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Algebraic varieties with small Chow groups

By

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Introduction

Let X be a smooth complete n-dimensional variety over a field k, let $A^iX=A_{n-i}X$ denote the Chow groups of X, and let $A^i_{\mathrm{hom}}X=A^{\mathrm{hom}}_{n-i}X$ denote the kernel of the cycle class map

$$\mathrm{cl}^i: A^iX \to H^{2i}X$$

to a fixed Weil cohomology theory. The group of 0-cycles of degree 0, denoted by $A_0^{\mathrm{hom}}X$, is called finite dimensional if there exist a universal domain $\Omega \supset k$ and an integer $m \in N$ such that the natural map

$$S^mX_\Omega \times S^mX_\Omega \to A_0^{\mathrm{hom}}(X_\Omega)$$
$$(a,b) \mapsto a-b$$

is surjective, where S^m denotes mth symmetric power.

One of the cornerstones of the study of Chow groups is the following famous result of Mumford [Mum] [Bl 1, Lecture 1]:

(0.1) Theorem (Mumford). *Let X be a surface over an algebraically closed field. If $A_0^{\mathrm{hom}}X$ is finite dimensional, then H^2X is algebraic.*

Bloch conjectured that the converse holds [Bl 1, Lecture 1]:

(0.2) Conjecture (Bloch). If X is a surface over an algebraically closed field such that H^2X is algebraic, then $A_0^{\mathrm{hom}}X$ is finite dimensional.

Mumford's theorem is usually read as indicating that for a general variety, the Chow groups are "very large" in codimension >1. But another way of paraphrasing Mumford's theorem is that varieties with "small" Chow groups have very special properties.

This last idea is systematically explored by Bloch and Srinivas [B-S]. Observing that $A_0^{\mathrm{hom}}X$ is finite dimensional iff $A_0(X_\Omega)$ has support on a curve (for a universal

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domain Ω), they study varieties X for which $A_0(X_\Omega)$ is supported on some subvariety, i.e. for which there exists a closed (possibly singular and reducible) $Y \subset X_\Omega$ of dimension r such that there is a surjection

$$A_0 Y \twoheadrightarrow A_0(X_\Omega).$$

If r is small, Bloch and Srinivas show this has many interesting consequences: e.g. if $r<n$ then the geometric genus $p_g(X)=0$, if $r\leq 3$ then the Hodge conjecture in codimension 2 is true for X, if $r\leq 2$ then the algebraic equivalence coincides with the homological one for codimension 2 cycles on X, and so on.

The influence of all Chow groups A^* (not just A_0) on Weil cohomology H^* is further studied by Jannsen, who proves (in the beautiful survey article [Ja 2]):

(0.3) Theorem (Jannsen). *Let X be a smooth complete variety over a universal domain Ω. Suppose all cycle class maps $\mathrm{cl}^i: A^iX\otimes H^0\Omega \to H^{2i}X$ are injective. Then they are also surjective, i.e. there is a ring-isomorphism*

$$A^*X\otimes H^0\Omega \xrightarrow{\sim} H^*X.$$

In particular, if $\Omega = \boldsymbol{C}$, it follows that the Hodge numbers $h^{p,q}(X)$ vanish for $p\neq q$.

A similar result is proven by Esnault and Levine [E-L]:

(0.4) Theorem (Esnault-Levine). *Let X be a smooth complete variety over $\boldsymbol{C}$. Suppose all cycle class maps into Deligne cohomology*

$$\mathrm{cl}^i_{\mathscr{D}}: A^iX_{\boldsymbol{Q}} \to H^{2i}_{\mathscr{D}}(X, \boldsymbol{Q}(i))$$

are injective. Then they are also surjective, and $h^{p,q}(X)=0$ for $|p-q|>1$.

In this paper, the main goal is to unify (and mildly generalize) these results of Mumford, Bloch-Srinivas, Jannsen and Esnault-Levine. Motivated by the Bloch-Srinivas approach, the following definition seems natural: We say that

$$\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}})\leq r$$

if there exists a closed (possibly singular and reducible) $Y\subset X$ of codimension $i-r$ such that push-forward induces a surjection

$$A_{n-i}(Y)_{\boldsymbol{Q}} \twoheadrightarrow A^i(X)_{\boldsymbol{Q}}.$$

The cases $r=0$ and $r=1$ correspond to the injectivity of cl^i and $\mathrm{cl}^i_{\mathscr{D}}$ respectively, as assumed in (0.3) and (0.4), cf. (1.5).

The main result is that over a universal domain, the niveau of Chow groups $A^*_{\boldsymbol{Q}}$ influences the niveau of other cohomology theories $H^*(-,*)$ (see (1.7) and (1.9) for precise statements). A particular case is that if X is a smooth complete

n-dimensional variety over $\boldsymbol{C}$ with

$$\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}}) \leq r \qquad \forall i \leq \frac{n-r}{2},$$

then one has $h^{p,q}(X)=0$ if $|p-q|>r$, i.e. X has a small Hodge diamond. Note that this is a Mumford-type result.

Another special case of our main result is that over a universal domain, the niveau of Chow groups in high degree influences the niveau of low degree Chow groups, cf. (1.8.1).

The conjectural existence of a category of mixed motives ([Be],[Ja 2], [SaS]) has led people ([Ja 2],[Pa]) to conjecture that the converse is true: over a universal domain, the niveau of Chow groups should in its turn be determined by the niveau of Weil cohomology, in particular the vanishing $h^{p,q}(X)=0$ for $|p-q|>r$ should imply that

$$\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}}) \leq r,$$

cf. (1.11) for a more precise statement. Note that this is a Bloch-type conjecture.

It should be stressed that the methods of proof in this paper are far from being new. The main idea, viz. that small Chow groups give a decomposition of the diagonal

$$\Delta \in A^n(X_\Omega \times X_\Omega)_{\boldsymbol{Q}},$$

and that this decomposition has consequences for other cohomology theories H^* since the diagonal acts as correspondence on H^*, can also be found in the afore-mentioned works of Bloch-Srinivas, Jannsen and Esnault-Levine. This idea makes its first apperance in Bloch's book [Bl 1, Appendix to Lecture 1], where it is attributed to Colliot-Thélène.

In a second section, we give several applications of this approach; most of these are straightforward generalizations of applications in [B-S]. The principle of these applications is that if a smooth complete variety X over a universal domain satisfies

$$\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}}) \leq r \qquad \forall i,$$

then (the Chow motive of) X behaves in every way as (the Chow motive of) an r-dimensional variety. For instance, if all Chow groups of X have niveau ≤ 3, then the Hodge conjecture for X is true since it can be reduced to the known cases of curves and divisors, cf. §2.2.

A new application is given in a sequel to this article [La 2]; hare we verify Murre's conjectures (on a motivic decomposition of the Chow motive, [Mur]) for 3-and 4-folds with $\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}}) \leq 2$.

1. Main result

The following definition is inspired by the notion of a "twisted Poincaré duality theory" [B-O]:

(1.0) Definition. Let $\mathcal{VAR}_k$ be the category whose objects are smooth complete varieties over the field k, and with arbitrary morphisms of varieties as arrows. Let R be a ring.

A *good cohomology theory with values in R on $\mathcal{VAR}_k$* is a contravariant functor

$$H^*(-,*): \mathcal{VAR}_k \to \{\text{bigraded } R\text{-algebras}\}$$

satisfying:

(i) Every $X \in \mathcal{VAR}_k$ has a canonical element

$$[X] \in H^0(X, 0),$$

the *fundamental class*;

(ii) For every $X \in \mathcal{VAR}_k$ there is a ring-structure

$$H^i(X,j) \otimes (X,l) \to H^{i+k}(X,j+l)$$

for which $[X]$ is a unit and which is compatible with pull-backs;

(iii) For a proper morphism $p: X \to Y$ between equidimensional varieties in $\mathcal{VAR}_k$ there exists a functorial push-forward

$$p_*: H^i(X,j) \to H^{i+2d}(Y, j+d)$$

where $d := \dim X - \dim Y$;

(iv) (*Projection formula*) For a proper morphism $p: X \to Y$, one has

$$p_*(\alpha \cdot p^*\beta) = p_*\alpha \cdot \beta$$

for any $\alpha \in H^i(X,j)$, $\beta \in H^k(Y,l)$;

(v) (*Base change*) For a Cartesian diagram of projections

$$\begin{array}{ccc} X \times X' \times X'' & \xrightarrow{p'} & X \times X' \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ X' \times X'' & \xrightarrow{p} & X' \end{array}$$

one has $p^* q_* = (q')_* (p')^*$;

(vi) There exists a "cycle class" natural transformation of contravariant functors

$$\mathrm{cl}^i: \mathrm{A}^i_{\mathbb{Q}} \to H^{2i}(-, i),$$

compatible with product and proper push-forward;

(vii) (*Vanishing*) For $X \in \mathcal{VAR}_k$ of pure dimension n, one has

$$H^i(X,j)=0 \quad \text{if} \quad i<0 \quad \text{or} \quad i>2n.$$

The following is a weak version of the notion of "Weil cohomology" that can be found in the literature:

(1.1) Definition. A *Weil cohomology* is a contravariant functor

$$H^*:\mathscr{V}\mathscr{A}\mathscr{R}_k \to \{\text{graded } R\text{-algebras}\}$$

($R=\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{Q}_l$) satisfying:
(i) $H^i(-,j)=H^i$ defines a good cohomology theory;
(ii) Each H^iX is a finitely generated R-module, and for any n-dimensional variety X, $H^{2n}X$ is generated by the irreducible components of X;
(iii) (*Poincaré duality*) For any n-dimensional X, intersection defines a perfect pairing

$$H^iX\times H^{2n-i}X\to H^{2n}X;$$

(iv) (*Weak Lefschetz*) If X is projective and $Y\subset X$ is a smooth hyperplane section, then the homomorphisms

$$H^{i-2}Y\to H^iX$$

are surjective for $i>n:=\dim X$.

(1.2) Examples. Every twisted Poincaré duality theory [B-O] satisfying the vanishing (vii) gives a good cohomology theory; in particular we have singular cohomology with rational coefficients for $k=\boldsymbol{C}$, Deligne cohomology $H^*_{\mathscr{D}}(-,\boldsymbol{Q}(*))$ for $k=\boldsymbol{C}$, étale cohomology with values in $\boldsymbol{Q}_l$ for k algebraically closed of characteristic prime to l, DeRham cohomology for k algebraically closed.

Singular, étale and DeRham cohomology are the main examples of Weil cohomologies.

Over any field k, a trivial example of a good cohomology theory is given by

$$H^i(X,j)=\begin{cases}A^j(X)_{\boldsymbol{Q}} & \text{if } i=2j;\\ 0 & \text{otherwise.}\end{cases}$$

Extending this last example, it is expected that higher Chow groups [Bl 2] form a good cohomology theory after a renumbering (indeed, it is even expected they are the universal good cohomology theory), but for the vanishing (vii) the Beilinson-Soulé conjecture [So] is needed.

The next definition is motivated by Grothendieck's coniveau filtration [Gro 1][B-O] and by the work of Bloch-Srinivas [B-S]:

(1.3) Definition. Let $X\in\mathscr{V}\mathscr{A}\mathscr{R}_k$.
(i) We say that

$$\mathrm{Niveau}(A^i(X)_{\mathbf{Q}})\leq r$$

if there exists a closed reduced subscheme $Y\subset X$ of codimension $\geq i-r$ such that one has $A^i(X\setminus Y)_{\mathbf{Q}}=0$ (equivalently, such that push-forward induces a surjection

$$A_{n-i}(Y)_{\mathbf{Q}}\twoheadrightarrow A^i(X)_{\mathbf{Q}}).$$

(ii) For any good cohomology theory $H^*(-,*)$, we say that

$$\mathrm{Niveau}(H^i(X,j))\leq r$$

if there exists a smooth complete variety Y of dimension $d\leq n+(r-i)/2$ and a proper morphism $Y\to X$ inducing a surjection

$$H^{i+2d-2n}(Y,j+d-n)\twoheadrightarrow H^i(X,j).$$

(1.4) Remarks. 1. For a Weil cohomology H^*, it is immediate that one has

$$\mathrm{Niveau}(H^iX)\leq r\Leftrightarrow H^iX=N^{\lceil\frac{i-r}{2}\rceil}H^iX,$$

where N^* denotes the *coniveau filtration* on H^* [Gro 1] [Gro 2] [B-O], i.e.

$$N^lH^iX:=\bigcup_{\substack{p:Y\to X\text{ proper,}\\ Y\text{ smooth of dim. }n-l.}}\mathrm{Im}(H^{i-2l}Y\to H^iX).$$

2. Suppose $k=\mathbf{C}$, and H^* is singular cohomology. The above definition of the coniveau filtration coincides with the following one:

$$N^lH^iX:=\bigcup_{Y\subset X\text{ closed of codim. }l}\mathrm{Im}(H_{2n-i}Y\to H^iX),$$

as can be seen using resolution of singularities.

It is not hard to see that

$$\mathrm{Niveau}(H^iX)\leq r\Rightarrow h^{p,q}(X)=0\text{ for }p+q=i,\ |p-q|>r,$$

where $h^{p,q}$ denotes the Hodge numbers. In fact, it is expected that these two statements are equivalent; the right-to-left implication is a consequence of Grothendieck's generalized Hodge conjecture [Gro 2].

The following lemma further motivates definition (1.2):

(1.5) Lemma. *Let X be a smooth complete variety of dimension n over a universal domain Ω, and let s be a non-negative integer.*

(i) $$\text{Niveau}(A^i(X)_{\mathbf{Q}}) \le 0 \quad \forall i \ge n-s$$
$$\Leftrightarrow A^i_{\text{hom}}(X)_{\mathbf{Q}} = 0 \quad \forall i \ge n-s$$
$$\Leftrightarrow A^i(X)_{\mathbf{Q}} \text{ has finite rank } \forall i \ge n-s.$$

(ii) *Let* $\Omega = \mathbf{C}$. *Then*

$$\text{Niveau}(A^i(X)_{\mathbf{Q}}) \le 1 \quad \forall i \ge n-s \quad \Leftrightarrow \quad A^i_{\text{AJ}}(X)_{\mathbf{Q}} = 0 \quad \forall i \ge n-s.$$

Proof. (i) Clearly the second statement implies the last, and the last implies the first. That the first statement implies the second follows from (1.7) with $r=0$, cf. (2.1).
(ii) This follows from the fact that both statements are equivalent to the existence of a decomposition of the diagonal as in (1.7) (ii) with $r=1$ (for the equivalence between this decomposition and the right-hand-side of (1.5)(ii), cf. [E-L]).

(1.6) Remarks. 1. The question whether the equivalence of lemma (1.5) holds for any individual index i still seems open. Also, I don't know whether (1.5) holds without tensoring by $\mathbf{Q}$.

2. Beilinson and Murre have conjectured the existence of an $i+1$-step filtration F^* on $A^i(X)_{\mathbf{Q}}$, of which the first two steps should be homological and Abel-Jacobi equivalence [Be][Mur][Ja 2][Ja 3]. In terms of this conjectural filtration, the condition $\text{Niveau}(A^i(X)_{\mathbf{Q}}) \le r$ should correspond to $F^{r+1}A^i(X)_{\mathbf{Q}} = 0$.

With the above terminology, the main result of this paper is:

(1.7) Theorem. *Let X be a smooth complete variety of dimension n defined over the field k, let $\Omega \supset k$ be a universal domain. For any two given non-negative integers r and s, the following statements are equivalent:*

(i) $$\text{Niveau}(A^i(X_\Omega)_{\mathbf{Q}}) \le r \quad \text{for all } i \ge n-s;$$

(ii) *There exist closed and reduced subschemes $V_0, \cdots, V_s$ and $W_0, \cdots, W_{s+1}$ of X_Ω such that* $\dim V_j \le j+r (j=0, \cdots, s)$, $\dim W_j \le n-j (j=0, \cdots, s+1)$, *and such that the diagonal $\Delta \in A^n(X_\Omega \times X_\Omega)_{\mathbf{Q}}$ has a decomposition*

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_s + \Delta^{s+1},$$

with Δ_j in the image

$$A_n(V_j \times W_j)_{\mathbf{Q}} \to A^n(X_\Omega \times X_\Omega)_{\mathbf{Q}}$$

$(j=0, \cdots, s)$, *and Δ^{s+1} in the image*

$$A_n(X_\Omega \times W_{s+1})_{\mathbf{Q}} \to A^n(X_\Omega \times X_\Omega)_{\mathbf{Q}};$$

(iii) $$\text{Niveau}(H^i(X_\Omega, l)) \le \begin{cases} 2r & \textit{for all } i > 2(n-s-1); \\ \max(2r, i-2s-2) & \textit{for all } i \end{cases}$$

for all good cohomology theories.

Proof. (iii) $\Rightarrow$ (i): This is trivial, since for $H^*(X, *)$ we are allowed to take the Chow groups (1.2).
(i) $\Rightarrow$ (ii): The hypothesis

$$\mathrm{Niveau}(A^n(X_\Omega))_{\mathbf{Q}} \leq r$$

means that there exists $Y \subset X_\Omega$ of dimension $\leq r$ such that

$$A^n(X_\Omega \setminus Y)_{\mathbf{Q}} = 0.$$

Taking k to be the smallest field of definition of X and Y, and using Bloch's result that for a field extension $K \subset L$ the application

$$A^*(M_K)_{\mathbf{Q}} \to A^*(M_L)_{\mathbf{Q}}$$

is injective [Bl 1, Appendix to Lecture 1], we can suppose that k is finitely generated over its prime subfield and that

$$\mathrm{Niveau}(A^n(X_K))_{\mathbf{Q}} \leq r$$

for any finitely generated $K \supset k$.

Consider now the restriction

$$A^n(X_k \times X_k)_{\mathbf{Q}} \to A^n(X_{k(X)})_{\mathbf{Q}}.$$

The last group has niveau $\leq r$ by assumption, i.e. there exists $V_0 \subset X_{k(X)}$ of dimension $\leq r$ and a surjection

$$A_0(V_0)_{\mathbf{Q}} \twoheadrightarrow A^n(X_{k(X)})_{\mathbf{Q}}.$$

In particular, the restriction of the diagonal to $X_{k(X)}$ comes from a cycle on V_0. Let $\Delta_0 \in A^n(X \times X)_{\mathbf{Q}}$ be the closure of this cycle.

By construction, the cycle

$$\Delta^1 = \Delta - \Delta_0 \in A^n(X \times X)_{\mathbf{Q}}$$

maps to 0 in

$$A^n(X_{k(X)})_{\mathbf{Q}} = \varinjlim_{U \subset X \text{ open}} A^n(X \times U)_{\mathbf{Q}},$$

so that it maps to 0 when restricted to some sufficiently small U. Denoting by $W_1 \subset X$ the complement of such a U, we find by localization that the cycle Δ^1 comes from $A_n(X \times W_1)_{\mathbf{Q}}$.

If $s = 0$, we have found a decomposition $\Delta = \Delta_0 + \Delta^1$ satisfying (ii), where

$W_0=X$. If $s\geq 1$, we apply the same reasoning to

$$\Delta^1\in A_n(X\times W_1)_{\mathbf{Q}}$$

and the restriction

$$A_n(X\times W_1)_{\mathbf{Q}}\to A^{n-1}(X_{k(W_1)})_{\mathbf{Q}}.$$

After $s+1$ steps, we arrive at a decomposition satisfying (ii).
(ii) $\Rightarrow$ (iii): We consider the action of the correspondence

$$\Delta=\Delta_0+\Delta_1+\cdots+\Delta_s+\Delta^{s+1}$$

on the R-module $H^i(X,l)$.

Let $\tilde{V}_j\to V_j$, $\tilde{W}_j\to W_j$ be generically finite proper morphisms with $\tilde{V}_j$ and $\tilde{W}_j$ smooth (these exist by de Jong's work [dJ]); let

$$\tilde{\Delta}_j\in A_n(\tilde{V}_j\times\tilde{W}_j)_{\mathbf{Q}}=A^r(\tilde{V}_j\times\tilde{W}_j)_{\mathbf{Q}}$$
$$\tilde{\Delta}^{s+1}\in A^{n-s-1}(X\times\tilde{W}_{s+1})_{\mathbf{Q}}$$

be cycles mapping to Δ_j, Δ^{s+1}.

First, let's consider the action of the correspondence Δ_j, which will be denoted $(\Delta_j)_*$, for $j=0,\cdots,s$. This action fits into the following commutative diagram:

$$\begin{array}{ccc}
H^i(\tilde{V}_j\times\tilde{W}_j,l) & \xrightarrow{\cdot[\tilde{\Delta}_j]} & H^{i+2r}(\tilde{V}_j\times\tilde{W}_j,l+r)\\
\uparrow & & \downarrow\\
H^i(\tilde{V}_j,l) & & H^{i-2j}(\tilde{W}_j,l-j)\\
\uparrow & & \downarrow\\
H^i(X,l) & \xrightarrow{(\Delta_j)_*} & H^i(X,l),
\end{array}$$

where the left (resp. right) vertical maps are the obvious pull-backs (resp. push-forwards). (Commutativity of this diagram follows from the axioms defining good cohomology: If $f_j\colon\tilde{V}_j\to X$, $g_j\colon\tilde{W}_j\to X$ denotes the natural proper morphisms, p_1 resp. $p_2\colon X\times X\to X$ denotes projection on the first resp. second factor, and $p_{\tilde{V}}$ resp. $p_{\tilde{W}}$ denotes projection from $\tilde{V}_j\otimes\tilde{W}_j$ on the first resp. second factor, then for any $\alpha\in H^i(X,l)$:

$$\begin{aligned}
(\Delta_j)_*\alpha &:=(p_2)_*((p_1)^*\alpha\cdot[\Delta_j])\\
&=(p_2)_*((p_1)^*\alpha\cdot(f_j\times g_j)_*[\tilde{\Delta}_j])\\
&=(p_2)_*(f_j\times g_j)_*((f_j\times g_j)^*(p_1)^*\alpha\cdot[\tilde{\Delta}_j])\\
&=(g_j)_*(p_{\tilde{W}})_*((p_{\tilde{V}})^*(f_j)^*\alpha\cdot[\tilde{\Delta}_j])
\end{aligned}$$

— We found this argument in [E-L,Lemma 2.1], where it is stated for Deligne cohomology.)

Note that $H^i(\tilde{V}_j, l)=0$ for $i>2\cdot\dim\tilde{V}_j=2(j+r)$, so the above diagram implies $(\Delta_j)_* H^i(X,l)$ comes from $\tilde{W}_j$ of dimension $n-j\leq n+\frac{2r-i}{2}$, i.e.

$$\operatorname{Niveau}((\Delta_j)_* H^i(X,l))\leq 2r.$$

Next we consider the action of the correspondence Δ^{s+1}. There is a commutative diagram similar to the above one:

$$\begin{tikzcd}
H^i(X\times\tilde{W}_{s+1}, l) \arrow[r, "\cdot[\tilde{\Delta}^{s+1}]"] & H^{i+2(n-s-1)}(X\times\tilde{W}_{s+1}, l+n-s-1) \arrow[d] \\
 & H^{i-2(s+1)}(\tilde{W}_{s+1}, l-s-1) \arrow[d] \\
H^i(X,l) \arrow[uu] \arrow[r, "(\Delta^{s+1})_*"] & H^i(X,l),
\end{tikzcd}$$

which implies

$$\operatorname{Niveau}((\Delta^{s+1})_* H^i(X,l))\leq i-2s-2.$$

Altogether, since $\Delta=\Delta_0+\ +\Delta_s+\Delta^{s+1}$ acts as the identity, we find that

$$\operatorname{Niveau}(H^i(X,l))=\operatorname{Niveau}(\Delta_* H^i(X,l))\leq\max(2r, i-2s-2).$$

To get the bound on the niveau in case $i>2(n-s-1)$, we apply the same reasoning to the correspondence

$$\Delta={}^t\Delta={}^t\Delta_0+\cdots+{}^t\Delta_s+{}^t\Delta^{s+1}$$

(where ^{t} denotes the transpose); vanishing of cohomology now gives that ${}^t\Delta^{s+1}$ acts as 0, and the conclusion follows.

(1.8) Remarks. 1. Here are some particular cases of theorem (1.7). Suppose X is defined over a universal domain $k=\Omega$, and that $A_0(X)_{\boldsymbol{Q}}\cong\boldsymbol{Q}$, i.e. $\operatorname{Niveau}(A^n(X)_{\boldsymbol{Q}})=0$. Then it follows from (1.7) that $A^1_{\text{hom}}(X)\otimes\boldsymbol{Q}=0$. More generally,

$$A_0^{\text{hom}}(X)_{\boldsymbol{Q}}=A_{d-1}^{\text{hom}}(X)_{\boldsymbol{Q}}=\cdots=A_s^{\text{hom}}(X)_{\boldsymbol{Q}}=0$$

implies

$$A^1_{\text{hom}}(X)_{\boldsymbol{Q}}=A^2_{\text{hom}}(X)_{\boldsymbol{Q}}=\cdots=A^{s+1}_{\text{hom}}(X)_{\boldsymbol{Q}}=0$$

(here I have used lemma (1.5)(i)).

Likewise,

$$A_0^{\text{AJ}}(X)_{\boldsymbol{Q}}=A_1^{\text{AJ}}(X)_{\boldsymbol{Q}}=\cdots=A_s^{\text{AJ}}(X)_{\boldsymbol{Q}}=0$$

implies

$$A^2_{\text{AJ}}(X)_{\boldsymbol{Q}}=A^3_{\text{AJ}}(X)_{\boldsymbol{Q}}=\cdots=A^{s+2}_{\text{AJ}}(X)_{\boldsymbol{Q}}=0$$

(using lemma (1.5)(ii)).

In particular, to have injectivity for all cycle class maps cl^i (resp. $cl^i_{\mathscr{D}}$) in Jannsen's theorem (0.3)(resp. in Esnault-Levine's theorem (0.4)), it suffices to have injectivity of a bit less than half of them.

I like to consider this influence of A^i for i large on A^i for i small as a kind of "crypto-Poincaré duality" on the level of Chow groups.

2. Here is another corollary of theorem (1.7): Let X, X' be two smooth complete varieties over a universal domain Ω, and suppose that

$$\text{Niveau}(A_i X) \leq r \quad \forall i \leq s,$$

$$\text{Niveau}(A_i X') \leq r' \quad \forall i \leq s.$$

Then one has

$$\text{Niveau}(A_i(X \times X')) \leq r + r' \quad \forall i \leq s$$

(as follows from the equivalence (i)$\Leftrightarrow$(ii) in (1.7)). For 0-cycles this corollary is easily proved directly, but for $i > 0$ it seems to be non-trivial. If a good category of mixed motives $\mathscr{M}\mathscr{M}_k$ exists, this corollary could be deduced from the Künneth formula for the Weil cohomology and Beilinson's formula, cf. remark (1.12)(ii) below; as such this corollary presents some evidence in favour of $\mathscr{M}\mathscr{M}_k$.

3. As many people have stressed in this context [Ja 2][Sc], the hypothesis that Ω be a "very large" field is essential in (1.7). For instance, if k is a finite field it is known that

$$\text{Niveau}(A_0(X_k)_{\mathbf{Q}} \leq 1$$

for any variety X [K-M]; the same is expected to hold for number fields k [Ja 1]. But of course, a variety over a finite field does not necessarily have an algebraic H^2.

4. In view of applications, it would be interesting to know whether theorem (1.7) holds without tensoring A^* by $\mathbf{Q}$. An application in the style of our second section, but not ignoring torsion, is given by Colliot-Thélène [Co, Theorem 4.3. 10].

In case of a Weil cohomology H^*, one can give a better bound for Niveau(H^*) than the one appearing in (1.7):

(1.9) Theorem. *Let X be a smooth complete variety of dimension n defined over a universal domain Ω. Suppose*

$$\text{Niveau}(A^i(X)_{\mathbf{Q}}) \leq r \text{ for all } i \geq n - s.$$

Then for any Weil cohomology H^* one has

$$\text{Niveau}(H^i X) \leq \max(r, i - 2s - 2),$$

i.e.

$$H^iX = N^lH^iX$$

for $l := \min(s+1, \lceil \frac{i-r}{2} \rceil)$.

Proof. It follows from (1.7) that the diagonal decomposes as

$$\Delta = \Delta_0 + \cdots + \Delta_s + \Delta^{s+1}$$

(notation as in (1.7)), and we consider the action of Δ on $\mathrm{Gr}^l_N H^iX$.

The action of Δ_j factors as

$$\begin{array}{ccc}
\mathrm{Gr}^l_N H^i(\tilde{V}_j \times \tilde{W}_j) & \xrightarrow{[\Delta_j]} & \mathrm{Gr}^{l+r}_N H^{i+2r}(\tilde{V}_j \times \tilde{W}_j) \\
\uparrow & & \downarrow \\
\mathrm{Gr}^l_N H^i \tilde{V}_j & & \mathrm{Gr}^{l-j}_N H^{i-2j} \tilde{W}_j \\
\uparrow & & \downarrow \\
\mathrm{Gr}^l_N H^iX & \xrightarrow{(\Delta_j)_*} & \mathrm{Gr}^l_N H^iX.
\end{array}$$

Clearly $\mathrm{Gr}^{l-j}_N = 0$ if $l < j$. On the other hand, it follows from weak Lefschetz that $\mathrm{Gr}^l_N H^i \tilde{V}_j = 0$ for $l < i - \dim \tilde{V}_j = i - j - r$. Putting these two inequalities together, we find that Δ_j acts as 0 if $l < [\frac{i-r}{2}]$.

Likewise, the correspondence Δ^{s+1} acts as 0 on $\mathrm{Gr}^l_N H^iX$ for $l < s+1$.

(1.10) Corollary. *Let X be a smooth complete n-dimensional veriety over $\boldsymbol{C}$, and suppose that*

$$\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}}) \le r \quad \text{for all } i \ge n-s.$$

Then

$$h^{p,q}(X) = 0 \quad \text{if } |p-q| > r \text{ and } p \le s.$$

The results (1.9) and (1.10) are "Mumford type" theorems. Inspired by Bloch's conjecture, several people [Pa][Ja 2] have conjectured the converse implication:

(1.11) Conjecture. The converses of (1.9) and (1.10) hold. In particular, for $X \subset \boldsymbol{P}^{n+1}(\boldsymbol{C})$ a degree d hypersurface, this conjecture predicts that $A_i(X)_{\boldsymbol{Q}} = \boldsymbol{Q}$ for all $i < \lfloor \frac{n+1}{d} \rfloor$.

(1.12) Remarks. 1. The case $s=0$, $r=1$ of theorem (1.9) is Mumford's theorem (0.1). The case $r=1$ of (1.10) is proven by Esnault-Levine [E-L]. A weaker version of (1.9) is proven by Paranjape [Pa], who also makes the conjecture (1.11).

Results closely related to (1.10) have been obtained by Lewis [Le 1] and Schoen [Sc], but only under the hypothesis of the generalized Hodge conjecture or some standard conjecture.

2. Philosophically speaking, in view of remark (1.6) one also expects a Mumford type theorem for the Beilinson-Murre filtration on Chow groups. That is, suppose such a filtration F^* exists. Then if X is an n-dimensional variety with $p_g(X)>0$, one should have

$$F^n A^n(X)_Q \neq 0,$$

i.e. the filtration has maximal length.

3. The converse of (1.9) would follow from the existence of a category of mixed motives $\mathcal{M}\mathcal{M}_k$ in which the so-called *Beilinson formula* holds:

$$\mathrm{Gr}^r_F A^i(X)_Q \cong \mathrm{Ext}^r_{\mathcal{M}\mathcal{M}_k}(h(\mathrm{Spec}\,k),\ h^{2i-r}(X)(i));$$

here F is the conjectural filtration on Chow groups alluded to in (1.6), and h denotes motives for homological equivalence. This argument is explained in detail in [Ja 2, 3.3 and 3.4].

The converse of (1.10) would follow from the converse of (1.9) in conjunction with the generalized Hodge conjecture, cf. (1.4).2.

4. Voisin has proven conjecture (1.11) for certain "well-formed" hypersurfaces [Vo 2]. Another result in the direction of (1.11) is proven by Esnault-Levine-Viehweg [E-L-V].

2. Applications

§2.1. Surjectivity

(2.1) Proposition. *Let X be a smooth complete n-dimensional variety defined over a universal domain Ω.*
(i) *Let H^* be a Weil cohomology. Suppose the cycle class map*

$$\mathrm{cl}^i: A^iX \otimes H^0\Omega \to H^{2i}X$$

is injective for all $i \geq n-s$. Then cl^i is an isomorphism for $i \geq n-s$ and for $i \leq s+1$.
(ii) *Suppose $\Omega = C$ and suppose the map*

$$\mathrm{cl}^i_{\mathscr{D}}: A^i(X)_Q \to H^{2i}_{\mathscr{D}}(X, Q(i))$$

is injective for all $i \geq n-s$. Then $\mathrm{cl}^i_{\mathscr{D}}$ is an isomorphism for $i \geq n-s$ and for $i \leq s+2$.

Proof. (i) From (1.5) and (1.7) it follows that the diagonal of X decomposes as

$$\Delta = \Delta_0 + \cdots + \Delta_s + \Delta^{s+1} \in A^n(X \times X) \times H^0(\Omega),$$

where the Δ_j have support on lower-dimensional varieties $V_j \times W_j$ as in (1.7). Consider now the action of Δ on H^{2i}, for some $i \leq s+1$. The action of Δ_j factors as

$$\begin{array}{ccc} H^{2i}(\tilde{V}_j \times \tilde{W}_j) & \xrightarrow{\cdot[\tilde{\Delta}_i]} & H^{2k}(\tilde{V}_j \times \tilde{W}_j) \\ \uparrow & & \downarrow \\ H^{2i}(\tilde{V}_j) & & H^{2i-2j}(\tilde{W}_j) \\ \uparrow & & \downarrow \\ H^{2i}(X) & \xrightarrow{(\Delta_j)_*} & H^{2i}(X) \end{array}$$

(notation as in (1.7)). But the group $H^{2i}(\tilde{V}_j)$ vanishes if $i > \dim \tilde{V}_j = j$, and for $i \leq j$ the group $H^{2i-2j}(\tilde{W}_i)$ is either 0 or generated by cycles. A similar diagram shows $(\Delta^{s+1})_* H^{2i}$ to be generated by cycles (here the assumption $i < s+1$ comes in), and we conclude that

$$H^{2i}(X) = \Delta_* H^{2i}(X)$$

is generated by cycles.

In case $i \geq n-s$, we use the transpose of the diagonal

$$\Delta = {}^t\Delta = {}^t\Delta_0 + \cdots + {}^t\Delta_s + {}^t\Delta^{s+1}.$$

(ii) Similar to the above.

(2.2) Corollary. *Let X be a smooth complete Fano variety over $\mathbf{C}$. Then the Abel-Jacobi map*

$$\mathrm{AJ}^2 : A^2_{\mathrm{hom}}(X) \to J^2$$

is an isomorphism modulo torsion.

Proof. A Fano variety X is rationally connected [Ca][Ko], so has $A_0(X)_{\mathbf{Q}} \cong \mathbf{Q}$.

(2.3) Remarks. 1. For quartic 3-folds, Bloch proves that AJ^2 is an isomorphism also on the torsion parts [Bl 1, Lecture 3].

2. It follows from (2.2) that every Fano hypersurface whose J^2 is non-trivial modulo torsion is an exception to the Noether-Lefschetz theorem. These exceptions (cubic and quartic 3-folds) are also noted by Green [Gre].

§2.2. Hodge conjecture

(2.4) Proposition. *Let X be a smooth complete n-dimensional variethy over $\mathbf{C}$.*
(i) Suppose $\mathrm{Niveau}(A_i(X)_{\mathbf{Q}}) \leq 3$ for $i = 0, 1, \cdots, s$. Then the Hodge conjecture for X is verified in codimensions $\leq s+2$ and $\geq n-s-2$, i.e. the map

$$\mathrm{cl}^i : A^i(X)_{\mathbf{Q}} \to H^{i,i}(X, \mathbf{Q})$$

is surjective for $i \leq s+2$ and for $i \geq n-s-2$;
(ii) Suppose $\mathrm{Niveau}(A_i(X)_{\mathbf{Q}}) \leq 2$ for $i = 0, 1, \cdots, s$. Then the generalized Hodge conjecture for X is verified in degrees $i \leq 2s+4$ and $\geq 2n-2s-3$, i.e. for these values

of i, every level $i-2l$ sub-Hodge structure of $H^i(X, \mathbb{Q})$ is contained in $N^l H^i(X, \mathbb{Q})$.

Proof. (i) We use the decomposition of the diagonal resulting from (1.7), and consider the action of Δ on

$$HC^i(X) := H^{i,i}(X, \mathbb{Q})/\mathrm{Im}\,\mathrm{cl}^i.$$

The action of Δ^{s+1} factors as

$$\begin{array}{ccc} HC^i(X\times \tilde{W}_{s+1}) & \rightarrow & HC^{i+n-s-1}(X\times \tilde{W}_{s+1}) \\ & & \downarrow \\ \uparrow & & HC^{i-s-1}(\tilde{W}_{s+1}) \\ & & \downarrow \\ HC^i(X) & \xrightarrow{(\Delta^{s+1})_*} & HC^i(X). \end{array}$$

Since the Hodge conjecture is known for curves and divisors, the group $HC^{i-s-1}(\tilde{W}_{s+1})$ vanishes for $i \leq s+2$, i.e. Δ^{s+1} acts as 0 on $HC^i(X)$ for these values of i.

The action of Δ_j ($j=0, \cdots, s$) factors as

$$\begin{array}{ccc} HC^i(\tilde{V}_j\times \tilde{W}_j) & \rightarrow & HC^{i+3}(\tilde{V}_j\times \tilde{W}_j) \\ \uparrow & & \downarrow \\ HC^i(\tilde{V}_j) & & HC^{i-j}(\tilde{W}_j) \\ \uparrow & & \downarrow \\ HC^i(X) & \xrightarrow{(\Delta_j)_*} & HC^i(X). \end{array}$$

The Hodge conjecture being known for curves, $HC^i(\tilde{V}_j)=0$ if $i \geq \dim \tilde{V}_j - 1 = j+2$. But since the Hodge conjecture is known for divisors, $HC^{i-j}(\tilde{W}_j)=0$ if $i \leq j+1$.

We conclude that Δ acts trivially on $HC^i(X)$ for $i \leq s+2$, so these groups are 0, i.e. the Hodge conjecture holds in this range.

For $i \geq n-s-2$, we use the transpose of the diagonal.

(ii) Follows as above, using the fact that the generalized Hodge conjecture is known in degrees ≤ 2 and $\geq 2n-1$.

(2.5) Corollary. (i) *The Hodge conjecture is completely verified for: uniruled* 4-*folds; rationally connected* 4-*and* 5-*folds* (*in particular Fano* 4-*and* 5-*folds*);

(ii) *The generalized Hodge conjecture is completely verified for: uniruled* 3-*folds; rationally connected* 3-*and* 4-*folds* (*in particular Fano* 3-*and* 4-*folds*); *cubics of dimension at most* 6; *a variety of dimension at most* 6 *which is the intersection of a quadric and a cubic; a variety of dimension at most* 8 *which is the intersection of two quadrics.*

Proof. (i) Obviously uniruled 4-folds have $\mathrm{Niveau}(A_0(X)) \leq 3$, and rationally connected varieties have $\mathrm{Niveau}(A_0(X)) \leq 0$.

(ii) Obviously uniruled 3-folds have $\mathrm{Niveau}(A_0(X)) \leq 2$.

For cubic 5-and 6-folds, the conclusion follows from the fact that they verify

$$A_0(X)_{\boldsymbol{Q}} \cong A_1(X)_{\boldsymbol{Q}} \cong \boldsymbol{Q}$$

(i.e. these two Chow groups have niveau ≤ 0), which is proven by Paranjape [Pa] and by Kollár [Ko], generalizing Roitman's work on A_0 [Ro].

The intersection of a quadric and a cubic also has A_0 and A_1 of rank one; this is proven by Esnault-Levine-Viehweg [E-L-V].

The intersection of two quadrics has

$$A_0(X)_{\boldsymbol{Q}} \cong A_1(X)_{\boldsymbol{Q}} \cong A_2(X)_{\boldsymbol{Q}} \cong \boldsymbol{Q},$$

this is again proven in [E-L-V].

(2.6) Remarks. 1. For uniruled 4-folds, the Hodge conjecture was first proven by Conte and Murre [C-M]; it has since been reproven in many different ways [St][SaM, Remark 1.8].

2. The case $s=0$ of (2.4)(i) was proven by Bloch and Srinivas (only they forgot to mention that the Hodge conjecture is also verified in codimension $n-2$).

§2.3. Algebraic and homological equivalence

(2.7) Proposition. *Let X be a smooth complete n-dimensional variety over a universal domain Ω, and suppose* $\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}}) \leq 2$ for $i=0,\cdots,s$. *Then the Griffiths group*

$$\mathrm{Gr}^i(X)_{\boldsymbol{Q}} := Z^i_{\mathrm{hom}}(X) \otimes \boldsymbol{Q} / Z^i_{\mathrm{alg}}(X) \otimes \boldsymbol{Q}$$

is 0 *for $i \leq s+2$ and for $i \geq n-s-1$.*

Proof. Let $\Delta = \Delta_0 + \cdots + \Delta_s + \Delta^{s+1}$ act on $\mathrm{Gr}^i(X)_{\boldsymbol{Q}}$.

The action of Δ_j factors as

$$\begin{array}{ccc}
\mathrm{Gr}^i(\tilde{V}_j \times \tilde{W}_j)_{\boldsymbol{Q}} & \to & \mathrm{Gr}^{i+2}(\tilde{V}_j \times \tilde{W}_j)_{\boldsymbol{Q}} \\
\uparrow & & \downarrow \\
\mathrm{Gr}^i(\tilde{V}_j)_{\boldsymbol{Q}} & & \mathrm{Gr}^{i-j}(\tilde{W}_j)_{\boldsymbol{Q}} \\
\uparrow & & \downarrow \\
\mathrm{Gr}^i(X)_{\boldsymbol{Q}} & \xrightarrow{(\Delta_j)_*} & \mathrm{Gr}^i(X)_{\boldsymbol{Q}}.
\end{array}$$

Since homological and algebraic equivalence coincide for 0-cycles, the group $\mathrm{Gr}^i(\tilde{V}_j)_{\boldsymbol{Q}}$ vanishes for $i \geq \dim \tilde{V}_j = j+2$. Since homological and algebraic equivalence coincide for divisors, the group $\mathrm{Gr}^{i-j}(\tilde{W}_j)_{\boldsymbol{Q}}$ vanishes for $i \leq j+1$. It follows that Δ_j acts as 0 on $\mathrm{Gr}^i(X)_{\boldsymbol{Q}}$.

Similarly, we find that Δ^{s+1} acts as 0 if $i \leq s+2$; this ends the proof for $i \leq s+2$.

For $i \geq n-s-1$, we use the transpose $\Delta = {}^t\Delta = {}^t\Delta_0 + \cdots + {}^t\Delta_s + {}^t\Delta^{s+1}$.

(2.8) Remarks.1. Proposition (2.7) is inspired by Bloch-Srinivas, who prove the case $s=0$. In fact, using Merkuriev-Suslin on K_2, they prove vanishing of $\mathrm{Gr}^2(X)$ not neglecting torsion [B-S, Theorem 1].

2. By way of example: every rationally connected 3-fold or 4-fold has torsion Griffiths groups; the same holds for cubic 5-folds and 6-folds; a cubic 7-fold X has

$$\mathrm{Gr}^i(X)_{\boldsymbol{Q}}=0 \quad \text{for} \quad i\leq 3 \text{ or } i\geq 5$$

(for these examples, cf. the proof of (2.5)). This last result is optimal since Albano and Collino have proven that a cubic 7-fold has a non-finitely generated $\mathrm{Gr}^4(X)_{\boldsymbol{Q}}$[A-C].

§2.4. Chow-Lefschetz conjecture

(2.9) The *Chow-Lefchetz conjecture* [Ha] asserts that if $X\subset Z$ is an inclusion of smooth complete varieties such that the complement $Z\backslash X$ is affine, then pull-back induces an isomorphism

$$A^iZ \xrightarrow{\sim} A^iX \quad \text{for} \quad i<\dim X/2=:n/2.$$

The case $i=1$ has been settled by Grothendieck [SGA2], but apart from this little progress has been made, not even for $Z=\boldsymbol{P}^{n+1}$. Note that the truth of the conjecture would follow from the truth of Beilinson's fomula mentioned in (1.12).3.

(2.10) Proposition. *Let $X\subset Z$ be as in* (2.9), *defined over a universal domain Ω. Suppose that $A^*(Z)_{\boldsymbol{Q}}\to H^*Z$ is an isomorphism.*
(i) *Suppose $A_i^{\mathrm{hom}}(X)_{\boldsymbol{Q}}=0$ for $i=0,\cdots,s<\frac{n}{2}-2$. Then there are isomorphisms*

$$A^i(Z)_{\boldsymbol{Q}} \xrightarrow{\sim} A^i(X)_{\boldsymbol{Q}} \quad \text{for } i\leq s+2;$$

(ii) *Let $\Omega=\boldsymbol{C}$. Suppose $A_i^{\mathrm{AJ}}(X)_{\boldsymbol{Q}}=0$ for $i=0,\cdots,s<\frac{n}{2}-4$.* Then there are isomorphisms

$$A^i(Z)_{\boldsymbol{Q}} \xrightarrow{\sim} A^i(X)_{\boldsymbol{Q}} \quad \text{for } i\leq s+3.$$

Proof. (i) Immediate from (1.8).1 and the weak Lefschetz theorem for the Weil cohomology (1.1)(iv).
(ii) Immediate from (1.8).1 and the weak Lefschetz theorem for Deligne cohomology.

(2.11) Examples. Let $X\subset \boldsymbol{P}^{n+1}(\boldsymbol{C})$ be a smooth hypersurface of degree d, and suppose $d\leq n>8$. Then $A^i(X)_{\boldsymbol{Q}}=\boldsymbol{Q}$ for $i\leq 3$.

For cubics we can actually do better: suppose $d=3$ in the above, then

$$A^i(X)_{\boldsymbol{Q}}=\boldsymbol{Q} \quad \text{for} \quad i<\min(L+3,\tfrac{n}{2}),$$

$$A^i(X)_{\boldsymbol{Q}}=\boldsymbol{Q} \quad \text{for} \quad i<\min(L+4,\tfrac{n}{2}-1),$$

where L is defined as the largest integer satisfying $(L+2)(L+3)\leq 2n+2$. (This last result follows from (2.10) combined with the fact that cubics have $A^i(X)=\boldsymbol{Q}$ for $i\leq L$ [Ko] [E-L-V].)

§2.5, Decomposability

(2.12) Definition. Let X be a smooth complete variety over a field k, and let $\mathscr{K}_l$ denote the Zariski sheaves on X associated to higher algebraic K-theory. We say that the group $H^i(X,\mathscr{K}_l)$ is *decomposable* if the cokernel of the natural map

$$H^i(X,\mathscr{K}_i)\otimes K_{l-i}k\rightarrow H^i(X,\mathscr{K}_l)$$

is torsion.

Likewise, we say that the higher Chow group $A^i(X,l)$ [Bl 2] is *decomposable* if the natural map

$$A^{i-l}X\otimes A^l(k,l)\rightarrow A^i(X,l)$$

has torsion cokernel.

(2.13) Proposition. *Let X be a smooth complete n-dimensional variety over a universal domain Ω.*
(i) *Suppose that* $\mathrm{Niveau}(A^i(X)_{\boldsymbol{Q}})\leq 1$ *for all $i\leq s$. Then there are isomorphisms*

$$A^{i-1}X\otimes A^1(\Omega,1)\otimes\boldsymbol{Q}\xrightarrow{\sim}A^i(X,1)\otimes\boldsymbol{Q}$$

for $i\leq s+2$ and for $i\geq n-s$;
(ii) *Suppose that $A^{\mathrm{hom}}_i(X)_{\boldsymbol{Q}}=0$ for all $i\leq s$. Then $A^i(X,2)$ is decomposable for $i\leq s+2$ and for $i\geq n-s+1$.*

Proof. (i) Suppose first $i\leq s+2$. To prove decomposability, consider the action of

$$\Delta=\Delta_0+\cdots+\Delta_s+\Delta^{s+1}$$

on $A^i(X,1)$.

The action of Δ_j factors over $A^i(\tilde{V}_j,1)$ (which by (2.14) is decomposable for $i\geq\dim\tilde{V}_j+1=j+2$) and over $A^{i-j}(\tilde{W}_j,1)$ (which by (2.14) is decomposable for $i\leq j+1$), so it sends $A^i(X,1)$ into its decomposable part.

The action of Δ^{s+1} factors over $A^{i-s-1}(\tilde{W}_{s+1},1)$, so goes into the decomposable part for $i\leq s+2$.

To prove injectivity, consider the action of Δ on

$$\mathrm{Ker}(A^{i-1}X\otimes A^1(\Omega,1)\otimes\boldsymbol{Q}\rightarrow A^i(X,1))$$

and use lemma (2.14).

In case $i \geq n-s$, use the transpose of the diagonal.
(ii) Similar to (i)

(2,.14) Lemma. *Let M be a smooth m-dimensional variety over a field k. Then the natural map determines isomorphisms*

$$A^i M \otimes A^1(k,1) \xrightarrow{\sim} A^{i+1}(M,1)$$

for $i=0$ and for $i=m$.

Proof. The $i=0$ case follows from Bloch's computation $A^1(M,1)=k^*$ [Bl 2, Theorem 6.1].

For $i=m$, surjectivity is obvious. To prove injectivity, note that by the truth of Gersten's conjecture [Qu, §7 Prop. 5.14] [Bl 2, §10], $A^{m+1}(M,1) \cong H^m(M, \mathscr{K}_{m+1})$ equals

$$\operatorname{Coker}\left(\bigoplus_{x \in M^{(m-1)}} A^2(k(x),2) \to \bigoplus_{x \in M^{(m)}} A^1(k,1) \right)$$

(where as usual $M^{(i)}$ denotes codimension i points of M).

Also $A^m M \otimes A^1(k,1)$ equals

$$\operatorname{Coker}\left(\bigoplus_{x \in M^{(m-1)}} A^1(k(x),1) \otimes A^1(k,1) \to \bigoplus_{x \in M^{(m)}} A^1(k,1) \right),$$

and the exterior product map factors over the groups inside the parentheses, so injectivity follows from surjectivity of

$$A^1(k(x),1) \otimes A^1(k,1) \to A^2(k(x),2).$$

(2.5) Corollary. *Let X be a smooth complete variety over $\mathbf{C}$, and suppose $A_i^{\mathrm{AJ}}(X)_{\mathbf{Q}}=0$ for $i=0,\cdots,s$. Then the cycle class map*

$$A^i(X,1)_{\mathbf{Q}} \to H_{\mathscr{D}}^{2i-1}(X, \mathbf{Q}(i))$$

is surjective with kernel $A_{\mathrm{hom}}^{i-1} X \otimes A^1(\mathbf{C},1)$ for $i \leq s+2$ and for $i \geq n-s$.

Proof. Applying the diagonal to Deligne cohomology, we find that $H_{\mathscr{D}}^{2i-1}(X, \mathbf{Q}(i))$ is decomposable for the indicated i, i.e. there is a surjection

$$p: H_{\mathscr{D}}^{2i-1}(X, \mathbf{Q}(i-1)) \otimes H_{\mathscr{D}}^1(\mathbf{C}, \mathbf{Q}(1)) \to H_{\mathscr{D}}^{2i-1}(X, \mathbf{Q}(i));$$

this proves surjectivity of the cycle class map.

To prove the statement about the kernel, it suffices to prove that

$$\operatorname{Ker} p = \operatorname{Ker}(H_{\mathscr{D}}^{2i-2}(X, \mathbf{Q}(i-1)) \to H^{i-1,i-1}(X, \mathbf{Q})) \otimes H_{\mathscr{D}}^1(\mathbf{C}, \mathbf{Q}(1))$$

for the indicated i. This last statement follows from the $i=\dim X+1$-case [E-L, Lemma 2.2] after applying the diagonal.

(2.16) Corollary. *Let X be a smooth complete n-dimensional variety over a universal domain.*
(i) *Under the assumption of* (2.13)(*i*), $H^{i-1}(X,\mathscr{K}_i)$ *is decomposable for* $i\leq s+2$ *and for* $i\geq n-s$;
(ii) *Under the assumption of* (2.13)(*ii*), $H^{i-2}(X,\mathscr{K}_i)$ *is decomposable for* $i\leq s+2$ *and for* $i\geq n-s+1$.

Proof. This is immediate from (2.13) and the existence of functorial "Bloch formula" isomorphisms

$$A^i(X,l)_{\mathbf{Q}}\cong H^{i-1}(X,\mathscr{K}_i)\otimes\mathbf{Q}\quad\text{for}\quad l\leq 2$$

[La 1, 2.27].

(2.17) Remarks. 1. The decomposability of $A^2(S,1)$ for a surface S with $A_0^{\mathrm{AJ}}(S)_{\mathbf{Q}}=0$ was proven by Coombes-Srinivas [C-S]. The decomposability of $A^i(X,1)$ for X and i as in (2.13)(i) was proven by Esnault-Levine [E-L, §4]; the isomorphism in (2.13)(i) answers a question about the kernel asked by Müller -Stach [Mü].

2. The notion of "decomposable $H^i(X,\mathscr{K}_l)$" as given in (2.12) is more restrictive than Esnault-Levine's definition [E-L]; however the two notions coincide for $l-i=1$ [E-L, §4].

3. In contrast to (2.15), Voisin has proven [Vo 1, 1.6] that if $X\subset\mathbf{P}^3(\mathbf{C})$ is a general hypersurface of general type, then the cycle class map

$$A^2(X,1)_{\mathbf{Q}}\to H^3_{\mathscr{D}}(X,\mathbf{Q}(2))$$

is not surjective.

4. As a special case of (2.16), we find that $H^1(X,\mathscr{K}_2)$, $H^{n-1}(X,\mathscr{K}_n)$, $H^0(X,\mathscr{K}_2)$ and $H^{n-1}(X,\mathscr{K}_{n+1})$ are decomposable for any Fano variety X of dimension n; for more examples of varieties satisfying the assumptions, cf. (2.5).

§**2.6. Murre conjectures.** In a sequel to this article [La 2], we prove that Murre's conjectures (on a decomposition of the Chow motive of a variety, [Mur]) hold for varieties X over a universal domain verifying

$$\dim X\leq 4,\qquad \text{Niveau}(A^i(X)_{\mathbf{Q}})\leq 2.$$

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