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Blow-ups of $\mathbf{P}^2$ and root systems of type D

By

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1. Introduction

Nonsingular cubic surfaces in $\mathbf{P}^3(\mathbf{C})$ are obtained by blowing up 6 points on $\mathbf{P}^2$. Also it is well known that geometry of cubic surfaces is closely related to the root system and Weyl group of type E_6; (i) The symmetry of the 27 lines on nonsingular cubic surface can be described by the root system and Weyl group of type E_6. (ii) In the middle homology lattice of cubic surface, the orthogonal complement of the class of canonical divisor is isomorphic to the root lattice of type E_6. (iii) The semi-universal deformation of simple singularity of type E_6 can be described by a Cartan subalgebra of Lie algebra of type E_6. Furthermore a nonsingular cubic surface can be regarded as a compactification of a generic fiber of this deformation.

For certain class of rational surfaces, the geometry of surfaces is closely related to infinite root systems and the moduli space for the surfaces are constructed in terms of root systems and periods [9].

In this paper, we construct rational surfaces related to the root system and Weyl group of type D_m. We discuss the moduli problem of the surfaces.

In sections 2 and 3, we show the relation between surfaces X_m obtained by blowing up m points on $\mathbf{P}^2$ and the root systems and Weyl groups of type D_m. In section 4, we prove the theorem of Torelli type for the pairs of X_m and a certain anticanonical divisor of X_m in terms of the root systems and Weyl groups of type D_m. In section 5, we construct a family $\varphi : \mathfrak{X} \to S$ of the surfaces X_{2n+3}, where the base space S is the quotient space of the Cartan subalgebra of simple Lie algebra of type D_{2n+3} by its Weyl group.

The nonsingular fiber $\mathfrak{X}_s$ can be regarded as a compactification of the fiber of semi-universal deformation of the simple singularity of type D_{2n+3}. So the relation between X_m and the simple singularity of type D_{2n+3} is similar to that between Del Pezzo surfaces and the simple singularities of type E (see Remark 5.10). In section 6, we show the relation between the surface Z_{2n+2} obtained by blowing up X_{2n+2} and the root system of type D_{2n+2}. Also we can construct a family $\varphi : \mathfrak{X} \to S$ of these surfaces Z_{2n+2}, where the base space S is the quotient space of the Cartan subalgebra of simple Lie algebra of type D_{2n+2} by its Weyl group. The

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fiber can be also regarded as a compactification of the fiber of semi-universal deformation of the simple singularity of type D_{2n+2}.

The period mapping of semi-universal deformation $\mathfrak{W} \to S$ of simple singularity of type D_m is studied by Looijenga and Saito ([8], [15]). We give a concrete description of the period mapping for the families constructed in section 5 and 6 in terms of the root system and Weyl group of type D_{2n+3} and D_{2n+2}.

In sections 5 and 6, we define a meromorphic 2-form ω on $\mathfrak{X}$. Denote by $\Delta \subset S$ the discriminant variety of φ and by $\mathfrak{D}_s (s \in S \backslash \Delta)$ the anticanonocal divisor on $\mathfrak{X}_s$ such that the restriction of ω to $\mathfrak{X}_s$ has poles along $\mathfrak{D}_s$. Then the monodromy group of $\pi_1(S\backslash\Delta)$ on $H_2(\mathfrak{X}_s \backslash \mathfrak{D}_s; \mathbf{Z})$ is isomorphic to the Weyl group of type D_m and $\pi_1(S\backslash\Delta)$ acts on the period domain as a reflection group which is isomorphic to the Weyl group of type D_m.

2. $\mathbf{P}^2$ with several points blown up

Let C be a conic in $\mathbf{P}^2$, L a line tangent to C and P a point on $L\backslash C$. By $\mathfrak{C}$ we denote the set of all such pairs (C, L, P) and an elements of $\mathfrak{C}$ is said to be the framing. Assume $m \geq 4$ in this paper.

Definition 2.1. For a framing (C, L, P), we say that m points $P_1, \dots, P_m$ on $C\backslash L$ are in general position if m points $P_1, \dots, P_m$ are distinct and if P and any two of them are not collinear.

Let $P_1, \dots, P_m$ be m points on $C\backslash L$ in general position. Let

$$p : X_m \to \mathbf{P}^2$$

be the blowing up of $\mathbf{P}^2$ at $P_1, \dots, P_m$ and P. Then put $E_P = p^{-1}(P), E_1 = p^{-1}(P_1), \dots, E_m = p^{-1}(P_m)$. Let $\bar{L}, \bar{C}$ be the proper transforms of L and C. Then $D = \bar{L} + \bar{C}$ is an anticanonical divisor on X_m.

Definition 2.2. Let (C, L, P) (resp. $(C', L', P')) \in \mathfrak{C}$. Let X_m (resp. X'_m) be surface obtained by blowing up P (resp. P') and m points on $C\backslash L$ (resp. $C'\backslash L'$) in general position. Put $D = \bar{L} + \bar{C}$ (resp. $D' = \bar{L}' + \bar{C}'$).

Then we say that the pairs (X_m, D) and (X'_m, D') are isomorphic if there exists an isomorphism $\phi : X_m \to X'_m$ such that

$$\phi(\bar{C}) = \bar{C}', \phi(\bar{L}) = \bar{L}'.$$

Lemma 2.3. *Let $(x : y : z)$ be homogeneous coordinate of $\mathbf{P}^2$, C a conic defined by $z^2 = xy$, L a line defined by $x = 0$ and $P = (0 : 0 : 1)$.*

Let $(C', L', P'), X'_m, D'$ be as above. Then there exist m points $P_1, \dots, P_m$ in general position with respect to the framing (C, L, P) which have the following property;

Let X_m be the surface obtained by blowing up $P, P_1, \dots, P_m$, then there exists an isomorphism $\Phi : X_m \to X'_m$ such that

$$\Phi(\bar{C}) = \bar{C}', \qquad \Phi(\bar{L}) = \bar{L}'.$$

By this lemma, we may assume (C, L, P) as in the lemma.

Proposition 2.4. *Let* (C, L, P) *be as above. Let* $P_1, \dots, P_m$ *(resp.* $P'_1, \dots, P'_m$*) be* m *points on* $C \backslash L$ *in general position respectively. Let* $p : X_m \to \mathbf{P}^2$ *(resp.* $p' : X'_m \to \mathbf{P}^2$*) be the blowing up* $P, P_1, \dots, P_m$ *(resp.* $P'_1, \dots, P'_m$*).*

Put $E_i = p^{-1}(P_i)$ *(resp.* $E'_i = p'^{-1}(P'_i)$*). Put* $D = \bar{L} + \bar{C}$ *(resp.* $D' = \bar{L}' + \bar{C}'$*), where* $\bar{L}, \bar{C}$ *(resp.* $\bar{L}', \bar{C}'$*) are proper transforms of* L, C. *Let* $P_i = (1 : s_i^2 : s_i)$ *(resp.* $P'_i = (1 : s_i'^2 : s'_i)$*).*

Then there exists isomorphism $\Phi : (X_m, D) \to (X'_m, D')$ *such that* $\Phi(E_i) = E'_i$ $(i = 1, \dots, m)$ *if and only if there exists* $\alpha \in \mathbf{C}^*$ *such that*

$$s_i = \alpha s'_i \qquad (i = 1, \dots m).$$

Proof. Let A be an element of $PGL(3, \mathbf{C})$ such that $A(C) = C$, $A(L) = L$, $A(P) = P$. Then line defined by $y = 0$ is tangent to C at $(1 : 0 : 0)$ and passes through P. Therefore A maps the point $(1 : 0 : 0)$ to itself. Since A also satisfies that $A((0 : 0 : 1)) = (0 : 0 : 1)$ and $A((1 : 0 : 0)) = (1 : 0 : 0)$, we have

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Since $A(C) = C$, we have $c^2 = ab$. Therefore

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

The result follows from this.

Proposition 2.5. *Let* $p : X_m \to \mathbf{P}^2$ *be as above and* L_i $(1 \le i \le m)$ *the proper transform of the line passing through* P *and* P_i. *Let* I *be a subset of* $\{1, \dots, m\}$ *and assume the number* $\# I$ *is even. Then we have the Hirzebruch surface* Σ_1 *of degree* 1 *by contracting* L_i *for* $i \in I$ *and* E_j *for* $j \in \{1, \dots, m\} - I$. *Contracting* (-1)*-section of* Σ_1 *further, we get another framing* $(C', L', P') \in \mathfrak{C}$, *where* C', L' *are the images of* $\bar{C}, \bar{L}$ *and* P' *is the image of* (-1)*-section of* Σ_1.

Proof. By contracting L_i for $i \in I$ and E_j for $j \in \{1, \dots, m\} - I$, we have $\mathbf{P}^1$ bundle over $\mathbf{P}^1$. Therefore the resulting surface is isomorphic to the Hirzebruch surface Σ_r of degree r for some r.

$$p' : X_m \to \Sigma_r.$$

Let $f, s (\in H_2(\Sigma_r; \mathbf{Z}))$ be the classes of a fiber and the $(-r)$-section S of Σ_r. Let $\bar{c}' = af + bs$ be the class of $p'(\bar{C})$, then $b = 2$, because a fiber of Σ_r intersect $p'(\bar{C})$ at 2 points. Since

$$0 \le p'(\bar{C}) \cdot S = (af + 2s) \cdot s = a - 2r,$$

we have

$$a \geq 2r.$$

Also

$$4 = p'(\bar{C}) \cdot p'(\bar{C}) = 4a - 4r.$$

Thus we have $r = a - 1$. Since $a \geq 2r$, we have $r \leq 1$. Therefore r is 0 or 1.

Let D be the section of Σ_r. Since the class of D is $xf + s$ $(x \in \mathbf{N})$ and

$$D \cdot D = 2x - r,$$

we have

$$D \cdot D \equiv S \cdot S \qquad (\operatorname{mod} 2).$$

If we take $D = p'(E_P)$, then

$$p'(E_P) \cdot p'(E_P) = -1 + \# I$$

is odd. Therefore $S \cdot S$ must be odd. Hence we have $r = 1$.

The remaining part of proposition is obvious.

3. Homology and root system

In this section, we shall study the exceptional curves of the first kind on X_m and the homology groups of X_m and $X_m \backslash D (D = \bar{C} + \bar{L})$. The root systems of type D_m can be realized in the middle homology group of X_m. The Weyl group can be regarded as the automorphism group of the configuration of the exceptional curves of the first kind. It is similar to the realization of the root systems and Weyl groups of type E_6 in that of cubic surfaces.

Let $e_P, e_1, \dots, e_m \in H_2(X_m; \mathbf{Z})$ be the classes of the exceptional curves E_P, $E_1, \dots, E_m$ defined in section 2. Let $l \in H_2(X_m; \mathbf{Z})$ be the class of total transform of line. Then we have next proposition.

Propositon 3.1. (1) *$H_2(X_m; \mathbf{Z})$ is generated by l, $e_P, e_1, \dots, e_m$.*
(2) *The intersection pairing on X_m is given by*

$$l^2 = 1, \qquad e_P^2 = -1, \qquad e_i^2 = -1 \qquad (i = 1, \dots, m),$$

$$l \cdot e_P = 0, \qquad l \cdot e_i = 0 \qquad (i = 1, \dots, m),$$

$$e_i \cdot e_P = 0 \qquad (i = 1, \dots, m), \qquad e_i \cdot e_j = 0 \qquad (i \neq j \text{ and } i, j = 1, \dots, m).$$

(3) *The class of canonical divisor on X_m is $k_m = -3l + e_P + e_1 + \cdots + e_m$.*

Now we consider the homology exact sequence:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_3(X_m; \mathbf{Z}) & \longrightarrow & H_3(X_m, X_m \backslash D; \mathbf{Z}) \\ & & & & \| \\ & & & & 0 \\ \xrightarrow{\partial_*} & H_2(X_m \backslash D; \mathbf{Z}) & \xrightarrow{i_*} & H_2(X_m; \mathbf{Z}) & \xrightarrow{j_*} & H_2(X_m, X_m \backslash D; \mathbf{Z}) \\ \longrightarrow & \cdots \end{array}$$

The intersection pairing in $H_2(X_m;\mathbf{Z})$ can be extended to the bilinear form on $H_2(X_m;\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{R}$. Put

$$Q = \ker j_* \subset H_2(X_m;\mathbf{Z}),$$
$$R = \{\alpha \in Q \mid \alpha\cdot\alpha = -2\}.$$

Lemma 3.2. *Let Q and R be as above. Then we have*

$$H_2(X_m\backslash D;\mathbf{Z}) \cong Q.$$

Proof. By the definition of Q, we have a following short exact sequences

$$0 \longrightarrow H_3(X_m, X_m\backslash D;\mathbf{Z}) \xrightarrow{\partial_*} H_2(X_m\backslash D;\mathbf{Z}) \xrightarrow{i_*} Q \to 0.$$

Therefore we have only to prove that $H_3(X_m, X_m\backslash D;\mathbf{Z}) = 0$. By the duality,

$$\begin{aligned} H_3(X_m, X_m\backslash D;\mathbf{Z}) &\cong H^1(D;\mathbf{Z}) \\ &\cong H_1(D;\mathbf{Z})^*. \end{aligned}$$

Since $D = \bar{C}+\bar{L}, \bar{C}\cong S^2, \bar{L}\cong S^2$ and $\bar{C}\cap\bar{L} = \{pt\}$, then $H_1(D;\mathbf{Z}) = 0$.

Proposition 3.3. *The lattice Q is given by*

$$Q = \left\{ \alpha \in H_2(X_m;\mathbf{Z}) \,\middle|\, \begin{array}{l} \alpha\cdot(2l - e_1 - \cdots - e_m) = 0 \\ \alpha\cdot(l - e_P) = 0 \end{array} \right\} \tag{3.1}$$

and R is a root system of type D_m in $Q\otimes_{\mathbf{Z}}\mathbf{R}$ and generates Q. The set

$$\prod = \{e_1 - e_2, \ldots, e_{m-1} - e_m, -(l - e_P - e_{m-1} - e_m)\}$$

is a basis of R.

Proof. By the duality $H_2(X_m, X_m\backslash D;\mathbf{Z}) \cong H^2(D;\mathbf{Z})$, $\ker j_*$ is the lattice whose elements are orthogonal to the classes of the components of D. Since the classes of $\bar{C}$ and $\bar{L}$ are $\bar{c} = 2l - e_1 - \cdots - e_m$ and $\bar{l} = l - e_P$ respectively, we have (3.1).

Let $\alpha = al + b_Pe_P + b_1e_1 + \cdots + b_me_m \in Q$. By (3.1),

$$\begin{aligned} 2a + b_1 + \cdots + b_m &= 0, \\ a + b_P &= 0. \end{aligned}$$

Thus

$$Q = \left\{ \alpha \in H_2(X_m;\mathbf{Z}) \,\middle|\, \begin{array}{l} \alpha = a(l - e_P - e_1) + (a+b_1)e_1 + \cdots + b_me_m, \\ a + (a+b_1) + \cdots + b_m = 0 \end{array} \right\}. \tag{3.2}$$

Let $\alpha = al + b_Pe_P + b_1e_1 + \cdots + b_me_m \in R$. Then $a^2 - (b_P^2 + b_1^2 + \cdots + b_m^2) = -2$. It follows from $a + b_P = 0$ that $b_1^2 + \cdots + b_m^2 = 2$. Thus

$$R = \{\pm(e_i - e_j), \pm(l - e_P - e_i - e_j), i \neq j, i, j = 1, \dots, m\}.$$

Thus we have the proposition.

Proposition 3.4. *There are $2^{m-1} + 2m$ exceptional curves of the first kind on X_m. They are the exceptional curves of the blowing up $p : X_m \to \mathbf{P}^2$, the proper transforms of the lines passing through P and P_i, the proper transforms of the curves of degree a $(1 \leq a \leq [m/2])$ passing through $2a$ distinct points of $\{P_1, \dots, P_m\}$ and P with multiplicity $a - 1$.*

Proof. Let $\bar{c} = 2l - e_1 - \dots - e_m, \bar{l} = l - e_P$ be the classes of $\bar{C}, \bar{L}$. Let E be an exceptional curve of the first kind and $e = al - b_P e_P - \sum_{i=1}^m b_i e_i$ its class. Since E is exceptional curve of the first kind, we have

$$1 = -k_m \cdot e = 3a - b_P - \sum_{i=1}^m b_i, \tag{3.4}$$

$$-1 = e \cdot e = a^2 - b_P^2 - \sum_{i=1}^m b_i^2. \tag{3.5}$$

Also we have

$$\bar{C} \cdot D = 2a - \sum_{i=1}^m b_i \geq 0, \tag{3.6}$$

$$\bar{L} \cdot D = a - b_P \geq 0. \tag{3.7}$$

By (3.4) and (3.6) we have

$$1 = 3a - b_P - \sum_{i=1}^m b_i \geq a - b_P.$$

It follows from this and (3.7) that

$$b_P \leq a \leq b_P + 1.$$

Thus we have $a = b_P$ or $a = b_P + 1$.

(i) Suppose $a = b_P$. By (3.5) we have

$$\sum_{i=1}^m b_i^2 = 1.$$

Thus there exist i such that $b_i = \pm 1$ and $b_j = 0 (j \neq i)$. By (3.4) we have $2a - b_i = 1$ and $a = 1$ if $b_i = 1, a = 0$ if $b_i = -1$. Now we have $2m$ exceptional classes

$$e_1, \dots, e_m, l - e_P - e_1, \dots, l - e_P - e_m.$$

These are the classes of exceptional curves of the blowing up p and that of the proper transforms of the lines passing through P and P_i. Let $\mathscr{L}_1$ be the set of these classes.

(ii) Suppose $a = b_P + 1$. By (3.4) we have

$$\sum_{i=1}^{m} b_i = 2a.$$

By (3.5) we have

$$\sum_{i=1}^{m} b_i^2 = 2a.$$

Thus

$$0 = \sum_{i=1}^{m} b_i^2 - \sum_{i=1}^{m} b_i = \sum_{i=1}^{m} b_i(b_i - 1).$$

Since $b_i(b_i - 1) \geq 0$, we have $b_i(b_i - 1) = 0$ for all i. Therefore

$$b_i = 0, 1.$$

Now we have exceptional classes

$$al - (a-1)b_P - e_{i_1} - \cdots - e_{i_{2a}}.$$

Let $\mathscr{L}_2$ be the set of these classes. Since

$$\sum_{i=0}^{[m/2]} \binom{m}{2i} = 2^{m-1},$$

we have $\# \mathscr{L}_2 = 2^{m-1}$.

If we take m skew classes $l_1, \dots l_m$ in $\mathscr{L}_1$, then there exists only one class l_{m+1} in $\mathscr{L}_2$ such that $l_{m+1} \cdot l_i = 0$ $(1 \leq i \leq m)$. This class is nothing but the class of the curve D on X_m whose image under the contraction p' in the proof of Proposition 2.5 is the (-1)-section of Hirzebruch surface Σ_1. Thus for every class e of $\mathscr{L}_2$, there exist the exceptional curve of the first kind on X_m whose class is e.

We next define 2-cycle of $X_m \backslash D$. Let $E_i = p^{-1}(P_i)$, $E_j = p^{-1}(P_j)$, $B_i = E_i \cap \bar{C}$, $B_j = E_j \cap \bar{C}$. Let T be a tubular neighborhood of $\bar{C}$ in X_m such that $T \cap E_i$ and $T \cap E_j$ are fibers. Let γ be a injective path in $\bar{C}$ from B_i to B_j and put

$$\Gamma_{i,j} = (E_i \backslash (E_i \cap T)) \cup \partial T|_{\gamma} \cup (E_j \backslash (E_j \cap T)). \tag{3.3}$$

Then we can take the orientation such that $i_*([\Gamma_{ij}]) = e_i - e_j$, where $[\Gamma_{ij}]$ is the homology class of Γ_{ij}.

Furthermore let $L_j \subset \mathbf{P}^2$ be a line passing P and P_j. Let $\bar{L}_j$ be its proper transform. Then the homology class of $\bar{L}_j$ is $l - e_P - e_j \in H_2(X_m; \mathbf{Z})$. Let $B_j' = L_j \cap \bar{C}$. Let γ' an injective path in $\bar{C}$ from B_i to B_j'. Then we can define Γ_{ij}' similarly.

$$\Gamma_{i,j}' = (E_i \backslash (E_i \cap T)) \cup \partial T|_{\gamma'} \cup (\bar{L}_j \backslash (\bar{L}_j \cap T)), \tag{3.3$'$}$$

$$i_*([\Gamma_{i,j}']) = e_i - (l - e_P - e_j).$$

Let $\alpha_1, \ldots, \alpha_m \in H_2(X_m \backslash D; \mathbf{Z})$ be the homology classes of $\Gamma_{1,2}, \ldots, \Gamma_{m-1,m}$, and $\Gamma'_{m-1,m}$.

Corollary 3.5. *$H_2(X_m \backslash D; \mathbf{Z})$ is generated by $\{\alpha_1, \ldots, \alpha_m\}$. The intersection paring is given by*

$$\alpha_i \cdot \alpha_j = \begin{cases} -2 & i = j, \\ 1 & |i-j| = 1, i, j \neq m, \\ 1 & \{i, j\} = \{m-2, m\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By Lemma 3.2 and Proposition 3.3, it is clear that $H_2(X_m \backslash D; \mathbf{Z})$ is generated by $\{\alpha_1, \ldots, \alpha_m\}$. Since $\alpha_i \cdot \alpha_j = i_*(\alpha_i) \cdot i_*(\alpha_j)$, the intersection paring is given as above.

It is well known that there is close relation between a cubic surface and a Weyl group of type E_6. We have the same relation between X_m and the Weyl group of type D_m.

Proposition 3.6. *The group*

$$W = \left\{ g \in \operatorname{Aut}(H_2(X_m; \mathbf{Z})) \,\middle|\, \begin{array}{l} g(\bar{c}) = \bar{c}, g(\bar{l}) = \bar{l} \\ g(\alpha) \cdot g(\alpha') = \alpha \cdot \alpha' \textit{ for } \alpha, \alpha' \in H_2(X_m; \mathbf{Z}) \end{array} \right\}.$$

is isomorphic to the Weyl group of type D_m.

Proof. It is clear that W contains the group generated by reflections with respect to the elements of R, which is isomorphic to the Weyl group of type D_m.

Let $g \in W$ and $g(e_i) = al + be_P + b_1 e_1 + \cdots + b_m e_m$. It follows from the condition $g(\alpha) \cdot g(\alpha') = \alpha \cdot \alpha' (\forall \alpha, \forall \alpha' \in H_2(X_m; \mathbf{Z}))$ that

$$g(e_i) \cdot g(\bar{c}) = e_i \cdot \bar{c} = 1,$$

$$g(e_i) \cdot g(\bar{l}) = e_i \cdot \bar{l} = 0,$$

$$g(e_i) \cdot g(e_i) = e_i \cdot e_1 = -1.$$

On the other hand, since $g(\bar{c}) = \bar{c}, g(\bar{l}) = \bar{l}$, we have

$$g(e_i) \cdot g(\bar{c}) = g(e_i) \cdot \bar{c} = 2a + b_1 + \cdots + b_m = 1,$$

$$g(e_i) \cdot g(\bar{l}) = g(e_i) \cdot \bar{l} = a + b = 0,$$

$$g(e_i) \cdot g(e_i) = a^2 - b^2 - b_1^2 - \cdots - b_m^2 = -1.$$

Thus we have

$$g(e_i) = \begin{cases} e_j, \\ l - e_P - e_j. \end{cases}$$

for some j. Therefore there exists an element σ of the symmetric group S_m such that

$$g(e_i) = \begin{cases} e_{\sigma(i)}, \\ l - e_P - e_{\sigma(i)}. \end{cases}$$

Since g satisfies that $g(\bar{c}) = \bar{c}$ and $g(\bar{l}) = \bar{l}$, g is determined uniquely. It follows from $g(\bar{c}) = \bar{c}$ that

$$2g(l) = 2l - (e_1 + \cdots + e_m) + \{g(e_1) + \cdots + g(e_m)\}.$$

Since the coefficient of l in the left-hand side is even, the number of the indices i that satisfy $g(e_i) = l - e_P - e_{\sigma(i)}$ must be even. Therefore the order of W is equal to that of the Weyl group of type D_m.

Let $\mathscr{L}(X_m)$ be the set of exceptional classes given in Proposition 3.4. The Weyl group $W(D_m)$ acts on $\mathscr{L}(X_m)$.

Propositon 3.7. (i) *There are 2 orbit of $\mathscr{L}(X_m)$ under the action of $W(D_m)$. One is $W(D_m)$-orbit $\mathscr{L}_1$ of e_1 and another is $W(D_m)$-orbit $\mathscr{L}_2$ of e_P.*

(ii) *Let $M = \{l_1, \dots, l_s\}$ be a maximal set of mutually skew classes, i.e. $l_i \cdot l_j = 0$ $(i \neq j)$. Then $s = m + 1$. The set M consists of m elements of $\mathscr{L}_1$ and one element of $\mathscr{L}_2$.*

(iii) *Let $\mathscr{E}(X_m)$ be the set of ordered set of mutually skew lines:*

$$\mathscr{E}(X_m) = \{(l_1, \dots, l_m; l_{m+1}) | l_i \cdot l_j = 0 \ (i \neq j), l_i \in \mathscr{L}_1 \ (1 \leq i \leq m), l_{m+1} \in \mathscr{L}_2\}.$$

Then the Weyl group $W(D_m)$ acts on $\mathscr{E}(X_m)$ simply transitively.

Proof. (i) Straightforward.

(ii) If $M \cap \mathscr{L}_2 = \varnothing$, then $M \subset \mathscr{L}_1$. In this case we have $\# M = m$. But there exists one element e of $\mathscr{L}_2$ such that e is skew to the elements of M. Thus we have $M \cap \mathscr{L}_2 \neq \varnothing$ and let $l_s \in M \cap \mathscr{L}_2$. By the action of $W(D_m)$, we may assume $l_s = e_P$. The set of the elements of $\mathscr{L}(X_m)$ that are skew to e_P is $M' = \{e_1, \dots, e_m\}$. Thus we have $s = m + 1$.

(iii) As in the proof of (ii), l_{m+1} determine the set $\{l_1, \dots, l_m\}$ uniquely. Therefore $W(D_m)$ acts on $\mathscr{E}(X_m)$ transitively. Since $\# \mathscr{L}_2 = 2^{m-1}$ by Propositon 3.4, $\# \mathscr{E}(X_m) = 2^{m-1} m!$. This is the order of $W(D_m)$. Thus we have (iii).

4. Torelli theorem for the pairs (X_m, D)

Let $(C, L, P) \in \mathfrak{C}$ be a framing defined in section 2. Let X_m be a surface obtained by blowing up $\mathbf{P}^2$ at P and m points $P_1, \dots, P_m$ on $C \backslash L$ in general position.

$$p : X_m \to \mathbf{P}^2$$

Put $D = \bar{C} + \bar{L}$. By Lemma 2.3, we may assume $C : z^2 = xy, L : x = 0$ and $P(0 : 0 : 1)$. We next define meromorphic 2-form ω_0 on X_m. Let V_1, V_2, V_3 be open sets of $\mathbf{P}^2$ defined by

$$V_1 = \{(1 : x_1 : y_1) \in \mathbf{P}^2\},$$

$$V_2 = \{(x_2 : 1 : y_2) \in \mathbf{P}^2\},$$

$$V_3 = \{(x_3 : y_3 : 1) \in \mathbf{P}^2\}.$$

Then we define a meromorphic 2-form ω_0' on $\mathbf{P}^2$ by

$$\omega_0' = \begin{cases} \dfrac{dx_1 \wedge dy_1}{2\pi\sqrt{-1}(y_1^2 - x_1)} & on\ V_1, \\ \dfrac{dx_2 \wedge dy_2}{2\pi\sqrt{-1}x_2(x_2 - y_2^2)} & on\ V_2, \\ \dfrac{dx_3 \wedge dy_3}{2\pi\sqrt{-1}x_3(1 - x_3 y_3)} & on\ V_3. \end{cases} \tag{4.1}$$

Put

$$\omega_0 = p^*\omega_0'.$$

Lemma 4.1. *Let $\Gamma_{i,j}, \Gamma_{i,j}'$ be the 2-cycles defined by* (3.3), (3.3)$'$ *and $P_i = (1 : s_i^2 : s_i)$ $(i = 1, \dots, m)$. Then we have*

$$\int_{\Gamma_{i,j}} \omega_0 = s_i - s_j,$$

$$\int_{\Gamma_{i,j}'} \omega_0 = s_i + s_j.$$

Proof. Since $E_i = p^{-1}(P_i), E_j = p^{-1}(P_j)$, we have

$$\int_{E_i \setminus (E_i \cap T)} \omega_0 = \int_{E_j \setminus (E_j \cap T)} \omega_0 = 0.$$

Therefore

$$\int_{\Gamma_{i,j}} \omega_0 = \int_{\partial T|_\gamma} \omega_0.$$

The point of $C \setminus L$ can be parameterized by $(1 : s^2 : s), s \in \mathbf{C}$. Then by the residue formula, we have

$$\begin{aligned} \int_{\partial T|_\gamma} \omega_0 &= 2\pi\sqrt{-1} \int_\gamma \mathrm{Res}_{\tilde{C}}\omega_0 \\ &= \int_\gamma ds \\ &= \int_{s_j}^{s_i} ds \\ &= s_i - s_j. \end{aligned}$$

We next calculate $\int_{\Gamma'_{i,j}} \omega_0$. Since $\int_{L_j} \omega'_0 = 0$, we have

$$\int_{\bar{L}_j \setminus (\bar{L}_j \cap T)} \omega_0 = 0.$$

Since $L_j \cap C = \{(1 : s_j^2 : s_j), (1 : s_j^2 : -s_j)\}$,

$$\begin{aligned}\int_{\Gamma'_{i,j}} \omega_0 &= \int_{\partial T|_{\gamma'}} \omega_0 \\ &= 2\pi\sqrt{-1} \int_{\gamma'} \mathrm{Res}_{\bar{C}} \omega_0 \\ &= \int_{\gamma'} ds \\ &= \int_{-s_j}^{s_i} ds \\ &= s_i + s_j.\end{aligned}$$

Thus the lemma follows.

Let ω be a meromorphic 2-form such that ω has poles only along D. We can define a map

$$\chi_\omega : Q \to \mathbf{C}$$

by

$$\chi_\omega(\alpha) = \int_\Gamma \omega, \qquad \alpha \in Q \tag{4.2}$$

where Γ is a 2-cycle of $X_m \setminus D$ such that α is the image of the class of Γ under i_*. Now we have the theorem of Torelli type for our framed surfaces.

Theorem 4.2. *Let (C, L, P) (resp. (C', L', P')) be an element of $\mathfrak{C}$ and $p : X_m \to \mathbf{P}^2$ (resp. $p' : X'_m \to \mathbf{P}^2$) the morphism obtained by blowing up P (resp. P') and m points $P_1, \ldots, P_m$ (resp. $P'_1, \ldots, P'_m$) on $C \setminus L$ (resp. $C' \setminus L'$) in general position. Put $D = \bar{C} + \bar{L}$ (resp. $D' = \bar{C}' + \bar{L}'$). Let ω (resp. ω') be one of meromorphic 2-forms on X_m (resp. X'_m) which has poles only along D (resp. D'). For ω (resp. ω'), let $\chi_\omega : Q \to \mathbf{C}$ (resp. $\chi'_{\omega'} : Q' \to \mathbf{C}$) be the mapping defined as* (4.2).

If $\phi : H_2(X_m; \mathbf{Z}) \to H_2(X'_m; \mathbf{Z})$ is an isometry such that
(1) $\phi(\bar{c}) = \bar{c}', \phi(\bar{l}) = \bar{l}'$,
(2) *there exists $\varrho \in \mathbf{C}^*$ such that $\phi^*(\chi_{\omega'}) = \varrho\chi_\omega$,*
then there exists an isomorphism $\Phi : (X_m, D) \to (X'_m, D')$ which induces ϕ and maps $\bar{C}$ to $\bar{C}'$ and $\bar{L}$ to $\bar{L}'$.

Proof. It follows from the condition (1) and Proposition 3.6 that there exists $\sigma \in S_m$ such that $\phi(e_i) = e'_{\sigma(i)}$ or $l - e'_P - e'_{\sigma(i)}$ and that the number of i such that $\phi(e_i) = l - e'_P - e'_{\sigma(i)}$ is even.

Let $L_i (i = 1, \dots, m)$ be the line in $\mathbf{P}^2$ which passing through P and P_i. Let $\bar{L}_i$ be a proper transform of L_i. It follows from the Proposition 2.5 that we may assume that $\phi(e_P) = e_{P'}, \phi(e_i) = e'_i, (i = 1, \dots, m)$. By Lemma 2.3, we may assume that conics C, C' are given by $z^2 = xy$, lines L, L' are the line given by $x = 0$ and that P and P' are the point $(0:0:1)$, where $(x:y:z)$ is a homogeneous coordinate of $\mathbf{P}^2$.

Let $(1 : s_i^2 : s_i), (1 : s_j'^2 : s_j')$ be coordinates of P_i, P'_j respectively. Then by Lemma 4.1 and the condition (2) of the theorem, we have

$$s'_i = \varrho s_i \qquad i = 1, \dots, m.$$

Thus the theorem follows from Propositon 2.4.

5. A family of $\mathbf{P}^2$ with $2n+3$ points blown up

Let $\mathfrak{H}$ be a Cartan subalgebra of simple Lie algebra $\mathfrak{so}(2(2n+3), \mathbf{C})$ of type D_{2n+3} and W its Weyl group. Then $S = \mathfrak{H}/W \cong \mathbf{C}^{2n+3}$. In this section, we construct a family of the surfaces X_{2n+3} whose base space is S. To do it, we construct a family $\mathfrak{Y}$ of surfaces whose general fiber is double covering of Hirzebruch surface of degree n branched along a hyperelliptic curve. The general fiber is also isomorphic to a blowing up of X_{2n+3} at one point.

Let

$$\begin{aligned} \mathfrak{H} &= \{(h_1, \dots, h_m, -h_1, \dots, -h_m) \in \mathbf{C}^{2m}\} \\ &= \{(h_1, \dots, h_m) \in \mathbf{C}^m\}. \end{aligned}$$

The quotient

$$\mathfrak{H} \to S = \mathfrak{H}/W$$

is given by

$$h = (h_1, \dots, h_m) \mapsto (a, b_1, b_2, \dots, b_{2n+2}),$$

where $a = h_1 \cdots h_m, b_i = (-1)^i \sigma_i(h_1^2, \dots, h_m^2), \sigma_i$ is the i-th elementary symmetric polynomial.

For $s = (a, b_1, b_2, \dots, b_{2n+2}) \in S$, put

$$f_s = x^{2n+2} + b_1 x^{2n+1} + \cdots + b_{2n+2}.$$

Then

$$F_1(x_1, y_1, z_1, s) = z_1^2 + x_1 y_1^2 + 2a y_1 + f_s(x_1) = 0$$

is the semi-universal deformation of singularities of type D_m.

Put

$$\mathscr{U}_i = \{(x_i, y_i, z_i, s) \in \mathbf{C}^3 \times S \mid F_i(x_i, y_i, z_i, s) = 0\}, \qquad 1 \le i \le 4,$$

where

$$F_1(x_1, y_1, z_1, s) = z_1^2 + x_1 y_1^2 + 2ay_1 + f_s(x_1),$$

$$F_2(x_2, y_2, z_2, s) = z_2^2 + x_2 y_2^2 + 2ax_2^{n+2} y_2 + x_2^{2n+2} f_s(x_2^{-1}),$$

$$F_3(x_3, y_3, z_3, s) = z_3^2 + x_3 + 2ax_3^{n+2} y_3 + y_3^2 x_3^{2n+2} f_s(x_3^{-1}),$$

$$F_4(x_4, y_4, z_4, s) = z_4^2 + x_4 + 2ay_4 + y_4^2 f_s(x_4).$$

We can glue $\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3, \mathscr{U}_4$ as follows and denote by $\mathfrak{Y}$:

$$x_1 = x_4, \qquad x_2 = x_3, \qquad x_1 x_2 = 1,$$

$$y_1 y_4 = 1, \qquad y_2 y_3 = 1, \qquad y_1 x_2^n = y_2,$$

$$z_1 x_2^{n+1} = z_2, \qquad z_2 = z_3 y_2, \qquad z_4 = z_1 y_4.$$

For $s \in S$ denote by $\mathfrak{Y}_s$ the fiber of the projection $\mathfrak{Y} \to S$.

Remark 5.1. The glueing formula for (x_i, y_i) is same as (5.2) below for Hirzebruch surface Σ_n.

Lemma 5.2. *Put*

$$\Delta = \{s \in S | x f_s(x) - a^2 = 0 \textit{ has multiple roots.}\}.$$

If $s \in S \backslash \Delta$, there exists $(C, L, P) \in \mathfrak{C}$ and $2n+3$ points $P_1, \ldots, P_{2n+3}$ on $C \backslash L$ in general position such that $\mathfrak{Y}_s$ is isomorphic to a surface obtained by blowing up $P, Q, P_1, \ldots, P_{2n+3}$, where $C \cap L = \{Q\}$.

Proof. We fix $s \in S \backslash \Delta$. Put $U_i = \mathscr{U}_i \cap \mathfrak{Y}_s (i = 1, \ldots, 4)$. If $a \neq 0$, $\mathfrak{Y}_s$ has no singularity on $x_1 = 0$. Thus we assume $x_1 \neq 0$. Since

$$x_1 F_1 = x_1 z_1^2 + \left(x_1 \left(y_1 + \frac{a}{x_1} \right) \right)^2 + x_1 f_s(x_1) - a^2 \tag{5.1}$$

and $X_1 = x_1$, $Y_1 = x_1 \left(y_1 + \dfrac{a}{x_1} \right)$ and $Z_1 = \sqrt{x_1} z_1$ is a local coordinate, $\mathfrak{Y}_s$ has no singularity on U_1.

If $a = 0$, we can prove $\mathfrak{Y}_s$ has no singularity on U_1 when $x_1 \neq 0$. If $x_1 = 0$, we have

$$F_1(0, y_1, z_1, s) = z_1^2 + f_s(0).$$

Since $s \in S \backslash \Delta$, $f_s(0) \neq 0$. Therefore since $\dfrac{\partial F_1}{\partial z_1} = 2z_1 \neq 0$, $\mathfrak{Y}_s$ has no singularity on U_1.

Furthermore since

$$\left. \frac{\partial F_4}{\partial x_4} \right|_{y_4 = 0} = 1,$$

then $\mathfrak{Y}_s$ has no singularity on U_4.

Since $U_2 \backslash U_1$ is the set defined by $x_2 = 0$ and $\left.\dfrac{\partial F_2}{\partial z_2}\right|_{x_2=0} = 2z_2, F_2(0, y_2, 0, s) = 1$, $\mathfrak{Y}_s$ has no singularity on U_2.

Since $U_3 \backslash (U_2 \cup U_4)$ is the set defined by $x_3 = y_3 = 0$ and

$$\left.\frac{\partial F_3}{\partial x_3}\right|_{x_3=y_3=0} = 1,$$

$\mathfrak{Y}_s$ has no singularity on U_3.

Therefore $\mathfrak{Y}_s$ is nonsingular. Let Σ_n be the Hirzebruch surface of degree n,

$$\Sigma_n = \{(\zeta_0 : \zeta_1 : \zeta_2)(u' : v') \in \mathbf{P}^2 \times \mathbf{P}^1 \,|\, s^n\zeta_0 = t^n\zeta_1\}.$$

Let W_1, W_2, W_3, W_4 be open coverings of Σ_n,

$$\left.\begin{array}{ll} W_1 = \{(\zeta_0 : \zeta_1 : \zeta_2)(u' : v') \in \Sigma_n | v' \neq 0, \zeta_0 \neq 0\} & (x_1, y_1) = (u'/v', \zeta_2/\zeta_0) \\ W_2 = \{(\zeta_0 : \zeta_1 : \zeta_2)(u' : v') \in \Sigma_n | u' \neq 0, \zeta_1 \neq 0\} & (x_2, y_2) = (v'/u', \zeta_2/\zeta_1) \\ W_3 = \{(\zeta_0 : \zeta_1 : \zeta_2)(u' : v') \in \Sigma_n | u' \neq 0, \zeta_2 \neq 0\} & (x_3, y_3) = (v'/u', \zeta_1/\zeta_2) \\ W_4 = \{(\zeta_0 : \zeta_1 : \zeta_2)(u' : v') \in \Sigma_n | v' \neq 0, \zeta_2 \neq 0\} & (x_4, y_4) = (u'/v', \zeta_0/\zeta_2) \end{array}\right\} \tag{5.2}$$

Let H be a curve on Σ_n defined as follows:

$$\begin{aligned} x_1 y_1^2 + 2ay_1 + f_s(x_1) &= 0 \qquad \textit{on } W_1, \\ x_2 y_2^2 + 2ax_2^{n+2} y_2 + x_2^{2n+2} f_s(x_2^{-1}) &= 0 \qquad \textit{on } W_2, \\ x_3 + 2ax_3^{n+2} y_3 + y_3^2 x_3^{2n+2} f_s(x_3^{-1}) &= 0 \qquad \textit{on } W_3, \\ x_4 + 2ay_4 + y_4^2 f_s(x_4) &= 0 \qquad \textit{on } W_4. \end{aligned}$$

It follows from the way to glue U_1, U_2, U_3, U_4 that $\mathfrak{Y}_s$ is a double covering of Σ_n branched along H

$$\nu : \mathfrak{Y}_s \to \Sigma_n.$$

The curve H is a hyperelliptic curve of genus $n+1$ ramified at the points $(\beta_1, -a/\beta_1), \dots, (\beta_{2n+3}, -a/\beta_{2n+3}) \in W_1$ and $(0, 0) \in W_3$, where $\beta_1, \dots, \beta_{2n+3}$ are the roots of the equation $xf_s(x) - a^2 = 0$ (If $a = \beta_i = 0$, we take $(0, 0) \in W_4$ instead of $(\beta_i, -a/\beta_i) \in W_1$).

Let F_i $(i = 1, \dots, 2n+3)$ be the fiber of Σ_n defined by $u'/v' = \beta_i$ and F_∞ the fiber defined by $v'/u' = 0$. Then for $i = 1, \dots, 2n+3, \infty$, the inverse image $\nu^{-1}(F_i)$ is a union of two lines;

$$\nu^{-1}(F_i) = F_{i,1} \cup F_{i,2}, \qquad F_{i,1}, F_{i,2} \simeq \mathbf{P}^1.$$

Since $\nu^{-1}(F_i) \cdot \nu^{-1}(F_i) = 0, F_{i,1} \cdot F_{i,1} = F_{i,2} \cdot F_{i,2} = -1$. Since $F_{i,1} \cong F_{i,2} \cong \mathbf{P}^1, F_{i,1}$, $F_{i,2}$ $(i = 1, \dots, 2n+3, \infty)$ are exceptional curves of the first kind.

We next blow down the exceptional curves $F_{i,j(i)}(j(i) \in \{1,2\}, i = 1, \dots, 2n+3)$, and $F_{\infty,1}$. Then we have $\mathbf{P}^1$-bundle over $\mathbf{P}^1$. Therefore it is the Hirzebruch surface Σ_r of degree r for some r. Then we need the following lemma.

Lemma 5.3. $r = 0, 1$. *We can choose* $j(i)$ *such that* $r = 1$.

$$\mu : \mathfrak{Y}_s \to \Sigma_1.$$

Proof of Lemma 5.3. Let

$$S^{(n)} = \{(\zeta_0 : \zeta_1 : \zeta_2)(u' : v') \in \Sigma_n | \zeta_0 = \zeta_1 = 0\}$$

be $(-n)$-section of Σ_n. Let $\tilde{S}^{(n)}$ be the inverse image of $S^{(n)}$ under the covering ν. Then $\mu(\tilde{S}^{(n)}) \cdot \mu(\tilde{S}^{(n)}) = 4$ and $\mu(\tilde{S}^{(n)})$ intersects a general fiber at two points. Let $f^{(r)}$ and $s^{(r)}$ be the linear equivalence classes of a fiber of Σ_r and $(-r)$-section respectively. Then

$$\mu(\tilde{S}^{(n)}) \cdot f^{(r)} = 2, \mu(\tilde{S}^{(n)}) \cdot s^{(r)} \geq 0, \mu(\tilde{S}^{(n)}) \cdot \mu(\tilde{S}^{(n)}) = 4. \tag{5.3}$$

This shows $r \leq 1$. If $r = 0$, by exchanging $F_{1,1}$ and $F_{1,2}$, we have $\mu(\mathfrak{Y}_s) = \Sigma_1$.

Now we blow down (-1)-section of Σ_1

$$\mu' : \mathfrak{Y}_s \to \mathbf{P}^2.$$

Since we have $\mu(\tilde{S}^{(n)}) \sim 2f^{(1)} + 2s^{(1)}$ by (5.3), $\mu(\tilde{S}^{(n)})$ doesn't intersect (-1)-section of Σ_1. Therefore $\mu'(\tilde{S}^{(n)}) \cdot \mu'(\tilde{S}^{(n)}) = 4$.

Then since $\mu'(\tilde{S}^{(n)}) \cong \mathbf{P}^1$, $\mu'(\tilde{S}^{(n)})$ is a conic C on $\mathbf{P}^2$. The image of $F_{\infty,2}$ is a line L tangent to $\mu'(\tilde{S}^{(n)})$ and the image of (-1)-section of Σ_1 is a point P on $L \backslash C$. Thus we have Lemma 5.2.

Remark 5.4. For $h = (h_1, \dots, h_m) \in \mathfrak{H}$, we have

$$F_1 = z_1^2 + x_1 y_1^2 + 2ay_1 + \frac{\prod_{i=1}^m (x_1 - h_i^2) + h_1^2 \cdots h_m^2}{x_1}.$$

Thus the roots $\beta_1, \dots, \beta_m$ of the equation $x_1 f_s(x_1) - a^2 = 0$ are $h_1^2, \dots, h_m^2$.

Proposition 5.5. *The manifold* $\mathfrak{Y}$ *is nonsingular and satisfies the following conditions.*

(1) *If* $s \in S \backslash \Delta$, *the fiber* $\mathfrak{Y}_s$ *is nonsingular, and there exists a framing* $(C, L, P) \in \mathfrak{C}$ *and* $2n+3$ *points* $P_1, \dots, P_{2n+3}$ *on* $C \backslash L$ *in general position such that* $\mathfrak{Y}_s$ *is isomorphic to the surface obtained from* $\mathbf{P}^2$ *by blowing up* $P, Q, P_1, \dots, P_{2n+3}$, *where* $C \cap L = \{Q\}$.

(2) *If* $s \in \Delta$ *and* $a \neq 0$, *the fiber* $\mathfrak{Y}_s$ *has singularity. Put*

$$x f_s(x) - a^2 = (x - d_1)^{k_1} \cdots (x - d_r)^{k_r} \qquad d_i \neq d_j, \ (i \neq j).$$

Then $\mathfrak{Y}_s$ *has simple singularities of type* $A_{k_i - 1} (i = 1, \dots, r)$.

(3) *If $s \in \Delta$ and $a = 0$, the fiber $\mathfrak{Y}_s$ has singularity. Put*

$$f_s(x) = x^{k_0}(x - d_1)^{k_1} \cdots (x - d_{r'})^{k_{r'}}, \qquad d_i \neq d_j \ (i \neq j), \ d_i \neq 0.$$

Then $\mathfrak{Y}_s$ has simple singularities of type A_{k_i-1} $(i = 1, \dots, r')$ and simple singularity of type D_{k_0+1}. (If $k_0 = 1, 2$, then $D_1 = A_1, D_2 = A_3$).

Proof. Since $\partial F_1/\partial b_0 = 1$, $\mathfrak{Y}$ has no singularity on $\mathscr{U}_1$. By the proof of Lemma 5.2, it is clear that $\mathfrak{Y}$ has no singularity on $\mathfrak{Y}\backslash\mathscr{U}_1$. Thus $\mathfrak{Y}$ is non-singular. Lemma 5.2 shows (1). By (5.1), we have (2).

Put $s \in S\backslash\Delta$, $a = 0$. Then $\mathfrak{Y}_s$ has a simple singularity of type A_{k_i-1} $(i = 1, \dots, r')$.

Put $f_s(x) = x^{k_0}h(x)$. Then $h(0) \neq 0$. Since

$$F_1 h(x_1)^{-1} = z_1^2 h(x_1)^{-1} + x_1 y_1^2 h(x_1)^{-1} + x_1^{k_0} = 0$$

and take $(X_1', Y_1', Z_1') = (x_1, y_1/\sqrt{h(x_1)}, z_1/\sqrt{h(x_1)})$ as a local coordinate for a neighbor-hood U of $x_1 = 0$, we have

$$Z_1'^2 + X_1' Y_1'^2 + X_1'^{k_0} = 0.$$

(a) If $k_0 = 1$, U has singularity only at the points $(0, \pm\sqrt{-1}, 0) \in U$. Since

$$Z_1'^2 + X_1' Y_1'^2 + X_1'^{k_0} = Z_1'^2 + X_1'(Y_1'^2 + 1) = 0$$

and $(X_1', Y_1'^2 + 1, Z_1')$ is a local coordinate near the singularities, U has a simple singularity of type A_1 at the points $(0, \pm\sqrt{-1}, 0) \in U$.

(b) If $k_0 = 2$, U has singularity only at the point $(0, 0, 0) \in U$. Since

$$Z_1'^2 + X_1' Y_1'^2 + X_1'^{k_0} = Z_1'^2 + \left(X_1' + \tfrac{1}{2} Y_1'\right)^2 - \tfrac{1}{4} Y_1'^4 = 0$$

and $\left(X_1' + \frac{1}{2} Y_1', Y_1', Z_1'\right)$ is a local coordinate near the point, U has a simple singularity of type A_3 at $(0 : 0 : 0) \in U$.

(c) If $k_0 \geq 3$, U has singularity only at $(0 : 0 : 0) \in U$. It is clear that U has a simple singularity of type D_{k_0+1} at $(0, 0, 0) \in U$.

The proposition is proved.

We have constructed a family of the surfaces obtained by blowing up at one point on X_{2n+3}. We next construct a family of the surfaces X_{2n+3}. Put

$$f_s(x) = x^{2n+2} + b_1 x^{2n+1} + \cdots + b_{2n+2},$$

$$A = \{z \in \mathbf{C} \mid |z - 1| < 1/2\},$$

$$\mathscr{A} = \{(x, s) \in \mathbf{C} \times S \mid -(x^{2n+2} f_s(x^{-1}) - a^2 x^{2n+3}) \in A\}.$$

Then there exists holomorphic function $g_s(x)$ on $\mathscr{A}$ such that

$$(g_s(x))^2 = -(x^{2n+2}f_s(x^{-1}) - a^2x^{2n+3}).$$

We fix $g_s(x)$. We need the following lemma to construct the family.

Lemma 5.6. *Fix $s \in S$. Let*

$$U_i' = \{(x_i, y_i, z_i) \in U_i \mid (x_i, s) \in \mathscr{A}\} \qquad (i = 2, 3),$$

where $U_j = \mathfrak{Y}_s \cap \mathscr{U}_j$ $(j = 1, \ldots, 4)$. Let $\pi' : U_2' \cup U_3' \to \mathbf{C} \times \mathbf{P}^2$ be a morphism defined as follows.

$$\pi'(w) = \begin{cases} (x_2, (z_2 - g_s(x_2) : y_2 + ax_2^{n+1})) \in V, & w = (x_2, y_2, z_2) \in U_2', z_2 - g_s(x_2) \neq 0, \\ (x_2, (-x_2(y_2 + ax_2^{n+1}) : z_2 + g_s(x_2))) \in V, & w = (x_2, y_2, z_2) \in U_2', z_2 + g_s(x_2) \neq 0, \\ (x_2, (z_2 - g_s(x_2) : y_2 + ax_2^{n+1})) \in V, & w = (x_2, y_2, z_2) \in U_2', x_2 = 0, \\ (x_3, (z_3 - g_s(x_3)y_3 : 1 + ax_3^{n+1}y_3)) \in V, & w = (x_3, y_3, z_3) \in U_3', z_3 - g_s(x_3)y_3 \neq 0, \\ (x_3, (-x_3(1 + ax_3^{n+1}y_3) : z_3 + g_s(x_3)y_3)) \in V, & w = (x_3, y_3, z_3) \in U_3', z_3 + g_s(x_3)y_3 \neq 0, \\ (x_3, (z_3 - g_s(x_3)y_3 : 1 + ax_3^{n+1}y_3)) \in V, & w = (x_3, y_3, z_3) \in U_3', x_3 = 0, \end{cases}$$

where

$$V = \{(x, (u : v) \in \mathbf{C} \times \mathbf{P}^1 | (x, s) \in \mathscr{A}\}.$$

The open set $U_2' \cup U_3'$ has no singularity and π' is blowing down an exceptional curve E_∞, where $E_\infty = \{(x_2, y_2, z_2) \in U_2 | x_2 = 0, z_2 - g_s(x_2) = 0\} \cup \{(x_3, y_3, z_3) \in U_3 | x_3 = 0, z_3 - g_s(x_3)y_3 = 0\}$.

Proof. We have only to prove $U_1 \cap (U_2' \cup U_3')$ has no singularity to show that $U_2' \cup U_3'$ has no singularity (see the proof of Lemma 5.2). Singularity on $U_1 \cap (U_2 \cup U_3)$ is on the set defined by $x_1 f_s(x_1) - a^2 = 0$ (see (5.1)). By the definition of $\mathscr{A}$, $x_1 f_s(x_1) - a^2 \neq 0$ on $U_1 \cap (U_2' \cup U_3')$. Therefore $U_2' \cup U_3'$ has no singularity.

We next show that π' is blowing down of E_∞. By putting $y_2 = Y'/X'$, $z_2 = Z'/X'$, we have

$$U_2' \cup U_3'$$

$$\cong \{(x, (X' : Y' : Z')) \in \mathbf{C} \times \mathbf{P}^2 \mid Z'^2 + xY'^2 + 2ax^{n+2}X'Y' + x^{2n+2}f_s(x^{-1})X'^2 = 0\}.$$

Since

$$Z'^2 + xY'^2 + 2ax^{n+2}X'Y' + x^{2n+2}f_s(x^{-1})X'^2$$
$$= (Z' - g_s(x)X')(Z' + g_s(x)X') + x(Y' + ax^{n+1}X')^2,$$

we have

$$U_2' \cup U_3' \cong \{(x, (X : Y : Z)) \in \mathbf{C} \times \mathbf{P}^2 \mid XZ + xY^2 = 0\},$$

where $X = Z' - g_s(x)X'$, $Y = Y' + ax^{n+1}X'$, $Z = Z' + g_s(x)X'$. The exceptional curve E_∞ is given by $x = X = 0$. Now we have

$$\pi'(x,(X:Y:Z)) = \begin{cases} (x,(X:Y)) & \text{if } X \neq 0, \\ (x,(-xY:Z)) & \text{if } Z \neq 0. \end{cases}$$

Put $U' = \{(x,(X:Y:Z)) \in U_2' \cup U_3' | (X:Y:Z) \neq (1:0:0)\}$. Then U' is a open neighborhood of E_∞. We have a coordinate transformation ξ of U' as follows:

$$\xi : U' \to \{(x,y) \in \mathbf{C}^2 \mid (x,s) \in \mathscr{A}\} \times \mathbf{P}^1,$$

$$\xi(x,(X:Y:Z)) = \begin{cases} \left(x, \dfrac{X}{Y}\right)(Z:Y) & \text{if } Y \neq 0, \\ \left(x, -\dfrac{Y}{Z}x\right)(Z:Y) & \text{if } Z \neq 0. \end{cases}$$

This shows π' is nothing but a blowing down of the exceptional curve E_∞.

We construct a family of X_{2n+3} by Lemma 5.6. Put

$$\mathscr{U}_1 = \{(x_1,y_1,z_1,s) \in \mathbf{C}^3 \times S \mid F_1(x_1,y_1,z_1,s) = 0\},$$

$$\mathscr{V} = \{(x,s)(u:v) \in \mathbf{C} \times S \times \mathbf{P}^1 \mid (x,s) \in \mathscr{A}\},$$

$$\mathscr{U}_4 = \{(x_4,y_4,z_4,s) \in \mathbf{C}^3 \times S \mid F_4(x_4,y_4,z_4,s) = 0\}.$$

We can glue $\mathscr{U}_1, \mathscr{V}, \mathscr{U}_4$ as follows (see Lemma 5.6) and denote it by $\mathfrak{X}$:

(1) $x_1 = x_4,\ y_1 y_4 = 1,\ y_4 z_1 = z_4$

(2) Put $\mathscr{U}_1' = \{(x_1,y_1,z_1,s) \in \mathscr{U}_1 | (x_1^{-1},s) \in \mathscr{A}\}$. If $(x_1,y_1,z_1,s) \in \mathscr{U}_1'$,

$$x = x_1^{-1}$$

$$(u:v) = \begin{cases} (z_1 - x_1^{n+1} g_s(x_1^{-1}) : x_1 y_1 + a) & \text{if } z_1 - x_1^{n+1} g_s(x_1^{-1}) \neq 0 \\ (-(x_1 y_1 + a) : x_1(z_1 + x_1^{n+1} g_s(x_1^{-1}))) & \text{if } z_1 + x_1^{n+1} g_s(x_1^{-1}) \neq 0 \end{cases}$$

(3) Put $\mathscr{U}_4' = \{(x_4,y_4,z_4,s) \in \mathscr{U}_4 | (x_4^{-1},s) \in \mathscr{A}\}$. If $(x_4,y_4,z_4,s) \in \mathscr{U}_4'$,

$$x = x_4^{-1},$$

$$(u:v)$$
$$= \begin{cases} (z_4 - x_4^{n+1} y_4 g_s(x_4^{-1}) : x_4 + a y_4) & \text{if } z_4 - x_4^{n+1} y_4 g_s(x_4^{-1}) \neq 0, \\ (-(x_4 + a y_4) : x_4(z_4 + x_4^{n+1} y_4 g_s(x_4^{-1}))) & \text{if } z_4 + x_4^{n+1} y_4 g_s(x_4^{-1}) \neq 0. \end{cases}$$

Let

$$\varphi : \mathfrak{X} \to S$$

be the projection to S. For $s \in S \backslash \Delta$, let

$$\pi : \mathfrak{Y}_s \to \mathfrak{X}_s$$

be a morphism defined as follows:

$$\pi(w) = \begin{cases} (x_1, y_1, z_1) \in U_1, & w = (x_1, y_1, z_1) \in U_1, \\ (x_2, (z_2 - g_s(x_2)) : y_2 + ax_2^{n+1})) \in V, & w = (x_2, y_2, z_2) \in U_2 z_2 - g_s(x_2) \neq 0, \\ (x_2, (-x_2(y_2 + ax_2^{n+1})) : z_2 + g_s(x_2))) \in V, & w = (x_2, y_2, z_2) \in U_2, z_2 + g_s(x_2) \neq 0, \\ (x_3, (z_3 - g_s(x_3))y_3 : 1 + ax_3^{n+1}y_3)) \in V, & w = (x_3, y_3, z_3) \in U_3, z_3 - g_s(x_3)y_3 \neq 0, \\ (x_3(-x_3(1 + ax_3^{n+1}y_3) : z_3 + g_s(x_3)y_3)) \in V, & w = (x_3, y_3, z_3) \in U_3, z_3 + g_s(x_3)y_3 \neq 0, \\ (x_3, (z_3 - g_s(x_3)y_3 : 1 + ax_3^{n+1}y_3)) \in V, & w = (x_3, y_3, z_3) \in U_3, x_3 = 0, \\ (x_4, y_4, z_4) \in U_4. & w = (x_4, y_4, z_4) \in U_4. \end{cases}$$

Then this is blowing down of the exceptional curve E_∞.

Proposition 5.7. *$\mathfrak{X}$ is nonsingular. Put*

$$\Delta = \{s \in S | xf_s(x) - a^2 = 0 \textit{ has multiple roots.}\}$$

(1) *If $s \in S \backslash \Delta$, the fiber $\mathfrak{X}_s = \varphi^{-1}(s)$ is nonsingular and there exists a framing $(C, L, P) \in \mathfrak{C}$ and $2n+3$ points $P_1, \dots, P_{2n+3}$ on $C \backslash L$ in general position such that $\mathfrak{X}_s$ is isomorphic to the surface obtained from $\mathbf{P}^2$ by blowing up $P, P_1, \dots, P_{2n+3}$.*

(2) *If $s \in \Delta$ and $a \neq 0$, the fiber $\mathfrak{X}_s$ has singularities. Put*

$$xf_s(x) - a^2 = (x - d_1)^{k_1} \cdots (x - d_r)^{k_r}, \qquad d_i \neq d_j \ (i \neq j).$$

Then $\mathfrak{X}_s$ has simple singularities of type $A_{k_i - 1}$ $(i = 1, \dots, r)$.

(3) *If $s \in \Delta$ and $a = 0$, the fiber $\mathfrak{X}_s$ has singularities. Put*

$$f_s(x) = x^{k_0}(x - d_1)^{k_1} \cdots (x - d_{r'})^{k_{r'}}, \qquad d_i \neq d_j (i \neq j), \ d_i \neq 0$$

Then $\mathfrak{X}_s$ has simple singularities of type $A_{k_i - 1}$ $(i = 1, \dots, r')$ and simple singularity of type D_{k_0+1}. (If $k_0 = 1, 2$, then $D_1 = A_1, D_2 = A_3$).

Proof. It is clear that $\mathscr{V}$ has no singularity by definition of $\mathscr{V}$ and we have that $\mathscr{U}_1$ and $\mathscr{U}_4$ have no singularity (see Proposition 5.5). Therefore $\mathfrak{X}$ is nonsingular.

By Proposition 5.5 and Lemma 5.6, we have (1). In the proof of Lemma 5.2, we showed that $\mathfrak{Y}_s$ has no singularity outside $U_1 = \mathfrak{Y}_s \cap \mathscr{U}_1$. Therefore the statements (2) and (3) follow from Proposition 5.5.

Remark 5.8. The fiber of $\mathfrak{Y}_s$ defined by $v'/u' = 0$ is union of two exceptional curves. There are two choices of sign for fixing the function $g_s(x)$ at the beginning of construction of $\mathfrak{X}$. This corresponds to the choice of the exceptional curve that is blown down in Lemma 5.6.

We next consider a meromorphic 2-form ω on $\mathfrak{X}$ defined as follows:

$$\omega = \begin{cases} \dfrac{dx_1 dy_1}{2\pi\sqrt{-1}z_1} & \text{on } \mathscr{U}_1, \\ \dfrac{dx dv_1}{\pi\sqrt{-1}x(v_1^2 + x)} & \text{on } \mathscr{V}_1 = \{(x,s)(v_1 : 1) \in \mathscr{V}\}, \\ -\dfrac{dx dv_2}{\pi\sqrt{-1}x(1 + xv_2^2)} & \text{on } \mathscr{V}_2 = \{(x,s)(1 : v_2) \in \mathscr{V}\}, \\ -\dfrac{dx_4 dy_4}{2\pi\sqrt{-1}y_4 z_4} & \text{on } \mathscr{U}_4. \end{cases} \tag{5.4}$$

Let $\mathfrak{D}$ be the pole divisor of ω. Put $\mathfrak{D}_s = \mathfrak{D} \cap \mathfrak{X}_s$.

Proposition 5.9. *If $s \in S \backslash \Delta$, there exists a framing $(C, L, P) \in \mathfrak{C}$ satisfying the following conditions.*
(i) *$\mathfrak{X}_s$ is the surface obtained from $\mathbf{P}^2$ by blowing up P and $2n+3$ points in general position on $C \backslash L$.*
(ii) *$\mathfrak{D}_s = \bar{C} + \bar{L}$, where $\bar{C} + \bar{L}$ are the proper transforms of C, L.*

Proof. By the proof of Lemma 5.2 and 5.3, that we have only to show $\mathfrak{D}_s = \pi(\tilde{S}^{(n)}) + \pi(F_{\infty,2})$. On U_1 we have

$$2\pi\sqrt{-1}\omega_s = \frac{dx_1 dy_1}{z_1} = -2\frac{dx_1 dz_1}{\dfrac{\partial F_1}{\partial y_1}} = 2\frac{dy_1 dz_1}{\dfrac{\partial F_1}{\partial x_1}}.$$

Since $\mathfrak{X}_s$ is nonsingular, ω doesn't have pole on U_1. On U_4

$$\begin{aligned} \mathfrak{D}_s \cap U_4 &= \{(x_4, y_4, z_4, s) \in U_4 | y_4 = 0\} \\ &= \pi(\tilde{S}^{(n)}) \cap U_4. \end{aligned}$$

On $V_1 = \mathscr{V}_1 \cap \mathfrak{X}_s$, ω_s has pole along $x = 0$ or $v_1^2 + x = 0$. Since

$$y_4 = -\frac{x(x + v_1^2)}{av_1^2 + 2v_1 x^{n+2} g_s(x^{-1}) + ax},$$

then

$$\{(x, (1 : v_1)) \in V_1 | x(v_1^2 + x) = 0\} = (\pi(\tilde{S}^{(n)}) \cup \pi(F_{\infty,2})) \cap V_1.$$

Therefore

$$\mathfrak{D}_s \cap V_1 = (\pi(\tilde{S}^{(n)}) \cup \pi(F_{\infty,2})) \cap V_1.$$

Similarly on $V_2 = \mathscr{V}_2 \cap \mathfrak{X}_s$,

$$\mathfrak{D}_s \cap V_2 = (\pi(\tilde{S}^{(n)}) \cup \pi(F_{\infty,2})) \cap V_2.$$

Remark 5.10. Let $\mathfrak{Z} \to S$ be the semi-universal deformation of simple surface singularity of type E_m $(m = 6, 7, 8)$. Then there exists a family $\bar{\mathfrak{Z}} \to S$

whose general fibers are Del Pezzo surfaces and compactifications of general fibers of $\mathfrak{Z} \to S$ ([14]).

The family $\varphi|_{\mathscr{U}_1} : \mathscr{U}_1 \to S$ is the semi-universal deformation of simple surface singularity of type D_{2n+3} and general fiber $\mathfrak{X}_s$ is compactification of general fiber of semi-universal deformation of simple singularity of type D_{2n+3}.

It is well known that $\overline{\mathfrak{Z}}_s$ is a surface obtained by blowing up m points in general position on $\mathbf{P}^2$. Furthermore $D' = \overline{\mathfrak{Z}}_s \backslash \mathfrak{Z}_s$ is an anticanonical divisor of $\overline{\mathfrak{Z}}_s$ and

$$R' = \{\alpha \in H_2(\overline{\mathfrak{Z}}_s; \mathbf{Z}) | \alpha \cdot [D'] = 0, \alpha \cdot \alpha = -2\}$$

is the root system of type E_m ([6], [10]).

The surfaces X_{2n+3} and the family $\mathfrak{X}$ have same properties.

6. A family of $\mathbf{P}^2$ with $2n+2$ points blown up

In the previous section, we constructed a family of surfaces related to a simple singularity of type D_{2n+3}. In this section, we construct a family of surfaces related to a simple singularity of type D_{2n+2}. Let $m = 2n+2$ in this section.

Let $(C, L, P) \in \mathfrak{C}$ be a framing defined in section 2 and $\{Q_0\} = L \cap C$. Blow up P, Q_0 and m points $P_1, \dots, P_m$ on $C \backslash L$ in general position. Let E_P, E_{Q_0} and E_i be the exceptional curves cooresponding to these points. Let Q_1 be the intersection point of $\bar{C}$ and E_{Q_0}, where $\bar{C}$ is the proper transform of C. Blow up Q_1 further. Then we have the surface Z_m

$$p : Z_m \to \mathbf{P}^2.$$

Let $\bar{E}_{Q_0}$ be the proper transform of E_{Q_0} and E_Q the exceptional curve.

Propositon 6.1. *The divisor*

$$D = \bar{L} + \bar{C} + \bar{E}_{Q_0} + 2E_{Q_1}$$

is an anticanonical divisor of Z_m.

We next define isomorphism of the pair (Z_m, D).

Definition 6.2. Let (C, L, P) (resp. $(C', L', P')) \in \mathfrak{C}$. Let Z_m (resp. Z'_m) be the surface obtained as above. Put $D = \bar{L} + \bar{C} + \bar{E}_{Q_0} + 2E_{Q_1}$ (resp. $D' = \bar{L}' + \bar{C}' + \bar{E}'_{Q_0} + 2E'_{Q_1}$). Then we say that (Z_m, D) and (Z'_m, D') are isomorphic if there exists an isomorphism $\phi : Z_m \to Z'_m$ such that

$$\phi(\bar{C}) = \bar{C}', \qquad \phi(\bar{L}) = \bar{L}', \qquad \phi(\bar{E}_{Q_0}) = \bar{E}'_{Q_0}, \qquad \phi(E_{Q_1}) = E'_{Q_1}$$

From now on, we assume that $C : z^2 = xy$, $L : x = 0$, $P = (0:0:1)$ (see Lemma 2.3).

Proposition 6.3. *Let m points $P_1, \dots, P_m$ (resp. $P'_1, \dots, P'_m$) $\in C \backslash L$ be in general position and Z_m (resp. Z'_m) the surface obtained by blowing up P, Q_0, Q_1,*

$P_1, \dots, P_m$ (resp. $P', Q_0', Q_1', P_1', \dots, P_m'$). Put $D = \bar{L} + \bar{C} + \bar{E}_{Q_0} + 2E_{Q_1}$ (resp. $D' = \bar{L}' + \bar{C}' + \bar{E}_{Q_0}' + 2E_{Q_1}'$). Then there exists an isomorphism $\Phi : (Z_m, D) \to (Z_m', D')$ such that $\Phi(E_i) = E_i'$ $(i = 1, \dots, m)$ if and only if there exists $\alpha \in \mathbf{C}^*$ such that $s_i = \alpha s_i'$ $(i = 1, \dots, m)$, where $P_i = (1 : s_i^2 : s_i)$ $(i = 1, \dots, m)$ (resp. $P_i' = (1 : s_i'^2 : s_i')$ $(i = 1, \dots, m)$).

Proof. The proof is same as the proof of Proposition 2.4.

We consider homology exact sequence.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_3(Z_m; \mathbf{Z}) & \longrightarrow & H_3(Z_m, Z_m \backslash D; \mathbf{Z}) \\ & & \| & & \\ & & 0 & & \\ \xrightarrow{\partial_*} & H_2(Z_m \backslash D; \mathbf{Z}) & \xrightarrow{i_*} & H_2(Z_m; \mathbf{Z}) & \xrightarrow{j_*} & H_2(Z_m, Z_m \backslash D; \mathbf{Z}) \\ \longrightarrow & \cdots & & & & \end{array}$$

We can extend the intersection pairing in $H_2(Z_m; \mathbf{Z})$ to a bilinear form on $H_2(Z_m; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R}$. Put

$$Q = \ker j_* \subset H_2(Z_m; \mathbf{Z}),$$

$$R = \{\alpha \in Q \mid \alpha \cdot \alpha = -2\}.$$

Lemma 6.4. *Let Q and R as above. Then we have*

$$H_2(Z_m \backslash D; \mathbf{Z}) \cong Q.$$

Proof. Put $D = \bar{C} + \bar{L} + \bar{E}_{Q_0} + 2E_{Q_1}$. The curves $\bar{C}$, $\bar{L}$, $\bar{E}_{Q_0}$, E_{Q_1} are homeomorphic to 2-sphere. Therefore $H_1(D; \mathbf{Z}) = 0$. Then $H_3(Z_m, Z_m \backslash D; \mathbf{Z}) = 0$ and the result follows.

Let l be the homology class of total transform of line and $e_P, \bar{e}_{Q_0}, e_{Q_1}, e_1, \dots, e_m$ the classes of $E_P, \bar{E}_{Q_0}, E_{Q_1}, E_1, \dots, E_m$ respectively. Let $\bar{c}$ and $\bar{l}$ be the classes of $\bar{C}$ and $\bar{L}$.

Proposition 6.5. *Let Q, R be as above. Then*

$$Q = \left\{ \alpha \in H_2(Z_m; \mathbf{Z}) \;\middle|\; \begin{array}{l} \alpha \cdot (2l - e_1 - \cdots - e_m - \bar{e}_{Q_0} - 2e_{Q_1}) = 0 \\ \alpha \cdot (l - e_P - \bar{e}_{Q_0} - 2e_{Q_1}) = 0 \\ \alpha \cdot \bar{e}_{Q_0} = \alpha \cdot e_{Q_1} = 0 \end{array} \right\} \tag{6.1}$$

and R is the root system of type D_m in $Q \otimes_{\mathbf{Z}} \mathbf{R}$. Furthermore R generates Q. Put

$$\Pi = \{e_1 - e_2, \dots, e_{m-1} - e_m, -(l - e_P - e_{m-1} - e_m)\},$$

then Π is a basis of R.

Proof. Since $D = \bar{L} + \bar{C} + \bar{E}_{Q_0} + 2E_{Q_1}$, $\bar{l} = l - e_P - \bar{e}_{Q_0} - 2e_{Q_1}$ and $\bar{c} = 2l - e_1 - \cdots - e_m - \bar{e}_{Q_0} - 2e_{Q_1}$, we have (6.1).

Put $\alpha = al + b_P e_P + b_1 e_1 + \cdots + b_m e_m + c_0 \bar{e}_{Q_0} + c_1 e_{Q_1} \in Q$, then

$$\begin{cases} 2a + b_1 + \cdots + b_m + c_1 = 0, \\ a + b_P + c_1 = 0, \\ -2c_0 + c_1 = 0, \\ c_0 - c_1 = 0. \end{cases}$$

Therefore since

$$\begin{cases} 2a + b_1 + \cdots + b_m = 0, \\ a + b_P = 0, \\ c_0 = 0, \\ c_1 = 0, \end{cases}$$

then the result follows from Proposition 3.3.

We take 2-cycles $\Gamma_{i,j}$, $\Gamma'_{i,j}$ as (3.3) and (3.3)$'$. Let $\alpha_1, \ldots, \alpha_m \in H_2(Z_m \backslash D; \mathbf{Z})$ be the classes of $\Gamma_{1,2}, \ldots, \Gamma_{m-1,m}, \Gamma'_{m-1,m}$.

Corollary 6.6. *$H_2(Z_m \backslash D; \mathbf{Z})$ is generated by $\{\alpha_1, \ldots, \alpha_m\}$ and the intersection paring is given by*

$$\alpha_i \cdot \alpha_j = \begin{cases} -2 & i = j \\ 1 & |i - j| = 1, i, j \neq m \\ 1 & \{i, j\} = \{m - 2, m\} \\ 0 & \textit{otherwise} \end{cases}$$

The next proposition follows from Proposition 3.6 and Proposition 6.5.

Proposition 6.7. *Put*

$$W = \left\{ g \in \operatorname{Aut}(H_2(Z_m; \mathbf{Z})) \;\middle|\; \begin{array}{l} g(\bar{c}) = \bar{c}, g(\bar{l}) = \bar{l} \\ g(\bar{e}_{Q_0}) = \bar{e}_{Q_0}, g(e_{Q_1}) = e_{Q_1} \\ g(\alpha) \cdot g(\alpha') = \alpha \cdot \alpha' \textit{ for } \alpha, \alpha' \in H_2(Z_m; \mathbf{Z}) \end{array} \right\}.$$

Then W is isomorphic to the Weyl group of type D_m.

We have the theorem of Torelli type from these results and the same discussion in section 4.

Theorem 6.8. *Let (C, L, P) (resp. (C', L', P')) $\in \mathfrak{C}$ and $\{Q_0\} = L \cap C$, (resp. $\{Q'_0\} = L' \cap C'$). Let $P_1, \ldots, P_m$ (resp. $P'_1, \ldots, P'_m$) be m points on $C \backslash L$ (resp. $C' \backslash L'$) in general position.*

Let $p : Z_m \to \mathbf{P}^2$ (resp. $p' : Z'_m \to \mathbf{P}^2$) be the morphism obtained by blowing up $P, Q_0, P_1, \ldots, P_m$ (resp. $P', Q'_0, P'_1, \ldots, P'_m$) and infinitely near point Q_1 of Q_0 (resp.

Q_0'), where Q_1 (resp. Q_1') is the intersection point of the exceptional curve of blowing up of Q_0 (resp. Q_0') and the proper transform of C.

Put $D = \bar{C} + \bar{L} + \bar{E}_{Q_0} + 2E_{Q_1}$ (resp. $D' = \bar{C}' + \bar{L}' + \bar{E}_{Q_0'} + 2E_{Q_1'}$). Let ω (resp. ω') be a meromorphic 2-form on Z_m (resp. Z_m') such that ω (resp. ω') has poles only along D (resp. D'). Then as (4.2), *we can define the mapping $\chi_\omega : Q \to \mathbf{C}$ (resp. $\chi'_{\omega'} : Q' \to \mathbf{C}$), where $Q = \ker j_*$ (resp. $Q' = \ker j'_*$) is the root lattice.*

If $\phi : H_2(Z_m; \mathbf{Z}) \to H_2(Z_m'; \mathbf{Z})$ is an isometry satisfying the following conditions (1) *and* (2),

(1) $\phi(\bar{c}) = \bar{c}'$, $\phi(\bar{l}) = \bar{l}'$, $\phi(\bar{e}_{Q_0}) = \bar{e}'_{Q_0}, \phi(e_{Q_1}) = e'_{Q_1}$,

(2) *there exists $\varrho \in \mathbf{C}^*$ such that $\phi^*(\chi_{\omega'}) = \varrho\chi_\omega$.*

Then there exists an isomorphism $\Phi : (Z_m, D) \to (Z_m', D')$ such that Φ induces ϕ.

In the remaining of this section, we construct a family of these surfaces. Let $\mathfrak{H}$ be a Cartan subalgebra of simple Lie algebra $\mathfrak{so}(2(2n+2), \mathbf{C})$ of type D_{2n+2} and W its Weyl group. The quotient

$$\mathfrak{H} \to S = \mathfrak{H}/W \cong \mathbf{C}^{2n+2}$$

is given as in section 5. For $s = (a, b_1, \ldots, b_{2n+1}) \in S$, put

$$f_s(x) = x^{2n+1} + b_1 x^{2n} + \cdots + b_{2n+1},$$

$$h_s(x) = x^{2n} f_s(x^{-1}) - x^{-1}.$$

Let H' be a curve on Σ_n defined by as follows:

$$\begin{aligned}
&x_1 y_1^2 + 2a y_1 + f_s(x_1) &&= 0 \quad \text{on } W_1,\\
&x_2(y_2^2 + 2a x_2^{n+1} y_2 + x_2^{2n+1} f_s(x_2^{-1})) &&= 0 \quad \text{on } W_2,\\
&x_3(1 + 2a x_3^{n+1} y_3 + y_3^2 x_3^{2n+1} f_s(x_3^{-1})) &&= 0 \quad \text{on } W_3,\\
&x_4 + 2a y_4 + y_4^2 f_s(x_4) &&= 0 \quad \text{on } W_4,
\end{aligned}$$

where W_i, $i = 1, 2, 3, 4$ is open sets of Σ_n defined by (5.2).

In Lemma 5.2, we consider the double covering of Σ_n branched along a nonsingular curve H. But H' has singularities at $(0, \pm\sqrt{-1}) \in W_3$. Therefore we blow up these singularities and take double covering branched along the proper transform H'' of H'.

Put

$$F_1(x_1, y_1, z_1, s) = z_1^2 + x_1 y_1^2 + 2a y_1 + f_s(x_1),$$

$$F_2(x_2, y_2, z_2, s) = z_2^2 + x_2 y_2^2 + 2a x_2^{n+2} y_2 + x_2 \cdot x_2^{2n+1} f_s(1/x_2),$$

$$G_1(x_3, u, z_3, s) = z_3^2 + u(ux_3 + 2\sqrt{-1}) + 2ax_3^n(ux_3 + \sqrt{-1}) + (ux_3 + \sqrt{-1})^2 h_s(x_3),$$

$$G_2(v, w, z_3', s) = z_3'^2 + \{w + 2av^n(vw - 2\sqrt{-1})^n(vw - \sqrt{-1}) + (vw - \sqrt{-1})^2 h_s(v(vw - 2\sqrt{-1}))\},$$

$$G_3(t, y_3, z_3'', s) = z_3''^2 + t\{1 + 2ay_3t^{n+1}(y_3^2 + 1)^n + ty_3^2 h_s(t(y_3^2 + 1))\},$$

$$F_4(x_4, y_4, z_4, s) = z_4^2 + x_4 + 2ay_4 + y_4^2 f_s(x_4).$$

Let $\mathfrak{X}$ be a manifold obtained by gluing the following open sets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{U}_4$ as follows:

$$\mathcal{U}_1 = \{(x_1, y_1, z_1, s) \in \mathbf{C}^3 \times S \mid F_1(x_1, y_1, z_1, s) = 0\},$$

$$\mathcal{U}_2 = \{(x_2, y_2, z_2, s) \in \mathbf{C}^3 \times S \mid F_2(x_2, y_2, z_2, s) = 0, (x_2, y_2, z_2) \neq (0, \pm\sqrt{-1}, 0)\},$$

$$\mathcal{Y}_1 = \{(x_3, u, z_3, s) \in \mathbf{C}^3 \times S \mid G_1(x_3, u, z_3, s) = 0\},$$

$$\mathcal{Y}_2 = \{(v, w, z_3', s) \in \mathbf{C}^3 \times S \mid G_2(v, w, z_3', s) = 0\},$$

$$\mathcal{Y}_3 = \{(t, y_3, z_3'', s) \in \mathbf{C}^3 \times S \mid G_3(t, y_3, z_3'', s) = 0\},$$

$$\mathcal{U}_4 = \{(x_4, y_4, z_4, s) \in \mathbf{C}^3 \times S \mid F_4(x_4, y_4, z_4, s) = 0\},$$

$$x_1 = x_4, \qquad x_2 = x_3, \qquad x_1x_2 = 1,$$

$$y_1y_4 = 1, \qquad y_2y_3 = 1, \qquad y_1 = x_1^n y_2,$$

$$z_1 x_2^{n+1} = z_2, \qquad z_4 = z_1 y_4,$$

$$y_3 - \sqrt{-1} = ux_3, \qquad x_3 = v(y_3 - \sqrt{-1}),$$

$$y_3 + \sqrt{-1} = wv, \qquad v = t(y_3 + \sqrt{-1}),$$

$$z_2/y_2 = x_3z_3 = v(vw - 2\sqrt{-1})z_3' = (y_3^2 + 1)z_3''.$$

The glueing formulas for (x_i, y_i) are same as (5.2) and that for u, v, w, t, x_3 and y_3 are nothing but blowing up of $(x_3, y_3) = (0, \pm\sqrt{-1})$. Rewriting F_1 by these formulas, we have F_2, G_1, G_2, G_3 and F_4.

Proposition 6.9. *$\mathfrak{X}$ is nonsingular. Put*

$$\Delta = \{s \in S \mid xf_s(x) - a^2 = 0 \text{ has multiple roots}\}.$$

Then

(1) *If $s \in S\backslash\Delta$, the fiber $\mathfrak{X}_s$ is nonsingular and there exists a framing $(C, L, P) \in \mathfrak{C}$ such that $\mathfrak{X}_s$ is isomorphic to the surface obtained from $\mathbf{P}^2$ by blowing up P, $2n+2$ points $P_1, \ldots, P_{2n+2}$ on $C\backslash L$ in general position, Q_0 and infinitely near point Q_1 of Q_0 as in Theorem 6.8.*

(2) *If* $s \in \Delta, a \neq 0$, *the fiber* $\mathfrak{X}_s$ *has singularity. Put*

$$x f_s(x) - a^2 = (x - d_1)^{k_1} \cdots (x - d_r)^{k_r}, \qquad d_i \neq d_j \ (i \neq j).$$

Then $\mathfrak{X}_s$ *has simple singularities of type* $A_{k_i - 1}$ $(i = 1, \ldots, r)$.

(3) *If* $s \in \Delta, a = 0$, *the fiber* $\mathfrak{X}_s$ *has singularity. Put*

$$f_s(x) = x^{k_0}(x - d_1)^{k_1} \cdots (x - d_{r'})^{k_{r'}}, \qquad d_i \neq d_j (i \neq j), \ d_i \neq 0.$$

Then $\mathfrak{X}_s$ *has simple singularities of type* D_{k_0+1} *and of type* A_{k_i-1} $(i = 1, \ldots, r')$ *(if* $k_0 = 1, 2$, *then* $D_1 = A_1, D_2 = A_3$*).*

Proof. There is no singularity on $\mathcal{U}_1$ and $\mathcal{U}_4$ (see the proof of Proposition 5.5). The complement $\mathcal{U}_2 \backslash \mathcal{U}_1$ is defined by $x_2 = 0$. We have

$$\frac{\partial F_2}{\partial x_2} = y_2^2 + 2a(n+2)x_2^{n+1} y_2 + x_2^{2n+1} f_s(x_2^{-1}) + x_2 \frac{\partial}{\partial x_2}(x_2^{2n+1} f_s(x_2^{-1})),$$

$$\frac{\partial F_2}{\partial z_2} = 2z_2.$$

Therefore on the set defined by $x_2 = 0$, we have

$$\frac{\partial F_2}{\partial x_2} = \frac{\partial F_2}{\partial z_2} = 0 \quad \Leftrightarrow \quad \begin{cases} y_2 = \pm\sqrt{-1} \\ z_2 = 0. \end{cases}$$

Thus $\mathfrak{X}$ has no singularity on $\mathcal{U}_2$.

On $\mathfrak{Y}_1$, if $x_3 \neq 0$, then there is no singularity. Since

$$\left.\frac{\partial G_1}{\partial u}\right|_{x_3=0} = 2\sqrt{-1},$$

$\mathfrak{Y}_1$ has no singularity.

On $\mathcal{Y}_2$, if $v \neq 0$, then there is no singularity. Since

$$\left.\frac{\partial G_2}{\partial w}\right|_{v=0} = 1,$$

there is no singularity on $\mathcal{Y}_2$.

Then we have only to prove $\mathfrak{X}$ has no singularity on the subset of $\mathcal{Y}_3$ defined by $y_3 = 0$. Since

$$\frac{\partial G_3}{\partial z_3''} = 2z_3'', \qquad G_3(t, 0, 0, s) = t, \qquad \left.\frac{\partial G_3}{\partial t}\right|_{t=y_3=z_3''=0} = 1,$$

then $\mathfrak{X}_s$ has no singularity on $\mathcal{Y}_3$. Therefore $\mathfrak{X}$ is nonsingular. The proof of (2) and (3) is same as Proposition 5.5. If $s \in S \backslash \Delta$, then $\mathfrak{X}_s$ is nonsingular (see Proposition 5.5). We next prove (1). Put $s \in S \backslash \Delta$ and

$$U_i = \mathfrak{X}_s \cap \mathcal{U}_i, \qquad i = 1, 2, 4, \qquad Y_i = \mathfrak{X}_s \cap \mathcal{Y}_i, \qquad i = 1, 2, 3.$$

The fiber $\mathfrak{X}_s$ is a double covering of blowing up of Σ_n at two points with branching along H'',

$$\nu' : \mathfrak{X}_s \to \Sigma_n.$$

The irreducible components of H'' are heperelliptic cuve H_1'' and $\mathbf{P}^1$. The hyperelliptic curve H_1'' is ramified at $(\beta_1, -a/\beta_1), \dots, (\beta_{2n+2}, -a/\beta_{2n+2}) \in W_1$, where $\beta_1, \dots, \beta_{2n+2}$ be the roots of the equation $xf_s(x) - a^2 = 0$ (if $a = \beta_i = 0$, then $(0,0) \in W_4$).

Let F_i $(i = 1, \dots, 2n+2)$ be the fiber of Σ_n defined by $u'/v' = \beta_i$. Then put

$$\nu'^{-1}(F_i) = F_{i,1} \cup F_{i,2}, \qquad F_{i,1}, F_{i,2} \simeq \mathbf{P}^1.$$

For points $(0, \pm\sqrt{-1}) \in W_3$, put $E_+ = \nu'^{-1}((0, \sqrt{-1}))$, $E_- = \nu'^{-1}((0, -\sqrt{-1}))$ and let $\tilde{F}$ be the inverse image of proper transform of the fiber F defined by $x_2 = 0$ in W_2 (see (5.2)). The self-intersection number of $F_{i,j}$ $(i = 1, \dots, 2n+2, j = 1, 2)$ is -1 (see the proof of Lemma 5.2). Furthermore the self-intersection number of E_+ and E_- is -2 and that of $\tilde{F}$ is -1.

Since $F_{i,j}$ $(i = 1, \dots, 2n+2, j = 1, 2), E_-, E_+$, and $\tilde{F}$ are isomorphic to $\mathbf{P}^1$, $F_{i,j}$ $(i = 1, \dots, 2n+2, j = 1, 2)$ and $\tilde{F}$ are exceptional curves of the first kind. Then we blow down $F_{i,j(i)}$ $(i = 1, \dots, 2n+2, j(i) = 1$ or $2)$, $\tilde{F}$ and the image of E_+. Then we have $\mathbf{P}^1$-bundle Σ_r over $\mathbf{P}^1$

$$\eta : \mathfrak{X}_s \to \Sigma_r.$$

We may assume $r = 1$ (see Lemma 5.3).

We next blow down (-1)-section of Σ_1. We have a morphism

$$\eta' : \mathfrak{X}_s \to \mathbf{P}^2.$$

Put $\tilde{S}^{(n)} = \nu'^{-1}(S^{(n)})$, where $S^{(n)}$ is $(-n)$-section of Σ_n. Since the self-intersection number of $\tilde{S}^{(n)}$ is $-2n$, self-intersection number of $\eta'(\tilde{S}^{(n)})$ is 4. Then $\eta'(\tilde{S}^{(n)})$ is a conic because it is isomorphic to $\mathbf{P}^1$ and its self-intersection number is 4. Then $\eta'(E_-)$ is a line tangent to $\eta'(\tilde{S}^{(n)})$. Thus we have the statement (1).

Remark 6.10. (i) We can choose $j(i)$ such that a surface obtained by blowing down $F_{i,j(i)}$ $(i = 1, \dots, 2n+2, j(i) \in \{1, 2\})$, $\tilde{F}$ and E_- is Σ_1.
(ii) For $h = (h_1, \dots, h_m) \in \mathfrak{H}$, we have

$$F_1 = z_1^2 + x_1 y_1^2 + 2ay_1 + \frac{\prod_{i=1}^m (x_1 - h_i^2) + h_1^2 \cdots h_m^2}{x_1}.$$

Thus the roots $\beta_1, \dots, \beta_m$ of the equation $x_1 f_s(x_1) - a^2 = 0$ are $h_1^2, \dots, h_m^2$.

We next define a meromorphic 2-form ω on $\mathfrak{X}$.

$$\omega = \begin{cases} \dfrac{dx_1dy_1}{2\pi\sqrt{-1}z_1} & \text{on } \mathscr{U}_1, \\ -\dfrac{dx_2dy_2}{2\pi\sqrt{-1}x_2z_2} & \text{on } \mathscr{U}_2, \\ \dfrac{dx_3du}{2\pi\sqrt{-1}x_3z_3(x_3u+\sqrt{-1})} & \text{on } \mathscr{Y}_1, \\ -\dfrac{dvdw}{2\pi\sqrt{-1}vz_3'(vw-2\sqrt{-1})(vw-\sqrt{-1})} & \text{on } \mathscr{Y}_2, \\ \dfrac{dtdy_3}{2\pi\sqrt{-1}tz_3''y_3(y_3^2+1)} & \text{on } \mathscr{Y}_3, \\ -\dfrac{dx_4dy_4}{2\pi\sqrt{-1}y_4z_4} & \text{on } \mathscr{U}_4. \end{cases} \tag{6.2}$$

Put $\mathfrak{D}$ be a pole divisor of ω.

Proposition 6.11. *If $s \in S\backslash\Delta$, there exists a framinf $(C, L, P) \in \mathfrak{C}$ which has the following properties:*

(i) *$\mathfrak{X}_s$ is isomorphic to the surface obtained by blowing up P, $2n+2$ points on $C\backslash L$ in general position, Q_0 and infinitely near point Q_1 of Q_0 as in Theorem* 6.8.

(ii) $\mathfrak{D}_s = \bar{C} + \bar{L} + \bar{E}_{Q_0} + 2E_{Q_1}$.

Proof. By Proposition 6.9, we have only to show $\mathfrak{D}_s = \tilde{S}^{(n)} + E_- + E_+ + 2\tilde{F}$. It is clear that ω_s has no pole on U_1. It is also clear that $\dfrac{dx_2dy_2}{z_2}$ has no pole on U_2. Therefore ω_s has poles only on the set defined by $x_2 = 0$ in U_2. Put $F_2(x_2, y_2, z_2, s) = z_2^2 + x_2\kappa_2(x_2, y_2, s)$. Then since in the neighborhood of any point of $x_2 = 0, \kappa_2 \neq 0$ and (x_2, y_2, z_2') is a local coordinate, where $z_2' = \dfrac{z_2}{\sqrt{\kappa_2}}$.

Since U_2 is defined by ${z_2'}^2 + x_2 = 0$, (y_2, z_2') is a coordinate of U_2. It follows from

$$\frac{dx_2dy_2}{x_2z_2} = 2\frac{dy_2dz_2'}{\sqrt{\kappa_2}{z_2'}^2},$$

that ω_s has poles along $x_2 = 0$ with multiplicity 2. We have

$$\tilde{S}^{(n)} \cap U_2 = \varnothing,$$

$$\tilde{F} \cap U_2 = \{(x_2, y_2, z_2) \in U_2 \mid x_2 = 0\},$$

$$E_+ \cap U_2 = \varnothing,$$

$$E_- \cap U_2 = \varnothing.$$

Therefore $\mathfrak{D}_s \cap U_2 = (\tilde{S}^{(n)} + E_- + E_+ + 2\tilde{F}) \cap U_2$. Similarly we have the followings. Let $Y_i = \mathscr{Y}_i \cap \mathfrak{X}_s$.

(a) On Y_1, ω_s has poles only along $x_3 = 0$ and $ux_3 + \sqrt{-1} = 0$.

$$\tilde{S}^{(n)} \cap Y_1 = \{(x_3, u, z_3) \in Y_1 \mid ux_3 + \sqrt{-1} = 0\},$$
$$\tilde{F} \cap Y_1 = \varnothing,$$
$$E_+ \cap Y_1 = \{(x_3, u, z_3) \in Y_1 \mid x_3 = 0\},$$
$$E_- \cap Y_1 = \varnothing.$$

(b) On Y_2, ω_s has poles only along $v = 0$, $vw - 2\sqrt{-1} = 0$ and $vw - \sqrt{-1} = 0$.

$$\tilde{S}^{(n)} \cap Y_1 = \{(v, w, z_3') \in Y_2 \mid vw - \sqrt{-1} = 0\},$$
$$\tilde{F} \cap Y_1 = \varnothing,$$
$$E_+ \cap Y_1 = \{(v, w, z_3') \in Y_1 \mid vw - 2\sqrt{-1} = 0\},$$
$$E_- \cap Y_1 = \{(v, w, z_3') \in Y_2 \mid v = 0\}.$$

(c) On Y_3, ω_s has poles only along $y_3 = 0$, $y_3 - \sqrt{-1} = 0$, $y_3 + \sqrt{-1} = 0$ with multiplicity 1 and $t = 0$ with multiplicity 2.

$$\tilde{S}^{(n)} \cap Y_1 = \{(v, w, z_3'') \in Y_2 \mid y_3 = 0\},$$
$$\tilde{F} \cap Y_1 = \{(v, w, z_3'') \in Y_2 \mid t = 0\},$$
$$E_+ \cap Y_1 = \{(v, w, z_3'') \in Y_1 \mid y_3 - \sqrt{-1} = 0\},$$
$$E_- \cap Y_1 = \{(v, w, z_3'') \in Y_2 \mid y_3 + \sqrt{-1} = 0\}.$$

(d) On U_4, ω_s has poles only along $y_4 = 0$.

$$\tilde{S}^{(n)} \cap Y_1 = \{(x_4, y_4, z_4) \in Y_1 \mid y_4 = 0\},$$
$$\tilde{F} \cap Y_1 = \varnothing,$$
$$E_+ \cap Y_1 = \varnothing,$$
$$E_- \cap Y_1 = \varnothing.$$

Thus we have $\mathfrak{D}_s = \tilde{S}^{(n)} + E_- + E_+ + 2\tilde{F}$.

Remark 6.12. It is clear that the general fiber of $\mathfrak{X} \to S$ is a compactification of the general fiber of semi-universal deformation of a simple singularity of type D_{2n+2} (see Remark 5.10).

7. Monodromy representation of $\pi_1(S \backslash \Delta)$ on $H_2(\mathfrak{X}_s \backslash \mathfrak{D}_s; \mathbf{Z})$

Let $\varphi : \mathfrak{X} \to S$ be the family and $\mathfrak{D}$ the divisor defined in section 5 if $m = 2n + 3$ or in section 6 if $m = 2n + 2$. Put

$$S' = S \backslash \Delta,$$

$$\mathfrak{X}' = \mathfrak{X} \backslash (\mathfrak{D} \cup \varphi^{-1}(\Delta)).$$

Then $\varphi' = \varphi|_{\mathfrak{X}'} : \mathfrak{X}' \to S'$ is a locally trivial fiber bundle with the fiber $\mathfrak{X}_t \backslash \mathfrak{D}_t$ ($t \in S'$). Therefore $\pi_1(S \backslash \Delta)$ acts on $H_2(\mathfrak{X}_s \backslash \mathfrak{D}_s; \mathbf{Z})$ as a monodromy. Put

$$\begin{aligned}\mathfrak{H} &= \{(s_1, \dots, s_m, -s_1, \dots, -s_m) \in \mathbf{C}^{2m}\} \\ &= \{(s_1, \dots, s_m) \in \mathbf{C}^m\},\end{aligned}$$

$$\mathfrak{H}_{reg} = \left\{ (s_1, \dots, s_m) \in \mathfrak{H} \;\middle|\; \prod_{i \neq j} (s_i - s_j)(s_i + s_j) \neq 0 \right\}.$$

Then

$$S' \cong \mathfrak{H}_{reg} / W,$$

where W is the Weyl group of type D_m.

Theorem 7.1 ([4], [5]). *The fundamental group $\pi_1(S')$ has a presentation with generators $\sigma_0, \dots, \sigma_{m-1}$ and relations:*

$$\underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m_{i,j} times} = \underbrace{\sigma_j \sigma_i \sigma_j \cdots}_{m_{i,j} times},$$

where

$$m_{i,j} = \begin{cases} 1 & i = j, \\ 3 & |i - j| = 1, i, j \neq m, \\ 3 & (i, j) = (m - 2, m), (m, m - 2), \\ 2 & \text{otherwise}. \end{cases}$$

The loop corresponding to the generators $\sigma_0, \dots, \sigma_{m-1}$ can be given as follows (see [5]).

Put

$$H_{i,j} = \{(s_1, \dots, s_m) \in \mathbf{R}^m \mid s_i - s_j = 0\}$$

and

$$H'_{i,j} = \{(s_1, \dots, s_m) \in \mathbf{R}^m \mid s_i + s_j = 0\}.$$

Then

$$\mathfrak{H}_{reg} = \mathfrak{H} - \bigcup_{i \neq j} (H_{i,j} + \sqrt{-1} H_{i,j} \cup H'_{i,j} + \sqrt{-1} H'_{i,j}).$$

The set

$$C_0 + \sqrt{-1}\mathbf{R}^m \subset \mathfrak{H}_{reg}$$

is a fundamental region of W in $\mathfrak{H}_{reg}$, where

$$C_0 = \{(s_1, \dots, s_m) \in \mathbf{R}^m \mid s_1 - s_2 < 0, \dots, s_{m-1} - s_m < 0, s_{m-1} + s_m > 0\}.$$

Put

$$u_{s_0} = (1, 2, \dots, m) \in C_0 + \sqrt{-1}\mathbf{R}^m$$

and let s_0 be a class of u_{s_0} in S'. We define paths in $\mathfrak{H}_{reg}$ which induces loops in S' as follows:

$$\gamma_1^{(i)} : [0, 1] \to \mathfrak{H}_{reg} \qquad (i = 1, 2),$$

$$\gamma_1^{(1)}(t) = (1 - t)u_{s_0} + t(2, 2 + \sqrt{-1}, 3, 4, \dots, m),$$

$$\gamma_1^{(2)}(t) = (1 - t)(2, 2 + \sqrt{-1}, 3, 4, \dots, m) + t(2, 1, 3, \dots, m).$$

Put $\gamma_1 = \gamma_1^{(2)} \cdot \gamma_1^{(1)}$, then γ_1 is a path from u_{s_0} to the image of u_{s_0} by the reflection in $H_{1,2}$. Similarly, we define $\gamma_i (i = 2, \dots, m-1)$ with respect to $H_{i,i+1}$ $(i = 2, \dots, m-1)$. We also define the path γ_m in $\mathfrak{H}_{reg}$ from u_{s_0} to the image of u_{s_0} by the reflection in $H'_{m-1,m}$ as follows.

$$\gamma_m^{(1)}(t) = (1 - t)u_{s_0} + t(1, 2, \dots, m-2, m-1 + \sqrt{-1}, m + \sqrt{-1}),$$

$$\begin{aligned}\gamma_m^{(2)}(t) = {} & (1 - t)(1, 2, \dots, m-2, m-1 + \sqrt{-1}, m + \sqrt{-1}) \\ & + t(1, 2, \dots, -m, -m+1),\end{aligned}$$

and put $\gamma = \gamma_m^{(2)}\gamma_m^{(1)}$.

Let $\bar{\gamma}_i$ $(i = 1, \dots, m)$ be the loops in S' given by these paths γ_i $(i = 1, \dots, m)$ and $\sigma_1, \dots, \sigma_m$ the classes of $\pi_1(S', s_0)$ induced by $\bar{\gamma}_i$ $(i = 1, \dots, m)$. Then $\pi_1(S', s_0)$ is generated by $\sigma_1, \dots, \sigma_m$.

We next define generators of $H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{S_0}; \mathbf{Z})$ corresponding to $\sigma_1, \dots, \sigma_m$. If $s = s_0$, it follows from Remark 5.4 and 6.10 that the roots of the equation $xf_s(x) - a^2 = 0$ are $1^2, 2^2, \dots, m^2$. We may assume $\beta_1, \dots, \beta_m$ is $1^2, 2^2, \dots, m^2$ respectively and

$$E_{i,j} \cap \bar{S}^{(n)} = \{(i^2, 0, (-1)^{j-1}\sqrt{-1}i) \in U_4\} \qquad (i = 1, \dots, m, j = 1, 2),$$

where

$$E_{i,j} = \begin{cases} \pi(F_{i,j}) & m = 2n+3 \\ F_{i,j} & m = 2n+2 \end{cases}, \qquad \bar{S}^{(n)} = \begin{cases} \pi(\tilde{S}^{(n)}) & m = 2n+3 \\ \tilde{S}^{(n)} & m = 2n+2 \end{cases}$$

When we blow down $\mathfrak{X}_{s_0}$ to Σ_1, we may assume all indices j of curves $E_{i,j}$ which should be contracted are 1 (see Remark 5.8, 6.10). Put

$$U_i = \mathscr{U}_i \cap \mathfrak{X}_{s_0}, \qquad (i = 1, 4),$$

$\mathscr{T}$: *a closed tubular neighborhood of* $\bar{S}^{(n)}$ *in* $\mathfrak{X}_{s_0}$.

Define paths τ_i $(i = 1, \dots, m-1)$ from $(i^2, 0, \sqrt{-1}i)$ to $(i^2, 0, \sqrt{-1}(i+1))$ as follows

$$\tau_i(t) = (((1-t)i + t(i+1))^2, 0, (1-t)\delta_{i,1} + t\delta_{i+1,1}) \in U_4.$$

Also

$$\tau_m(t) = (((1-t)(m-1) + tm)^2, 0, (1-t)\delta_{m-1,1} + t\delta_{m,2} \in U_4,$$

where $\delta_{i,j} = (-1)^{j-1}\sqrt{-1}i$. Then we can construct $\Gamma_{i,i+1}$ $(i = 1, \dots, m)$ and $\Gamma'_{m-1,m}$ as (3.3) and (3.3)$'$.

$$\Gamma_{i,i+1} = (E_{i,1} \backslash (E_{i,1} \cap \mathscr{T})) \cup \partial\mathscr{T}|\tau_i \cup (E_{i+1,1} \backslash (E_{i+1,1} \cap \mathscr{T})),$$

$$\Gamma'_{m-1,m} = (E_{m-1,1} \backslash (E_{m-1,1} \cap \mathscr{T})) \cup \partial\mathscr{T}|_{\tau_m} \cup (E_{m,2} \backslash (E_{m,2} \cap \mathscr{T})).$$

Let $\alpha_1, \dots, \alpha_m$ be classes of $\Gamma_{1,2}, \dots, \Gamma_{m-1,m}, \Gamma'_{m-1,m}$ in $H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z})$ respectively. It follows from Corollary 3.4 that $H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z})$ is generated by $\alpha_1, \dots, \alpha_m$ and we have following theorem.

Theorem 7.2. *Let $s_0, \sigma_1, \dots, \sigma_m, \alpha_1, \dots, \alpha_m$ be as above. Let*

$$\rho : \pi_1(S', s_0) \to \mathrm{Aut}(H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}))$$

be the monodromy of the fibration $\phi : \mathfrak{X}' \to S'$. Then

$$\rho(\sigma_i)(x) = x - \frac{2x \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i, \tag{1}$$

$$\rho(\sigma_i\sigma_j)(x) = \rho(\sigma_i)\rho(\sigma_j)(x). \tag{2}$$

This shows the monodromy group $\rho(\pi_1(S', s_0))$ is isomorphic to the Weyl group of type D_m.

Proof. The condition (2) is clear. At first we prove (1) for $\sigma_i = \sigma_1$. We consider diffeomorphic mapping induced by $\bar{\gamma}_1$:

$$\eta(t) : \mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0} \to \mathfrak{X}_{\bar{\gamma}_1(t)} \backslash \mathfrak{D}_{\bar{\gamma}_1(t)}, \qquad t \in [0, 1].$$

Then

$$\eta(t)((i^2, 0, \sqrt{-1}i)) = (s_i(t)^2, 0, \pm\sqrt{-1}s_i(t)) \in \mathfrak{X}_{\bar{\gamma}_1(t)} \backslash \mathfrak{D}_{\bar{\gamma}_1(t)} \cap \mathscr{U}_4,$$

where $\gamma_1(t) = (s_1(t), \dots, s_m(t))$. Since $s_i(t) \neq 0$ and $\eta(t)$ is continuous, $\eta(t)((i^2, 0, \sqrt{-1}i)) = (s_i(t)^2, 0, \sqrt{-1}s_i(t))$. Therefore we have

$$\eta(1)(i^2, 0, \sqrt{-1}i) = \begin{cases} (2^2, 0, 2\sqrt{-1}) & i = 1, \\ (1^2, 0, \sqrt{-1}) & i = 2, \\ (i^2, 0, \sqrt{-1}i) & i \neq 1, 2. \end{cases}$$

Similarly we have

$$\eta(1)(i^2, 0, -\sqrt{-1}i) = \begin{cases} (2^2, 0, -2\sqrt{-1}) & i = 1, \\ (1^2, 0, -\sqrt{-1}) & i = 2, \\ (i^2, 0, -\sqrt{-1}i) & i \neq 1, 2. \end{cases}$$

Thus we have

$$\rho(\sigma_1)(\alpha_i) = \begin{cases} -\alpha_1 & i = 1, \\ \alpha_1 + \alpha_2 & i = 2, \\ \alpha_i & i \neq 1, 2, \end{cases}$$

and

$$\rho(\sigma_1)(x) = x - \frac{2x \cdot \alpha_1}{\alpha_1 \cdot \alpha_1} \alpha_1.$$

We can prove (1) for $\sigma_2, \ldots, \sigma_m$ in the same way.

Remark 7.3. We showed the monodromy group $\rho(\pi_1(S', s_0))$ is isomorphic to the Weyl group of the root system of type D_m. But it is well known that the monodromy group of the locally trivial fiber bundle induced by semi-universal deformation of simple singularity is isomorphic to the Weyl group of the root system corresponding to its singularity. ([1, Volume II, Theorem 3.14])

8. Period mapping for the fibration $\varphi' : \mathfrak{X}' \to S'$

The notation is as in section 7. For $u_{s_0} = (1, \ldots, m)$, put

$$\Omega = \mathrm{Hom}_{\mathbf{Z}}(H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}), \mathbf{C}).$$

Then $\pi_1(S', s_0)$ acts on Ω.

$$\rho^* : \pi_1(S', s_0) \to \mathrm{Aut}(\Omega).$$

For $\alpha_1, \ldots, \alpha_m$, we define $\alpha_1^*, \ldots, \alpha_m^*$ as follows:

$$\alpha_i^*(x) = \alpha_i \cdot x, \qquad x \in H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}), \ i = 1, \ldots, m.$$

Put

$$V^* = \sum_{i=1}^m \mathbf{R}\alpha_i^*.$$

Then we have

$$\Omega = V^* + \sqrt{-1}V^*.$$

We shall define a non-degenerate bilinear form on V^* by

$$\langle x^*, y^* \rangle = \left(\sum_{i=1}^m x_i\alpha_i\right) \cdot \left(\sum_{i=1}^m y_i\alpha_i\right), \qquad x^* = \sum_{i=1}^m x_i\alpha_i^*, \ y^* = \sum_{i=1}^m y_i\alpha_i^*.$$

Let $w_{\alpha_i^*} \in \Omega^*$ be the reflection in the hyperplane orthogonal to α_i^* and W^* the group generated by $w_{a_1^*}, \ldots, w_{a_m^*}$. Let

$$R^* = \{w^*(\alpha_i^*) \in V^* \mid w^* \in W^*, i = 1, \ldots, m\}.$$

We shall define a period mapping for the family $\varphi' : \mathfrak{X}' \to S'$. There is one-to-one correspondence between the equivalence class of covering spaces of S' and the conjugacy class of $\pi_1(S', s_0)$. Let

$$\iota : \hat{S}' \to S'$$

be the covering space of S' corresponding to $\ker \rho$. Then $\hat{S}'$ is a regular covering of S' and its covering transformation group is $G = \rho(\pi_1(S', s_0))$.

Put $\hat{s}_0 = (s_0, [e])$, where $[e]$ is the unit of $\rho(\pi_1(S', s_0))$. For any $\hat{s} \in \hat{S}'$, we can define a diffeomorphism of $\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}$ to $\mathfrak{X}_s \backslash \mathfrak{D}_s$ induced by one of the paths from s_0 to s in S' which corresponds to $\hat{s}$, where $s = \iota(\hat{s})$.

This diffeomorphism induces the isomorphism of homology groups

$$(\hat{s})_* : H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}) \to H_2(\mathfrak{X}_s \backslash \mathfrak{D}_s; \mathbf{Z}).$$

This isomorphism does not depend on the choice of representative of homotopy. Therefore for any $\hat{s} \in \hat{S}'$, we define $\lambda_{\hat{s}} \in \mathrm{Hom}_{\mathbf{Z}}(H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}), \mathbf{C})$ uniquely by

$$\lambda_{\hat{s}}([c]) = \int_{\hat{s}_*(c)} \omega_s,$$

where $[c]$ is the homology class of 2-cycle c.

Then we define a period mapping $\mathscr{P}$ for $\varphi' : \mathfrak{X}' \to S'$

$$\mathscr{P} : \hat{S}' \to \mathrm{Hom}_{\mathbf{Z}}(H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}), \mathbf{C})$$

by $\mathscr{P}(\hat{s}) = \lambda_{\hat{s}}$.

Put

$$H_{\alpha^*} = \{v^* \in V^* \mid \langle \alpha^*, v^* \rangle = 0\}, \qquad \alpha^* \in R^*,$$

$$\Omega' = \Omega - \bigcup_{\alpha^* \in R^*} (V^* + \sqrt{-1} H_{\alpha^*}).$$

Then $G \subset \mathrm{Aut}(H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}))$ acts on Ω' by

$$\theta \cdot \alpha(x) = \alpha(\theta^{-1}(x)), \qquad \theta \in G, \ \alpha \in \Omega', \ x \in H_2(\mathfrak{X}_{s_0} \backslash \mathfrak{D}_{s_0}; \mathbf{Z}),$$

(see Theorem 7.2). We have the following theorem.

Theorem 8.1. *The mapping*

$$\mathscr{P} : \hat{S}' \to \Omega'$$

is surjective and biholomorphic. The monodromy group G acts on $\hat{S}'$ as covering transformation group and Ω' as a reflection group. The period mapping $\mathscr{P}$ is equivariant with these actions. Thus we have isomorphism

$$S' \cong \hat{S}'/G \cong \Omega'/G.$$

Proof. We have $G \cong W$ by Theorem 7.2. Since $C_0 + \sqrt{-1}\mathbf{R}^m$ is a fundamental region of W in $\mathfrak{H}_{reg}$. Any element $\hat{s} \in \hat{S}'$ can be represented by an element $u_{\hat{s}} \in C_0 + \sqrt{-1}\mathbf{R}^m$ and an element $w_{\hat{s}} \in W$ uniquely. Then $\hat{s} = (u_{\hat{s}}, w_{\hat{s}})$.

Let $u_{\hat{s}} = (s_1, \dots, s_m)$ and $s = \iota(\hat{s})$. Let $\tau_{i,\hat{s}}$ $(i = 1, \dots, m)$ be the paths in $U_4 = \mathscr{U}_4 \cap \mathfrak{X}_s$ given by

$$\tau_{i,s}(t) = (((1-t)s_i + ts_{i+1})^2, 0, \sqrt{-1}((1-t)s_i + ts_{i+1})), \qquad i = 1, \dots, m-1,$$

$$\tau_{m,s}(t) = (((1-t)s_{m-1} + ts_m)^2, 0, \sqrt{-1}((1-t)s_{m-1} - ts_m)).$$

This path $\tau_{i,s}$ gives 2-cycle $\Gamma_i(\hat{s})$ as in section 7. Let $\alpha_i(s)$ be the class of $\Gamma_i(\hat{s})$ in $H_2(\mathfrak{X}_s \backslash \mathfrak{D}_s; \mathbf{Z})$. Then

$$\hat{s}_*(\alpha_i(s_0)) = w_{\hat{s}}^{-1}(\alpha_i(s)).$$

Since $\alpha_i(\hat{s}_0) = \alpha_i$, we have

$$\begin{aligned}\mathscr{P}(\hat{s})(\alpha_i) &= \int_{\hat{s}_*(\alpha_i)} \omega_s \\ &= \int_{w_{\hat{s}}^{-1}(\alpha_i(s))} \omega_s,\end{aligned}$$

where α_i is as in section 7.

Thus we have that $\mathscr{P}$ is equivariant with action of G. Put

$$\hat{S}_0 = \{\hat{s} \in \hat{S} \mid \hat{s} = (u_{\hat{s}}, [e]),\ u_{\hat{s}} \in C_0 + \sqrt{-1}\mathbf{R}^m\},$$
$$C_0^* = \{v^* \in V^* \mid \langle v^*, \alpha_i^* \rangle \langle 0 (i = 1, \dots, m-1, \langle v^*, \alpha_m^* \rangle \rangle 0\}.$$

Then $\hat{S}_0, \sqrt{-1}C_0^* + V^*$ is a fundamental region of $\hat{S}'$, Ω' for the action of G respectively. Therefore we have only to prove that

$$\mathscr{P}|_{\hat{S}_0} : \hat{S}_0 \to \sqrt{-1}C_0^* = V^*$$

is bijection to prove that $\mathscr{P}$ is a bijective mapping.

Put $\hat{s} = (u_{\hat{s}}, [e])$. Since

$$\begin{aligned}\omega_s &= -\frac{dx_4 dy_4}{2\pi\sqrt{-1}\, y_4 z_4} \\ &= -\frac{1}{2\pi\sqrt{-1}\, y_4 z_4} \frac{\partial F_4/\partial z_4}{\partial F_4/\partial x_4} dy_4 dz_4 \\ &= \frac{dy_4 dz_4}{\pi\sqrt{-1}\, y_4(1 + y_4^2 \partial f_s/\partial x_4)},\end{aligned}$$

we have

$$\begin{aligned}\mathscr{P}(\hat{s})(\alpha_i) &= \int_{\alpha_i(s)} \omega_s \\ &= 2\pi\sqrt{-1}\int_{\tau_{i,s}} \mathrm{Res}_{\bar{S}^{(n)}}\, \omega_s \\ &= 2\int_{\sqrt{-1}s_i}^{\sqrt{-1}s_{i+1}} dz_4 \\ &= 2(\sqrt{-1}s_{i+1} - \sqrt{-1}s_i)\end{aligned}$$

for $i = 1, \dots, m-1$. If $i = m$, we have

$$\begin{aligned}\mathscr{P}(\hat{s})(\alpha_m) &= \int_{\alpha_i(s)} \omega_s \\ &= 2\int_{-\sqrt{-1}s_m}^{\sqrt{-1}s_{m-1}} dz_4 \\ &= 2(\sqrt{-1}s_{m-1} + \sqrt{-1}s_m).\end{aligned}$$

Thus $\mathscr{P}|_{\hat{S}_0}$ is bijective. It is clear that $\mathscr{P}$ is biholomorphic. Thus the theorem is proved.

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