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Removable singularities for semilinear degenerate elliptic equations and its application

Dedicated to Professor Norio Shimakura on the occasion of his sixtieth birthday

By

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0. Introduction

Let $N \geq 1$ and $p > 1$. Let Ω be a bounded open set with smooth boundary and F be a compact set satisfying $F \subset \Omega \subset \boldsymbol{R}^N$. We also set $\Omega' = \Omega \backslash \partial F$, where $\partial F = F \backslash \operatorname{Int} F$. We assume that the measure of ∂F is zero. Define

$$P = -\operatorname{div}(A(x)\nabla \cdot), \tag{0-1}$$

where $A(x) \in C^1(\Omega')$ is positive in $\Omega \backslash F$ and vanishes in $\operatorname{Int} F$. First we shall consider removable singularities of solutions for degenerate semilinear elliptic equations. Assume that $u \in C^0(\Omega') \cap C^2(\Omega \backslash F)$ satisfies the differential inequality

$$Pu + B(x)Q(u) \leq C(x), \quad \text{in } \Omega', \tag{0-2}$$

for some nonnegative functions $B(x)$ and $C(x)$. Here $Q(t)$ is continuous and strictly monotone increasing on $\boldsymbol{R}$ satisfying the growth condition (1-6). For instance we can adopt $|t|^{p-1}t$ with $p > 1$ and $(e^{|t|} - 1)\operatorname{sgn}(t)$ for $Q(t)$. Then we shall show under some additional conditions on $A(x)$, $B(x)$, $C(x)$ and $Q(t)$ that

$$\limsup_{x \to \partial F} u(x) < +\infty. \tag{0-3}$$

From this result we can deduce that if $u \in C^0(\Omega') \cap C^2(\Omega \backslash F)$ satisfies

$$Pu + B(x)Q(u) = f(x), \quad \text{in } \Omega' \tag{0-4}$$

for $f/B \in L^\infty(\Omega)$, then there is a bounded function in Ω which coincides with u in $\Omega' = \Omega \backslash \partial F$.

This result was established by H. Brezis and L. Veron, under the assumptions that F consists of finite points, $Q(t) = |t|^{p-1}t$ and $A(x)$, $B(x)$, $C(x)$ are positive constants. More precisely they proved in [BV] that if u satisfies (0-2) with some

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additional assumptions on p, u can possess only removable singularities on F. (See also [VV1], [VV2] and [V]). In this paper we generalize their results for an arbitrary compact set F in place of finite set and for wider class of (degenerate) elliptic operators P. Roughly speaking, the operators P considered here are permitted not only to vanish infinitely on a compact set $F\subset\Omega$, but also remain unbounded on F if $|F|=0$. The main tools for this aim are similar to those in [BV], namely a comparison principle, Kato's inequality and a weak maximum principle. Since the operators P are rather general, we need to modify them suitably. As a result we are able to derive a pointwise estimate of u. We also prove the sharpness of our results for the removability of singularities in the special case that F is either a set of finite points or an m-dimensional compact Lipschitz submanifolds $(0<m\leq N-1)$ of $\boldsymbol{R}^N$, and

$$\text{(0–5)}\qquad \begin{cases} A(x)=d(x)^{2\alpha}, B(x)=d(x)^{2\beta}, C(x)=d(x)^{2\gamma}, \\ Q(t)=|t|^{p-1}t, \quad d(x)=dist(x,\partial F), \end{cases}$$

where $p>1$ and α, β and γ are real numbers.

Secondly as an application, we shall consider the Dirichlet boundary problem for genuinely degenerate semilinear elliptic operators:

$$\text{(0–6)}\qquad \begin{cases} Pu+B(x)Q(u)=f(x), \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega. \end{cases}$$

Then we shall establish the existence and uniqueness of bounded solutions u for this problem with $f/B\in L^\infty(\Omega)$. When P is uniformly elliptic on $\bar{\Omega}$, this problem has been treated by many authors. In [S], G. Stampacchia considered the linear case. In [BS] H. Brezis and W.A. Strauss proved the existence and uniqueness of solution for $f\in L^1(\Omega)$ with a monotone increasing non-linear term in u (possibly multi-valued). See also [BBC] and [BG]. The quasi-linear case has been also considered in L^1- framework by many researchers, for instance [LL], [LM], [BGDM], [R1], [R2], [R3], and so on. But the development of the theory seems to be rather limitted in the study of genuinely degenerate operators.

This paper is organized in the following way. In §1 we shall describe our precise framework and main results which consisit of the removability of singularities and the unique existence of solutions for Dirichlet boundary problem (0–6). In §1 we shall also construct examples showing that in certain respects Theorem 1 gives best possible results. §2 is devoted to prepare auxiliary lemmas. In §3 we shall prove Theorem 1 by the use of weak maximum principle in Orlicz space. Theorem 2 will be finally established in §4 as an application of Theorem 1.

1. Main results and Applications

In this section we describe our precise framework and main results. Let $N\geq 1$. Let F and Ω be a compact set and a bounded open set with smooth boundary respectively, satisfying $F\subset\Omega\subset\boldsymbol{R}^N$, and set

$$\Omega' = \Omega \backslash \partial F, \tag{1-1}$$

where ∂F is defined as $\partial F = F \backslash$Interior of F. In this paper we assume that the measure of ∂F is zero. For example if F is a smooth compact subset of $\boldsymbol{R}^N$, then the measure of ∂F is zero.

In the next we define a modified distance to ∂F.

Definition 1. Let $d(x) \in C^\infty(\Omega')$ be a nonnegative function satisfying

$$C(0) \leq \frac{d(x)}{dist(x, \partial F)} \leq 1, \ x \in \Omega', \tag{1-2}$$

$$|\partial^\gamma d(x)| \leq C(|\gamma|) dist(x, \partial F)^{1-|\gamma|}, \ x \in \Omega', \gamma \neq 0,$$

where γ is an arbitrary multi-index and $C(|\gamma|)$ is positive number depending on each $|\gamma|$. For the construction of a modified distance $d(x)$, see [T] for example.

First we assume the following [H-1] on the nonnegative functions $A(x)$, $B(x)$ and $C(x)$.

[H-1].

$$\begin{cases} A(x) \in C^1(\Omega') \cap L^1_{loc}(\Omega), \\ A(x) = 0 \text{ in Int } F = F \backslash \partial F, \\ A(x) > 0 \text{ in } \Omega \backslash F, \end{cases} \tag{1-3}$$

$$\begin{cases} B(x) \in L^\infty_{loc}(\Omega') \cap L^1_{loc}(\Omega), \\ B(x) > 0 \quad \text{in } \Omega' = \Omega \backslash \partial F, \end{cases} \tag{1-4}$$

and

$$\begin{cases} C(x) \in L^\infty_{loc}(\Omega') \cap L^1_{loc}(\Omega), \\ C(x) \geq 0 \quad \text{in } \Omega. \end{cases} \tag{1-5}$$

Secondly we assume the following [H-2] on the nonlinear term $Q(t)$.

[H-2]

$Q(t)$ is strictly monotone increasing and continuous on $\boldsymbol{R}$ such that $Q(0) = 0$ and $Q(t)t > 0$ for any $t \in \boldsymbol{R} \backslash \{0\}$. Moreover we assume that there is a positive number δ_0 such that

$$\limsup_{|t| \to +\infty} \frac{|t|^{1+\delta_0}}{|Q(t)|} < +\infty. \quad \text{(Super-linearity)} \tag{1-6}$$

We need more notations.

Definition 2. Let us set for any $x \in \Omega' = \Omega \backslash \partial F$

$$\text{(1–7)} \qquad \begin{cases} \tilde{A}(x)=A(x)+d(x)|\nabla A(x)|, \\ \Phi(x)=\text{ess-sup}_{|y-x|<\frac{d(x)}{2}} \frac{\tilde{A}(y)}{B(y)}, \\ \Psi(x)=\text{ess-sup}_{|y-x|<\frac{d(x)}{2}} \frac{A(y)}{B(y)}. \end{cases}$$

Then we assume that:

[H-3]. For the same positive number $\delta_0>0$ as in [H-2], it holds that

$$\text{(1–8)} \qquad \tilde{A}(x)\left[\left(\frac{\Phi(x)}{d(x)^2}\right)^{\frac{1}{\delta_0}}+1\right]\frac{1}{d(x)}\in L^1_{loc}(\Omega)$$

and

$$\text{(1–9)} \qquad \liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\tilde{A}(x)\left[\left(\frac{\Phi(x)}{d(x)^2}\right)^{\frac{1}{\delta_0}}+1\right]\frac{dx}{d(x)}<+\infty.$$

We also assume that:

[H-4].

$$\text{(1–10)} \qquad \sup_{x\in\Omega}\frac{C(x)}{B(x)}<+\infty.$$

In the application, it will be useful to introduce the following subclass consisting of admissible weight functions $A(x)$. Namely:

Definition 3. Let $L\geq 1$. $A(x)\in C^1(\Omega')$ is said to belong to the class S_L if there is a positive number C such that $d(x)^L|\nabla A(x)|\leq C\cdot A(x)$, in Ω' where C is independent of each x.

It is easy to see that if $A\in S_L$ for $L\geq 1$, then the following [H-5] implies (1-9) in [H-3] with the same δ_0.

[H-5]. There is a positive number δ_0 such that

$$\text{(1–11)} \qquad \liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}A(x)\left[\left(\frac{\Psi(x)}{d(x)^{L+1}}\right)^{\frac{1}{\delta_0}}+1\right]\frac{dx}{d(x)^L}<+\infty.$$

Remark 1. In Definition 3, if F is assumed to be smooth and A vanishes on ∂F uniformly in finite order, then by the mean-value theorem we may take $L=1$. On the other hand if A vanishes infinitely on ∂F, then we see $L\geq 1$. For instance, $F=\{0\}$, $A(x)=\exp(-1/|x|^\alpha)$, $\alpha\in \boldsymbol{R}^+$, then $L=\alpha+1$. If $A\in S_L$ with $L=1$, then Φ is equivalent to Ψ.

Remark 2. The conditions (1–9) in [H-3] and [H-5] mean that $B(x)$ does not vanish much faster than $A(x)$. If $1\leq N\leq 2$, then either $\tilde{A}$ or Φ must vanish on ∂F in order to satisfy (1–9) in [H-3]. See also [H-5].

Remark 3. If we assume the following condition [H-6], then [H-3] is satisfied. In fact, the both conditions (1–8) and (1–9) in [H-3] are weaker than (1–12) in [H-6]. We shall give an example in Theorem 3 which does not satisfy [H-6] but satisfies [H-3]. (See Remark 6 in §2.)

[H-6]. For the same positive number $\delta_0>0$ as in [H-2], it holds that

$$\tilde{A}(x)\left[\left(\frac{\Phi(x)}{d(x)^2}\right)^{\frac{1}{\delta_0}}+1\right]\frac{1}{d(x)^2}\in L^1_{loc}(\Omega). \tag{1–12}$$

Let D be an open subset of Ω. In order to state our main results, we prepare more definitions.

Definition 4. For u, uA and $u|\nabla A|\in L^1_{loc}(D)$, we set

$$\langle \hat{P}u,\varphi\rangle \equiv \langle -\Delta(uA)+\nabla(u\nabla A),\varphi\rangle \tag{1–13}$$

for all $\varphi\in C_0^\infty(D)$.

Definition 5. For u, $|\nabla u|$ and $A|\nabla u|\in L^1_{loc}(D)$, we set

$$\langle Pu,\varphi\rangle \equiv \langle -\nabla(A\nabla u),\varphi\rangle \tag{1–14}$$

for all $\varphi\in C_0^\infty(D)$.

Here we note that from [H-1] and [H-3] we have A and $|\nabla A|\in L^1_{loc}(\Omega)\cap L^\infty_{loc}(\Omega')$. Therefore if u is bounded on Ω, then $\hat{P}u$ is well defined by (1–13) as a distribution on D. It is also obvious to see that if A is a smooth function, then (1–13) coincides with the usual definition of the distribution. Moreover we show the following.

Lemma 1-1. *Assume* [H-1] *and* [H-3]. *Let* $u\in L^1_{loc}(\Omega')$ *and let* $\hat{P}u$ *be a distribution defined by Definition* 4 *with* $D=\Omega'$. *Suppose that* $\hat{P}u\in L^1_{loc}(\Omega')$, *then* $A|\nabla u|\in L^1_{loc}(\Omega')$ *and the distribution* Pu *defined by Definition* 5 *with* $D=\Omega'$ *coincides with* $\hat{P}u$. *In particular* $Pu\in L^1_{loc}(\Omega')$ *and we have*

$$\langle Pu,\varphi\rangle=\int_{\Omega'}Pu\cdot\varphi dx,\quad \textit{for all } \varphi\in C_0^\infty(\Omega'). \tag{1–15}$$

Proof. From [H-1] we see that the operator P is elliptic in Ω'. Hence we see that if $u\in L^1_{loc}(\Omega')$ and $Pu\in L^1_{loc}(\Omega')$, then $|\nabla u|\in L^1_{loc}(\Omega')$. (For the detailed proof, see [K; Lemma 1] for example.) Since A and $|\nabla A|\in L^1_{loc}(\Omega)\cap L^\infty_{loc}(\Omega')$, we see that $A|\nabla u|\in L^1_{loc}(\Omega')$. Hence we see for $\varphi\in C_0^\infty(\Omega')$

$$<\hat{P}u,\varphi>=-\int_{\Omega'}u(A\Delta\varphi+\nabla A\cdot\nabla\varphi)dx=\int_{\Omega'}uP\varphi dx=\langle Pu,\varphi\rangle.$$

This proves the statement.

Now we are able to state our main results.

Theorem 1. *Assume* [H-1], [H-2], [H-3] *and* [H-4]. *Assume that* $u \in L^\infty_{loc}(\Omega')$ *satisfies* $\hat{P}u \in L^1_{loc}(\Omega')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in \Omega'; u(x) \geq 0\}$,

$$\hat{P}u + B(x)Q(u) \leq C(x). \tag{1–16}$$

Then we have $u_+ \in L^\infty_{loc}(\Omega)$, *that is to say,* $\text{ess-sup}_{x \to \partial F} u_+(x) < +\infty$. *Here* $u_+(x) = \max(u(x), 0)$.

Remark 4. It follows from Lemma 1–1 that under the assumptions in Theorem 1, the distribution $\hat{P}u$ on Ω' definesd by (1–13) is the function $Pu \in L^1_{loc}(\Omega')$. Then, Theorem 1 says that $u_+ \in L^\infty_{loc}(\Omega')$ can be extended as a locally bounded function on a whole Ω. Since the measure of ∂F is zero, this extension coincides with u_+ except on a set of measure zero.

The following is a direct consequence of this Theorem.

Corollary 1. *Assume* [H-1], [H-2] *and* [H-3]. *Instead of* [H-4], *assume that* $f(x) \in L^\infty_{loc}(\Omega') \cap L^1_{loc}(\Omega)$ *satisfies for some positive number* C

$$|f(x)| \leq C \cdot B(x), \quad \textit{for almost all } x \in \Omega. \tag{1–17}$$

Assume that $u \in L^\infty_{loc}(\Omega')$ *satisfies*

$$\hat{P}u + B(x)Q(u) = f, \quad \textit{in } \mathscr{D}'(\Omega'). \tag{1–18}$$

Then there exists a function $v \in L^\infty_{loc}(\Omega)$ *such that*

$$\begin{cases} \hat{P}v + B(x)Q(v) = f, & \textit{in } \mathscr{D}'(\Omega) \\ v|_{\Omega'} = u. \end{cases} \tag{1–19}$$

Here by $\mathscr{D}'(\Omega')$ *and* $\mathscr{D}'(\Omega)$ *we denote the set of distributions on* Ω' *and* Ω *respectively.*

Proof. From Theorem 1 we have $u_+ \in L^\infty_{loc}(\Omega)$. The function $-u$ satisfies (1–18) with replacing f and $Q(t)$ by $-f$ and $-Q(-t)$ respectively. Since $-Q(-t)$ satisfies the same assumption as the one for $Q(t)$, we see in a similar way $u_- \in L^\infty_{loc}(\Omega)$. According to Remark 4, u is extended as a locally bounded function on Ω. By v we denote this extension of u to a whole Ω. Thus $v \in L^\infty_{loc}(\Omega)$ and $v|_{\Omega'} = u$. From [H-1] and [H-2] we also see that $B \cdot Q(v) \in L^\infty_{loc}(\Omega)$. Here we note that since $A(x) = 0$ on $F \backslash \partial F$, $u(x) = v(x) = Q^{-1}(f(x)/B(x))$ on $F \backslash \partial F$. Then it follows form Lemma 2–3 in §2 that v is extended as a solution of the same equation on a whole Ω. Here we remark that the uniqueness of solutions of (1–19) in $L^\infty_{loc}(\Omega)$ follows from the same argument in the proof of Theorem 2 in §4.

As an application we consider the Dirichlet boundary value problem for degenerate semi-liniear elliptic equation:

$$\begin{cases} \hat{P}u + B(x)Q(u) = f & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1–20}$$

We prepare more notations. Let D be an open subset of $\boldsymbol{R}^N$. Let $q \geq 1$ and let j be a positive integer. By $H^{j,q}(D)$ we denote the spaces of all functions on D, whose generalized derivatives $\partial^\gamma u$ of order $\leq j$ satisfy

$$\|u\|_{j,q} = \sum_{|\gamma| \leq j} \left(\int_D |\partial^\gamma u(x)|^q dx \right)^{1/q} < +\infty. \tag{1–21}$$

Also, $H^{j,q}_{loc}(D)$ is a local version of $H^{j,q}(D)$, and by $\|u\|_\infty$ we denote the essential supremum of u. By $H^{1,q}_0(D)$ we denote the completion of $C^\infty_0(D)$ with respect to the norm defined by (1–21). Conventionally we set $H^1(D) = H^{1,2}(D)$, $H^1_{loc}(D) = H^{1,2}_{loc}(D)$, and $H^1_0(D) = H^{1,2}_0(D)$. Then we have

Theorem 2. *Assume* [H-1], [H-2] *and* [H-3]. *Instead of* [H-4] *assume that* $f(x) \in L^\infty(\Omega)$ *satisfies for some positive number* C

$$|f(x)| \leq C \cdot B(x), \text{ for almost all } x \in \Omega. \tag{1–22}$$

Moreover we assume that $A(x)$, $B(x) \in C^0(\bar{\Omega})$. *Then there exists a unique function*

$$u \in L^\infty(\Omega) \cap H^1_{loc}(\bar{\Omega} \backslash F) \tag{1–23}$$

which satisfies the homogeneous Dirichlet boundary value problem (1–20) *in the distribution sense and satisfies*

$$\int_\Omega [A(x)|\nabla u|^2 + B(x)Q(u)u] dx \leq C'(\|f/B\|^\lambda_\infty + \|f\|_\infty). \tag{1–24}$$

Here $\lambda = \frac{2+\delta_0}{1+\delta_0}$ *and* C' *is a positive number independent of each function* f.

Remark 5. If Q is Lipschitz continuous, then $u \in H^2_{loc}(\Omega \backslash F)$ as well. For the proof of Theorem 2 we shall regularize the problem. By virtue of Theorem 1, we shall prove that the solutions of this approximating nonlinear elliptic equations converge to the unique bounded solution of the original equation. The monotonicity of the nonlinear term Q on $\boldsymbol{R}$ will be needed to establish the uniqueness of solutions. Therefore it suffices to assume in Theorem 1 that there is a positive number C such that $Q(t)$ is monotone increasing for $t \in \boldsymbol{R} \backslash [-C, C]$.

Counter examples to Theorem 1. In the rest of this subsection we shall construct examples showing that in certain respects Theorem 1 gives best possible results. Let F be either the origin 0 or an m-dimensional C^∞ compact submanifolds in $\boldsymbol{R}^N$ without boundary for $0 < m \leq N-1$, and let $d(x)$ be a modified distance function defined by Definition 1. If F consists of the origin 0, then we put $d(x) = |x|$. We set for some positive smooth functions $b(x)$ and $c(x)$,

$$\begin{cases} L_\alpha u = -\mathrm{div}(d(x)^{2\alpha} \nabla u), \\ Q(u) = |u|^{p-1}u, \ B(x) = b(x) \cdot d(x)^{2\beta} \text{ and } C(x) = c(x) \cdot d(x)^{2\gamma}. \end{cases} \tag{1–25}$$

Assume that real numbers α, β and γ satisfy the following conditions. First we assume (h-1) which is equivalent to [H-1].

$$\text{(h-1)} \qquad \beta > -\frac{N-m}{2} \text{ and } \gamma > -\frac{N-m}{2}.$$

Here we note that the condition on α is included in (h-2) below.

Let us set for $0 \le m \le N-1$

$$\text{(1–26)} \qquad p_m^* = \begin{cases} 1+2\dfrac{1-\alpha+\beta}{N+2\alpha-2-m}, & \text{if } \alpha < \beta+1, \\ 1, & \text{if } \alpha \ge \beta+1. \end{cases}$$

Then we assume (h-2) which is equivalent to [H-3].

$$\text{(h-2)} \qquad \begin{cases} p \ge p_m^*, & \text{if } \alpha < \beta+1, \\ p > p_m^* = 1, & \text{if } \alpha \ge \beta+1, \\ \alpha > -\dfrac{N-m-2}{2}. \end{cases}$$

Lastly we assume (h-3) which is equivalent to [H-4].

$$\text{(h-3)} \qquad \beta \le \gamma.$$

Let us set $u_+ = \max[0, u]$ and $u_- = \max[0, -u]$.

By $\hat{L}_\alpha u$ we denote the distribution defined by Definition 4 with $P = L_\alpha$ and $A = d(x)^{2\alpha}$.

Theorem 3. *Let F be either the origin or an m-dimensional C^∞ compact submanifolds in $\boldsymbol{R}^N$ without boundary for $0 < m \le N-1$. Assume* [h-1], [h-2] *and* [h-3]. *Assume that $u \in L^\infty_{loc}(\Omega')$ satisfies $\hat{L}_\alpha u \in L^1_{loc}(\Omega')$ in the distribution sense. Moreover we assume that for almost all $x \in \{x \in \Omega'; u(x) \ge 0\}$*

$$\text{(1–27)} \qquad \hat{L}_\alpha u + b(x)d(x)^{2\beta}u^p \le c(x)d(x)^{2\gamma},$$

for some positive smooth functions $b(x)$ and $c(x)$. Then we have $u_+ \in L^\infty_{loc}(\Omega)$.

Proof. Since $Q(u) = |u|^{p-1}u$, we can put $\delta_0 = p-1$ to obtain (1–6). Putting $A(x) = d(x)^{2\alpha}$, $B(x) = b(x)d(x)^{2\beta}$ and $C(x) = c(x)d(x)^{2\gamma}$, we shall apply Theorem 1 and the remark just after it. Then these obviously satisfy [H-1] and [H-2]. Hence, it suffices to show that the condition [H-3] is satisfied. A direct caclulation leads us to

(1–28)

$$\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\left(d(x)^{\frac{-2(1-\alpha+\beta)}{\delta_0}}+1\right)d(x)^{2\alpha-1}dx$$

$$=\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\left(d(x)^{\frac{2\alpha-1}{p-1}(p-\frac{2\beta+1}{2\alpha-1})}+d(x)^{2\alpha-1}\right)dx$$

$$=\frac{1}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon}d\rho\int_{d(x)=\rho}\left(d(x)^{\frac{2\alpha-1}{p-1}(p-\frac{2\beta+1}{2\alpha-1})}+d(x)^{2\alpha-1}\right)dH^{N-1}(x)$$

$$\leq C diam(F)^m\frac{1}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon}\left(\rho^{\frac{2\alpha-1}{p-1}(p-\frac{2\beta+1}{2\alpha-1})}+\rho^{2\alpha-1}\right)\rho^{N-m-1}d\rho$$

$$=C' diam(F)^m\left(\varepsilon^{\frac{N+2\alpha-m-2}{p-1}(p-\frac{N+2\beta-m}{N+2\alpha-m-2})}+\varepsilon^{2\alpha+N-m-2}\right)$$

$$=O(1).\quad \text{(h-1) and (h-2)}$$

This proves the assertion. Here $H^{N-1}(x)$ is the $(N-1)$-dimensional Hausdorff measure, and we used the fact: There is a positive number C such that we have

(1–29) $$|\{0<d(x)<\varepsilon\}|\leq C\varepsilon^{N-m}diam(F)^m,\quad 0<\varepsilon<1.$$

Remark 6. Here we note that if $p=p_m^*$, then [H-6] is not satisfied in this example.

The following is also a direct consequence of this whose proof is omitted.

Corollary 2. *Assume* [h-1] *and* [h-2]. *Instead of* [h-3], *assume that $b(x)$ is a positive smooth function on Ω and $f(x)\in L^\infty_{loc}(\Omega')\cap L^1_{loc}(\Omega)$ satisfies for some positive number C*

(1–30) $$|f(x)|\leq C\cdot d(x)^{2\beta},\quad \textit{for almost all } x\in\Omega.$$

Assume that $u\in L^\infty_{loc}(\Omega')$ satisfies

(1–31) $$\hat{L}_\alpha u+b(x)d(x)^{2\beta}|u|^{p-1}u=f,\quad \textit{in } \mathscr{D}'(\Omega').$$

Then there exists a unique funciton $v\in L^\infty_{loc}(\Omega)$ such that

(1–32) $$\begin{cases}\hat{L}_\alpha v+b(x)d(x)^{2\beta}|v|^{p-1}v=f, & \textit{in } \mathscr{D}'(\Omega)\\ v|_{\Omega'}=u.\end{cases}$$

Here by $\mathscr{D}'(\Omega')$ and $\mathscr{D}'(\Omega)$ we denote the set of distributions on Ω' and Ω respectively.

Counter-examples to Theorem 3. We shall see that Theorem 3 is best possible in certain respects, provided $1\leq p$. Let F be either the origin or an m-dimensional C^∞ compact submanifolds without boundary in $\boldsymbol{R}^N$ for $0<m\leq N-1$. We note that $F=\partial F$ holds. Since it suffices to construct the counter-examples in a small ball

contained in some neighborhood of F, we may assume $d(x)=dist(x, F)$ and $d(x)$ is smooth on F^c so that we have $|\nabla d(x)|=1$ near F. Let W be a small neighborhood of F. Now we consider a funciton U in $W\backslash F$ of the form

(1–33) $$U(x)=d(x)^{-M}, \text{ for } M>0.$$

Let U be a solution of the following equation for some $M>0$.

(1–34) $$\hat{L}_\alpha U(x)+b(x)d(x)^{2\beta}U(x)^p=0, \quad \text{in } W\backslash F.$$

Equivalently we have

(1–35) $$M(d(x)\Delta d(x)+2\alpha-1-M)+b(x)d(x)^{2(1-\alpha+\beta)-M(p-1)}=0.$$

Since $d(x)$ is smooth and $d(x)\Delta d(x)$ is bounded, in order to get an unbounded solution it suffices to make M, p, α, β and $b(x)$ satisfy

(1–36) $$p=1+\frac{2(1-\alpha-\beta)}{M}, \; M(d(x)\Delta d(x)+2\alpha-1-M)+b(x)=0.$$

Here we note that $b(x)$ becomes a smooth positive function in $W\backslash F$ as desired, if it holds that

(1–37) $$M>d(x)\Delta d(x)+2\alpha-1.$$

For sake of simplicity, we examine (1–37) when F is a plane. Let us set $F=F_m$, where

(1–38) $$\begin{cases} F_m=\{x=(x_1, x_2, \cdots, x_N): x_{m+1}=\cdots=x_N=0\}, \\ \text{for } 1\le m\le N-1, \\ F_0=\{0\} \end{cases}$$

Then we immediately see that

(1–39) $$dist(x, F_m)=\sqrt{\sum_{l=m+1}^{N} x_l^2} \text{ and } dist(x, F_m)\Delta dist(x, F_m)=N-m-1.$$

Remark 7. When F is an m-dismensional compact smooth manifolds without boundary, we can also show that

(1–40) $$\lim_{x\to F} dist(x, F)\Delta dist(x, F)=N-m-1.$$

For the proof of this formula, see Lemma 2–2 in [V] for example.

Then the condition (1–37) becomes in a sufficiently small neighborhoods of F,

(1–41) $$M>N-m+2\alpha-2.$$

Therefore we see that $U(x)=dist(x, F_m)^{-M}$ becomes an unbounded solution to (1–34) if (1–36) and (1–41) hold. After all we get

Proposition 1-1. *Assume that* $F=F_m$ *for* $0\leq m\leq N-1$. *Moreover we assume that* [h-1]. *Then for the validity of Theorem* 3, *the assumptions* [h-2] *is necessary.*

Proof. Assume that [h-1]. If $\alpha\neq\beta+1$, then from the previous consideration $U(x)=dist(x,F_m)^{-M}$ becomes a counter-example for a suitable $b(x)$ provided $1<p<p_m^*$. If $\alpha=\beta+1$, then $p_m^*=1$. Therefore we put $p=1$. Since the equation is linear, a fundamental solution or a good parametrix exists, and it becomes a counter-example. Here we note that if $\alpha\leq-\frac{N-m-2}{2}$, then $U(x)$ becomes an counter-example for any p, M, $b(x)$ satisfying (1–36) and (1–41).

Lastly we consider [h-3]. We can show the following:

Proposition 1-2. *Let us set* $F=F_m$ *for* $0\leq m\leq N-1$. *Assume* [h-1]. *Then for the validity of Theorem* 3, *the assumption* $\beta\leq\gamma$ ([h-3]) *is necessary if* $\alpha\geq\gamma+1$. *If* $\alpha<1+m/2$, *then* $\beta<\gamma+p(\frac{N-2-m}{2}+\alpha)$ *is necessary as well.*

Proof. Assume that $\beta>\gamma$. Let us set $U(x)=-\log d(x)$. Then it is easy to see that $U(x)$ becomes a counter-example, provided $\alpha\geq\gamma+1$. Secondly we assume $\alpha<1+m/2$. Since $d(x)^{-(N-m-2+2\alpha)}\in L^1_{loc}$ and $L_\alpha d(x)^{-(N-m-2+2\alpha)}=0$ in $\Omega\backslash F$, we see that $\beta<\gamma+p(\frac{N-m-2}{2}+\alpha)$ is needed to avoid this null solution.

2. Lemmas

We shall prepare auxiliary lemmas which will be needed to establish Theorem 1 in §1.

Lemma 2-1 (Kato's inequality). *Assume that* $u\in L^1_{loc}(\Omega')$ *and* $\hat{P}u\in L^1_{loc}(\Omega')$. *Then we have*

$$\hat{P}u_+\leq(\hat{P}u)\mathrm{sgn}^+u, \text{ in } \mathscr{D}'(\Omega'), \tag{2–1}$$

$$\text{where} \quad \mathrm{sgn}^+u=\begin{cases}1, \text{ for } u>0,\\ 1/2, \text{ for } u=0,\\ 0, \text{ for } u<0.\end{cases}$$

Proof. This follows from Kato's inequality. Let us set

$$M(x,\partial_x)=\sum_{j,k=1}^{N}\partial_{x_j}(a_{jk}(x)\partial_{x_k}\cdot), \tag{2–2}$$

where $a_{jk}(x)\in C^1(\Omega\backslash F)$ is positive definite.

Then we have, for u and $M(x,\partial_x)u\in L^1_{loc}(\Omega\backslash F)$,

$$M(x,\partial_x)|u|\geq(M(x,\partial_x)u)\mathrm{sgn}\,u \quad \text{in } \mathscr{D}'(\Omega\backslash F). \tag{2–3}$$

Since P is elliptic in $\Omega\backslash F$ and identically zero in $\mathrm{Int}F$ and $2u_+=|u|+u$, we get the desired inequality. For the detailed proof of Kato's inequality see [K].

Next we make sure that $P\varphi$ is a distribution on Ω for any $\varphi\in C_0^\infty(\Omega)$.

Lemma 2-2 (**$P\varphi$ with $\varphi\in C_0^\infty(\Omega)$**). *Assume* [H-1] *and* [H-3]. *Then it holds that $P\varphi\in L^1(\Omega)$ for any $\varphi\in C_0^\infty(\Omega)$. In particular, $P\varphi$ satisfies*

$$(2\text{–}4) \qquad \langle\hat{P}\varphi,\psi\rangle=\int_\Omega \varphi P\psi dx=\int_\Omega A\nabla\varphi\cdot\nabla\psi dx=\langle P\varphi,\ \psi\rangle,$$

for any $\psi\in C_0^\infty(\Omega)$.

Remark 8. If $\psi\in C_0^\infty(\Omega')$, then this is already proved in Lemma 1–1.

Proof. First we show that $P\varphi\in L^1(\Omega)$ for any $\varphi\in C_0^\infty(\Omega)$. By the definition of the operator P and Definition 2, we see that for some positive number C

$$(2\text{–}5) \qquad |P\varphi|=|A\Delta\varphi+\nabla A\cdot\nabla\varphi|\leq C\frac{\tilde{A}}{d}\ (|\Delta\varphi|+|\nabla\varphi|).$$

Then it follows form [H-3] that $P\varphi\in L^1(\Omega)$. Since φ, $|\nabla\varphi|$, $A|\nabla\varphi|$ and $|\nabla A|\varphi\in L^1_{loc}(\Omega)$ we can define both $\hat{P}\varphi$ and $P\varphi$, and have

$$(2\text{–}6) \qquad \begin{aligned}\langle\hat{P}\varphi,\psi\rangle&=-\int_\Omega \varphi(A\Delta\psi+\nabla A\nabla\psi)dx\\ &=-\int_\Omega \varphi\cdot\mathrm{div}(A\nabla\psi)dx=\int_\Omega A\nabla\varphi\cdot\nabla\psi dx=\langle P\varphi,\psi\rangle.\end{aligned}$$

Therefore we see $\hat{P}\varphi=P\varphi$.

In the next we shall show that $\hat{P}u$ with u being a bounded function on Ω becomes a distribution on Ω in a canonical way.

Lemma 2-3 (Extension). *Assume* [H-1] *and* [H-3]. *Let $f\in L^1_{loc}(\Omega)$ and $u\in L^\infty_{loc}(\Omega)$. Let $\hat{P}u$ be a distribution defined by* (1–13). *Assume that u satisfies*

$$(2\text{–}7) \qquad \hat{P}u=f, \qquad in\ \mathscr{D}'(\Omega').$$

Then we have

$$(2\text{–}8) \qquad \hat{P}u=f, \qquad in\ \mathscr{D}'(\Omega).$$

Here by $\mathscr{D}'(\Omega')$ and $\mathscr{D}'(\Omega)$ we denote the set of distributions on Ω' and Ω respectively.

Remark 9. We assume that [H-1], [H-3] and $f\in L^1_{loc}(\Omega)$. In particular, it follows from Lemma 2–3 that if $u\in L^\infty(\Omega')$ for $\Omega'=\Omega\backslash\partial F$ satisfies $\hat{P}u=f$ in $\mathscr{D}'(\Omega')$, then u can be extended as a solution of the same equation on a whole Ω. If the operator P is not singular but uniformly elliptic on Ω having smooth coefficients, then this result is already known provided that the set ∂F is sufficiently small. By u^* we denote the zero-extension of u to Ω, that is, $u^*=u$ in Ω' and $u^*=0$ otherwise. From

Definition 4 we see that $\hat{P}(u^*)$ is also a distribution on Ω in a canonical way. Then u^* equals u a.e. and satisfies the assumptions of Lemma 2–3. Therefore Lemma 2–3 imlpies, roughly speaking, the set ∂F is so small in some sense that the support of $\hat{P}(u^*)$ as a distribution and the set ∂F have no point in common. As a result we also have

Lemma 2-3' (Extension). *Assume* [H-1] *and* [H-3]. *Let* $f\in L^1_{loc}(\Omega)$ *and* $u\in L^\infty(\Omega')$. *Let* $\hat{P}u$ *be a distribution defined by* (1–13). *Assume that* u *satisfies* (2–7). *Then there is a function* $v\in L^\infty(\Omega)$ *such that* $v=u$ *in* Ω' *and* v *satisfies the same equation on a whole* Ω *in the distribution sense.*

Proof of Lemma 2–3. Now we take a smooth function $\eta(t)\in C^\infty(\boldsymbol{R})$ such that for some positive number C_0

(2–9) $$0\leq\eta(t)\leq 1,\ \max[\eta'(t)|,|\eta''(t)|]\leq C_0 \text{ and } \eta(t)=\begin{cases} 1, & \text{for } t\geq 1, \\ 0, & \text{for } t\leq\frac{1}{2}. \end{cases}$$

Then we put for any $\varepsilon>0$

(2–10) $$\eta_\varepsilon(x)=\eta(d(x)/\varepsilon).$$

Then by the definition of $\hat{P}u$, Lemma 1–1 and (2–7) we have for any test function $\varphi\in C_0^\infty(\Omega)$ and $\varepsilon>0$,

(2–11) $$\langle\hat{P}u,\eta_\varepsilon\varphi\rangle=\int_\Omega uP(\eta_\varepsilon\varphi)dx=\int_\Omega f\eta_\varepsilon\varphi dx.$$

We also have

(2–12) $$P(\eta_\varepsilon\varphi)=[P,\eta_\varepsilon]\varphi+\eta_\varepsilon P\varphi,\quad \text{for } [P,\eta_\varepsilon]\varphi=P(\eta_\varepsilon\varphi)-\eta_\varepsilon P\varphi.$$

and

(2–13) $$|u[P,\eta_\varepsilon]\varphi|\leq C(\varepsilon d(x))^{-1}|u|\tilde{A}(x),\quad \text{for } \varepsilon/2<d(x)<\varepsilon,$$

where C is a positive number independent of each x and ε. Then we have

(2–14) $$\left|\int_\Omega u[P,\eta_\varepsilon]\varphi dx\right|\leq\frac{C}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\tilde{A}(x)|u|\frac{dx}{d(x)},$$

and

(2–15)
$$\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\tilde{A}(x)|u|\frac{dx}{d(x)}\leq\left[\int_{\varepsilon/2<d(x)<\varepsilon}|u|^{\delta_0+1}B(x)dx\right]^{\frac{1}{1+\delta_0}}$$

$$\times\left[\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\tilde{A}(x)\left(\frac{\Phi(x)}{d(x)^2}\right)^{\frac{1}{\delta_0}}\frac{dx}{d(x)}\right]^{\frac{\delta_0}{1+\delta_0}}$$

$$\leq C\left[\int_{\varepsilon/2<d(x)<\varepsilon}|u|^{\delta_0+1}B(x)dx\right]^{\frac{1}{1+\delta_0}}\to 0, \quad ([\text{H-1}] \text{ and } [\text{H-3}]),$$

as $\varepsilon\to 0$.

Since $P\varphi\in L^1(\Omega)$ and $u\in L^\infty_{loc}(\Omega)$, we have by letting $\varepsilon\to 0$ in (2–11) that

(2–16) $$\int_\Omega u\cdot P\varphi dx=\int_\Omega f\varphi dx \qquad \text{for all } \varphi\in C_0^\infty(\Omega).$$

This proves the assertion.

Lemma 2-4. *Assume that $u\in L^\infty_{loc}(\Omega')$ satisfies $\hat{P}u\in L^1_{loc}(\Omega')$ in the distribution sense. Assume [H-1]-[H-4]. Moreover we assume that for almost all $x\in\{x\in\Omega'; u(x)\geq 0\}$*

(2–17) $$\hat{P}u+B(x)Q(u)\leq C(x).$$

Then we have, for some positive numbers C and ε_0,

(2–18) $$u(x)\leq C[\Phi(x)^{\frac{1}{\delta_0}}d(x)^{\frac{-2}{\delta_0}}+1], \qquad \text{for } x \text{ with } 0<d(x)\leq\varepsilon_0.$$

Remark 10. In particular if F is an m-dimensional compact smooth submanifolds of $\boldsymbol{R}^N$ and if $A(x)=d(x)^{2\alpha}$, $B(x)=d(x)^{2\beta}$, $C(x)=d(x)^{2\gamma}$ and $\delta_0=p-1$ in this lemma, then we get under the assumptions [h-1]-[h-3],

(2–19) $$\begin{cases} u(x)\leq Cd(x)^{-s}, \text{ for } d(x)<\varepsilon_0, \\ s=\max[s_1,s_2], \\ s_1=2\frac{1-\alpha+\beta}{p-1} \quad s_2=2(\beta-\gamma)\frac{1}{p}. \end{cases}$$

In particular if $\alpha<\beta+1$ and $p=p_m^*$, then we have

(2–20) $$s_1=N-2+2\alpha-m \quad \text{and} \quad s_2=2(\beta-\gamma)\,\frac{N-2+2\alpha-m}{N+2\beta-m}\,.$$

Proof. Let δ satisfy

(2–21) $$\delta\cdot\delta_0=2.$$

Let $x_0\in\Omega\backslash F$, with $0<d(x_0)<1/2$. For $R=d(x_0)/2$ and $r=|x-x_0|$, we set

(2–22) $$G=\{x\in\boldsymbol{R}^N; |x-x_0|<R\}.$$

We also put

$$(2\text{-}23)\qquad \begin{cases} \dfrac{\mu}{2}=Q^{-1}\left(3\sup_{x\in\Omega}\frac{C(x)}{B(x)}\right), \\ v(x)=\lambda w^{-\delta}+\mu, \quad w=R^2-r^2 \quad x\in G. \end{cases}$$

Now we determine a constant λ so that v satisfies

$$(2\text{-}24)\qquad \hat{P}v+B(x)Q(v)\geq C(x), \quad \text{in } G.$$

Then

$$\begin{aligned}(2\text{-}25)\qquad Pv &= -(A(x)\Delta+\nabla A(x)\cdot\nabla)v \\ &= -A(x)\left(\frac{\partial^2 v}{\partial r^2}+(N-1)\frac{1}{r}\frac{\partial v}{\partial r}\right)-\nabla A(x)\cdot\nabla v \\ &= -2\lambda\delta w^{-\delta-2}\times \\ &[A(x)(NR^2+(2\delta+2-N)r^2)+w\nabla A(x)\cdot(x-x_0)] \\ &\geq -2\lambda\delta CR^2(A(x)+d(x)|\nabla A(x)|)w^{-\delta-2},\end{aligned}$$

where C is a positive number independent of x_0, x, and R. From the monotonicity of Q and the definition of μ we have

$$(2\text{-}26)\qquad Q(v)\geq\frac{C(x)}{B(x)}+\frac{1}{3}(Q(\mu/2)+Q(\lambda w^{-\delta})).$$

Then we have

$$\begin{aligned}(2\text{-}27)\qquad &\hat{P}v+B(x)Q(v) \\ &\geq C(x)-2\lambda\delta CR^2\tilde{A}(x)w^{-\delta-2}+\frac{1}{3}B(x)[Q(\mu/2)+Q(\lambda w^{-\delta})] \\ &\geq C(x)+\frac{1}{3}\lambda^{-\delta_0}B(x)Q(\lambda w^{-\delta})\left(\lambda^{\delta_0}-C'\frac{\tilde{A}(x)}{B(x)}\frac{(\lambda w^{-\delta})^{\delta_0+1}}{Q(\lambda w^{-\delta})}\cdot R^2\right) \\ &+\frac{1}{3}B(x)Q(\mu/2), \quad \text{in } G.\end{aligned}$$

Here we note that $\delta+2=\delta(\delta_0+1)$, $C'=6\delta C$, and C is a positive number independent of each x.

Now we put

$$(2\text{-}28)\qquad \lambda^{\delta_0}=C'\Phi(x_0)R^2\max\left[\sup_{t\geq 1}\frac{t^{\delta_0+1}}{|Q(t)|}, \frac{1}{|Q(\mu/2)|}\right].$$

If $\lambda w^{-\delta}\geq 1$, then we immediately get the desired inequality (2–24). On the other hand, if $\lambda w^{-\delta}<1$, then we use the inequalities

(2–29) $$\lambda^{\delta_0} \geq C'\Phi(x_0)R^2 \frac{1}{|Q(\mu/2)|} \text{ and } \lambda w^{-\delta-2} \leq \lambda^{-\delta_0}.$$

Then we see

(2–30) $$\frac{1}{3} BQ(\mu/2) \geq (2C\delta)R^2 \tilde{A}\lambda w^{-\delta-2}.$$

Therefore we have the desired conclusion.

Be virtue of (2–17), (2–24) and Lemma 2–1, we have

(2–31)
$$\begin{aligned} -\hat{P}(u-v)_+ &\geq -\hat{P}(u-v)\cdot \mathrm{sgn}^+(u-v) \\ &\geq B(x)(Q(u)-Q(v))\cdot \mathrm{sgn}^+(u-v) \geq 0 \quad \text{in } \mathscr{D}'(G). \end{aligned}$$

Since $(u-v)_+ = 0$ near ∂G, it follows from a usual maximum principle that

(2–32)
$$\begin{aligned} u(x_0) \leq v(x_0) &= \lambda R^{-2\delta} + \mu \\ &= C\Phi(x_0)R^{\delta}\cdot R^{-2\delta} + C \\ &\leq C\Phi(x_0)^{\frac{1}{\delta_0}} d(x_0)^{-\frac{2}{\delta_0}} + C, \end{aligned}$$

and this proves the assertion. Here we used [H-2], $d(x_0) = R/2$ and $R \leq d(x) \leq 3R$ in G, and C is a positive number independent of R.

Lemma 2-5. *Assume* [H-1], [H-2] *and* [H-3]. *Assume that* $u \in L^{\infty}_{loc}(\Omega')$ *satisfies* $\hat{P}u \in L^1_{loc}(\Omega')$ *in the distribution sense. Moreover assume that for almost all* $x \in \{x \in \Omega'; u(x) \geq 0\}$

(2–33) $$\hat{P}u + B(x)Q(u) \leq C(x).$$

Then we have

(2–34) $$B(x)Q(u_+) \in L^1_{loc}(\Omega).$$

Proof. First we have from (2–33) and Lemma 2–1

(2–35) $$\hat{P}u_+ + B(x)Q(u_+) \leq C(x), \quad \text{in } \mathscr{D}'(\Omega').$$

For a nonnegative test function $\varphi \in C_0^{\infty}(\Omega)$, we have

(2–36)
$$\begin{aligned} &\langle \hat{P}u_+ + B(x)Q(u_+), \eta_\varepsilon \varphi \rangle \\ &= \int_{\Omega} [u_+ P(\eta_\varepsilon \varphi) + B(x)Q(u_+)\eta_\varepsilon \varphi)] dx \leq \int_{\Omega} C(x)\eta_\varepsilon \varphi dx. \end{aligned}$$

Here $\{\eta_\varepsilon\}_{\varepsilon>0}$ is a family of nonnegative smooth functions defined in (2–9) and (2–10) in the proof of Lemma 2–3.

As in Lemma 2–3, we have, for some positive number C,

$$\left|\int_\Omega u_+ P(\eta_\varepsilon\varphi)dx\right| \le C\left[\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}(d(x)^{-1}A(x)+|\nabla A(x)|)|u_+|dx\right.$$
$$\left.+\int_{\mathrm{supp}\varphi}(A(x)+|\nabla A(x)|)|u_+|dx\right] \tag{2-37}$$

and by Lemma 2–4 and [H-3],

$$\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}(d(x)^{-1}A(x)+|\nabla A(x)|)|u_+|dx \tag{2-38}$$
$$\le\frac{C}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\tilde{A}(x)\left[\left(\frac{\Phi(x)}{d(x)^2}\right)^{\frac{1}{\delta^0}}+1\right]\frac{dx}{d(x)}=O(1).$$

Similarly

$$\int_{\mathrm{supp}\varphi}(A(x)+|\nabla A(x)|)|u_+|dx<+\infty. \quad ([\text{H-3}]) \tag{2-39}$$

Then by taking $\varepsilon\to 0$ we see that $B(x)Q(u_+)\in L^1_{loc}(\Omega)$.

3. Proof of Theorem 1

Let us set $\mu=Q^{-1}(\sup_{x\in\Omega}\frac{C(x)}{B(x)})$. Then we have, as in the proof of Lemma 2–4,

$$\hat{P}(u-\mu)+B(x)(Q(u)-Q(\mu))\le 0, \qquad \text{for } x\in\{u(x)\ge 0\}. \tag{3-1}$$

Then we have from Lemma 2–1

$$\hat{P}(u-\mu)_+ + B(x)\ \mathrm{sgn}^+(u-\mu)\ (Q(u)-Q(\mu))\le 0, \qquad \text{in } \mathscr{D}'(\Omega'). \tag{3-2}$$

Now we assume that without loss of generality $\{x: d(x)<1\}\subset\Omega$ and $\mu\ge\sup_{1/2<d(x)<1}u(x)$, then we shall see

$$u(x)\le\mu, \qquad \text{for } d(x)<1/2. \tag{3-3}$$

In fact a weak maximum principle works in this case since the operator P is elliptic in $\Omega\backslash F$. To see this we set $\phi=(u-\mu)_+$ if $d(x)<3/4$, and $\phi=0$ otherwise, and set $\phi_j=\min(\phi,j)$ for $j=1,2,\cdots$. Here we note that by virtue of Lemma 2–5 $BQ(u)$ is locally integrable on $\{x\in\Omega; u\ge 0\}$. Therefore we see that $BQ(\phi)\in L^1(\Omega)$. Then we approximate ϕ_j by a sequence of smooth functions $\phi_j^m\in C_0^\infty(\Omega)$ for $m=1,2,\cdots$. After all we have as $j\to\infty$

$$\phi_j\to\phi=(u-\mu)_+ \text{ in } \Omega \text{ (a.e.)}, \quad BQ(\phi_j)\to BQ(\phi), \quad \text{in } L^1(\Omega) \tag{3-4}$$

and as $m\to\infty$ for each j

$$\phi_j^m\to\phi_j \text{ in } \Omega \text{ a.e.}, \quad BQ(\phi_j^m)\to BQ(\phi_j), \quad \text{in } L^1(\Omega). \tag{3-5}$$

For any nonnegative $\psi\in C_0^\infty(\Omega)$, we have

$$(3\text{-}6)\qquad \langle \hat{P}\phi_j^m, \psi\rangle + \int_\Omega B(x)\mathrm{sgn}^+(u-\mu)(Q(u)-Q(\mu)\psi dx$$

$$= \int_\Omega \phi_j^m P\psi dx + \int_{\{u\geq\mu\}} B(x)|Q(u)-Q(\mu)|\psi dx$$

$$= \int_\Omega (\phi_j^m - \phi_j)P\psi dx + \int_\Omega (\phi_j - \phi)P\psi dx$$

$$+ \int_\Omega \phi P\psi dx + \int_{\{u\geq\mu\}} B(x)|Q(x)-Q(\mu)|\psi dx.$$

From [H-2], for any $\varepsilon>0$ there is a positive number C_ε such that

$$(3\text{-}7)\qquad t\leq C_\varepsilon Q(t)^{\frac{1}{\delta_0+1}}+\varepsilon, \quad \text{for any } t\geq 0.$$

Then we have

$$(3\text{-}8)\qquad \left|\int_\Omega (\phi_j-\phi)\cdot P\psi dx\right|$$

$$\leq C_\varepsilon\left|\int_{\mathrm{supp}\varphi} Q(|\phi_j-\phi|)^{\frac{1}{\delta_0+1}}\cdot \tilde{A}(x)\,\frac{dx}{d}\right| + \varepsilon\int_\Omega |P\psi|dx$$

$$\leq C_\varepsilon\left(\int_{\mathrm{supp}\varphi} B(x)Q(|\phi_j-\phi|)dx\right)^{\frac{1}{\delta_0+1}}$$

$$\times\left(\int_{\mathrm{supp}\varphi} \tilde{A}(x)\left(\frac{\Phi(x)}{d^2}\right)^{\frac{1}{\delta_0}} d^{\frac{1}{\delta_0}-1}dx\right)^{\frac{\delta_0}{\delta_0+1}} + \varepsilon\int_\Omega |P\psi|dx$$

$$\to\varepsilon\int_\Omega |P\psi|dx, \quad \text{as } j\to\infty. \qquad ([\text{H-3}])$$

Since ε is an arbitrary, we see that

$$(3\text{-}9)\qquad \left|\int_\Omega (\phi-\phi_j)\cdot P\psi dx\right|\to 0, \quad \text{as } j\to\infty.$$

In a similar way we have for each j

$$(3\text{-}10)\qquad \left|\int_\Omega (\phi_j^m-\phi_j)\cdot P\psi dx\right|\to 0, \quad \text{as } m\to\infty.$$

Hence we are able to make the first and second terms in the right-hand side of (3–6) arbitrarily small by letting $j\to\infty$ and $m\to\infty$. Now we deal with the rest of terms. For a sufficiently small $\varepsilon>0$, let η_ε be defined in the proof of Lemma 2–3. Then,

(3–11)

$$\int_{\Omega}\phi P\psi dx+\int_{\{u\geq\mu\}}B(x)|Q(u)-Q(\mu)|\psi dx$$

$$=\int_{\Omega}(1-\eta_\varepsilon)\phi P\psi dx+\int_{\{u\geq\mu\}}(1-\eta_\varepsilon)B(x)|Q(u)-Q(\mu)|\psi dx$$

$$+\int_{\Omega}\phi P(\eta_\varepsilon\psi)dx+\int_{\{u\geq\mu\}}\eta_\varepsilon B(x)|Q(u)-Q(\mu)|\psi dx$$

$$+\int_{\Omega}\phi[P,\eta_\varepsilon]\psi dx.$$

By using the similar estimate to (3–8), we see from [H-3] and Lemm 2–5 that $\phi P\psi\in L^1(\Omega)$. From Lemma 2–5 we also have for $B(Q(u)-Q(\mu))\mathrm{sgn}^+(u-\mu)\psi\in L^1(\Omega)$ for $\mu>0$. Therefore it follows from the dominated comvergence theorem that the first and second terms converge to zero as $\varepsilon\to0$. Since $\mathrm{supp}(\eta_\varepsilon\psi)\subset\Omega'$, we see from (3–2) the sum of the third and forth terms in the right-hand side is non-positive. Now we show that the last term also converges to zero as $\varepsilon\to0$. From (2–13), for some positive number C we have

(3–12)

$$\left|\int_{\Omega}\phi[P,\eta_\varepsilon]\psi dx\right|\leq C\frac{1}{\varepsilon}\int_{\{\varepsilon/2\leq d\leq\varepsilon\}\cap\{u\geq\mu\}}\tilde{A}(x)|u_+|\frac{dx}{d(x)}$$

$$\leq C\frac{1}{\varepsilon}\int_{\varepsilon/2\leq d\leq\varepsilon}\tilde{A}(x)Q(u_+)^{\frac{1}{1+\delta_0}}\frac{dx}{d(x)}\qquad([\text{H-2}])$$

$$\leq C\left[\int_{\varepsilon/2<d(x)<\varepsilon}Q(u_+)B(x)dx\right]^{\frac{1}{1+\delta_0}}\times$$

$$\times\left[\frac{1}{\varepsilon}\int_{\varepsilon/2<d(x)<\varepsilon}\tilde{A}(x)\left(\frac{\Phi(x)}{d(x)^2}\right)^{\frac{1}{\delta_0}}\frac{dx}{d(x)}\right]^{\frac{\delta_0}{1+\delta_0}}\qquad(\text{Hölder inequality})$$

$$\leq C\left[\int_{\varepsilon/2<d(x)<\varepsilon}Q(u_+)B(x)dx\right]^{\frac{1}{1+\delta_0}}\to0,\ ([\text{H-1}],\ [\text{H-3}],\ \text{Lemma 2–5}),$$

as $\varepsilon\to0$.

After all, we see that for any $\kappa>0$ and any nonnegative $\psi\in C_0^\infty(\Omega)$ it holds that for some sufficiently large numbers j and m

(3–13) $$\langle\hat{P}\phi_j^m,\psi\rangle+\int_{\Omega}B(x)\mathrm{sgn}^+(u-\mu)(Q(u)-Q(\mu))\psi dx<\kappa.$$

Now we choose a ψ so that $\psi=1$ on the set $\{d(x)\leqq 3/4\}$. Since $\operatorname{supp}\phi, \operatorname{supp}\phi_j^m \subset \{d(x)\leq 3/4\}$, we see the first term (3–13) equals to zero, so that we have

$$(3\text{–}14) \qquad 0\leq \int_{\{u>\mu\}\cap\{d(x)<3/4\}} B(x)\cdot|Q(u)-Q(\mu)|dx\leq\kappa.$$

Since κ is arbitrary, (3–3) is derived from the positivity of $B(x)$ and the strict monotonicity of $Q(\cdot)$.

4. Proof of Theorem 2

Uniqueness. First we prove the uniqueness of solutions in

$$(4\text{–}1) \qquad T(\Omega)=L^{\infty}(\Omega)\cap H^1_{loc}(\bar{\Omega}\backslash F).$$

Assume that u and v are solutions to the homogeneous Dirichlet boundary value problem (1–20) in the space $T(\Omega)$. By subtraction we get

$$(4\text{–}2) \qquad \hat{P}(u-v)+B(x)(Q(u)-Q(v))=0, \quad \text{in } \mathscr{D}'(\Omega).$$

Since $f, B\cdot Q(u)\in L^1(\Omega)$ it follows from Lemma 2–1 that

$$(4\text{–}3) \qquad \hat{P}(u-v)_+ + B(x)\operatorname{sgn}^+(u-v)(Q(u)-Q(v))\leq 0, \quad \text{in } \mathscr{D}'(\Omega').$$

Since it holds that $\hat{P}(v+\varepsilon)+BQ(v+\varepsilon)\geq \hat{P}v+BQ(v)=f$ for any $\varepsilon>0$, we also have

$$(4\text{–}4) \qquad \hat{P}(u-v-\varepsilon)_+ + B(x)\operatorname{sgn}^+(\mathrm{u}-\mathrm{v}-\varepsilon)(Q(u)-Q(v+\varepsilon))\leq 0, \quad \text{in } \mathscr{D}'(\Omega').$$

Here we note that $\operatorname{supp}(u-v-\varepsilon)_+\cap\partial\Omega=\phi$. As in the proof of Theorem 1, for any $\kappa>0$ and $\psi\in C_0^\infty(\Omega)$ we can approximate a bounded function $(u-v-\varepsilon)_+$ by a sequence of smooth functions $\{\phi_j\}_{j=1}^\infty\subset C_0^\infty(\Omega)$ and we have for sufficiently large numbers j

$$(4\text{–}5) \qquad \langle P\phi_j, \psi\rangle + \int_\Omega B(x)\operatorname{sgn}^+(u-v-\varepsilon)(Q(u)-Q(v+\varepsilon))\psi dx<\kappa.$$

Now we choose a ψ so that $\psi=1$ on the support of ϕ_j. Then we have

$$(4\text{–}6) \qquad 0\leq \int_{u>v+\varepsilon} B(x)|Q(u)-Q(v+\varepsilon)|dx\leq\kappa.$$

Since Q is monotone and ε is arbitrary, we see that $u\leq v$. In a similar way we see $u\geq v$, so that we have $u=v$. Thus the uniqueness holds.

Existence. We assume that $N>1$. If $N=1$ the proof below still works with obvious modifications. First we shall regularize the problem by approximating the operator P by uniformly elliptic operators $\{P_\varepsilon\}_{\varepsilon>0}$ in the following way. If P is uniformly elliptic, the existence of solutions to (1–20) in $H_0^1(\Omega)$ is well known. Let us set for $\varepsilon>0$

$$P_\varepsilon = -\mathrm{div}[(\varepsilon+A(x))\nabla\cdot], \tag{4–7}$$

and consider the Dirichlet problem:

$$\begin{cases} \hat{P}_\varepsilon u+B(x)Q(u)=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{4–8}$$

Then we prepare a lemma which concerns the existence and regularity of solutions of (4–8). We shall sketch the proof for convenience.

Lemma 4-1. *Let* $N>1$. *Assume that the same assumptions as those in Theorem* 2. *Then there is a unique* $u_\varepsilon \in H_0^1(\Omega)$ *which satisfies* (4–8) *in the distribution sense. Moreover* u_ε *satisfies*

$$BQ(u_\varepsilon), BQ(u_\varepsilon)u_\varepsilon \in L^1(\Omega). \tag{4–9}$$

Sketch of Proof. This proposition can be shown in the following way. We replace Q for $Q_n(u)=\min(|Q(u)|, n)\mathrm{sgn}\, u$ and consider the truncated equation below;

$$\begin{aligned} &\hat{P}_\varepsilon u_n + B(x)Q_n(u_n)=f, \quad \text{in } \Omega. \\ &u_n=0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4–10}$$

Then we prove the existence of bounded solutions in $H_0^1(\Omega)$ by the use of Schauder's theorem. It is easy to see that $\{u_n\}_{n=1}^\infty$ is bounded in $H_0^1(\Omega)$. As we make n tend to infinity, we show the weak convergence of solutions in $H_0^1(\Omega)$ using a priori estimates for a fixed $\varepsilon>0$. Then by the compactness argument we see the limit u_ε satisfies (4–8) and (4–9).

Remark 11. For each compact set $K\subset\bar{\Omega}\backslash F$, it holds that $u_\varepsilon \in H^1(K)$ and $BQ(u_\varepsilon)u_\varepsilon \in L^1(\Omega)$. Since P is uniformly elliptic on K and $A(x)\in C^0(\bar{\Omega})$, we see that for some positive number $C(K)$ independent of each $\varepsilon>0$ such that

$$\sup_{x\in K}|u_\varepsilon(x)|\le C(K). \tag{4–11}$$

Moreover we can show that u_ε is Hölder continuous on K and its Hölder norm is uniformly bounded with respect to ε. If Q is unifomly Lipschitz continuous, then we see $u_\varepsilon \in H_{loc}^2(\Omega)\backslash F)$ as well.

End of the proof of Theorem 2. By u_ε we denote the solutions to (4–8) as before. From Lemma 4–1 and its remark we see $u_\varepsilon \in H_0^1(\Omega)$ and $BQ(u_\varepsilon)u_\varepsilon \in L^1(\Omega)$. First we prove that u_ε satisfies (1–24) unifiormly in $\varepsilon>0$. We set

$$a=\frac{2+\delta_0}{1+\delta_0}, \quad \text{and} \quad b=2+\delta_0. \tag{4–12}$$

From (1–6) in [H-2], we see $|u_\varepsilon| \le C[(|u_\varepsilon\|Q(u_\varepsilon)|)^{\frac{1}{2+\delta_0}}+1]$ for some positive number C. By Young's inequality we have for any positive number h

$$\int_\Omega |f\|u|dx \le C\int_\Omega \frac{|f|}{B}(|u_\varepsilon\|Q(u_\varepsilon)|)^{\frac{1}{2+\delta_0}}Bdx + C\int_\Omega |f|dx \tag{4–13}$$

$$\le Ca^{-1}h^{-a}\int_\Omega \left(\frac{|f|}{B}\right)^a Bdx + Cb^{-1}h^b\int_\Omega |u_\varepsilon\|Q(u_\varepsilon)|Bdx + C\int_\Omega |f|dx.$$

Multilpying u_ε to the bothside of (4–8) and integrating over Ω, we get

$$\int_\Omega (\varepsilon + A)|\nabla u_\varepsilon|^2 dx + (1 - Cb^{-1}h^b)\int_\Omega\int_\Omega B|u_\varepsilon\|Q(u_\varepsilon)|dx \tag{4–14}$$

$$\le Ca^{-1}h^{-a}\int_\Omega \left(\frac{|f|}{B}\right)^a Bdx + C\int_\Omega |f|dx.$$

Now we put $h^b = b(2C)^{-1}$, then we have the desired inequality.

Secondly, by the method of a priori estimate and compactness, we derive a subsequence $\{u_{\varepsilon_j}\}_{j=1}^\infty$ from $\{u_\varepsilon\}_{\varepsilon>0}$ which converges weakly to some element $\bar{u} \in H^1_{loc}(\bar{\Omega}\setminus F)$ and u_ε converges $\bar{u}$ a.e. in $\Omega\setminus F$. Then by virtue of Fatou's lemma and a weakly lower semicontinuity of L^2-norm, we get for some positive number C'

$$\int_{\Omega\setminus F} A|\nabla\bar{u}|^2 dx + \int_{\Omega\setminus F} BQ(\bar{u})\bar{u}dx \le C'[\|f/B\|_\infty^\lambda + \|f\|_\infty]. \tag{4–15}$$

Now we show that $BQ(u_{\varepsilon_j}) \to BQ(\bar{u})$ in $\mathscr{D}'(\Omega\setminus F)$. From the definition of weak convergence of $\{u_{\varepsilon_j}\}_{j=1}^\infty$ and the estimates (4–14) and (4–15), we see that $f - P_{\varepsilon_j}u_{\varepsilon_j} \to f - \hat{P}\bar{u}$ in $\mathscr{D}'(\Omega\setminus F)$. Therefore the limit of $BQ(u_{\varepsilon_j})$ in $\mathscr{D}'(\Omega\setminus F)$ as $j\to\infty$ exists. Hence it suffices to show that

$$\int_\Omega B(Q(u_\varepsilon) - Q(\bar{u}))\varphi dx \to 0, \quad \text{for all } \varphi \in C_0^\infty(\Omega\setminus F). \tag{4–16}$$

From Remark 11 just after the proof of Lemma 4–1, $\sup_{x\in \mathrm{supp}\varphi}|u_{\varepsilon_j}|$ is uniformly bounded on the support of φ, so that $BQ(u_{\varepsilon_j})$ is uniformly bounded with respect to ε_j. Since $u_{\varepsilon_j} \to \bar{u}$ a.e. in $\Omega\setminus F$, (4–16) follows from the dominated covergence theorem. After all we see that $\bar{u}$ satisfies (1–20) in $\Omega\setminus$F in the sense of distribution. Now we define

$$u(x) = \begin{cases} \bar{u}(x), & \text{if } x \in \Omega\setminus F, \\ Q^{-1}(f(x)/B(x)), & \text{if } x \in F\setminus\partial F. \end{cases} \tag{4–17}$$

Then u clearly satisfies (1–20) in $\Omega\setminus\partial$F in the sense of distribution. In $\Omega\setminus F$ the operator P is elliptic and the right-hand side of (1–20) belongs to $L^\infty(\Omega)$. Hence we see that $u\in L^\infty_{loc}(\Omega')$ Then it follows from Theorem 1 that u is bounded in Ω'. From Corollary 1 we see that there exists a unique function $v\in L^\infty(\Omega)$ which satisfies (1–19). Since $v=u$ in $\Omega\backslash\partial F$, we see that $v\in T(\Omega)$ is a unique solution to (1–20) in $\mathscr{D}'(\Omega)$ and v satisfies (1–24) for some positive number C.

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