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The Hopf algebra structure of the cohomology of the 3-connective fibre space over the special unitary group

By

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1. Introduction

Fix a prime p and let $\widetilde{SU}(n)$ be the 3-connective fibre space over $SU(n)$ for $n = 2, 3, \dots, \infty$. Note that $\widetilde{SU}(n)$ is a Hopf space with a inverse since the product and the inverse of $SU(n)$ induce those of $\widetilde{SU}(n)$ respectively.

In this paper, we determine $H^*(\widetilde{SU}(n); \mathbf{F}_p)$ as a Hopf algebra over $\mathscr{A}_p$ the mod p Steenrod algebra. The results are stated in §2.

As a Hopf algebra over $\mathscr{A}_p$, $H^*(\widetilde{SU}(\infty); \mathbf{F}_p)$ can be easily determined by inspections of the cohomology Serre spectral sequences associated with the fiberings

$$\mathbf{CP}^\infty \longrightarrow \widetilde{SU}(\infty) \xrightarrow{q_\infty} SU(\infty),$$

$$\widetilde{SU}(\infty) \xrightarrow{q_\infty} SU(\infty) \longrightarrow K(\mathbf{Z}, 3)$$

except one cohomology operation

$$\wp^1 \tilde{y}_{2p+1} = \varepsilon_p \tilde{x}_{4p-1} \qquad (0 \neq \varepsilon_p \in \mathbf{F}_p)$$

where $\tilde{x}_{4p-1}$ is the generator of degree $4p-1$ in the image of the homomorphism q_∞^* induced from the covering projection q_∞, while $\tilde{y}_{2p+1}$ is the generator of degree $2p+1$ not in the image of q_∞^*. This action will be shown in §3 by use of the mod p decomposability of $SU(\infty)$ (Adams [1]) and the information of the homotopy groups.

For finite n, $H^*(\widetilde{SU}(n); \mathbf{F}_p)$ can be almost determined as a Hopf algebra over $\mathscr{A}_p$ by the results of $H^*(SU(n); \mathbf{F}_p)$ and $H^*(\widetilde{SU}(\infty); \mathbf{F}_p)$. However, major difficulties will be encountered if one wants to know the coproduct and the $\mathscr{A}_p$-action of $\tilde{y}_{2p^r}$ where r is an integer such that $p^{r-1} < n \leq p^r$. Here $\tilde{y}_{2p^r}$ is the only one generator of degree $2p^r$ which is neither in the image of q_n^* nor in that of $\tilde{i}_{n,\infty}^*$ where $q_n : \widetilde{SU}(n) \to SU(n)$ is the covering projection and $\tilde{i}_{n,\infty} : \widetilde{SU}(n) \to \widetilde{SU}(\infty)$ is the map induced from the usual inclusion $i_{n,\infty} : SU(n) \hookrightarrow SU(\infty)$. We shall determine the coproduct of $\tilde{y}_{2p^r}$ in §4 by computing the homomorphism induced from the commutator map of $\widetilde{SU}(n)$ in two manners and comparing them. On

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the one hand, we compute directly from the coproduct, on the other hand, we decompose the commutator map of $\widetilde{SU}(n)$ and apply the results of Bott [2] and Hamanaka [4]. Here, for a Hopf space with a inverse, the commutator map is defined as the one which maps (x, y) to $xyx^{-1}y^{-1}$. In §5, we shall deduce the cohomology operations to $\tilde{y}_{2p^r}$ from those to the coproduct of $\tilde{y}_{2p^r}$.

Remark 1.1. Let $\widetilde{Spin}(n)$ be the 3-connective fibre space over $Spin(n)$ for $n = 7, 8, 9, 10, \dots, \infty$ and $\widetilde{Sp}(n)$ the 3-connective fibre space over $Sp(n)$ for $n = 2, 3, \dots, \infty$. As a Hopf algebra over $\mathscr{A}_p$ where p is an odd prime, $H^*(\widetilde{Spin}(n); \mathbf{F}_p)$ and $H^*(\widetilde{Sp}(n); \mathbf{F}_p)$ can be determined by the results of this paper together with the natural inclusions and the p-equivalence

$$Spin(2k-1) \hookrightarrow SU(2k-1),$$

$$Spin(2k) \simeq_p Spin(2k-1) \times S^{2k-1} \qquad (k = 4, 5, 6, \dots);$$

$$Sp(n) \hookrightarrow SU(2n).$$

(Moreover, $H^*(\widetilde{Sp}(n); \mathbf{F}_2)$ can be also determined as a Hopf algebra over $\mathscr{A}_2$ quite easily.)

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2. Results

In this paper, for any Hopf algebra, the reduced coproduct map is denoted by $\bar{\mu}^*$. Let $\wp^k = Sq^{2k}$ if $p = 2$ and $\binom{a}{b} = 0$ if $b < 0$ or $a - b < 0$.

We shall show the following theorem.

Theorem 2.1. *Let p be a prime and n an integer such that $p^{r-1} < n \leq p^r$ for a positive integer r. As a Hopf algebra over $\mathscr{A}_p$, $H^*(\widetilde{SU}(n); \mathbf{F}_p)$ is given as follows.*
(i) *As an algebra,*

$$H^*(\widetilde{SU}(n); \mathbf{F}_p) = \mathbf{F}_p[\tilde{y}_{2p^r}] \otimes \Lambda(\tilde{x}_k(k \in A_{p,n}), \tilde{y}_k(k \in B_{p,n}))$$

where

$$A_{p,n} = \{2j+1 \mid 1 < j < n, j \neq p, p^2, \dots, p^{r-1}\},$$

$$B_{p,n} = \{2j+1 \mid j = p, p^2, \dots, p^r\}$$

and $\deg \tilde{x}_k = \deg \tilde{y}_k = k$. *(In particular, $H^*(\widetilde{SU}(\infty); \mathbf{F}_p)$ is an exterior algebra.)*
(ii) *The coproducts are given as*
(a) *if p is an odd prime,* $\bar{\mu}^*(\tilde{y}_{2p^r}) = \sum_{\substack{k, k' \in A_{p,n} \\ k+k'=2p^r}} \tilde{x}_k \otimes \tilde{x}_{k'}$ *and other generators are primitive,*

(b) *if* $p = 2$, $\bar{\mu}^*(\tilde{y}_{2^{r+1}}) = \sum_{\substack{k,k' \in A_{2,n} \\ k+k'=2^{r+1} \\ k<k'}} \tilde{x}_k \otimes \tilde{x}_{k'}$ *and other generators are primitive.*

(In particular, $H^*(\widetilde{SU}(\infty); \mathbf{F}_p)$ *is primitively generated.)*

(iii) *The cohomology operations are given as*

(a) *for any* $k \in A_{p,n}$, $\beta\tilde{x}_k = 0$ *and*

$$\wp^j \tilde{x}_k = \begin{cases} \dbinom{\frac{k-1}{2}}{j} \tilde{x}_{k+2j(p-1)} & (k + 2j(p-1) \in A_{p,n}), \\ 0 & (\textit{otherwise}), \end{cases}$$

(b) $\beta\tilde{y}_{2p+1} = 0$,

(c) $\wp^1 \tilde{y}_{2p+1} = \begin{cases} \varepsilon_p \tilde{x}_{4p-1} & (n \geq 2p), \\ 0 & (n < 2p) \end{cases}$ *where* $0 \neq \varepsilon_p \in \mathbf{F}_p$,

(d) $\tilde{y}_{2p^k+1} = \wp^{p^{k-1}} \wp^{p^{k-2}} \cdots \wp^p \tilde{y}_{2p+1}$ $(k = 2, 3, \ldots, r)$,

(e) $\beta\tilde{y}_{2p^r} = \tilde{y}_{2p^r+1}$,

(f) $$\wp^{p^k} \tilde{y}_{2p^r} = \begin{cases} \tilde{y}^p_{2p^r} & (k = r), \\ \sum_{j=1}^{e_k} \tilde{x}_{d_{(k,j)}} \tilde{x}_{d'_{(k,j)}} & (p = 2, 2 \leq k \leq r-2), \\ 0 & (\textit{otherwise}) \end{cases}$$

where $e_k = \min\{2^{k-1} - 1, n - 2^{r-1} - 2^{k-1}\}$, $d_{(k,j)} = 2^r + 2^k + 1 - 2j$ *and* $d'_{(k,j)} = 2^r + 2^k - 1 + 2j$.

3. Proof for $n = \infty$

As stated in the introduction, $H^*(\widetilde{SU}(\infty); \mathbf{F}_p)$ is easily determined as a Hopf algebra over $\mathscr{A}_p$ except $\wp^1 \tilde{y}_{2p+1} = \varepsilon_p \tilde{x}_{4p-1}$ where $0 \neq \varepsilon_p \in \mathbf{F}_p$. In this section, we prove this cohomology operation. Let $K\langle n \rangle$ be the n-connective fibre space over K for any space K.

According to Adams [1],

$$SU(\infty)_{(p)} \simeq X_1 \times X_2 \times \cdots \times X_{p-1}$$

where for $j \geq 1$, $\pi_{2j+1}(X_k) = \mathbf{Z}_{(p)}$ if $j \equiv k \pmod{p-1}$ and $\pi_{2j+1}(X_k) = 0$ otherwise. Put $Y = X_2 \times \cdots \times X_{p-1}$. Then, we have

$$\begin{aligned} SU(\infty)\langle 3 \rangle_{(p)} &\simeq SU(\infty)_{(p)}\langle 3 \rangle \\ &\simeq X_1\langle 3 \rangle \times Y \end{aligned}$$

and $H^*(X_1\langle 3 \rangle; \mathbf{F}_p) = \Lambda(\tilde{y}'_{2p+1}, \tilde{x}'_{4p-1}, \ldots)$ where $\tilde{y}'_{2p+1}$ and $\tilde{x}'_{4p-1}$ correspond to $\tilde{y}_{2p+1}$ and $\tilde{x}_{4p-1}$ respectively. Further, we have

$$SU(\infty)_{(p)}\langle 2p-1 \rangle = X_1\langle 3 \rangle \times Y\langle 2p-1 \rangle.$$

Assume that $\wp^1\tilde{y}_{2p+1} = 0$. By inspecting the cohomology Serre spectral sequence associated with the fibering

$$K(\mathbf{Z}_{(p)}, 2p) \to X_1\langle 2p+1\rangle \to X_1\langle 3\rangle,$$

we can easily show that $H^{4p-2}(X_1\langle 2p+1\rangle; \mathbf{F}_p) \neq 0$. It contradicts that $\pi_k(X_1\langle 2p+1\rangle) = 0$ $(k \leq 4p-2)$ because of Hurewicz theorem.

4. The coproduct of $\tilde{y}_{2p^r}$

As an algebra, $H^*(\widetilde{SU}(n); \mathbf{F}_p)$ is easily determined also for finite n by the fibering

$$\mathbf{CP}^\infty \longrightarrow \widetilde{SU}(n) \xrightarrow{q_n} SU(n).$$

As stated in the introduction, $H^*(\widetilde{SU}(n); \mathbf{F}_p)$ can be almost determined as a Hopf algebra over $\mathscr{A}_p$ by the results of $H^*(SU(n); \mathbf{F}_p)$ and $H^*(\widetilde{SU}(\infty); \mathbf{F}_p)$. Then, we shall argue the only two problems stated in the introduction. In this section, we shall determine the coproduct of $\tilde{y}_{2p^r}$. In §5, the last section, we shall determine the cohomology operations to $\tilde{y}_{2p^r}$. Clearly, it suffices to consider the case $n = p^r$ for each positive integer r.

Here let $G = SU(p^r)$ and $q = q_{p^r} : \tilde{G} \to G$, the covering projection. We consider the commutator map

$$c : G \times G \to G$$

which maps (x, y) to $xyx^{-1}y^{-1}$. For the definition of c, we can define a map

$$c' : G \wedge G \to G$$

as the one which makes the following diagram commute:

$$\begin{array}{ccc} G \times G & \xrightarrow{c} & G \\ {\scriptstyle \pi}\downarrow & \nearrow{\scriptstyle c'} & \\ G \wedge G & & \end{array}$$

where π is the natural projection. On the other hand, using the inverse map of $\tilde{G}$, we define a map

$$\tilde{c} : \tilde{G} \times \tilde{G} \to \tilde{G}$$

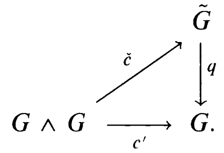
by $\tilde{c}(x, y) = xyx^{-1}y^{-1}$. The map $\tilde{c}$ satisfies the condition that it makes the following diagram commute up to homotopy:

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{c}} & \tilde{G} \\ {\scriptstyle q\times q}\downarrow & & \downarrow{\scriptstyle q} \\ G \times G & \xrightarrow[c]{} & G. \end{array}$$

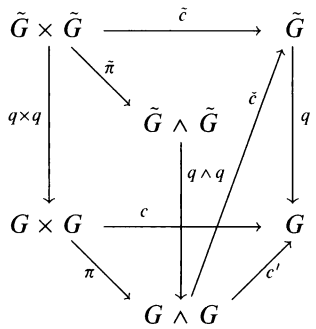
Note that any continuous map from $\tilde{G} \times \tilde{G}$ to $\tilde{G}$ which satisfies the above condition is homotopic to $\tilde{c}$ since $\tilde{G} \times \tilde{G}$ is 3-connected. Moreover, we define a map

$$\check{c} : G \wedge G \to \tilde{G}$$

as the one which makes the following diagram commute up to homotopy:



Note that $\check{c}$ certainly exists and is unique up to homotopy since $G \wedge G$ is 3-connected. Then, we have the following diagram:



where $\tilde{\pi}$ is the natural projection. We can show the following lemma which we need later.

Lemma 4.1. $\check{c} \circ (q \wedge q) \circ \tilde{\pi} \simeq \tilde{c}$.

In fact, it follows from

$$\begin{aligned} q \circ \check{c} \circ (q \wedge q) \circ \tilde{\pi} &\simeq c' \circ (q \wedge q) \circ \tilde{\pi} \\ &\simeq c' \circ \pi \circ (q \times q) \\ &\simeq c \circ (q \times q). \end{aligned}$$

Moving $\tilde{y}_{2p^r}$ modulo decomposable, we may put

$$\tilde{\mu}^*(\tilde{y}_{2p^r}) = \sum (\text{primitive}) \otimes (\text{primitive}).$$

Then, by the definition of $\tilde{c}$, we can directly compute that

$$\tilde{c}^*(\tilde{y}_{2p^r}) = \tilde{\mu}^*(\tilde{y}_{2p^r}) - \alpha^* \circ \tilde{\mu}^*(\tilde{y}_{2p^r}) \tag{4.1}$$

where $\alpha : A \times B \to B \times A$ is the switching map for any spaces A, B.

On the other hand, we can compute $\check{c}^*(\tilde{y}_{2p^r})$ as follows.

For $k = 2, 3, \dots, p^r - 1$, we define maps

$$c'_{(k)} : SU(k) \wedge SU(p^r + 1 - k) \to SU(p^r)$$

as the ones each of which is the composition of c' and the smash map of the natural inclusions. Similarly, we define maps

$$\check{c}_{(k)} : SU(k) \wedge SU(p^r + 1 - k) \to \widetilde{SU}(p^r)$$

as the ones each of which is the composition of $\check{c}$ and the smash map of the natural inclusions. For $k = 2, 3, \dots, p^r - 1$, the map $\check{c}_{(k)}$ satisfies the condition that it makes the following diagram commute up to homotopy:

$$\begin{array}{ccc} & & \widetilde{SU}(p^r) \\ & \overset{\check{c}_{(k)}}{\nearrow} & \downarrow q \\ SU(k) \wedge SU(p^r + 1 - k) & \xrightarrow[c'_{(k)}]{} & SU(p^r). \end{array}$$

Note that any continuous map from $SU(k) \wedge SU(p^r + 1 - k)$ to $\widetilde{SU}(p^r)$ which satisfies the above condition is homotopic to $\check{c}_{(k)}$ since $SU(k) \wedge SU(p^r + 1 - k)$ is 3-connected.

Recall the following homotopy fibre sequence:

$$\Omega S^{2p^r+1} \xrightarrow{\delta} SU(p^r) \to SU(p^r + 1) \to S^{2p^r+1}.$$

Since $SU(k)$ and $SU(p^r + 1 - k)$ commute in $SU(p^r + 1)$ up to homotopy, there exists a map

$$\lambda_{(k)} : SU(k) \wedge SU(p^r + 1 - k) \to \Omega S^{2p^r+1}$$

such that $\delta \circ \lambda_{(k)} \simeq c'_{(k)}$. Then, we have the following diagram:

$$\begin{array}{ccccc} \Omega S^{2p^r+1} & \xrightarrow{\tilde{\delta}} & \widetilde{SU}(p^r) & & \\ \| & & \downarrow q & \nwarrow \check{c}_{(k)} & \\ \Omega S^{2p^r+1} & \xrightarrow{\delta} & SU(p^r) & & \\ & \nwarrow \lambda_{(k)} & & \nwarrow c'_{(k)} & \\ & & & & SU(k) \wedge SU(p^r + 1 - k) \end{array}$$

where $\tilde{\delta}$ is induced from δ.

Lemma 4.2. $\check{c}_{(k)} \simeq \tilde{\delta} \circ \lambda_{(k)}$.

In fact, it follows from

$$\begin{aligned} q \circ \tilde{\delta} \circ \lambda_{(k)} &\simeq \delta \circ \lambda_{(k)} \\ &\simeq c'_{(k)}. \end{aligned}$$

Lemma 4.3. *$\tilde{\delta}^*(\tilde{y}_{2p^r}) = a\sigma(s_{2p^r+1})$ $(0 \neq a \in \mathbf{F}_p)$ where s_{2p^r+1} is the $\bmod p$ reduction of the generator of $H^*(S^{2p^r+1};\mathbf{Z})$ and σ is the cohomology suspension.*

Proof. Note that

$$\Omega S^{2p^r+1} \xrightarrow{\tilde{\delta}} \widetilde{SU}(p^r) \to \widetilde{SU}(p^r+1)$$

is a fibre space up to homotopy. By the Serre exact sequence, we have $H^{2p^r}(\widetilde{SU}(p^r+1);\mathbf{F}_p) \to H^{2p^r}(\widetilde{SU}(p^r);\mathbf{F}_p) \xrightarrow{\tilde{\delta}^*} H^{2p^r}(\Omega S^{2p^r+1};\mathbf{F}_p)$ (exact). Since no indecomposable element is in $H^{2p^r}(\widetilde{SU}(p^r+1);\mathbf{F}_p)$, $\tilde{y}_{2p^r}$ is not in $\operatorname{Ker}\tilde{\delta}^*$ and hence the lemma follows.

Moreover, we can show the following lemma by the results of Bott [2] in a similar manner to Hamanaka [4] lemma 2.4.

Lemma 4.4. $\lambda^*_{(k)}(\sigma(s_{2p^r+1})) = x_{2k-1} \otimes x_{2(p^r+1-k)-1}$.

By lemmas 4.2, 4.3 and 4.4, we have the following lemma.

Lemma 4.5. $\check{c}^*_{(k)}(\tilde{y}_{2p^r}) = a x_{2k-1} \otimes x_{2(p^r+1-k)-1}$.

Accordingly, we have

$$\check{c}^*(\tilde{y}_{2p^r}) = a \sum_{\substack{k,k' \geq 2 \\ k+k'=p^r+1}} x_{2k-1} \otimes x_{2k'-1}$$

by the definition of $\check{c}_{(k)}$. Therefore, we have by lemma 4.1 and (4.1)

$$\begin{aligned} \tilde{\pi}^* \circ (q \wedge q)^* \circ \check{c}^*(\tilde{y}_{2p^r}) &= a \sum_{\substack{k+k'=2p^r \\ k,k' \in A_{p,p^r}}} \tilde{x}_k \otimes \tilde{x}_{k'} \\ &= \bar{\mu}^*(\tilde{y}_{2p^r}) - \alpha^* \circ \bar{\mu}^*(\tilde{y}_{2p^r}). \end{aligned}$$

Consequently, moving $\tilde{y}_{2p^r}$ modulo decomposable again and multiplying $\tilde{y}_{2p^r}$ and $\tilde{y}_k (k \in B_{p,p^r})$ by non-zero scalar if we need, we can obtain the coproduct of $\tilde{y}_{2p^r}$ as stated in theorem 2.1.

5. The cohomology operations to $\tilde{y}_{2p^r}$

In this section, we shall determine the cohomology operations to $\tilde{y}_{2p^r} \in H^*(\widetilde{SU}(p^r);\mathbf{F}_p)$. We consider the non-trivial cases $\wp^{p^k}\tilde{y}_{2p^r}$ for $k \leq r-1$. Let M be a vector space over $\mathbf{F}_p$ and $a_l \in M$ for $l \in L$. Then the vector subspace generated by $\{a_l\}$ is denoted by $\langle a_l(l \in L)\rangle$.

Firstly assume that p is an odd prime. By the Cartan formula, $\bar{\mu}^*(\wp^{p^k}\tilde{y}_{2p^r})$ must be of the form

$$\bar{\mu}^*(\wp^{p^k}\tilde{y}_{2p^r}) = \sum (z \otimes z' + z' \otimes z),$$

while since

$$\wp^{p^k}\tilde{y}_{2p^r} \in \langle \tilde{x}_l\tilde{x}_{l'}(l, l' \in A_{p,p^r}, l+l' = 2p^r + 2p^k(p-1)) \rangle$$

and

$$\bar{\mu}^*(\tilde{x}_l\tilde{x}_{l'}) = \tilde{x}_l \otimes \tilde{x}_{l'} - \tilde{x}_{l'} \otimes \tilde{x}_l,$$

$\bar{\mu}^*(\wp^{p^k}\tilde{y}_{2p^r})$ also must be of the form

$$\bar{\mu}^*(\wp^{p^k}\tilde{y}_{2p^r}) = \sum (w \otimes w' - w' \otimes w).$$

Hence $\bar{\mu}^*(\wp^{p^k}\tilde{y}_{2p^r})$ is zero and so is $\wp^{p^k}\tilde{y}_{2p^r}$.

The case $p = 2$ is more complicated. By the Cartan formula,

$$\bar{\mu}^*(Sq^{2^{k+1}}\tilde{y}_{2^{r+1}}) = \sum_{\substack{l, l' \in A_{2,2^r} \\ l+l'=2^{r+1} \\ l<l'}} (Sq^{2^{k+1}}\tilde{x}_l \otimes \tilde{x}_{l'} + \tilde{x}_l \otimes Sq^{2^{k+1}}\tilde{x}_{l'})$$

since we can easily show the following lemma.

Lemma 5.1.

$$Sq^f\tilde{x}_l \otimes Sq^{2^{k+1}-f}\tilde{x}_{l'} = 0$$

where $l, l' \in A_{2,2^r}$, $l + l' = 2^{r+1}$, $l < l'$ *and* $0 < f < 2^{k+1}$.

Moreover, we can get

$$\bar{\mu}^*(Sq^{2^{k+1}}\tilde{y}_{2^{r+1}})$$
$$= \begin{cases} \sum_{j=1}^{2^{k-1}-1} \tilde{x}_{d'_{(k,j)}} \otimes \tilde{x}_{d_{(k,j)}} + \sum z \otimes z' & (2 \le k \le r-2), \\ \sum z \otimes z' & (k = 0, 1, r-1) \end{cases}$$

where $\deg z < \deg z'$. Since

$$Sq^{2^{k+1}}\tilde{y}_{2^{r+1}} \in \langle \tilde{x}_l\tilde{x}_{l'}(l, l' \in A_{2,2^r}, l + l' = 2^{r+1} + 2^{k+1}) \rangle,$$

we can obtain $Sq^{2^{k+1}}\tilde{y}_{2^{r+1}}$ as stated in theorem 2.1.

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