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Some categories of lattices associated to a central idempotent

By

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0. Let R be a noetherian integral domain with field of quotients K. An *R-lattice* is a finitely generated torsion free R-module. An *R-order* is an R-algebra Λ which is an R-lattice. For an R-order Λ, a *Λ-lattice* is a left Λ-module which is an R-lattice. Let lat Λ denote the category of Λ-lattices.

Let e be a central idempotent of the K-algebra $\tilde{\Lambda} := K \otimes_R \Lambda$, so that $e\Lambda$ is an R-order in the K-algebra $e\tilde{\Lambda}$. The category lat $e\Lambda$ can be viewed as a full subcategory of lat Λ via the ring homomorphism $\Lambda \to e\Lambda$, $(\lambda \mapsto e\lambda)$.

0.0. A purpose of this paper is to investigate the *quotient category* $\mathscr{C} := \operatorname{lat}\Lambda/\operatorname{lat}e\Lambda$. By definition, $\mathscr{C}$ has the same objects as $\operatorname{lat}\Lambda$, and $\operatorname{Hom}_{\mathscr{C}}(X, Y) = \operatorname{Hom}_{\Lambda}(X, Y)/I(X, Y)$, where $I(X, Y)$ is the totality of Λ-morphisms $f : X \to Y$ which factor through some object of lat $e\Lambda$. By 2.1.1, $\operatorname{Hom}_{\mathscr{C}}(X, Y) = (1-e)\operatorname{Hom}_{\Lambda}(X, Y)$ holds.

Let $\mathscr{P}$ be the full subcategory of $\mathscr{C}$ formed by $X \in \mathscr{C}$ satisfying the following condition (*).

(*) There exist a projective Λ-lattice P, $e\Lambda$-lattice Ω and an exact sequence $0 \to \Omega \to P \to X \to 0$ in $\operatorname{lat}\Lambda$.

0.1. Theorem (Proof in 2.5). *Assume that $\mathscr{P}$ has an additive generator Q (i.e. any object in $\mathscr{P}$ is isomorphic to a direct summand of $Q^n = Q \oplus \cdots \oplus Q$ for some n). Put $\Gamma := \operatorname{Hom}_{\mathscr{C}}(Q, Q)$, $FX := \operatorname{Hom}_{\mathscr{C}}(Q, X)$ for $X \in \operatorname{lat}\Lambda$. Then Γ is an R-order and F induces a categorical equivalence from $\mathscr{C} = \operatorname{lat}\Lambda/\operatorname{lat}e\Lambda$ to $\operatorname{lat}\Gamma$.*

0.2. Assume that R is a complete discrete valuation ring. Then $\operatorname{lat}\Lambda$ is a Krull-Schmidt category, and any $X \in \operatorname{lat}\Lambda$ has a projective cover $0 \to \Omega(X) \to P(X) \to X \to 0$. In this case, the above $\mathscr{P}$ can be described as $\{X \in \mathscr{C} \mid \Omega(X) \in \operatorname{lat}e\Lambda\}$.

Let $\operatorname{ind}\Lambda$ denote the set of isomorphism classes of indecomposable Λ-lattices and put

$$\mathscr{Q} := \{X \in \operatorname{ind}\Lambda - \operatorname{ind}e\Lambda \mid \Omega(X) \in \operatorname{lat}e\Lambda\}.$$

If $\mathscr{Q}$ is a finite set, by the additivity of projective cover, $\mathscr{P}$ has an additive generator $Q = \bigoplus_{X \in \mathscr{Q}} X$.

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0.2.0. Further assume that $\tilde{\Lambda}$ is a semi-simple K-algebra. Then lat Λ has almost split sequences and the Auslander translation τ. Since $K\otimes_R \Omega(X) \simeq K\otimes_R \tau X$ as $\tilde{\Lambda}$-module, the above $\mathcal{Q}$ can be described as

$$\mathcal{Q} = \{X \in \operatorname{ind}\Lambda - \operatorname{ind} e\Lambda \mid \tau X \in \operatorname{ind} e\Lambda \cup \{0\}\}$$
$$= (\tau^{-1}(\operatorname{ind} e\Lambda) \cup \operatorname{proj}\Lambda) - \operatorname{ind} e\Lambda.$$

Here, $\operatorname{proj}\Lambda$ is the set of projective lattices in $\operatorname{ind}\Lambda$. Hence we have a simple sufficient condition for the validity of 0.1.

0.2.1. If $\operatorname{ind} e\Lambda$ is finite, then $\mathcal{P}$ of 0.1 has an additive generator.

0.2.2. Recall that an R-order Δ is called an *Auslander order* (resp. *generalized Auslander order*) if it satisfies the following conditions (i), (ii) and (iii) (resp. (ii) and (iii)):

(i) $\tilde{\Delta}$ is semi-simple.

(ii) $\operatorname{gl.dim}\Delta \leq 2$.

(iii) For a minimal projective resolution $0 \to P_1 \to P_0 \to \operatorname{Hom}_R(\Delta, R) \to 0$, P_0 is an injective lattice.

By [AR], an R-order Δ is an Auslander order if and only if there exists an R-order Λ such that lat Λ has an additive generator L (i.e. Λ is of finite representation type) and $\Delta \simeq \operatorname{Hom}_\Lambda(L, L)$ as R-algebra.

0.2.3. Corollary of 0.1. *Let Δ be an Auslander order and ε be a central idempotent of $\tilde{\Delta} = K\otimes_R \Delta$. Then $\varepsilon\Delta$ is an Auslander order.*

Proof. Since Λ is of finite representation type, $\tilde{\Lambda}$ is necessarily semi-simple. Hence $\tilde{\Delta}$ is Morita equivalent with $\tilde{\Lambda}$, and a central idempotent of $\tilde{\Delta}$ can be naturally identified with a central idempotent of $\tilde{\Lambda}$. By this identification, put $e := 1 - \varepsilon \in \tilde{\Lambda}$. Then $\varepsilon\Delta = (1-e)\operatorname{Hom}_\Lambda(L, L) = \operatorname{Hom}_{\mathcal{C}}(L, L)$ with $\mathcal{C} = \operatorname{lat}\Lambda/\operatorname{lat} e\Lambda$.

By 0.1, there is an R-order Γ such that $\mathcal{C} \simeq \operatorname{lat}\Gamma$. Since L is an additive generator of lat Λ, L is also an additive generator of $\mathcal{C}$, hence of lat Γ, and we have $\varepsilon\Delta = \operatorname{Hom}_{\mathcal{C}}(L, L) \simeq \operatorname{Hom}_\Gamma(L, L)$, showing that $\varepsilon\Delta$ is an Auslander order.

0.3. Assume that R is a complete discrete valuation ring and $\tilde{\Lambda}$ is a semi-simple K-algebra. Let e be a central idempotent of $\tilde{\Lambda}$ such that $\mathcal{Q} := \{X \in \operatorname{ind}\Lambda - \operatorname{ind} e\Lambda \mid \tau X = \operatorname{ind} e\Lambda \cup \{0\}\}$ is a finite set, say $\operatorname{ind} e\Lambda$ is a finite set.

Let Γ be an R-order associated to (Λ, e) by Theorem 0.1, $\operatorname{lat}\Gamma \simeq \operatorname{lat}\Lambda/\operatorname{lat} e\Lambda$.

Then the Auslander-Reiten quiver $\mathfrak{A}(\Gamma)$ of Γ can be described from $\mathfrak{A}(\Lambda)$ by a very simple way (Proposition 3.1).

If ind $e\Lambda$ is small, then $\mathfrak{A}(\Gamma)$ is not much different from $\mathfrak{A}(\Lambda)$. For example, $\mathfrak{A}(\Lambda)$ of some Bäckström order Λ in a non-connected $\tilde{\Lambda}$ is very similar to $\mathfrak{A}(\Gamma)$ of some tiled order Γ in a simple $\tilde{\Gamma}$. This is, in fact, the first motivation of this study—to explain the reason why apparently very different orders have similar Auslander-Reiten quivers.

Several examples of such (Λ, e, Γ)'s will be given in §3.

0.4. Let R be a complete discrete valuation ring. For R-orders Λ and Λ', according to [DK], we say that Λ' is an *over ring* (resp. *over order*) of Λ if there is an R-algebra homomorphism $\phi : \Lambda \to \Lambda'$ such that $\tilde{\phi} : \tilde{\Lambda} \to \tilde{\Lambda}'$ is surjective (resp. bijective). If Λ' is an over ring of Λ, we can naturally consider as $\operatorname{lat}\Lambda' \subseteq \operatorname{lat}\Lambda$, $\operatorname{ind}\Lambda' \subseteq \operatorname{ind}\Lambda$.

Therefore it is a basic problem to characterize a subcategory $\mathscr{C}'$ of $\mathscr{C} := \operatorname{ind}\Lambda$ (or equivalently a subset $\mathscr{S}$ of $\operatorname{ind}\Lambda$) which has the form $\mathscr{C}' = \operatorname{ind}\Lambda'$ (resp. $\mathscr{S} = \operatorname{ind}\Lambda - \operatorname{ind}\Lambda'$) by some over ring Λ'. We call the problem as Rejection Lemma since it is a (wide) generalization of the Rejection Lemma of Drozd-Kirichenko ([DK]), which gives a solution when $\mathscr{S}$ is a singleton set.

0.4.1. Assume that $\tilde{\Lambda} = K \otimes_R \Lambda$ is semi-simple. Then general cases can be reduced to the following two fundamental cases.

(a0) Λ' is an over order of Λ.

(a1) $\Lambda' = e\Lambda$ by some central idempotent e of $\tilde{\Lambda}$.

Rejection Lemma for the case (a0) is given in our previous paper [I], where $\mathscr{S}$ is called a rejectable subset if it has the form $\mathscr{S} = \operatorname{ind}\Lambda - \operatorname{ind}\Lambda'$.

Restricting to the case where Λ is of finite representation type, a similar Rejection Lemma for the case (a1) will be given in Theorem 4.2, in terms of $\mathfrak{A}(\Lambda)$ and some numerical invariants.

In the final subsection 4.3, a few remarks on related topics, in particular a relation to a result of [RV], will be stated without proof.

1. In this section, let $\mathscr{C}$ denote an arbitrary additive category, and $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ will be denoted by $\mathscr{C}(X, Y)$. For $f \in \mathscr{C}(X, Y)$ and $g \in \mathscr{C}(Y, Z)$, we write their composition as $fg \in \mathscr{C}(X, Z)$.

1.0. Recall that g is a *cokernel* of f if the following sequence of abelian groups is exact for any T:

$$0 \to \mathscr{C}(Z, T) \xrightarrow{g\cdot} \mathscr{C}(Y, T) \xrightarrow{f\cdot} \mathscr{C}(X, T)$$

Dually, f is a *kernel* of g if the following is exact:

$$0 \to \mathscr{C}(T, X) \xrightarrow{\cdot f} \mathscr{C}(T, Y) \xrightarrow{\cdot g} \mathscr{C}(T, Z)$$

As is well known and easily seen, cokernel (resp. kernel) is unique up to isomorphism if exists, so that it is not too confusing to write $X \xrightarrow{f} Y \xrightarrow{\operatorname{cok} f} \operatorname{Cok} f$, indicating that $\operatorname{Cok} f$ is an object and $\operatorname{cok} f$ is a morphism which is a cokernel of f.

1.1. (i) An object $P \in \mathscr{C}$ will be called *quasi-projective* (*in* $\mathscr{C}$) if for any f and α, there is α' which makes the following diagram commutative:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\operatorname{cok} f} & \operatorname{Cok} f \\ & & & \nwarrow^{\alpha'} & \uparrow \alpha \\ & & & & P \end{array}$$

By definition, zero object in $\mathscr{C}$ is quasi-projective. Obviously, a direct sum $P = \bigoplus X$ is quasi-projective if and only if each summand X is quasi-projective.

(ii) A morphism $g : P \to X$ will be called a *Q-covering of X* if P is a direct summand of Q^n for some $n \geq 0$ and moreover g is a cokernel of some $h : Y \to P$.

1.2. In the rest of this section assume that $\mathscr{C}$ is a pre-abelian category (i.e. any morphism f has a kernel and cokernel). Fix a quasi-projective object Q of $\mathscr{C}$ and put

$$\Gamma := \mathscr{C}(Q,Q), \quad FX := \mathscr{C}(Q,X) \quad \text{for } X \in \mathscr{C},$$

$$Ff : FX \to FY \ (\phi \mapsto \phi f) \quad \text{for } f \in \mathscr{C}(X,Y).$$

Consequently, Γ is a ring, $FX \in \operatorname{Mod}\Gamma :=$ (the category of left Γ-modules) and $F : \mathscr{C} \to \operatorname{Mod}\Gamma$ is a functor.

1.3. For $f \in \mathscr{C}(P', P)$, put $g := \operatorname{cok} f$ and $a := \ker g$. Since $fg = 0$, there exists a unique g' which makes the following diagram commutative:

$$\begin{array}{ccccc} P' & \xrightarrow{f} & P & \xrightarrow{g} & X \\ \big\downarrow{\scriptstyle g'} & \nearrow{\scriptstyle a} & & & \\ X' & & & & \end{array}$$

Since Q is quasi-projective, we have

$$0 \longrightarrow FX' \xrightarrow{Fa} FP \xrightarrow{Fg} FX \longrightarrow 0 \quad \text{(exact)}.$$

Since $\operatorname{Cok} Fg' = FX'/\operatorname{Im} Fg' \simeq \operatorname{Im} Fa/\operatorname{Im} Ff \subseteq FP/\operatorname{Im} Ff = \operatorname{Cok} Ff$, we have

$$\operatorname{Cok} Fg' \subseteq \operatorname{Cok} Ff. \tag{1}$$

It is easily seen that the following conditions (2) and (3) are equivalent:

$$\operatorname{Cok} Ff \simeq FX \tag{2}$$

$$\operatorname{Cok} Fg' = 0 \tag{3}$$

1.4. Lemma. *Let $\mathscr{C}$ be a pre-abelian category, Q be a quasi-projective object of $\mathscr{C}$ and assume that any object X of $\mathscr{C}$ admits a Q-covering. Then the functor $F : \mathscr{C} \to \operatorname{Mod}\Gamma$ is fully faithful.*

1.4.1. Any object X of $\mathscr{C}$ is isomorphic to $\operatorname{Cok} f$ for some $f : P' \to P$, P and P' are isomorphic to direct summands of Q^n for some $n \geq 0$.

Proof. By assumption, there is a Q-covering $g : P \to X$, $g = \operatorname{cok} h$, $h : X' \to P$. Take a Q-covering $g' : P' \to X'$ and put $f := g'h$, then we have

$$0 \longrightarrow \mathscr{C}(X,T) \xrightarrow{g\cdot} \mathscr{C}(P,T) \xrightarrow{h\cdot} \mathscr{C}(X',T) \quad \text{(exact)}$$

$$0 \longrightarrow \mathscr{C}(X',T) \xrightarrow{g'\cdot} \mathscr{C}(P',T) \quad \text{(exact)}.$$

Hence we have

$$0 \longrightarrow \mathscr{C}(X,T) \xrightarrow{g\cdot} \mathscr{C}(P,T) \xrightarrow{f\cdot} \mathscr{C}(P',T) \quad \text{(exact)}.$$

Namely $g = \operatorname{cok} f$.

1.4.2. By 1.4.1, for given $X \in \mathscr{C}$, we can take $f : P' \to P$ such that $X \simeq \operatorname{Cok} f$, P and P' are isomorphic to direct summands of Q^n for some $n \geq 0$. Put $g = \operatorname{cok} f$.

(i) Firstly we assume that both of P and P' are direct summands of Q. We shall show that any $\alpha \in \operatorname{Hom}_\Gamma(FX, FY)$ has the form $\alpha(\phi) = \phi t$ ($\phi \in FX = \mathscr{C}(Q,X)$) by the unique $t \in \mathscr{C}(X, Y)$. Then we have $\mathscr{C}(X, Y) \simeq \operatorname{Hom}_\Gamma(FX, FY)$.

Let $p : Q \to P$, $p' : Q \to P'$ (resp. $i : P \to Q$) be splitting epimorphisms (resp. monomorphism) such that $ip = 1$. Since $fg = 0$, $0 = p'fg \in FX$ and $p'fi \in \Gamma$, we have $0 = \alpha(p'fg) = \alpha(p'fipg) = p'fi\alpha(pg)$, so that $fi\alpha(pg) = 0$. Since $g = \operatorname{cok} f$, there is some $t : X \to Y$ such that $i\alpha(pg) = gt$. For any $\phi \in FX = \mathscr{C}(Q,X)$, since Q is quasi-projective, there is some $q : Q \to P$ such that $\phi = qg$. Then $\alpha(\phi) = \alpha(qg) = \alpha(qipg) = qi\alpha(pg) = qgt = \phi t$.

If $\phi t = 0$ for any ϕ, taking $\phi = pg$, we have $pgt = 0$, so that $gt = 0$ and $t = 0$ since g is a cokernel.

(ii) In general, P and P' are direct summands of $Q' = Q^n$ by some n. Consider $\Gamma' := \mathscr{C}(Q', Q') \simeq M_n(\Gamma)$ and $F'X := \mathscr{C}(Q', X) \simeq (FX)^n$. By the same reasoning as (i), we have $\mathscr{C}(X, Y) \simeq \operatorname{Hom}_{\Gamma'}(F'X, F'Y)$. While the latter is isomorphic to $\operatorname{Hom}_\Gamma(FX, FY)$.

2. Let R, K, Λ, $\tilde{\Lambda} = K \otimes_R \Lambda$ and e be as in §0. For $X \in \operatorname{lat}\Lambda$, put

$$\tilde{X} := K \otimes_R X \simeq K \otimes_R (\Lambda \otimes_\Lambda X) \simeq (K \otimes_R \Lambda) \otimes_\Lambda X \simeq \tilde{\Lambda} \otimes_\Lambda X.$$

By the canonical injection $x \mapsto 1 \otimes x$, we identify as $X \subset \tilde{X}$, and put

$$\tilde{X}^1 := e\tilde{X}, \quad \tilde{X}^2 := (1-e)\tilde{X}, \quad X^1 := eX, \quad X^2 := (1-e)X,$$
$$X_1 := \tilde{X}^1 \cap X, \quad X_2 := \tilde{X}^2 \cap X.$$

In particular, $e\Lambda = \Lambda^1$, $(1-e)\Lambda = \Lambda^2$, $\tilde{\Lambda} = \tilde{\Lambda}^1 \oplus \tilde{\Lambda}^2$, Λ^i is an R-order of $\tilde{\Lambda}^i$, $\tilde{X}^i \in \operatorname{mod}\tilde{\Lambda}^i$ and X^i, $X_i \in \operatorname{lat}\Lambda^i$.

2.0. We have the following commutative diagram (4) of Λ-modules for $(i,j) = (1,2)$ or $(2,1)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{X}^j & \longrightarrow & \tilde{X}^1 \oplus \tilde{X}^2 & \longrightarrow & \tilde{X}^i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X^j & \longrightarrow & X^1 \oplus X^2 & \longrightarrow & X^i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & X_j & \longrightarrow & X & \longrightarrow & X^i & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X_j & \longrightarrow & X_1 \oplus X_2 & \longrightarrow & X_i & \longrightarrow & 0
\end{array} \qquad (4)$$

where horizontal arrow is the inclusion and each row is exact, and splits except perhaps the third one.

2.0.1. The correspondence $X \mapsto \tilde{X} = K \otimes_R X$ induces functors:

$$\operatorname{lat} \Lambda \to \operatorname{mod} \tilde{\Lambda} \simeq \operatorname{mod} \tilde{\Lambda}^1 \oplus \operatorname{mod} \tilde{\Lambda}^2 \to \operatorname{mod} \tilde{\Lambda}^i$$

For $f : X \to Y$, $\tilde{f} = 1 \otimes f : \tilde{X} \to \tilde{Y}$ is the unique extension of f, then $\tilde{f}$ uniquely splits into the direct sum $\tilde{f} = f^1 \oplus f^2$, $f^i : \tilde{X}^i \to \tilde{Y}^i$. One may identify $\operatorname{Hom}_\Lambda(X, Y)$ as a subset $\{\phi \mid X\phi \subseteq Y\}$ of $\operatorname{Hom}_{\tilde{\Lambda}}(\tilde{X}, \tilde{Y})$.

2.1. As in §0, let $\mathscr{C}$ be the quotient category $\operatorname{lat} \Lambda / \operatorname{lat} \Lambda^1$. By definition, $\mathscr{C}$ has the same objects as $\operatorname{lat} \Lambda$, and

$$\mathscr{C}(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/I(X, Y)$$

where $I(X, Y)$ is the submodule of $\operatorname{Hom}_\Lambda(X, Y)$ consisting of all morphisms which factor through some object of $\operatorname{lat} \Lambda^1$.

Let $f : X \to Y$ be a morphism in $\operatorname{lat} \Lambda$, and $\bar{f}$ be its image in $\mathscr{C}(X, Y)$.

2.1.1. $\bar{f} = 0$ in $\mathscr{C}$ if and only if $f^2 = 0$ in $\operatorname{mod} \tilde{\Lambda}^2$. Hence we have the following commutative diagram (5) of exact sequences of abelian groups:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I(X, Y) & \longrightarrow & \operatorname{Hom}_\Lambda(X, Y) & \longrightarrow & \mathscr{C}(X, Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_{\tilde{\Lambda}^1}(\tilde{X}^1, \tilde{Y}^1) & \longrightarrow & \operatorname{Hom}_{\tilde{\Lambda}}(\tilde{X}, \tilde{Y}) & \longrightarrow & \operatorname{Hom}_{\tilde{\Lambda}^2}(\tilde{X}^2, \tilde{Y}^2) & \longrightarrow & 0 \end{array} \tag{5}$$

Moreover, $\mathscr{C}(X, Y) = (1 - e)\operatorname{Hom}_\Lambda(X, Y)$, $\tilde{\mathscr{C}}(X, Y) = \operatorname{Hom}_{\tilde{\Lambda}^2}(\tilde{X}^2, \tilde{Y}^2)$, $\mathscr{C}(X, Y)$ is an R-lattice and $\mathscr{C}(X, X)$ is an R-order (or zero).

Proof. Obviously inclusion $\operatorname{Hom}_\Lambda(X, Y) \to \operatorname{Hom}_{\tilde{\Lambda}}(\tilde{X}, \tilde{Y})$ induces $I(X, Y) \to \operatorname{Hom}_{\tilde{\Lambda}^1}(\tilde{X}^1, \tilde{Y}^1)$. This shows that $\bar{f} = 0$ implies $f^2 = 0$. Conversely if $f^2 = 0$, $Y \supset Xf = X\tilde{f} = X(f^1 \oplus f^2) \subset \tilde{Y}^1$, so that $Xf \subset Y_1$, and f factors through $Y_1 \in \operatorname{lat} \Lambda^1$.

This implies $I(X, Y) = \operatorname{Hom}_\Lambda(X, Y) \cap \operatorname{Hom}_{\tilde{\Lambda}^1}(\tilde{X}^1, \tilde{Y}^1)$ and exactness of (5). In particular, we obtain $\mathscr{C}(X, Y) = (1 - e)\operatorname{Hom}_\Lambda(X, Y)$ and $\tilde{\mathscr{C}}(X, Y) = \operatorname{Hom}_{\tilde{\Lambda}^2}(\tilde{X}^2, \tilde{Y}^2)$.

We have to show that $\mathscr{C}(X, Y)$ is a finitely generated R-module. Take an exact sequence of R-modules $R^n \to X \to 0$, then $\operatorname{Hom}_\Lambda(X, Y) \subseteq \operatorname{Hom}_R(X, Y) \subseteq \operatorname{Hom}_R(R^n, Y) = Y^n$. Since R is noetherian, $\operatorname{Hom}_\Lambda(X, Y)$ is finitely generated, hence $\mathscr{C}(X, Y)$ is also finitely generated.

2.1.2. (i) $\bar{f}$ is an epimorphism (resp. monomorphism) in $\mathscr{C}$ if and only if f^2 is an epimorphism (resp. monomorphism) in $\operatorname{mod} \tilde{\Lambda}^2$.

(ii) If $\bar{f}$ is an isomorphism in $\mathscr{C}$, then so is f^2 in $\operatorname{mod} \tilde{\Lambda}^2$.

Proof. By definition, $\bar{f}$ is epic (resp. monic) in $\mathscr{C}$ iff the following $(*)$ is satisfied for any $a : Y \to T$ (resp. $a : T \to Y$).

$$(*) \qquad \bar{f}\bar{a} = 0 \text{ (resp. } \bar{a}\bar{f} = 0) \Rightarrow \bar{a} = 0$$

While, by 2.1.1, $\bar{f}\bar{a} = 0 \Leftrightarrow f^2a^2 = 0$, and $\bar{a} = 0 \Leftrightarrow a^2 = 0$. Hence $\bar{f}$ is epic iff f^2 is epic.

(ii) If $\bar{f}$ is an isomorphism, $\bar{f}$ is epic and monic. By (i), f^2 is epic and monic, so that f^2 is an isomorphism.

2.2. $\mathscr{C}$ is pre-abelian. Let $f : X \to Y$ be a morphism in $\operatorname{lat}\Lambda$, and $\tilde{f} = f^1 \oplus f^2$. Let $g^{(2)} : \tilde{Y}^2 \to V$ (resp. $h^{(2)} : W \to \tilde{X}^2$) be a cokernel (resp. kernel) of f^2 in $\operatorname{mod}\tilde{\Lambda}^2$.

2.2.1. Put $\gamma := 1 \oplus g^{(2)} : \tilde{Y} = \tilde{Y}^1 \oplus \tilde{Y}^2 \to \tilde{Y}^1 \oplus V$, $Z := Y\gamma \in \operatorname{lat}\Lambda$, $g := \gamma|_Y : Y \to Z$. Then we have

(i) g is onto. (ii) g^1 is bijective. (iii) g^2 is a cokernel of f^2 in $\operatorname{mod}\tilde{\Lambda}^2$.
(iv) $\bar{g}$ is a cokernel of $\bar{f}$ in $\mathscr{C}$.

Proof. (i) (ii) (iii) Immediate from definition since $g^1 = 1$, $g^2 = g^{(2)} = \operatorname{cok} f^2$.
(iv) Since g^2 is epic, so is $\bar{g}$ by 2.1.2.

Let $a : Y \to T$ be a morphism in $\operatorname{lat}\Lambda$ such that $\bar{f}\bar{a} = 0$. By 2.1.1, $f^2a^2 = 0$, and there is $b^{(2)} : V \to \tilde{T}^2$ such that $a^2 = g^{(2)}b^{(2)}$. Put $\beta := a^1 \oplus b^{(2)} : \tilde{Z} = \tilde{Y}^1 \oplus V \to \tilde{T}^1 \oplus \tilde{T}^2 = \tilde{T}$. Then $\gamma\beta = (1 \oplus g^{(2)})(a^1 \oplus b^{(2)}) = a^1 \oplus a^2 = \tilde{a}$, and $Z\beta = (Y\gamma)\beta = Y\tilde{a} = Ya \subseteq T$. Hence there is $b := \beta|_Z : Z \to T$ satisfies $gb = a$.

2.2.2. Put $\gamma := 1 \oplus h^{(2)} : \tilde{X}^1 \oplus W \to \tilde{X}^1 \oplus \tilde{X}^2 = \tilde{X}, Z :=$ (inverse image of X by γ), and $h := \gamma|_Z : Z \to X$. Then we have

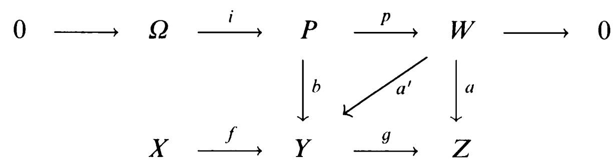
(i) h is one-to-one. (ii) h^1 is bijective. (iii) h^2 is a kernel of f^2 in $\operatorname{mod}\tilde{\Lambda}^2$.
(iv) $\bar{h}$ is a kernel of $\bar{f}$ in $\mathscr{C}$.

Proof. Similar to 2.2.1.

2.3. Recall that $\mathscr{P}$ is the full subcategory of $\mathscr{C}$ formed by $W \in \mathscr{C}$ which has an exact sequence $0 \to \Omega \xrightarrow{i} P \xrightarrow{p} W \to 0$ in $\operatorname{lat}\Lambda$ such that P is projective and $\Omega \in \operatorname{lat} e\Lambda$. Assume that $\mathscr{P}$ has an additive generator Q (i.e. any object in $\mathscr{P}$ is isomorphic to a direct summand of Q^n for some n).

2.3.1. Quasi-projectivity of Q. If $W \in \mathscr{P}$, then W is quasi-projective (1.1) in $\mathscr{C}$. In particular, Q is quasi-projective in $\mathscr{C}$.

Proof. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\bar{g} = \operatorname{cok}\bar{f}$ and $a : W \to Z$, we shall construct $a' : W \to Y$ such that $\bar{a} = \bar{a}'\bar{g}$.



By 2.2.1, we may assume that g is onto and g^1 is bijective.

Since P is Λ-projective and g is onto, there is $b : P \to Y$ such that $pa = bg$. Since $ibg = ipa = 0$, we have $(ib)^1 g^1 = (ibg)^1 = 0$. Since g^1 is bijective, we have $(ib)^1 = 0$, while $(ib)^2 = 0$ since $\Omega \in \operatorname{lat} \Lambda^1$. Consequently, $ib = 0$, and since $p = \operatorname{cok} i$ in $\operatorname{lat} \Lambda$, there is $a' : W \to Y$ such that $b = pa'$. Since p is onto, we have $a = a'g$ as wanted.

2.3.2. Existence of Q-covering. For $X \in \operatorname{lat} \Lambda$, take an exact sequence $0 \to \Omega \to P \xrightarrow{f} X \to 0$ such that P is Λ-projective. Let $p' : P \to W := P/\Omega_1$ be the canonical projection and $p : W \to X$ be the unique map such that $f = p'p$.
Then $\bar{p} : W \to X$ is a Q-covering of X in $\mathscr{C}$.

Proof. Let $i_1 : \Omega_1 \to \Omega$ be the inclusion, and we have the following commutative diagram in $\operatorname{mod} \Lambda$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_1 & \longrightarrow & P & \xrightarrow{p'} & W & \longrightarrow & 0 \quad \text{(exact)} \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \Omega & \longrightarrow & P & \xrightarrow{f} & X & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

By snake Lemma, we have $\operatorname{cok} i_1 \simeq \ker p$, while $\operatorname{cok} i_1 \simeq \Omega^2$ by 2.0, i.e. $0 \to \Omega^2 \xrightarrow{i} W \xrightarrow{p} X \to 0$. This implies, first of all, that W is a Λ-lattice. Moreover, since $\Omega_1 \in \operatorname{lat} \Lambda^1$, $W \in \mathscr{P}$.

We shall show that $\bar{p} = \operatorname{cok} \bar{i}$ in $\mathscr{C}$, which will complete the proof. Since p is epic in $\operatorname{lat} \Lambda$, $p^2 : \tilde{W}^2 \to \tilde{X}^2$ is epic in $\operatorname{mod} \tilde{\Lambda}^2$, hence $\bar{p} : W \to X$ is epic in $\mathscr{C}$ by (i) 2.1.2.

Let $a : W \to T$ be a Λ-morphism such that $\bar{i}\bar{a} = 0$. Then $(ia)^2 = 0$ by 2.1.1 and $(ia)^1 = 0$ by $\Omega^2 \in \operatorname{lat} \Lambda^2$, so that $ia = 0$. Since $p = \operatorname{cok} i$ in $\operatorname{lat} \Lambda$, there is $b : X \to T$ such that $a = pb$, and $\bar{a} = \bar{p}\bar{b}$ as required.

2.4. Assume that $\bar{g} = \operatorname{cok} \bar{f}$, $\bar{a} = \ker \bar{g}$ and $\bar{g}'\bar{a} = \bar{f}$.

$$\begin{array}{ccccc} P' & \xrightarrow{\bar{f}} & P & \xrightarrow{\bar{g}} & X \\ \downarrow{\scriptstyle \bar{g}'} & \nearrow{\scriptstyle \bar{a}} & & & \\ X' & & & & \end{array}$$

Then the cokernel object of $\mathscr{C}(Q, P') \xrightarrow{\cdot \bar{g}'} \mathscr{C}(Q, X')$ is a torsion R-module.

Proof. By 2.2.1 and 2.2.2, we can assume $g^2 = \operatorname{cok} f^2$, $a^2 = \ker g^2$ and $(g')^2 a^2 = f^2$ in $\operatorname{mod} \tilde{\Lambda}^2$. This obviously implies that $(g')^2$ is epic in $\operatorname{mod} \tilde{\Lambda}^2$.

Take an exact sequence $0 \to \Omega(Q) \to P(Q) \to Q \to 0$ such that $P(Q)$ is projective in $\operatorname{lat} \Lambda$ and $\Omega(Q) \in \operatorname{lat} \Lambda^1$. Then $\tilde{Q}^2 = \widetilde{P(Q)}^2$, hence $\tilde{Q}^2$ is projective in $\operatorname{mod} \tilde{\Lambda}^2$. By 2.1.1, $\widetilde{\operatorname{Cok}}(\cdot \bar{g}') = \operatorname{Cok}(\tilde{\mathscr{C}}(Q, P') \xrightarrow{\cdot \bar{g}'} \tilde{\mathscr{C}}(Q, X')) = \operatorname{Cok}(\operatorname{Hom}_{\tilde{\Lambda}^2}(\tilde{Q}^2, \tilde{P}'^2) \xrightarrow{\cdot (g')^2} \operatorname{Hom}_{\tilde{\Lambda}^2}(\tilde{Q}^2, \widetilde{X'^2})) = 0$. This shows $\operatorname{Cok}(\cdot \bar{g}')$ is torsion.

2.5. Proof of Theorem 0.1. Let $\mathscr{C}$ be the quotiont category $\operatorname{lat} \Lambda/\operatorname{lat} \Lambda^1$, Q be an additive generator of $\mathscr{P}$, $FX = \mathscr{C}(Q, X)$ for $X \in \mathscr{C}$, $\Gamma = \mathscr{C}(Q, Q)$ as in §0.

2.5.1. (i) $\mathscr{C}$ is a pre-abelian category by 2.2, Q is quasi-projective and any $X \in \mathscr{C}$ admits a Q-covering by 2.3.

By Lemma 1.4, the functor $F : \mathscr{C} \to \operatorname{Mod} \Gamma$ is fully faithful.

(ii) $\Gamma = \mathscr{C}(Q, Q)$ is an R-order and $FX = \mathscr{C}(Q, X)$ is a left Γ-lattice for any $X \in \mathscr{C}$ by 2.1.1. Thus in fact F is a functor from $\mathscr{C}$ to $\operatorname{lat} \Gamma$. Therefore it remains to see the following.

(∗) Any $\underline{L} \in \operatorname{lat} \Gamma$ is isomorphic to FX for some $X \in \mathscr{C}$.

2.5.2. For a given $\underline{L} \in \operatorname{lat} \Gamma$, take a Γ-projective resolution

$$\Gamma^n \xrightarrow{\underline{f}} \Gamma^m \longrightarrow \underline{L} \longrightarrow 0.$$

By 1.4, there exist $\bar{f} : Q^n \to Q^m$ such that $\underline{f} = F\bar{f}$. Put $X = \operatorname{Cok} \bar{f}$, $\bar{g} = \operatorname{cok} \bar{f}$, $\bar{a} = \ker \bar{g}$ and $\bar{g}'\bar{a} = \bar{f}$.

$$\begin{array}{ccccc} Q^n & \xrightarrow{\bar{f}} & Q^m & \xrightarrow{\bar{g}} & X \\ \downarrow{\scriptstyle \bar{g}'} & \nearrow{\scriptstyle \bar{a}} & & & \\ X' & & & & \end{array}$$

Since $\operatorname{Cok} F\bar{f} \simeq \underline{L}$ is R-torsion free, $\operatorname{Cok} F\bar{g}'$ is also R-torsion free by (1) 1.3. While $\operatorname{Cok} F\bar{g}'$ is R-torsion by 2.4, we get $\operatorname{Cok} F\bar{g}' = 0$. Equivalence of 1.3 (2) and (3) shows $\underline{L} = \operatorname{Cok} F\bar{f} \simeq FX$, establishing (∗).

3. Let R be a complete discrete valuation ring, (Λ, Γ) be as in 0.1 and $- : \operatorname{lat} \Lambda \to \operatorname{lat} \Gamma$ be the natural full functor.

For $X \in \operatorname{ind} \Lambda$, let $\tau X \xrightarrow{\nu} \theta X \xrightarrow{\mu} X$ be the complex of the sink map to X (i.e. μ is the sink map to X and ν is the kernel of μ) in $\operatorname{lat} \Lambda$.

3.1. Proposition. *We can obtain* $\mathfrak{A}(\Gamma)$ *from* $\mathfrak{A}(\Lambda)$ *by the following.*

Remove all vertices in $\operatorname{ind}(e\Lambda)$ *and all arrows which start or end in* $\operatorname{ind}(e\Lambda)$. *If* $\tau X \in \operatorname{ind}(e\Lambda)$ *(resp* $\tau^{-1} X \in \operatorname{ind}(e\Lambda)$*), we regard* X *is a projective (resp. injective) vertex in* $\mathfrak{A}(\Gamma)$.

Proof. We only have to show that for any $X \in \operatorname{ind} \Lambda - \operatorname{ind} e\Lambda$, the complex of the sink map to $\bar{X}$ in $\operatorname{lat} \Gamma$ is given by

$$\overline{\tau X} \xrightarrow{\bar{\nu}} \overline{\theta X} \xrightarrow{\bar{\mu}} \bar{X}.$$

(i) We will show $\operatorname{rad}_\Gamma(\bar{Y}, \bar{Z}) = \overline{\operatorname{rad}_\Lambda(Y, Z)}$ for any Y, Z.

We may assume $Y, Z \in \operatorname{ind} \Lambda$. If Y (resp. Z) is contained in $\operatorname{ind} e\Lambda$, then this is obvious since $\bar{Y} = 0$ (resp. $\bar{Z} = 0$). If $Y, Z \notin \operatorname{ind} e\Lambda$, $\operatorname{rad}_\Lambda(Y, Z) \supseteq I(Y, Z)$

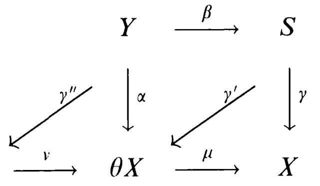
where $I(Y,Z)$ is as in 0.0. Hence preimages in $\mathrm{lat}\,\Lambda$ of any isomorphism in $\mathrm{lat}\,\Gamma$ are isomorphisms, this shows $\mathrm{rad}_\Gamma(\bar{Y},\bar{Z}) \supseteq \overline{\mathrm{rad}_\Lambda(Y,Z)}$. On the other hand, the image of any isomorphism in $\mathrm{lat}\,\Lambda$ is obviously an isomorphism in $\mathrm{lat}\,\Gamma$, this shows $\mathrm{rad}_\Gamma(\bar{Y},\bar{Z}) \subseteq \overline{\mathrm{rad}_\Lambda(Y,Z)}$.

(ii) From (i), $\bar{\mu}$ is right almost split.

(iii) We will show that the following is exact:

$$\mathrm{Hom}_\Gamma(\ ,\overline{\tau X}) \xrightarrow{\cdot\bar{\nu}} \mathrm{Hom}_\Gamma(\ ,\overline{\theta X}) \xrightarrow{\cdot\bar{\mu}} \mathrm{Hom}_\Gamma(\ ,\bar{X})$$

Obviously this is a complex. Assume $\alpha \in \mathrm{Hom}_\Lambda(Y,\theta X)$ satisfies $\alpha\mu \in I(Y,X)$. Then there exists $S \in \mathrm{lat}\,e\Lambda$, $\beta : Y \to S$, $\gamma : S \to X$ such that $\alpha\mu = \beta\gamma$. Then since $X \in \mathrm{ind}\,\Lambda - \mathrm{ind}\,e\Lambda$ and $S \in \mathrm{lat}\,e\Lambda$, $\gamma \in \mathrm{rad}_\Lambda(S,X)$, so there exists $\gamma' : S \to \theta X$ such that $\gamma = \gamma'\mu$.



Since $(\alpha - \beta\gamma')\mu = \beta\gamma - \beta\gamma'\mu = 0$, there exists $\gamma'' : Y \to \tau X$ such that $\alpha - \beta\gamma' = \gamma''\nu$. This shows $\bar{\alpha} = \bar{\gamma}''\bar{\nu}$.

(iv) To show the right minimality of $\bar{\mu}$, assume $\bar{g} \in \mathrm{Hom}_\Gamma(\overline{\theta X},\overline{\theta X})$ satisfies $\overline{g\mu} = \bar{\mu}$. Then (iii) shows there exists $\bar{g}'$ such that $\bar{g} - 1 = \bar{g}'\bar{\nu}$. Since $\bar{\nu} \in \mathrm{rad}_\Gamma(\overline{\tau X},\overline{\theta X})$, $\bar{g}$ is an automorphism of $\overline{\theta X}$. By (ii), $\overline{\theta X} \xrightarrow{\bar{\mu}} \bar{X}$ is the sink map.

(v) We will show that $\bar{\nu}$ is the kernel of $\bar{\mu}$.

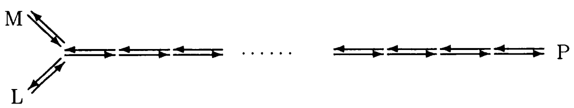
If $\overline{\tau X} = 0$, this is trivial by (iii).

Assume $\overline{\tau X} \neq 0$, so ν is the source map. $\bar{X} \neq 0$ and (ii) show $\overline{\theta X} \neq 0$. Hence $\bar{\nu} \neq 0$ since ν is irreducible and τX, $\theta X \notin \mathrm{lat}\ e\Lambda$. Let $\bar{K} \xrightarrow{\bar{k}} \overline{\theta X}$ be the kernel of $\bar{\mu}$. $\bar{\nu} \neq 0$ shows $\bar{K} \neq 0$. (iii) and the definition of kernel show that there exist $\bar{f}$ and $\bar{g}$ such that $\bar{k} = \bar{f}\bar{\nu}$ and $\bar{\nu} = \bar{g}\bar{k}$. Hence $\bar{k} = \bar{f}\bar{g}\bar{k}$, so $\bar{f}\bar{g} = 1_{\bar{K}}$. Since $\bar{K} \neq 0$ and $\overline{\tau X}$ is indecomposable, $\bar{f}$ is an isomorphism, hence $\bar{\nu}$ is the kernel of $\bar{\mu}$.

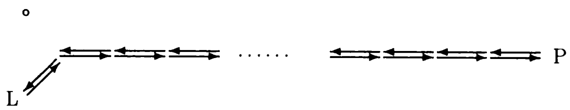
3.2. Examples. For a several pairs of (Λ, e), we shall exhibit what Γ is and how $\mathfrak{A}(\Gamma)$ differs from $\mathfrak{A}(\Lambda)$.

In examples, $\mathcal{O}$ or $\mathcal{O}_i$ always denote a maximal order in some division K-algebra and $\wp$ or $\wp_i$ denote the radical of $\mathcal{O}$ or $\mathcal{O}_i$.

(1) For $n > 0$, let $\Lambda = \{(s,t) \in \mathcal{O} \oplus \mathcal{O} \mid s \equiv t\ (\mathrm{mod}\ \wp^n)\}$. Then $\mathfrak{A}(\Lambda)$ is the following, where P is projective injective and $\tau L = M$, $\tau M = L$, and for any other vertices, τ is the identity. There are $n+2$ verices.

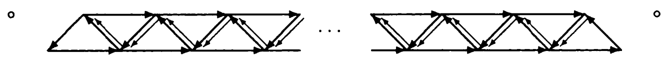


Let $e = (1,0)$. Then $\Gamma = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \wp^n & \mathcal{O} \end{pmatrix}$ and $\mathfrak{A}(\Gamma)$ is the following, where L is projective injective and removed vertex from $\mathfrak{A}(\Lambda)$ is M.



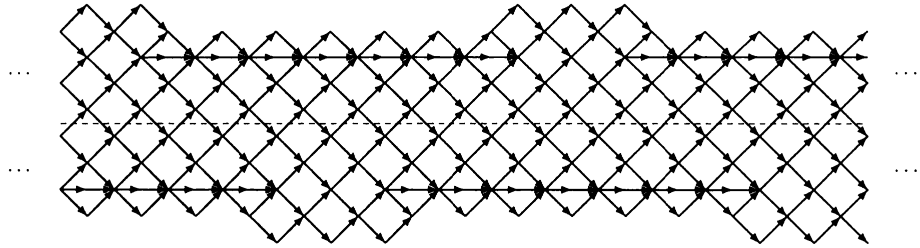
(2) For $n > 0$, let $\Lambda = \left\{ \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} (w) \right) \in \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \wp^n & \mathcal{O} \end{pmatrix} \oplus (\mathcal{O}) \,\middle|\, v \equiv w \ (\operatorname{mod} \wp) \right\}$.
Then $\mathfrak{A}(\Lambda)$ is the following, where thin arrows indicate τ, the L at the left end is identified with the L at the right end. There are $2n+2$ vertices.

(i) Let $e = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (1) \right)$. Then $\Gamma = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \wp^n & \mathcal{O} & \mathcal{O} \\ \wp^n & \wp & \mathcal{O} \end{pmatrix}$ and $\mathfrak{A}(\Gamma)$ is the following, where removed vertex from $\mathfrak{A}(\Lambda)$ is L.

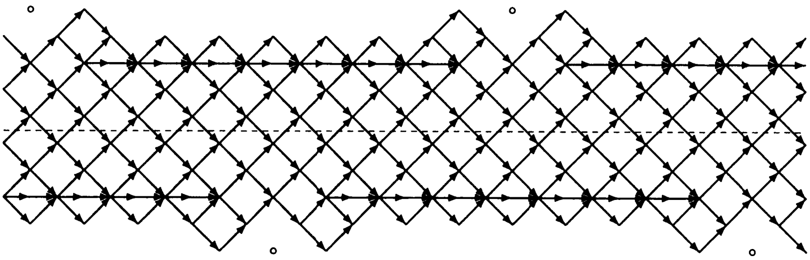


(ii) Let $e = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (0) \right)$. Then Γ is a basic hereditary order of matrix size $n+1$ and $\mathfrak{A}(\Gamma)$ is the following.

(3) For even $n > 0$, let $\Lambda = \left\{ \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} (w) \right) \in \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \wp^n & \mathcal{O} \end{pmatrix} \oplus (\mathcal{O}) \,\middle|\, v \equiv w \ (\operatorname{mod} \wp^2) \right\}$. Let Q be the following translation quiver, where τ is the left shift. Then $\mathfrak{A}(\Lambda) = Q/\langle \phi \rangle$, where ϕ is the composition of 'the reflection by the central dotted line' and $\tau^{(n+1)/2}$. There are $(n^2 + 3n + 4)/2$ vertices. (For simplicity, let $n = 8$.)



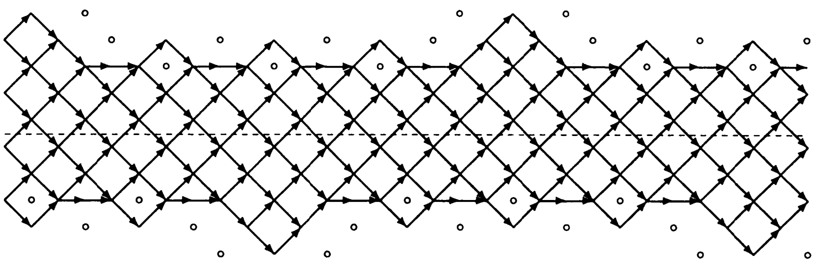
(i) Let $e = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(1)\right)$. Then $\Gamma = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \wp^n & \mathcal{O} & \mathcal{O} \\ \wp^n & \wp^2 & \mathcal{O} \end{pmatrix}$ and $\mathfrak{A}(\Gamma)$ is the following.



(ii) Let $e = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(0)\right)$. Then

$$\Gamma = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \wp & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \wp^2 & \wp & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \wp^2 & \wp^2 & \wp^2 & \cdots & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \wp^2 & \wp^2 & \wp^2 & \cdots & \wp & \mathcal{O} & \mathcal{O} \\ \wp^2 & \wp^2 & \wp^2 & \cdots & \wp^2 & \wp & \mathcal{O} \end{pmatrix}$$

(matrix size is $n+1$) and $\mathfrak{A}(\Gamma)$ is the following.



(4) For $i = 1, 2, 3, 4$, let $\mathcal{O}_i$ be a copy of $\mathcal{O}$ and $\wp_i$ be the maximal ideal of $\mathcal{O}_i$. Let $\Lambda = \mathcal{O}_1 \overset{1}{-} \mathcal{O}_2 \overset{1}{-} \mathcal{O}_3 \overset{1}{-} \mathcal{O}_4 := \{(s, t, u, v) \in \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \oplus \mathcal{O}_4 \mid s \equiv t \equiv u \equiv v \ (\mathrm{mod}\, \wp)\}$, $e = (1, 1, 1, 0) \in \tilde{\Lambda}$. It is well known that $\sharp \operatorname{ind} \Lambda = \infty$.

By 0.2.0, we can take $\mathcal{Q} = \tau^{-1}(\operatorname{ind} e\Lambda) \cup \operatorname{proj} \Lambda$, $Q = \bigoplus_{i=1}^{9} L_i$, where

$$L_1 = \mathcal{O}_2 \overset{1}{-} \mathcal{O}_3 \overset{1}{-} \mathcal{O}_4, \quad L_2 = \mathcal{O}_1 \overset{1}{-} \mathcal{O}_3 \overset{1}{-} \mathcal{O}_4, \quad L_3 = \mathcal{O}_1 \overset{1}{-} \mathcal{O}_2 \overset{1}{-} \mathcal{O}_4$$

$$L_4 = \mathcal{O}_3 \overset{1}{-} \mathcal{O}_4, \quad L_5 = \mathcal{O}_1 \overset{1}{-} \mathcal{O}_4, \quad L_6 = \mathcal{O}_2 \overset{1}{-} \mathcal{O}_4,$$

$$L_7 = \{(x, y, z, a, b) \in \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \oplus \mathcal{O}_4 \oplus \mathcal{O}_4 \mid x \equiv a,\ y \equiv b,\ x + y + z \equiv 0\},$$

$$L_8 = \mathcal{O}_4, \quad L_9 = \mathcal{O}_1 \overset{1}{-} \mathcal{O}_2 \overset{1}{-} \mathcal{O}_3 \overset{1}{-} \mathcal{O}_4.$$

Hence, we get

$$\Gamma = \begin{pmatrix} \mathcal{O}_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \mathcal{O}_4 & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \mathcal{O}_4 & \wp_4 & \mathcal{O}_4 & \mathcal{O}_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \mathcal{O}_4 & \mathcal{O}_4 & a & a' & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \wp_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \wp_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \wp_4 & b & \mathcal{O}_4 & \wp_4 & c & \wp_4 & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \wp_4 & b' & \wp_4 & \mathcal{O}_4 & \wp_4 & c' & \mathcal{O}_4 & \wp_4 \\ \wp_4 & \wp_4 & \wp_4 & \wp_4 & \wp_4 & \wp_4 & \wp_4 & \wp_4 & \mathcal{O}_4 & \wp_4 \\ \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 & \mathcal{O}_4 \end{pmatrix}$$

where a, a', b, b', c, c' are elements of $\mathcal{O}_4$ such that $a + a' \equiv 0$, $b \equiv b'$, $c \equiv c' \pmod{\wp_4}$.

4. Let R be a complete discrete valuation ring and Λ be an R-order in a semi-simple K-algebra $\tilde{\Lambda} = K \otimes_R \Lambda$. In this section, we consider a full subcategory $\mathscr{C}'$ of $\mathscr{C} = \operatorname{lat} \Lambda$, which is closed under isomorphism, direct sum and direct summand. In other words, such $\mathscr{C}'$ bijectively corresponds to a subset $\mathscr{S}'$ of $\operatorname{ind} \Lambda$ by $\mathscr{C}' = \operatorname{add} \mathscr{S}'$, $\mathscr{S}' = \mathscr{C}' \cap \operatorname{ind} \Lambda$.

Let $\underline{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_n\}$ be the complete set of orthogonal irreducible central idempotents of $\tilde{\Lambda}$, so that $\tilde{\Lambda}$ is the ring direct sum $\tilde{\Lambda} = \bigoplus \varepsilon_i \tilde{\Lambda}$ with $\varepsilon_i \tilde{\Lambda}$ simple.

Put $\underline{\varepsilon}' := \{\varepsilon \in \underline{\varepsilon} \mid \varepsilon \mathscr{S}' \neq \{0\}\}$ and $e := \sum_{\varepsilon \in \underline{\varepsilon}'} \varepsilon$. Then $\mathscr{C}' \subseteq \operatorname{lat} e\Lambda$, $\mathscr{S}' \subseteq \operatorname{ind} e\Lambda$. It is straightforward to observe:

(A) $\mathscr{C}' = \operatorname{lat} e\Lambda \Leftrightarrow \mathscr{S}' = \operatorname{ind} e\Lambda \Leftrightarrow$ For $X \in \operatorname{ind} \Lambda$, $(X \in \mathscr{S}' \Leftrightarrow (1-e)X = 0)$.

(B) If $\mathscr{C}' \neq \operatorname{lat} e\Lambda$ and $X \in \operatorname{ind} e\Lambda - \mathscr{S}'$, then $\operatorname{Hom}_{\mathscr{C}/\mathscr{C}'}(X, X)$ is a non-zero R-torsion module.

4.1. To apply a result of [I], we sometimes identify the category $\mathscr{C} = \operatorname{lat} \Lambda$ with the set of non-isomorphic objects of $\operatorname{lat} \Lambda$, so that $\operatorname{lat} \Lambda \supset \operatorname{ind} \Lambda$. By Krull-Schmidt Theorem, we consider $\operatorname{lat} \Lambda$ as a free monoid $\mathbf{N} \operatorname{ind} \Lambda$ generated by the set $\operatorname{ind} \Lambda$, embedded in its quotient group $\mathbf{Z} \operatorname{ind} \Lambda$, $\operatorname{lat} \Lambda = \mathbf{N} \operatorname{ind} \Lambda \subset \mathbf{Z} \operatorname{ind} \Lambda$.

Any map $\xi : \operatorname{ind} \Lambda \to A$ (resp. any monoid homomorphism $\xi : \operatorname{lat} \Lambda = \mathbf{N} \operatorname{ind} \Lambda \to A$) into an abelian group A uniquely extends to a group homomorphism $\xi : \mathbf{Z} \operatorname{ind} \Lambda \to A$.

By this convention, the complex of source map $X \to \theta^- X \to \tau^- X$ from $X \in \operatorname{ind} \Lambda$ determines $\mathbf{Z}$-endomorphisms θ^-, τ^- of $\mathbf{Z} \operatorname{ind} \Lambda$ and $\phi^- := 1 - \theta^- + \tau^- \in \operatorname{End}_{\mathbf{Z}}(\mathbf{Z} \operatorname{ind} \Lambda)$.

The map of rational length $\tilde{l} := (L \mapsto \operatorname{length}_{\tilde{\Lambda}}(\tilde{L}))$, the action of central idempotent $\varepsilon := (L \mapsto \varepsilon L)$ determine $\mathbf{Z}$-homomorphisms $\tilde{l} : \mathbf{Z} \operatorname{ind} \Lambda \to \mathbf{Z}$, $\varepsilon : \mathbf{Z} \operatorname{ind} \Lambda \to \mathbf{Z} \operatorname{ind} \varepsilon\Lambda$.

4.1.1. Lemma. *Assume that* $\operatorname{ind} \Lambda$ *is a finite set. Let* $l : \operatorname{lat} \Lambda = \mathbf{N} \operatorname{ind} \Lambda \to \mathbf{N}$ *be a monoid homomorphism such that* $l \circ \phi^- = 0$.

Then $l(L) = \sum_{i=1}^{n} a_i \tilde{l}(\varepsilon_i L)$ *for any* $L \in \operatorname{lat} \Lambda$, *by some* $a_i \in \mathbf{N}$.

Proof. Let Ω be a maximal order containing Λ. Then $\Omega = \bigoplus_{i=1}^{n} \varepsilon_i \Omega$, $\operatorname{Ind} \varepsilon_i \Omega = \{Y_i\}$ with $\tilde{l}(Y_i) = 1$.

Hence $\Omega L = \sum_{i=1}^{n} c_i Y_i$ by some $c_i \in \mathbf{N}$, and $\widetilde{\varepsilon_i L} = \widetilde{\varepsilon_i \Omega L} = c_i \tilde{Y}_i$ and $\tilde{l}(\varepsilon_i L) = c_i$.

Since $\operatorname{ind} \Lambda$ is finite, $\operatorname{ind} \Lambda - \operatorname{ind} \Omega$ is a rejectable subset and by Theorem 5.4 of [I], there are some $b_X \in \mathbf{N}$ such that

$$\Omega L = L - \sum_{X \in \operatorname{ind} \Lambda - \operatorname{ind} \Omega} b_X \phi^- X.$$

Since $l \circ \phi^- = 0$ by the assumption, we have $l(L) = l(\Omega L) = \sum_{i=1}^{n} c_i l(Y_i) = \sum_{i=1}^{n} \tilde{l}(\varepsilon_i L) a_i$ with $a_i := l(Y_i) \in \mathbf{N}$.

4.2. Theorem. *Let $\mathscr{C}' = \mathbf{N}\mathscr{S}'$ be a full subcategory of $\mathscr{C} = \operatorname{lat} \Lambda = \mathbf{N} \operatorname{ind} \Lambda$. Assume that $\operatorname{ind} \Lambda$ is a finite set. Then the following three conditions are equivalent.*

(i) There exists a central idempotent e of $\tilde{\Lambda}$ such that $\mathscr{C}' = \operatorname{lat} e\Lambda$.

(ii) There exists an R-order Γ such that $\mathscr{C}/\mathscr{C}' \simeq \operatorname{lat} \Gamma$.

(iii) There exists a map $l : \operatorname{ind} \Lambda \to \mathbf{N}$ such that 1) $l \circ \phi^- = 0$ and 2) $l(X) = 0 \Leftrightarrow X \in \mathscr{S}'$.

Proof. (i) $\Rightarrow$ (ii) : By Theorem 0.1.

(ii) $\Rightarrow$ (i) : If $\mathscr{C}/\mathscr{C}' \simeq \operatorname{lat} \Gamma$, $\operatorname{Hom}_{\mathscr{C}/\mathscr{C}'}(X, X)$ is torsion free for any $X \in \operatorname{ind} \Lambda - \mathscr{S}'$. Then by (B), $\mathscr{C}' = \operatorname{lat} e\Lambda$.

(i) $\Rightarrow$ (iii) : Put $l := \tilde{l} \circ (1 - e) : \operatorname{lat} \Lambda \to \mathbf{N}$. Then obviously l has the required property.

(iii) $\Rightarrow$ (i) : By 4.1.1, $l(L) = \sum_{i=1}^{n} a_i \tilde{l}(\varepsilon_i L)$ by some $a_i \in \mathbf{N}$. Put $e := \sum_{a_i = 0} \varepsilon_i$, then for $X \in \operatorname{ind} \Lambda$, $X \in \mathscr{S}' \Leftrightarrow l(X) = 0 \Leftrightarrow (1 - e)X = 0$, hence $\mathscr{S}' = \operatorname{ind} e\Lambda$ by (A).

4.3. Remarks. Rejection Lemma is also studied, explicitly or implicitly, in some cases other than (a0) or (a1) of 0.4.1:

(b0) R is a commutative artin ring. Λ is an artin algebra over R. Λ' is a quotient algebra of Λ. $\mathscr{C} = \operatorname{mod} \Lambda$ (resp. $\mathscr{C}' = \operatorname{mod} \Lambda'$) is the category of finitely generated Λ-modules (resp. Λ'-modules) ([I]).

(b1) R is a 2-dimensional integrally closed complete noetherian domain. Λ is a *tame R-order* (i.e. Λ is a reflexive R-module and moreover $R_{\wp} \otimes_R \Lambda$ is a hereditary $R_{\wp}$-order at any height one prime ideal $\wp$ of R). Λ' is a tame over order of Λ. $\mathscr{C} = \operatorname{ref} \Lambda$ (resp. $\mathscr{C}' = \operatorname{ref} \Lambda'$) is the category of finitely generated Λ-modules (resp. Λ'-modules) which are reflexive as R-modules ([RV]).

In these examples, one finds strong similarity in the relation of $\mathscr{C}$ with $\mathscr{C}'$. As a matter of fact, one can construct an abstract theory of Rejection Lemma which specializes to each of the above cases.

4.3.1. In the cases (a0) and (b0) (where $\mathscr{C}/\mathscr{C}'$ is of dimension zero), $\mathscr{C}/\mathscr{C}'$ is not exactly a category of modules, although it is quite similar to a category of modules.

While in (a1), where $\mathscr{C}/\mathscr{C}'$ is one-dimensional, as was shown in 4.2, $\mathscr{C}/\mathscr{C}'(\simeq \operatorname{lat}\Gamma)$ is a lattice category.

We can find a similar phenomenon also in the case (b1), which will be recorded in 4.3.2, without proof. To state the result, it is more convenient to consider an additive category as a ring in a usual way, then describe the corresponding condition in terms of generalized Auslander orders (0.2.2). Note also that the equivalence of (i) and (iii) in 4.3.2 is already proved for a general tame order Λ in [RV] Theorem 3.9.

4.3.2. Proposition. *Let k be a field, $R = k[[x, y]]$, Λ be a tame R-order, $\mathscr{C}' = \mathbf{N}\mathscr{S}'$ be a full subcategory of $\mathscr{C} = \operatorname{ref}\Lambda = \mathbf{N}\operatorname{ind}\Lambda$, where $\operatorname{ind}\Lambda$ is the set of isomorphism classes of indecomposable objects in $\operatorname{ref}\Lambda$. Assume that $\operatorname{ind}\Lambda$ is a finite set. Then the following three conditions are equivalent.*

(i) There exists a tame over order Λ' of Λ such that $\mathscr{C}' = \operatorname{ref}\Lambda'$.

(ii) There exists a complete discrete valuation ring R' such that $\mathscr{C}/\mathscr{C}'$ is a generalized Auslander order over R'.

(iii) There exists a map $l : \operatorname{ind}\Lambda \to \mathbf{N}$ such that 1) $l \circ \phi^{-} = 0$ and 2) $l(X) = 0 \Leftrightarrow X \in \mathscr{S}'$.

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