J. Math. Kyoto Univ. (JMKYAZ)
39-4 (1999) 715–724

On the bamboo-shoot topology of certain inductive limits of topological groups

By

Takashi EDAMATSU

§ 0. Introduction

Let $\{(G_n\tau_n), \phi_{n+1\,n}\}_{n\in N}$ be an inductive system of topological groups G_n with topology τ_n, each $\phi_{n+1\,n}$ being a continuous homomorphism of G_n into G_{n+1}. Put $G = \varinjlim G_n$ and $\tau_{\mathrm{ind}} = \varinjlim \tau_n$. N. Tatsuuma—H. Shimomura—T. Hirai [2] showed by two counter examples that τ_{ind} is not necessarily a group topology for G. They also showed that if the given inductive system fulfils the "PTA-condition", there exists for G the finest group topology that makes every canonical map ϕ_n of G_n into G continuous. Such a topology is, of course, coarser than τ_{ind}. They called such a topology the bamboo-shoot topology for G, denoted by τ_{BS}, and gave a τ_{BS}-neighbourhood base at the unity e of G as the collection of all sets

$$U[k] = \bigcup\nolimits_{n \geqq k} \phi_n(U_n)\phi_{n-1}(U_{n-1}) \cdots \phi_k(U_k)\phi(U_k) \cdots \phi_{n-1}(U_{n-1})\phi_n(U_n)$$

with $k = 1, 2, \dots$ and U_j's each of which runs over symmetric neighbourhoods of the unity e_j of (G_j, τ_j), $j \geqq k$. Here the PTA-condition is a moderate one and stated as follows:

$$\forall n,\ \forall U,\ \exists V \subseteq U, \quad V = V^{-1}, \quad \forall m > n,\ \forall W,\ \exists W', \tag{0.1}$$
$$W'\phi_{mn}(V) \subseteq \phi_{mn}(V)W,$$

where U, V (resp. W, W') denote neighbourhoods of the unity e_n of G_n (resp. e_m of G_m) and $\phi_{mn} = \phi_{mm-1} \circ \cdots \circ \phi_{n+1\,n}$. For instance, any inductive system consisting of locally compact Hausdorff groups fulfils this condition and in this case τ_{ind} happens to coincide with τ_{BS}. τ_{BS} in general seems to be a topological- group-theoretic analogue of the locally convex inductive topology of the inductive limit of locally convex vector spaces (see Propositions 3.1 and 3.2 in [2]).

Now let us bring an inductive system of Banach algebras A_n $(n \in N)$ with the limit algebra $A = \varinjlim A_n$ (in algebraic sense). Let τ_{lct} denote the locally convex inductive topology of A as the inductive limit of Banach spaces A_n. In an appropriate circumstance this system yields an inductive system of topological

Communicated by Prof. T. Hirai, April 26, 1999
Revised July 6, 1999

groups $G(A_n)$ having $G(A)$ as its limit group, where each $G(A_n)$ consists of all invertible elements in A_n and inherits the norm topology and $G(A)$ consists of all invertible elements in A. In the present paper we study the topology τ_{BS} of $G(A)$ and shows that τ_{BS} is just identical with τ_{lct} relativized to $G(A)$ (Theorem 1). We shall also obtain a similar result for another circumstance of A_n's (Theorem 2). Our treatment yields as a special case, in particular, the result obtained in A. Yamasaki [3] for the inductive system of toplogical groups $GL_n(C(X,\mathbf{C}))$, X being a compact Hausdorff space (for details see Example 5 below). Moreover it will be shown that for the inductive systems of topological groups dealt with in the present paper the topology τ_{ind} gives a group topology only when all A_n are finite-dimensional. This fact enables us to produce abundance of elementary examples for which τ_{ind} is not a group topology. Here we shall use the following criterion theorem due to Yamasaki [3].

Theorem Y. *For the system $\{(G_n\tau_n),\phi_{n+1\,n}\}_{n\in\mathbf{N}}$ suppose that each $(G_n\tau_n)$ is first countable and that each $\phi_{n+1\,n}$ is a topological isomorphism of $(G_n\tau_n)$ onto a closed subgroup of $(G_{n+1}\tau_{n+1})$. (The PTA-condition is not assumed.) Then, τ_{ind} is a group topology for G if and only if one of the following two conditions is fulfilled with some $n_0\in\mathbf{N}$:*

(C_1) *Each $(G_n\tau_n)$ $(n\geqq n_0)$ is locally compact;*

(C_2) *Each $\phi_{nn_0}(G_{n_0})$ $(n\geqq n_0)$ is open in $(G_n\tau_n)$.*

§ 1. Preliminary: Strict inductive limits of Banach algebras

Let

$$(1.1)\qquad (A_1\|\ \|_1)\xrightarrow{\psi_{21}}(A_2\|\ \|_2)\xrightarrow{\psi_{32}}(A_3\|\ \|_3)\xrightarrow{\psi_{43}}\cdots$$

be a strict inductive system of Banach algebras over $\boldsymbol{C}$ (or $\boldsymbol{R}$), each $\psi_{n+1\,n}$ being assumed to be a norm-preserving algebra isomorphism into. Let $A=\varinjlim A_n=\bigcup\psi_n(A_n)$ be its limit algebra in algebraic sense, ψ_n being the canonical imbedding isomorphism of A_n into A, and τ_{lct} be the locally convex inductive topology for A as the inductive limit of Banach spaces. As known from the theory of locally convex vector spaces ([1]), the following hold: (i) The space $(A\ \tau_{\mathrm{lct}})$ is Hausdorff and complete; (ii) τ_{lct} induces the norm topology of each $\psi_n((A_n\|\ \|_n))$, that is, each $\psi_n((A_n\|\ \|_n))$ is a closed topological vector subspace of $(A\ \tau_{\mathrm{lct}})$; (iii) A subset of A is τ_{lct}-bounded if and only if it is a bounded subset of some $\psi_n((A_n\|\ \|_n))$. In the sequel each $(A_n\|\ \|_n)$ and its ψ_n-image in A are identified and every $\|\ \|_n$ is denoted by $\|\ \|$.

Now, for each decreasing sequence ε: $\varepsilon_1>\varepsilon_2>\cdots>0$ of positive numbers, define a seminorm $\|\ \|_\varepsilon$ on A as

$$(1.2)\qquad \|a\|_\varepsilon=\inf\left\{\sum\nolimits_k\|a_k\|/\varepsilon_k;\ a_k\in A_k,\ a=\sum\nolimits_k a_k\ \text{(finite sum)}\right\}\qquad(a\in A),$$

and put

$$(1.3) \qquad U_\varepsilon = \{a \in A;\ \|a\|_\varepsilon < 1\}$$
$$= \left\{\sum\nolimits_k a_k \text{ (finite sum)};\ a_k \in A_k,\ \sum\nolimits_k \|a_k\|/\varepsilon_k < 1\right\}.$$

Lemma 1. *The family $\{U_\varepsilon\}_\varepsilon$ gives a neighbourhood base at 0 in $(A\ \tau_{\mathrm{lct}})$.*

The routine verification of this lemma is omitted. Note that in this lemma the sequences ε may be confined to such ones that $\sum_{k=1}^{\infty} \varepsilon_k < 1$.

Remark 1. It is easy to see that in finding the infimum in (1.1) the decomposition $a = \sum_k a_k$ $(a_k \in A_k)$ of each $a \in A$ may be confined to such ones that $k \leqq \min\{n;\ a \in A_n\}$ and non-zero a_k corresponding to each of such k appears at most once.

Lemma 2. *$U_\varepsilon U_\varepsilon \subseteq U_\varepsilon$ holds if $\varepsilon_1 < 1$.*

Proof. Let $a, b \in U_\varepsilon$. Choose their finite decompositions $a = \sum_k a_k$ $(a_k \in A_k)$, $b = \sum_l b_l$ $(b_l \in A_l)$ such that $\sum_k \|a_k\|/\varepsilon_k < 1$, $\sum_l \|b_l\|/\varepsilon_l < 1$. Putting $n(k, l) = \min\{n; a_k, b_l \in A_n\}$, we have $ab = \sum_{k,l} a_k b_l = \sum_j \sum_{n(k,l)=j} a_k b_l$. Hence

$$\|ab\|_\varepsilon \leqq \sum\nolimits_j \left\| \sum\nolimits_{n(k,l)=j} a_k b_l \right\| \Big/ \varepsilon_j$$
$$\leqq \sum\nolimits_j \sum\nolimits_{n(k,l)=j} \|a_k\|\,\|b_1\|/\varepsilon_k \varepsilon_l \qquad (\text{since } \varepsilon_k, \varepsilon_l \leqq \varepsilon_j < 1)$$
$$= \sum\nolimits_k \|a_k\|/\varepsilon_k \sum\nolimits_l \|b_l\|/\varepsilon_l < 1.$$

This proves the assertion.

Lemma 3. *The limit algebra A becomes a topological algebra with respect to τ_{lct}, that is, the multiplication is jointly continuous w.r.t. τ_{lct}.*

Proof. Given any $a, a' \in A$ and U_ε. Choose $U_{\varepsilon'}$ with $\varepsilon_1' < 1$ so that $U_{\varepsilon'} + U_{\varepsilon'} + U_{\varepsilon'} \subseteq U_\varepsilon$ and $\alpha \in (0\ 1)$ so that $\alpha a \in U_{\varepsilon'}$, $\alpha a' \in U_{\varepsilon'}$. Then, for the sequence $\varepsilon'' = \alpha\varepsilon'$: $\alpha\varepsilon_1' > \alpha\varepsilon_2' > \cdots > 0$, we have $U_{\varepsilon''} = \alpha U_{\varepsilon'}$ and so $aU_{\varepsilon''} = \alpha a U_{\varepsilon'} \subseteq U_{\varepsilon'}^2 \subseteq U_{\varepsilon'}$ by Lemma 2. Similarly $U_{\varepsilon''} a \subseteq U_{\varepsilon'}$, $a' U_{\varepsilon''} \subseteq U_{\varepsilon'}$, $U_{\varepsilon''} a' \subseteq U_{\varepsilon'}$. Hence

$$(a + U_{\varepsilon''})(a' + U_{\varepsilon''}) \subseteq aa' + U_{\varepsilon'} + U_{\varepsilon'} + U_{\varepsilon''}^2$$
$$\subseteq aa' + U_\varepsilon \qquad (\text{since } U_{\varepsilon''} \subseteq U_{\varepsilon'}).$$

This proves the joint continuity under question.

Lemma 4. *The algebra A has identity e if and only if, for some $n_0 \in \boldsymbol{N}$, each A_n $(n \geqq n_0)$ has identity e_n and $\psi_{n+1\,n}(e_n) = e_{n+1}$ holds. In this case $e_n = e$ $(n \geqq n_0)$ holds under the identification of A_n and $\psi_n(A_n)$.*

Proof. Since $A = \bigcup A_n$, the "only if" part is obvious. Conversely, by assumption, $\psi_{n+1}(e_{n+1}) = \psi_{n+1}(\psi_{n+1\,n}(e_n)) = \psi_n(e_n)$ $(n \geqq n_0)$. Hence, putting $e = \psi_n(e_n)$ $(n \geqq n_0)$, we have the identity of A.

§ 2. Results for the case of A with identity

As is well known, the invertible elements of a Banach algebra $\mathfrak{A}$ with identity e make a Hausdorff topological group, which is open in $\mathfrak{A}$, by inheriting the norm topology of $\mathfrak{A}$. In particular each element $e+a$ for $a \in \mathfrak{A}$ s.t. $\|a\| < 1$ has the inverse $(e+a)^{-1} = e - a + a^2 - \cdots$.

Now bring the strict inductive system (1.1) of Banach algebras A_n and its limit topological algebra $(A\ \tau_{\mathrm{lct}})$. In this section we assume that A has identity e, namely, by transfering to a cofinal subsystem if necessary, that all A_n $(n \geqq 1)$ have a common identity e and each $\psi_{n+1\,n}$ maps e to e (Lemma 4).

Notation. $G(A_n)$: the topological group consisting of all invertible elements of A_n inheriting the norm topology of A_n.
$G(A)$: the group in algebraic sense consisting of all invertible elements of A.

Proposition 1. *The group $G(A)$ is open in $(A\ \tau_{\mathrm{lct}})$ and becomes a topological group inheriting the topology τ_{lct} of A. The family $\{e + U_\varepsilon;\ \sum_{k=1}^{\infty} \varepsilon_k < 1\}$ gives a neighbourhood base at e of this topological group.*

Proof. Given a neighbourhood $e + U_\varepsilon$ of e in $(A\ \tau_{\mathrm{lct}})$, where $\sum_{k=1}^{\infty} \varepsilon_k < 1$. If $a \in U_\varepsilon$, there is a finite decomposition $a = \sum a_k$ $(a_k \in A_k)$ s.t. $\sum_k \|a_k\|/\varepsilon_k < 1$. Hence $\|a\| \leqq \sum_k \|a_k\| \leqq \sum_k \varepsilon_k < 1$. Therefore the inverse $(e+a)^{-1} = e - a + a^2 - \cdots$ exists in those A_n to which a belongs. Thus $e + U_\varepsilon \subseteq G(A)$. Now let $a \in \frac{1}{3} U_\varepsilon$ $(= U_{(1/3)\varepsilon} \subseteq U_\varepsilon)$. Then $a^n \in \dfrac{1}{3^n} U_\varepsilon$ $(n = 1, 2, \ldots)$ by Lemma 2 and so $\|\sum_{n=1}^{\infty}(-a)^n\|_\varepsilon \leqq \sum_{n=1}^{\infty} \|a^n\|_\varepsilon \leqq \sum_{n=1}^{\infty} \dfrac{1}{3^n} < 1$. Hence $\sum_{n=1}^{\infty}(-a)^n \in U_\varepsilon$. Therefore $(e + \frac{1}{3} U_\varepsilon)^{-1} \subseteq e + U_\varepsilon$. Since ε is arbitrary, this shows that the invertion operation in $G(A)$ is τ_{lct}-continuous at e. Next, for any $b \in G(A)$ and any neighbourhood $b^{-1}(e + U_\varepsilon)$ of b^{-1}, we have $((e + \frac{1}{3} U_\varepsilon)b)^{-1} \subseteq b^{-1}(e + U_\varepsilon)$. This proves that the invertion operation is τ_{lct}-continuous at b. In view of Lemma 3 the verification is now complete.

Since each $\psi_{n+1\,n}$ maps e to e, it is obvious that the inductive system (1.1) of Banach algebras gives rise to the inductive system

$$G(A_1) \xrightarrow{\psi_{21}} G(A_2) \xrightarrow{\psi_{32}} G(A_3) \xrightarrow{\psi_{43}} \cdots \tag{2.1}$$

of topological groups and that $\varinjlim G(A_n) = \bigcup G(A_n) = G(A)$ holds as set. More generally suppose that there is given a topological subgroup G_n of each $G(A_n)$ so that $G_n \subseteq G_{n+1}$. Then the system (2.1) further gives rise to an inductive system

$$G_1 \xrightarrow{\psi_{21}} G_2 \xrightarrow{\psi_{32}} G_3 \xrightarrow{\psi_{43}} \cdots \tag{2.2}$$

of topological groups. Needless to say, (2.1) is included in (2.2) as a special case.

Proposition 2. *The system* (2.2) *fulfils the PTA-condition.*

Proof. We check (0.1) for this system. For any n and any neighbourhood U of e in G_n we can choose a symmetric neighbourhood V of e in G_n so that $V \subseteq U \cap \{e + a;\ a \in A_n, \|a\| < 1/2\}$. Given any $m > n$ and any neibourhood W of e in G_m. Take $\delta > 0$ so that

$$\{e + a;\ a \in A_m, \|a\| < \delta\} \cap G_m \subseteq W,$$

and put

$$W' = \{e + b;\ b \in A_m, \|b\| < \delta/4\} \cap G_m.$$

Then, for $v \in V$ and $w' = e + b \in W'$, we have $w'v = v(v^{-1}w'v) = v(e + v^{-1}bv)$ and $\|v^{-1}bv\| \leqq \|v^{-1}\|\|b\|\|v\| < \delta$ (since $\|v^{-1}\|, \|v\| < 2$). Hence $w'v \in vW$ which implies $W'V \subseteq VW$.

To get the main results of the paper (Theorems 1 and 2 below) we set here the following technique

Lemma 5. *Let H be a subgroup of $G(A)$. Assume that for each $k \in N$ and each neighbourhood O_n of 0 in $A_n (n \geqq k)$, there can be chosen a neighbourhood Q_n of 0 in each A_n $(n \geqq k)$ so that $\{\bigcup_{n \geqq k}(Q_k + \cdots + Q_n)\} \cap H' \subseteq \bigcup_{n \geqq k}\{(O_k \cap H') + \cdots + (O_n \cap H')\}$, where $H' = H - e$. Then, for the system (2.2) with $G_n = G(A_n) \cap H$, the topology τ_{BS} of its limit group $\varinjlim G_n = \bigcup G_n = H$ coincides with the topology τ_{lct} of $\varinjlim A_n = A$ relativized to H. (Hence, in this case, a τ_{BS}-neighbourhood base at e in H is given by $\{(e + U_\varepsilon) \cap H; \sum_{k=1}^{\infty} \varepsilon_k < 1\}$ (see Proposition 1)).*

Proof. (This proof was suggested by Prof. H. Shimomura.) Since τ_{lct} relativized to H is a group topology by Proposition 1, it is coarser than τ_{BS}. We prove the converse. Given any τ_{BS}-neighbourhood $U[k] = \bigcup_{n \geqq k} U_n U_{n-1} \cdots U_k U_k \cdots U_{n-1} U_n$ of e in H, where each U_j is a neighbourhood of e in G_j (see §0). Since each $G(A_j)$ is open in A_j, we can choose a neighbourhood O_j of 0 in A_j $(j \geqq k)$ so that $U_j \supseteq (e + O_j) \cap H = e + (O_j \cap H')$. Then, obviously, $U[k] \supseteq \bigcup_{n \geqq k} U_k \cdots U_n \supseteq \bigcup_{n \geqq k}\{e + (O_k \cap H') + \cdots + (O_n \cap H')\}$. Therefore the assumption of the lemma enables us to choose a sequence ε: $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ so that $\sum_{l=1}^{\infty} \varepsilon_l < 1$ and $U[k] \supseteq \bigcup_{n \geqq k}\{e + (Q_k + \cdots + Q_n) \cap H'\}$, where $Q_j = \{a \in A_j; \|a\| < \varepsilon_{j-k+1}\}$ $(j \geqq k)$. Now suppose $a \in U_\varepsilon \cap H'$. Then $a = \sum_{l=1}^{N} a_l$ and $\sum_{l=1}^{N} \|a_l\|/\varepsilon_l < 1$ for some N and $a_l \in A_l$. Hence $a_l \in Q_{l+k-1}$ and $a \in (Q_k + \cdots + Q_{N+k-1}) \cap H'$. Thus after all $(e + U_\varepsilon) \cap H \subseteq \bigcup [k]$, which completes the proof.

Theorem 1. *The topology τ_{BS} of $G(A) = \varinjlim G(A_n)$ coincides with τ_{lct} relativized to $G(A)$.*

Proof. Since $G(A)$ is open in (A, τ_{lct}), the assumption of the lemma 5 is fulfilled for $H = G(A)$.

Indeed, $(G(A) - e) \cap A_j$ is open in $A_j (\forall j)$ and therefore, for given O_j's $(j \geqq k)$ in Lemma 5, one can take $O_j \cap H' = O_j \cap (G(A) - e)$ as Q_j's.

Proposition 3. *The topology* τ_{ind} *for* $G(A) = \varinjlim G(A_n)$ *is a group topology (namely,* $\tau_{\text{ind}} = \tau_{\text{BS}}$ *holds) if and only if all* A_n *are finite-dimensional.*

Proof. Each $G(A_n)$ is first countable and closed in $G(A_m)$ $(m > n)$. But it is not open in $G(A_m)$. In fact, A_n is not open in A_m because A_m is connected and A_n is closed in it. Therefore, for any $\delta \in (0\ 1)$, there can be chosen $a \in A_m \backslash A_n$ s.t. $\|a\| < \delta$. Then $e + a \in G(A_m) \backslash G(A_n)$. Therefore e is not an interior point of $G(A_n)$ in $G(A_m)$ and so $G(A_n)$ is not open in $G(A_m)$. Thus, by Theorem Y in §0, the following equivalency obtains: τ_{ind} is a group topology $\Leftrightarrow$ every $G(A_n)$ is locally compact $(n \geqq \exists n_0) \Leftrightarrow$ some closed ball $\{e + a; \|a\| \leqq \delta < 1\}$ in A_n is compact $(n \geqq \exists n_0) \Leftrightarrow$ every A_n is finite-dimensional.

Remark 2. The norms of A_n's altogether define obviously a norm on $A = \bigcup A_n$ and A becomes a normed algebra (incomplete). $G(A)$ is a topological group by this norm topology relativized, denoted by τ_{norm}, as well. One has $\tau_{\text{norm}} \leqq \tau_{\text{BS}}$ and the equivalency $\tau_{\text{norm}} = \tau_{\text{BS}} \Leftrightarrow \exists n_0, \forall n \geqq n_0, A_n = A_{n_0}$. This equivalency can be checked easily by the completeness of $(A\ \tau_{\text{lct}})$, Baire's category theorem and the definition of the topologies of $G(A)$.

Example 1. Let $X = \prod_{n=1}^{\infty} X_n$ be the product space of compact Hausdorff spaces X_n and $C(X, \mathbf{C})$ be the Banach algebra consisting of all $\mathbf{C}$-valued continuous functions on X equipped with the uniform norm. For each n let A_n be the Banach subalgebra of $C(X, \mathbf{C})$ consisting of the functions depending only on the variables $x_i \in X_i$ $(i = 1, \dots, n)$. Then a strict inductive system $A_1 \to A_2 \to A_3 \to \cdots$ of Banach algebras is obtained, where each $\to$ is the natural imbedding. All A_n and $A = \varinjlim A_n = \bigcup A_n$ have the constant function 1 as the common identity and each $\to$ maps 1 to 1. Thus the above results apply to this system. Note that each $G(A_n)$ is the totality of never-vanishing functions in A_n. It is easily seen by Proposition 1 that τ_{BS} for $G(A)$ is strictly finer than the norm topology of $C(X, \mathbf{C})$ relativized to $G(A)$. Proposition 3 shows that $\tau_{\text{ind}} = \tau_{\text{BS}}$ holds if and only if every X_n is a finite set.

Example 2. Given an inductive system $H_1 \to H_2 \to H_3 \to \cdots$ of Hausdorff groups H_n, where each $\to$ is a topologically isomorphic imbedding. Let $M_0(H_n)$ be the usual Banach algebra formed of all bouded complex Radon measures on H_n. For each $\mu_n \in M_0(H_n)$ define $\mu_{n+1} \in M_0(H_{n+1})$ by $\mu_{n+1}(B) = \mu_n(B \cap H_n)$, B being Borel sets of H_{n+1}. Now take a sequence of compact subsets K_l of H_n s.t. $\mu_n(\bigcap_{l=1}^{\infty} K_l^c) = 0$. The Borel structure on $K = \bigcup_{l=1}^{\infty} K_l$ induced from H_n and that induced from H_{n+1} coincide. Furthermore, for any real Radon measure μ on a completely regular space X, one has $\|\mu\| = \sup\{\mu(B) - \mu(B^c); B \text{ is a Borel set of } X\}$. Hence $\|\mu_n\| = \|\mu_{n+1}\|$ follows. Thus each $M_0(H_n)$ is imbedded into $M_0(H_{n+1})$ by identifying each μ_n with μ_{n+1}, and a strict inductive system $M_0(H_1) \to M_0(H_2) \to M_0(H_3) \to \cdots$ of Banach algebras is obtained. Here each $M_0(H_n)$ has the Dirac measure δ_e as identity (e denoting the common unity of all H_n), which is mapped to $\delta_e \in M_0(H_{n+1})$ by $\to$. Thus the preceding results apply to this system. Proposition 3 shows for $\varinjlim G(M_0(H_n)) = G\left(\varinjlim M_0(H_n)\right)$ that $\tau_{\text{ind}} = \tau_{\text{BS}}$ holds if and only if every H_n is a finite group.

§3. On the case of A without identity

It is essentially the following two cases that $A = \varinjlim A_n$ has not identity (Lemma 4):

Case 1. No A_n has identity.

Case 2. Every A_n has identity e_n but there exist infinitely many n such that $\psi_{n+1\,n}(e_n) \neq e_{n+1}$.

In either cases we introduce a new strict inductive system of Banach algebras with identity. That is, adding a formal common element $\tilde{e}$ to all A_n and A, we make the direct sums of vector spaces $\tilde{A}_n = A_n + \mathbf{C}\tilde{e}$, $\tilde{A} = A + \mathbf{C}\tilde{e}$ and define the multiplication in them by

$$(a_n + \alpha\tilde{e})(b_n + \beta\tilde{e}) = (a_n b_n + \alpha b_n + \beta a_n) + \alpha\beta\tilde{e} \qquad (a_n, b_n \in A_n,\ \alpha, \beta \in \boldsymbol{C})$$

and similarly for $\tilde{A}$. Then $\tilde{A}_n$, $\tilde{A}$ become algebras with identity $\tilde{e}$. Further each $\tilde{A}_n$ becomes a Banach algebra by the norm $\|a_n + \alpha\tilde{e}\| = \|a_n\| + |\alpha|$. Through this procedure the strict inductive system (1.1) of Banach algebras A_n is extended uniquely to a strict inductive system

$$\tilde{A}_1 \xrightarrow{\tilde{\psi}_{21}} \tilde{A}_2 \xrightarrow{\tilde{\psi}_{32}} \tilde{A}_3 \xrightarrow{\tilde{\psi}_{43}} \cdots \tag{3.1}$$

of Banach algebras $\tilde{A}_n$. Here each $\tilde{e} \in \tilde{A}_n$ is mapped to $\tilde{e} \in \tilde{A}_{n+1}$ by $\tilde{\psi}_{n+1\,n}$. It is of course that the limit algebra of this system coincides with $\tilde{A}$. $\tilde{A}$ is endowed with the locally convex inductive topology, denoted by $\tilde{\tau}_{\text{lct}}$, of this system. $\tilde{A}$ is then a topological algebra by Lemma 3. In this section we intend to apply the preceding results to the system (3.1)

Lemma 6. *$\tilde{\tau}_{\text{lct}}$ for $\tilde{A} = A + \mathbf{C}\tilde{e}$ coincides with the product topology of τ_{lct} for A and the usual topology of $\mathbf{C}\tilde{e}$ $(\cong \mathbf{C})$.*

Proof. The seminorms $\|\ \|_\varepsilon$ generating τ_{lct} are extended to the seminorms $\|a + \alpha\tilde{e}\|_\varepsilon = \|a\|_\varepsilon + |\alpha|$ on the space $\tilde{A} = A + \mathbf{C}\tilde{e}$. Let $\tilde{\tau}$ denote the stated product topology. Obviously $\tilde{\tau}$ is generalized by these extended seminorms. On the other hand, $\tilde{\tau}_{\text{lct}}$ is generalized by the seminorms

$$\|a + \alpha\tilde{e}\|_{\tilde{\varepsilon}} = \inf\left\{ \sum\nolimits_k \|a_k + \alpha_k\tilde{e}\|/\varepsilon_k \text{ (finite sum)}; \right.$$
$$\left. \sum\nolimits_k a_k = a \ (a_k \in A_k), \sum\nolimits_k \alpha_k = \alpha \right\},$$

each of which is another extention of $\|\ \|_\varepsilon$ on A. Here we have $\|a + \alpha\tilde{e}\|_{\tilde{\varepsilon}} \geqq \|a\|_\varepsilon + \varepsilon_1^{-1}|\alpha|$ since $\sum_k |\alpha_k|/\varepsilon_k \geqq \sum_k |\alpha_k|/\varepsilon_1 \geqq |\alpha|/\varepsilon_1$, and conversely $\|a + \alpha\tilde{e}\|_{\tilde{\varepsilon}} \leqq \|a\|_{\tilde{\varepsilon}} + \|\alpha e\|_{\tilde{\varepsilon}} = \|a\|_\varepsilon + \|e\|_{\tilde{\varepsilon}}|\alpha|$. Hence the assertion follows.

Now let us consider the topological subgroups

$$\tilde{G}_n = (A_n + \tilde{e}) \cap G(\tilde{A}_n), \qquad \tilde{G} = (A + \tilde{e}) \cap G(\tilde{A}) \tag{3.2}$$

of each $G(\tilde{A}_n)$ and $G(\tilde{A})$. Here note that $G(\tilde{A}_n) = (\mathbf{C}\backslash\{0\})\tilde{G}_n$, $G(\tilde{A}) = (\mathbf{C}\backslash\{0\})\tilde{G}$ and $\tilde{G}_n = G(\tilde{A}_n) \cap \tilde{G}$. Recall that an element a of an algebra $\mathfrak{A}$, having identity or not, is quasi-invertible by definition if there exists $b \in \mathfrak{A}$ s.t. $a + b + ab = a + b + ba = 0$. Let $qi(A_n)$ (resp. $qi(A)$) denote the totality of quasi-invertible elements in A_n (resp. A). Then it is evident that

$$\tilde{G}_n = \tilde{e} + qi(A_n), \qquad \tilde{G} = \tilde{e} + qi(A). \tag{3.2$'$}$$

The system (3.1) induces an inductive system

$$\tilde{G}_1 \xrightarrow{\tilde{\psi}_{21}} \tilde{G}_2 \xrightarrow{\tilde{\psi}_{32}} \tilde{G}_3 \xrightarrow{\tilde{\psi}_{43}} \cdots \tag{3.3}$$

of topological subgroups of $G(\tilde{A}_n)$'s, which fulfils the PTA-condition by Proposition 2. Its limit group $\varinjlim \tilde{G}_n = \bigcup \tilde{G}_n$ coincides with $\tilde{G}$.

Theorem 2. *Suppose A has not identity. The set $qi(A)$ is open in $A = \varinjlim A_n$ bearing τ_{lct}. The bamboo-shoot topology, denoted by $\tilde{\tau}_{\mathrm{BS}}$, on the limit group $\tilde{G} = \tilde{e} + qi(A)$ is induced from τ_{lct} for A. That is, a $\tilde{\tau}_{\mathrm{BS}}$-neighbourhood base at $\tilde{e}$ in $\tilde{G}$ is given by $\{\tilde{e} + U_\varepsilon; \sum_{k=1}^{\infty} \varepsilon_k < 1\}$, where each U_ε is the same as in* (1.3).

Proof. $\tilde{e} + qi(A) = (\tilde{e} + A) \cap G(\tilde{A})$ (see (3.2), (3.2$'$)), and $G(\tilde{A})$ is open in $(\tilde{A}, \tilde{\tau}_{\mathrm{lct}})$ by Proposition 1. Therefore, in virtue of Lemma 6, it is evident that qi(A) is open in (A, τ_{lct}). The remaining assertion of the theorem just means that $\tilde{\tau}_{\mathrm{BS}}$ coincides with $\tilde{\tau}_{\mathrm{lct}}$ relativized to $\tilde{G}$. So our task is to show that for the subgroup $H = \tilde{G} = \tilde{e} + qi(A)$ of $G(\tilde{A})$ the assumption of Lemma 5 is fulfilled. Given any $k \in \boldsymbol{N}$ and any neighbourhood $\tilde{O}_j$ of 0 in $\tilde{A}_j$ $(j \geqq k)$. It is obvious that the set $\tilde{O}_j \cap H' = \tilde{O}_j \cap qi(A)$ is open in A_j. Hence, as Q_j's in Lemma 5, the sets $\mathbf{C}\tilde{e} + (\tilde{O}_j \cap qi(A))$ (say) can be taken.

Proposition 4. *The inductive topology for $\tilde{G}$ as the limit of* (3.3), *denoted by $\tilde{\tau}_{\mathrm{ind}}$, coincides with $\tilde{\tau}_{\mathrm{BS}}$ if and only if all A_n are finite-dimensional.*

Proof. The verification goes in parallel with the proof of Proposition 3. We have only to replace $G(A_n)$ and e there by $\tilde{G}_n$ and $\tilde{e}$.

Here we give an example belonging to Case 1 above.

Example 3. Bring the inductive system of Hausdorff groups H_n in Example 2 but assume that every group H_n is infinite and discrete. Let H denote the limit group of this system bearing the bamboo-shoot topology, i.e., the discrete topology. For each n, consider the commutative Banach algebra $C_0(H_n)$, with uniform norm, of all $\mathbf{C}$-valued functions on H_n vanishing at infinity. It is obvious that each $C_0(H_n)$ can be imbeded in $C_0(H)$ by regarding each $f \in C_0(H_n)$ as the function in $C_0(H)$ s.t. $f \equiv 0$ on $H\backslash H_n$. Thus a strict inductive system $C_0(H_1) \to C_0(H_2) \to$

$C_0(H_3) \to \cdots$ of Banach algebras without identity is obtained. It is obvious that $\varinjlim C_0(H_n) = \bigcup C_0(H_n)$ is dense in the Banach algebra $C_0(H)$. Hence the role of $\tilde{e}$ must be played by the constant function 1 on H. For this system one has $\tilde{G} = 1 + \{f \in C_0(H); \mathrm{Range}(f) \not\ni -1\}$. By Theorem 2 $\tilde{\tau}_{\mathrm{BS}}$ for $\tilde{G}$ is induced from τ_{lct} for $C_0(H) = \varinjlim C_0(H_n)$. Furthermore Proposition 4 shows that $\tilde{\tau}_{\mathrm{ind}}$ differs from $\tilde{\tau}_{\mathrm{BS}}$ for the present case because every H_n is an infinite group and so $C_0(H_n)$ is infinite-dimensional.

Now let us consider Case 2. (Note that in this case each A_n has identity e_n but (2.1) never gives an inductive system of groups because $\psi_{n+1\, n}(e_n) \neq e_{n+1}$ for infinitely many n.) In this case we have equivalency $a \in qi(A_n) \Leftrightarrow e_n + a \in G(A_n)$. Hence $qi(A_n) = G(A_n) - e_n$ and so, by (3.2′),

$$\tilde{G}_n = G(A_n) + (\tilde{e} - e_n). \tag{3.2″}$$

Here note that $\tilde{e} - e_n$ is an idempotent element of $\tilde{A}_n$ and therefore it makes a single group contained in $\tilde{A}_n$

Proposition 5. *Suppose each A_n has identity e_n but A does not. Then each $\tilde{G}_n$ is given by (3.2″) and topologically isomorphic to the direct product of $G(A_n)$, which inherits the norm topology of A_n, with the single group $\{\tilde{e} - e_n\}$ in $\tilde{A}_n$. (Hence $\tilde{G} = \bigcup\{G(A_n) + (\tilde{e} - e_n)\}$.)*

Proof. Since $a(\tilde{e} - e_n) = (\tilde{e} - e_n)a = 0$ for $a \in A_n$, the assertion is obvious.

Example 4. Let $H = H_1 + H_2 + H_3 + \cdots$ be an orthogonal sum of countably many Hilbert spaces. Put $H^{(n)} = H_1 + \cdots + H_n$ for each n and consider the usual Banach algebra $B(H^{(n)})$ formed of all bounded linear operators on $H^{(n)}$. Each $B(H^{(n)})$ has identity $I^{(n)}$. By identifying each $T^{(n)} \in B(H^{(n)})$ with $T \in B(H)$ s.t. $T = T^{(n)}$ on $H^{(n)}$, and $=0$ on $H^{(n)\perp}$ in H, a strict inductive system of Banach algebras $B(H^{(n)})$ is obtained which belongs to Case 2. Note that $B(H^{(n)})$ is identified with $P^{(n)}B(H^{(n)})P^{(n)}$ as Banach space, $P^{(n)}$ denoting the projection of H onto $H^{(n)}$. $\varinjlim B(H^{(n)})(=\bigcup B(H^{(n)}))$ is strongly dense in $B(H)$ because $P^{(n)}TP^{(n)}$ converges strongly to T for every $T \in B(H)$. Hence the role of the common identify $\tilde{e}$ for this system must be played by I, the identity operator on H. Therefore, by (3.2″), $\tilde{G}_n = \{T \in G(B(H)); T \mid H^{(n)} \in G(B(H^{(n)})), T = I \text{ on } H^{(n)\perp}\}$, where $G(B(H))$ denotes the totality of regular elements in $B(H)$. Proposition 4 shows for $\tilde{G} = \bigcup \tilde{G}_n$ in this case that $\tilde{\tau}_{\mathrm{ind}} = \tilde{\tau}_{\mathrm{BS}}$ holds if and only if all H_n are finite-dimensional.

Example 5. Let A be a Banach algebra over $\boldsymbol{C}$ (or $\boldsymbol{R}$) with identity e, and $M_n(A) = \{a = (a_{ij})_{ij=1\cdots n}; a_{ij} \in A\}$ be the full matrix-algebra of n-th order with elements in $A (n = 1, 2, \ldots)$. Each $M_n(A)$ has identity $I_n = \begin{bmatrix} e & & \\ & \ddots & \\ & & e \end{bmatrix}$. Let A^n be the product Banach space of n copies of A, the norm of which is defined by $\|b\|_n = \max_l \|b_l\|$ $(b = (b_1, \ldots, b_n) \in A^n)$, and $B(A^n)$ be the Banach algebra formed of all bounded linear operators on Λ^n. Then it is easy to see that each $M_n(\Lambda)$ is a Banach subalgebra of $B(\Lambda^n)$. By identifying each $a=(a_{ij})\in M_n(\Lambda)$ with $\left[\begin{array}{c:c} a & 0 \\ \hdashline 0 & 0 \end{array}\right] \in M_\infty(\Lambda)$ a strict inductive system of Banach algebras $M_n(\Lambda)$ is obtained which belongs to Case 2. Put $M(\Lambda)=\varinjlim M_n(\Lambda)(=\bigcup M_n(\Lambda))$. It is obvious that the role of the common identity for $M_n(\Lambda)^{\sim}$ and $M(\Lambda)^{\sim}$ is played by the matrix $I=\begin{bmatrix} e & & \\ & e & \\ & & \ddots \end{bmatrix}$. By (3.2″) we have $\tilde{G}_n=\left\{\left[\begin{array}{c:ccc} a & & 0 & \\ \hdashline & e & & \\ 0 & & e & \\ & & & \ddots \end{array}\right];\ a\in \mathrm{GL}_n(\Lambda)\right\}$, where $GL_n(\Lambda)=G(M_n(\Lambda))$. As to a $\tilde{\tau}_{\mathrm{BS}}$-neighbourhood base at I in $\tilde{G}=\varinjlim \tilde{G}_n$ $(=\bigcup \tilde{G}_n)$, denoted by $\mathrm{GL}(\Lambda)$, Theorem 2 applies. Proposition 4 shows that $\tilde{\tau}_{\mathrm{ind}}=\tilde{\tau}_{\mathrm{BS}}$ holds if and only if Λ is finite-dimensional. The case of $\Lambda=C(X,\mathbf{C})$, X being a compact Hausdorff space, was treated in Yamasaki [3] in a direct manner. (Of course $C(X,\mathbf{C})$ represents for all commutative C^*-algebras with identity.)

§ Appendix

Let H_n $(n=1,2,\dots)$ be Hausdorff groups satisfying the first countability. Put $G_n=H_1\times\cdots\times H_n$ and let $\psi_{n+1\,n}$ be the canonical imbedding of G_n into G_{n+1}. For the inductive system $\{G_n,\psi_{n+1\,n}\}_{n\in N}$ of topological groups thus obtained, it is easily seen by Theorem Y that τ_{ind} is a group topology for $G=\varinjlim G_n$ if and only if all H_n are locally compact, or all but a finite number of H_n are discrete. The first counter example given in [2] is just the case $H_1=\boldsymbol{Q}$, $H_n=\boldsymbol{R}$ $(n\geqq 2)$, which satisfies neither of these requirements.

College of Science and Technology
Nihon University
Present Address: Shin-Shiraoka 1-24-4,
Saitama Pref., Japan 349-0212

References

[1] H. H. Schaefer, Topological vector spaces, Springer, New York, 1971.

[2] N. Tatsuuma, H. Shimomura and T. Hirai, On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms, J. Math. Kyoto Univ., **38** (1998), 551–578.

[3] A. Yamasaki, Inductive limit of general linear groups, J. Math. Kyoto Univ, **38** (1998), 769–779.

[4] L. Natarajan, E. Rodrigues-Carrington and J. A. Wolf, Locally convex Lie groups, Nova J. Algebra and Geometry, **2** (1993), 59–87.

Added in proof. Corollary A.11 in Appendix of [4] asserts that every strict inductive limit of topological groups is a topological group w.r.t. τ_{ind}. I am afraid this assertion, however, runs counter to the examples given in the present paper and [2].