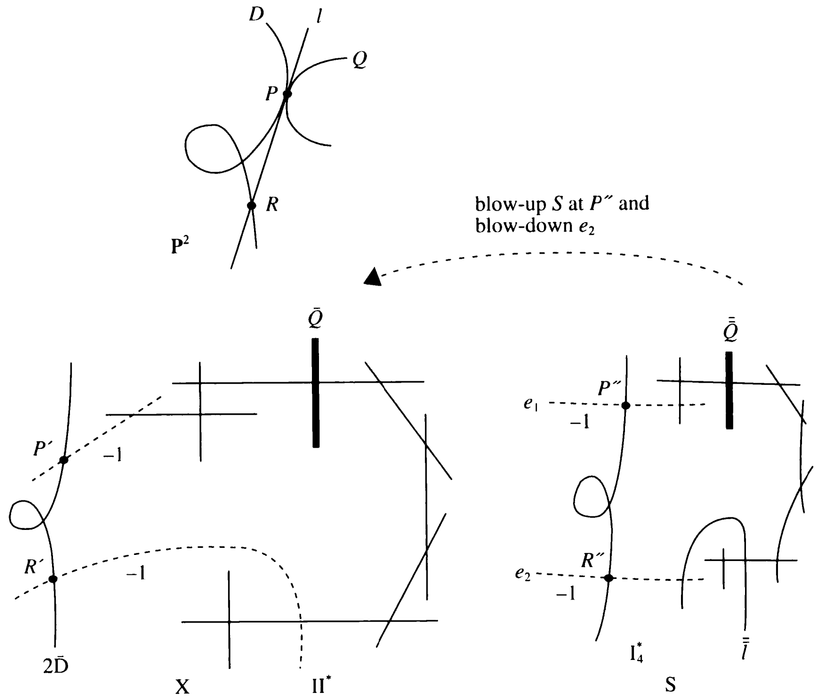

Fig. 1

The author wishes to express sincere thanks to the referee for many suggestions.

Example 1. *Step* 1. Take a nodal cubic D on $\mathbf{P}^2 : (X : Y : Z)$ and fix an inflexion point P_0 on D. The regular locus $D^* := D \backslash \{\text{node}\}$ should be given the natural group structure $\mathbf{C}^*$ with P_0 as the identity. Fix such an isomorphism $\varepsilon : D^* \simeq \mathbf{C}^*$.

Now, take the point P with $\varepsilon(P) = t$, where $t^3 = -1$. (Note that P is not an inflexion point of D.) Then there exists a smooth conic $\mathbf{Q}$ which is tangent to D at P with the full multiplicity 6. The pencil $\mathbf{L}$ generated by $2D$ and $3\mathbf{Q}$ has the unique base point P of multiplicity 36 and the generic member of $\mathbf{L}$ has infinitely near 9-fold double point. By blowing up $\mathbf{P}^2$ 9 times over P, all members of $\mathbf{L}$ can be seperated and the rational map $\Phi_{|\mathbf{L}|} : \mathbf{P}^2 \cdots \rightarrow \mathbf{P}^1$ extends to a morphism $f : X \rightarrow \mathbf{P}^1$ which gives X the structure of an elliptic surface with one multiple fiber of type ${}_2I_1$, II^* and two I_1-singular fibers. By this process, we obtain eight (-2)-curves, which form an A_8-configuration, and one (-1)-curve. See the figure 1.

The strict transform of $\bar{D}$ (resp. $\bar{\mathbf{Q}}$) of D (resp. $\mathbf{Q}$) is the support of the double fiber (resp. the irreducible component of the II^*-singular fiber with multiplicity 3.)

Step 2. Take a tangent line l at P, which intersects D transversally at another point R. For the three intersection points of D and a line, their product in the group $\mathbf{C}^*$ is equal to the identity. So we have $\varepsilon(R) = -t$ and R is another inflexion point of D.

Let $\mathbf{L}'$ be the pencil generated by the two cubics D and $\mathbf{Q} + l$. The rational elliptic surface S with sections can be obtained by seperating the member of $\mathbf{L}'$ by blowing up 8 times over P and once over R. S has two I_1 singular fibers and one I_4^*-singular fiber and the Mordell-Weil group of S is isomorphic to $\mathbf{Z}/_{2\mathbf{Z}}$ and consists of e_1 and e_2.

Step 3. If we blow up S at P'' and blow down the (-1)-curve e_2, we obtain the surface X in *Step* 1.

Remark (3). Let Y be a rational elliptic surface obtained from the pencil of the cubics [D, $3l_0$], where l_0 is the inflexion line of D at R. Y has one II^*-singular fiber and two I_1-singular fibers· and the Mordell-Weil group of Y is trivial. (See [N], 2.1.) If we perform logarithmic transformations of multiplicity two at one point on the base curve $\mathbf{P}^1$, we obtain X.

Example 2. Let S be the minimal resolution of the quotient of $\mathbf{P}^1 \times E$ under the involution $(t, \zeta) \to (-t, -\zeta)$, where E is a non-singular elliptic curve. The natural projection $\mathbf{P}^1 \times E \to \mathbf{P}^1$ induces on S the structure of an elliptic surface over $\mathbf{P}^1$ with two I_0^*-singular fibers. The 2-torsion points of E induces on S four sections e_i $(1 \leq i \leq 4)$. S is rational and the Mordell-Weil group of S is isomorphic to $\mathbf{Z}/_{2\mathbf{Z}} \oplus \mathbf{Z}/_{2\mathbf{Z}}$ and consists of e_i's. Let f be an arbitrary regular fiber and put $P_i' := f \cap e_i$. By blowing up S at P_2' and blowing down e_1, we obtain a new surface X. By the morphism $\Phi_{|2\bar{\mathbf{f}}|} : S \to \mathbf{P}^1$, S is an elliptic surface over $\mathbf{P}^1$ which has $2\bar{\mathbf{f}}$ as ${}_2I_0$-muptiple fiber and I_4^*-singular fiber. (See thick curves which do not intersect $\bar{\mathbf{f}}$.) The image of $\bar{\mathbf{f}}$ by the blowing-down to $\mathbf{P}^2$ is a non-singular cubic E in $\mathbf{P}^2$, P_1 is an inflexion point of E and P_2, P_3, P_4 are two torsion points of E. G_1 (resp. H_j) is the tangent line of E at P_1 (resp. P_j), three lines H_j $(1 \leq j \leq 3)$ intersect at one point P_1 and P_2, P_3, P_4 are on the same line Δ_2. (E can be endowed with the natural group structure with P_1 as the identity.)

Let $\mathbf{L}$ (resp. $\mathbf{L}'$) be the pencil of cubic (resp. sextic) curves in $\mathbf{P}^2$ generated by $H_2 + H_3 + H_4$ and $2\Delta_2 + G_1$ (resp. $2H_2 + H_3 + H_4 + 2\Delta_2$ and $2E$).

S (resp. X) can be obtained by blowing up $\mathbf{P}^2$ nine times until all the members of the pencil will be seperated.

Example 3. Let E be a non-singular elliptic curve with the period $(1, \tau)$, $\mathrm{Im}(\tau) > 0$ and consider a finite automorphism group G of $\mathbf{P}^1 \times E$ generated by $f : (s, [\zeta]) \to (-s, [\zeta + 1/2])$ and $g : (s, [\zeta]) \to (1/s, [-\zeta])$. Let Y be the minimal resolution of the quotient $Z := \mathbf{P}^1 \times E/_G$. The natural projection $\mathbf{P}^1 \times E \to \mathbf{P}^1$ gives rise to an elliptic fibration $Y \to \mathbf{P}^1/_G \simeq \mathbf{P}^1$.

On $\mathbf{P}^1 \times E$, the curve $0 \times E$ (resp. $1 \times E$) is mapped to $\infty \times E$ (resp. $-1 \times E$) by g (resp. f) and they give a support F of a mutiple fiber of type ${}_2I_0$ (resp. an irreducible curve $l_1 \simeq \mathbf{P}^1$) on Y. On $\mathbf{P}^1 \times E$, $(1, 0), (1, 1/2), (1, \tau/2), (1, 1/2 + \tau/2)$

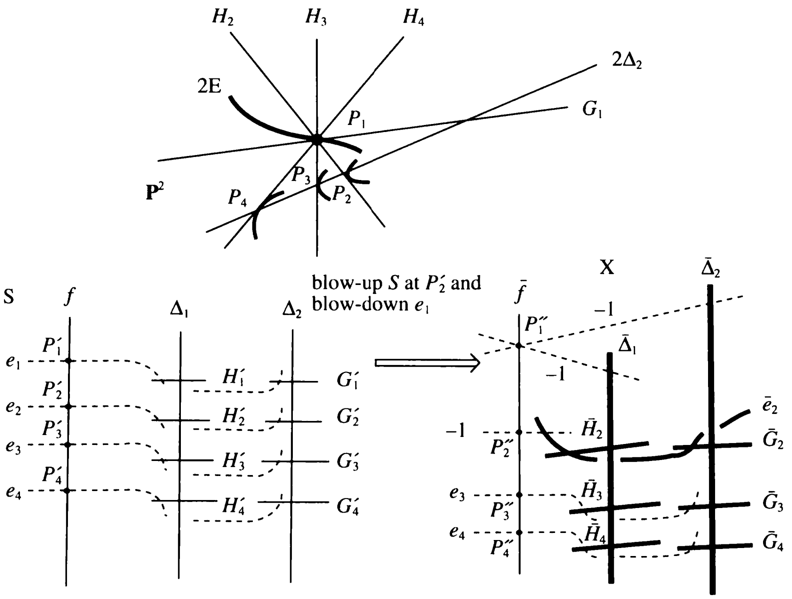

Fig. 2

$\in 1 \times E$, which are the fixed points by g, are mapped respectively to $\imath(-1, 1/2)$, $(-1,0)$, $(-1/,\tau/2)$, $(-1,\tau/2) \in -1 \times E$, which are also the fixed points of g. They give four A_1-singularity on Z and let G_1, G_2, G_3 and G_4 be the exceptional (-2)-curves of the resolution $Y \to Z$. Then $2l_1 + G_1 + G_2 + G_3 + G_4$ form a I_0^*-type singular fiber on Y.

The four disjoint sections on $\mathbf{P}^1 \times E$ defined by $[\zeta] = 0$, $\tau/2$, $1/4$, $1/4 + \tau/2$ are mapped to the four disjoint 2-sections e_j $(1 \le j \le 4)$, which are all (-1) curves.

Proof. Since $e_1 \simeq \mathbf{P}^1$ intersects transversally at one point with G_1 and G_2, we have $(e_1, 2F) = 2$. By the canonical bundle formula of Kodaira, we have $K_Y \sim -F$ and hence $(K_Y, e_1) = -1$. The rest are the same.

Another configurations of double sections and I_0^*-singular fibers are in Figure 3. If we consider each e_j as the double cover over the base curve $\mathbf{P}^1$, e_1 and e_2 (resp. e_3, e_4) are branched over Q_1 and Q_2 (resp. Q_1 and Q_3). Let $\tilde{\mathbf{P}}^1$ (resp. $\tilde{\tilde{\mathbf{P}}}^1$) be a double covering of $\mathbf{P}^1$ branched at Q_1, Q_3 (resp. Q_1, Q_2). If we take the normalization of the pull-back $Y \times_{\mathbf{P}^1} \tilde{\mathbf{P}}^1$ (resp. $Y \times_{\mathbf{P}^1} \tilde{\tilde{\mathbf{P}}}^1$) and blow down (-1)-curves in fibers, we obtain a rational elliptic surface S_1 (resp. S_2) with sections which are isomorphic to S in Example 2. Y and S_i are isogenous (i.e. there exists a finite rational map of degree two between X and S_i) and the four sections on S_1 (resp. S_2) are mapped to e_1, e_2 (resp. e_3, e_4) on Y under the quotient map.

By contracting eight (-1)-curves $e_1, G_1, e_2, G_3, e_3, e_4, H_1$ and H_3 in order, Y can be blown down to $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\mu : Y \to \mathbf{P}^1 \times \mathbf{P}^1$ be the contraction mor-

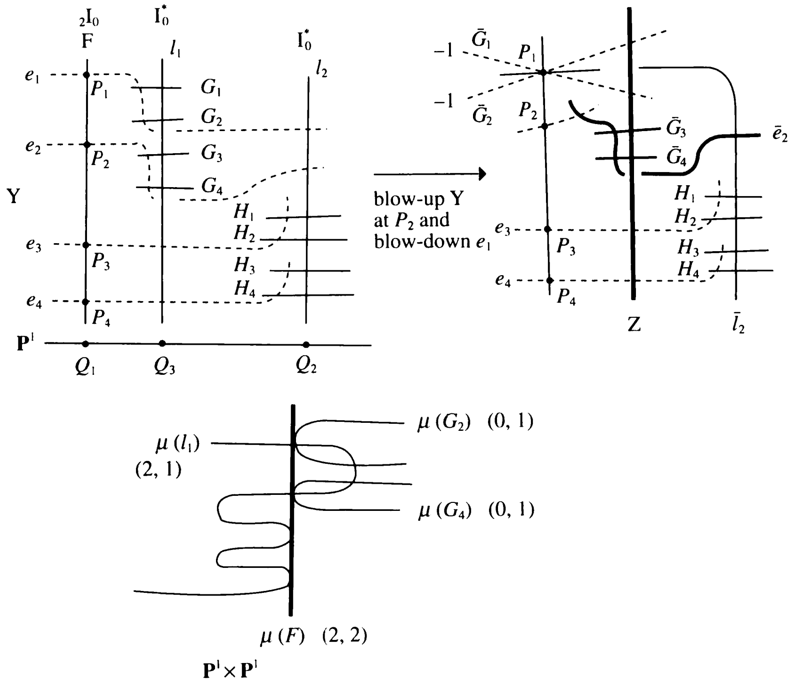

Fig. 3

phism. F (resp. l_1, G_2, G_4) are mapped to $(2,2)$ (resp. $(2,1)$, $(0,1)$, $(0,1)$) curves on $\mathbf{P}^1 \times \mathbf{P}^1$ (where $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1) \simeq \mathbf{Z} \oplus \mathbf{Z}$).

If we blow up Y at P_2 (resp. P_3 and P_4) and blow down e_1 (resp. e_1 and e_2), we obtain a new rational elliptic surface Z (resp. W) with sections and I_4 (resp. I_2^* and ${}_2I_0$-multiple) singular fiber. Z (resp. W) can be obtained as the minimal resolution of the pencil

$$[\mu(F), \mu(l_1) + \mu(G_4)]$$

(resp. $[2\mu(F), 2\mu(l_1) + \mu(G_2) + \mu(G_4)]$) on $\mathbf{P}^1 \times \mathbf{P}^1$.

Example 4. Take a non singular cubic E and 2 inflexion points P_1, P_3 on E. Let C_3 be the unique line passing through P_1 and P_3. C_3 intersects E transversally at another point (say, P_2). Let C_7 (resp. C_8) be the tangent line at P_3 (resp. P_1). Let $\mathbf{L}$ be the pencil generated by two cubics E and $C_3 + C_7 + C_8$. The rational elliptic surface S with sections can be obtained by blowing up $\mathbf{P}^2$ four times over P_1, P_3 and once at P_2 until all members of $\mathbf{L}$ will be seperated. S has three I_1-singular fibers and one I_9-singular fiber. The Mordell-Weil group of S is isomorphic to $\mathbf{Z}/_3$ and e_i $(1 \leq i \leq 3)$ are all the sections.

If we blow up S at P_2 and blow down e_1, we obtain a new surface X and $\Phi_{|3\bar{E}|} : X \to \mathbf{P}^1$ gives a new elliptic fibration with $3\bar{E}$ as a ${}_3I_0$-multiple fiber, one II^*-singular fiber and three I_1-singular fibers. X can be obtained from the pencil $\mathbf{L}$

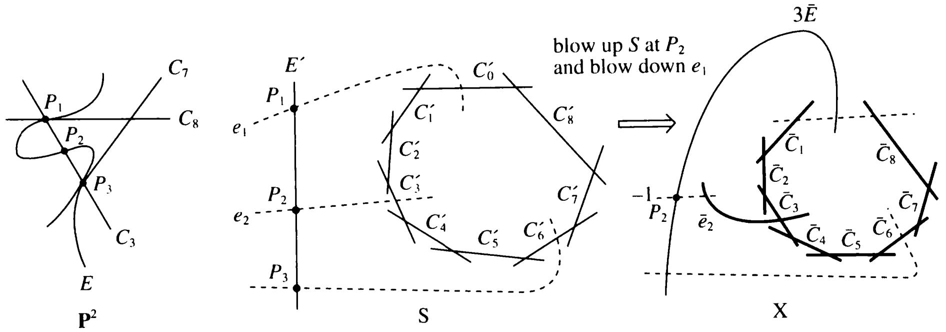

Fig. 4

generated by $3E$ and $6C_3 + 2C_7 + C_8$). The strict transform of C_3 (resp. C_7, C_8) is the irreducible component of the II^*-singular fiber with multiplicity 6 (resp. 2, 1.)

Remark (4). S can be endowed with the unique group scheme structure over $\mathbf{P}^1$ with the zero section e_1. Let G be the automorphism group of S generated by the translations (in the group law of the fibers) by the torsion section e_2. The only fixed points of G are the nodes of three I_1-singular fibers. The quotient space $S/_G$ has three A_2-singularity and the minimal resolution W of $S/_G$ is isomorphic to the elliptic modular surface $B_{\Gamma(3)}$ of level three structure in the sense of [S]. By the same way, it is easy to see that S and $B_{\Gamma(3)}$ are isogenous, i.e. there exist finite rational maps of degree three between S and $B_{\Gamma(3)}$.

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