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A duality theorem for homomorphisms between generalized Verma modules

By

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Introduction

Let K be a field of characteristic zero, $\mathfrak{g}$ a split semisimple Lie algebra over K, $\mathfrak{p}$ a parabolic subalgebra, and ε the half of the sum of roots whose root subspaces are contained in the nilpotent radical of $\mathfrak{p}$. Then -2ε gives a one dimensional $\mathfrak{p}$-module, which we denote by the same letter. For a finite dimensional simple $\mathfrak{p}$-module E, let E^* be its dual $\mathfrak{p}$-module. Put $M(E)=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}E$. The following duality theorem is attributed to G. Zuckerman (cf. [1,(4.9)]).

Duality Theorem. *For a finite dimensional simple $\mathfrak{p}$-modules E and F, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(M(E),M(F))\simeq \mathrm{Hom}_{\mathfrak{g}}(M(F^*\otimes(-2\varepsilon)),\ M(E^*\otimes(-2\varepsilon))).$$

In order to study the b-functions of semi-invariants and the generalized Verma modules [10], the author has come to need [1,(4.9)]. Since [1,(4.9)] seems difficult to understand correctly for non-experts, we give in this note a detailed proof, which follows a similar line as was indicated in [1,(4.9)], but is purely algebraic.

Convention. For an algebra A, an A-module means a left A-module, unless otherwise stated. Every vector space is considered over the base field K, and, Hom and $\otimes$ means Hom_K and $\otimes_K$. For a vector space V, V^* denotes its dual space, and $\langle\ \rangle$ the natural pairing of V and V^*. More generally, we sometimes denote the value of a (vector valued) function f at a point p by $\langle f,p\rangle$ or $\langle p,f\rangle$ for $f(p)$.

A Lie algebra character, say λ, of a Lie algebra $\mathfrak{g}$ gives a one dimensional $\mathfrak{g}$-module, which we shall denote by the same letter λ. We consider K as the trivial $\mathfrak{g}$-module, which is also denoted by 0 by the above convention.

When two objects are *naturally* isomorphic, we sometimes write $=$ for $\simeq$.

§1

The purpose of this section is to prove (1.7), which is used later in (3.7).

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1.1. We fix a field K of characteristic zero as the base field. For the sake of simplicity, we assume K to be algebraically closed. (Cf. [6, 7.2.2,(i)].) For a Lie algebra $\mathfrak{g}$ $U(\mathfrak{g})$ denotes the universal enveloping algebra. Let $u \to u^{\top}$ be the anti-automorphism of $U(\mathfrak{g})$ such that $x^{\top}=-x$ for $x\in\mathfrak{g}$. For a $U(\mathfrak{g})$-module M, the $U(\mathfrak{g})$-module structure of M^* is defined so that $\langle um,m^*\rangle=\langle m,u^{\top}m^*\rangle$ for $u\in U(\mathfrak{g})$, $m\in M$, and $m^*\in M^*$. Denote by $\hat{\mathfrak{g}}$ the set of isomorphism classes of finite dimensional simple $U(\mathfrak{g})$-modules.

1.2. Let $\mathfrak{k}$ be a reductive Lie algebra, $V\in\hat{\mathfrak{k}}$, $\{v(\xi)|\xi\in B(V)\}$ a linear basis of V parametrized by a set $B(V)$, and $\{v^*(\xi)|\xi\in B(V)\}$ the dual basis in V^*. For $k\in U(\mathfrak{k})$, put $c_{\xi\eta}(k)=\langle v^*(\xi),kv(\eta)\rangle$ and $c^*_{\xi\eta}(k)=\langle v(\xi),kv^*(\eta)\rangle$. Then $c^*_{\xi\eta}(k)=c_{\eta\xi}(k^{\top})$ and $c_{\xi\eta}, c^*_{\xi\eta}\in U(\mathfrak{k})^*$. Sometimes we write $c(\xi,\eta)=c_V(\xi,\eta)$ (resp. $c^*(\xi,\eta)=c^*_V(\xi,\eta)$) for $c_{\xi\eta}$ (resp. $c^*_{\xi\eta}$). Let $U(\mathfrak{k})^*_V$ be the linear span of $\{c_{\xi\eta}|\xi,\ \eta\in B(V)\}$. We can define a $U(\mathfrak{k})$-module structure of $U(\mathfrak{k})^*$ by $(kc)(k')=c(k^{\top}k')$ (resp. $(kc)(k')=c(k'k)$), which we shall denote by $U(\mathfrak{k})^*_L$ (resp. $U(\mathfrak{k})^*_R$). Let $U(\mathfrak{k})'_L$ (resp. $U(\mathfrak{k})'_R$) be the set of $U(\mathfrak{k})$-finite vectors in $U(\mathfrak{k})^*_L$ (resp. $U(\mathfrak{k})^*_R$). It is known [6, 2.7.12] that $U(\mathfrak{k})'_L=U(\mathfrak{k})'_R=\bigoplus_{V\in\hat{\mathfrak{k}}}U(\mathfrak{k})^*_V$, which we shall denote simply by $U(\mathfrak{k})'$. The two $U(\mathfrak{k})$-module structures of $U(\mathfrak{k})'$ commute each other, and give a $\mathfrak{k}\times\mathfrak{k}$-module structure in $U(\mathfrak{k})'$. More generally, for any $\mathfrak{k}$-module Z, let $Z_{(\mathfrak{k})}$ denote the $\mathfrak{k}$-submodule consisting of $z\in Z$ such that $\dim U(\mathfrak{k})\, z<\infty$.

1.3. Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{k}$ which is reductive in $\mathfrak{k}$ (cf. [6, 1.7.5]), and E a semisimple $\mathfrak{l}$-module (i.e., a sum of simple submodules) of finite dimension. Consider $U(\mathfrak{k})'_L\otimes E$ as a tensor product of two $\mathfrak{l}$-modules, and denote the subspace $(U(\mathfrak{k})'_L\otimes E)^{\mathfrak{l}}$ of $\mathfrak{l}$-invariant vectors by $Y(E)$. Then $Y(E)$ has a natural $\mathfrak{k}$-module structure coming from $U(\mathfrak{k})'_R$. Since $U(\mathfrak{k})^*_V$ is a $\mathfrak{k}$-submodule of $U(\mathfrak{k})'_L$ and $U(\mathfrak{k})'_R$, $(U(\mathfrak{k})^*_V\otimes E)^{\mathfrak{l}}=:Y(E)_V$ is a $\mathfrak{k}$-submodule of $Y(E)$. Obviously, $Y(E)=\bigoplus_{V\in\hat{\mathfrak{k}}}Y(E)_V$. We can identify $Y(E)$ with $\mathrm{coind}\,(E|\mathfrak{l}\to\mathfrak{k})_{(\mathfrak{k})}=\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{k}),E)_{(\mathfrak{k})}$ in a natural way. (See [6, 5.5] for the coinduction.) Here $U(\mathfrak{k})$ is considered as an $\mathfrak{l}$-module by the left multiplication of $\mathfrak{l}$. Thus, for $y=\Sigma c_i\otimes e_i\in Y(E)\subset U(\mathfrak{k})^*\otimes E$ and $k\in U(\mathfrak{k})$, we have $y(k)=\Sigma c_i(k)e_i$.

1.4. The coalgebra structure $k\to k\otimes 1+1\otimes k$ $(k\in\mathfrak{k})$ of $U(\mathfrak{k})$ gives a K-algebra structure $U(\mathfrak{k})^*\otimes U(\mathfrak{k})^*\to U(\mathfrak{k})^*$.

Let c_0 be the natural homomorphism $U(\mathfrak{k})\to U(\mathfrak{k})/U(\mathfrak{k})\mathfrak{k}=K$. Then $Kc_0=U(\mathfrak{k})^*_0$. (By convention, 0 denotes the trivial $U(\mathfrak{k})$-module.) Let τ be the composition of the projection $U(\mathfrak{k})'\to U(\mathfrak{k})^*_0$ and $U(\mathfrak{k})^*_0\ni c\to c(1)\in K$. The latter is an isomorphism because the image of c_0 is 1. Define a pairing of $U(\mathfrak{k})'\otimes E$ and $U(\mathfrak{k})'\otimes E^*$ by

$$\langle c\otimes e,c^*\otimes e^*\rangle=\tau(cc^*)\langle e,e^*\rangle$$

for $c,c^*\in U(\mathfrak{k})'$, $e\in E$, and $e^*\in E^*$. This bilinear form is $\mathfrak{k}\times\mathfrak{l}$-invariant and non-degenerate [6, 2.7.15]. (The $\mathfrak{k}$-action (resp. $\mathfrak{l}$-action) comes from $U(\mathfrak{k})'_L$ (resp. $U(\mathfrak{k})'_R$).) Since $U(\mathfrak{k})^*_V$ $(V\in\hat{\mathfrak{k}})$ is $\mathfrak{k}\times\mathfrak{k}$-simple, $U(\mathfrak{k})'$ is $\mathfrak{k}\times\mathfrak{k}$-semisimple. Since $\mathfrak{l}$ is reductive in $\mathfrak{k}$ and E is $\mathfrak{l}$-semisimple, $U(\mathfrak{k})'\otimes E$ and $U(\mathfrak{k})'\otimes E^*$ are semisimple $\mathfrak{k}\times\mathfrak{l}$-modules. Hence the above pairing gives non-degenerate pairings between respective isotypic parts. In particular it gives a non-degenerate pairing $\langle\ ,\ \rangle=\langle\ ,\ \rangle_Y$ between $Y(E)_V$ and $Y(E^*)_{V^*}$. The pairing $\langle\ ,\ \rangle_Y$ can be also obtained as the composition of the natural $\mathfrak{k}$-homomorphisms

$$(1) \qquad Y(E)\otimes Y(E^*)\to Y(E\otimes E^*)\to Y(K)=U(\mathfrak{k})'\xrightarrow{\tau} K.$$

See [6, 5.6.7] for the first arrow.

Let E and F be semisimple $\mathfrak{l}$-modules, Φ : $Y(E^*)\to Y(F^*)$ a $\mathfrak{k}$-homomorphism, and Φ_V : $Y(E^*)_{V^*}\to Y(F^*)_{V^*}$ ($V\in\hat{\mathfrak{k}}$) the $\mathfrak{k}$-homomorphism induced on the isotypic subspace. Since $Y(E^*)_{V^*}$ and $Y(F^*)_{V^*}$ are finite dimensional, we can consider the dual Ψ_V : $Y(F)_V\to Y(E)_V$ of Φ_V with respect to $\langle\ ,\ \rangle_Y$. Let $\Psi=\bigoplus_{V\in\hat{\mathfrak{k}}}\Psi_V$. Then Ψ : $Y(F)\to Y(E)$ is the dual $\mathfrak{k}$-homomorphism of Φ. Thus we get the following assertion.

Lemma 1.5. *Let $\mathfrak{k}$ be a reductive Lie algebra, $\mathfrak{l}$ a subalgebra of $\mathfrak{k}$ which is reductive in $\mathfrak{k}$, and, E and F semisimple $\mathfrak{l}$-modules. Then there exists a natural isomorphism (= transposition)*

$$T:\ \mathrm{Hom}_{\mathfrak{k}}(Y(E^*),Y(F^*))\to\mathrm{Hom}_{\mathfrak{k}}(Y(F),Y(E)).$$

1.6. Under the same assumptions as in (1.5), consider the following condition for $\Psi\in\mathrm{Hom}_{\mathfrak{k}}(Y(F),Y(E))$: *For any $e^*\in E^*$, there exists a finite family $\{k_i\in U(\mathfrak{k}),\ f_i^*\in F\mid i\in I\}$ such that*

$$(\#) \qquad \langle(\Psi y)(k),e^*\rangle=\sum_{i\in I}\langle y(k_ik),f_i^*\rangle$$

for any $y\in Y(F)$ and $k\in U(\mathfrak{k})$. Denote by $\mathrm{Hom}_{\mathfrak{k}}^{\#}(Y(F),Y(E))$ the totality of such Ψ. (The meaning of this condition will become clear in (2.8).)

Lemma 1.7. *Under the same assumptions as in* (1.5), *T induces an isomorphism*

$$T:\ \mathrm{Hom}_{\mathfrak{k}}^{\#}(Y(E^*),Y(F^*))\to\mathrm{Hom}_{\mathfrak{k}}^{\#}(Y(F),Y(E)).$$

The remainder of this section is devoted to the proof of this lemma. We assume the simplicity of E and F. The general case can be easily reduced to this case.

1.8. Let $\{h(\iota)\mid\iota\in B(E,V)\}$ (resp. $\{e(\alpha)\mid\alpha\in B(E)\}$) be a linear basis of $\mathrm{Hom}_{\mathfrak{l}}(E,V)$ (resp. E), where $B(E,V)$ (resp. $B(E)$) is a parameter set. Put

$$B(V)=\bigcup_{E\in\hat{\mathfrak{l}}}\{(\iota,\alpha)\mid\iota\in B(E,V),\ \alpha\in B(E)\}, \qquad \text{and}$$

$$v(\xi)=\langle h(\iota),e(\alpha)\rangle \text{ for } \xi=(\iota,\alpha)\in B(V).$$

Then $\{v(\xi)\mid\xi\in B(V)\}$ gives a linear basis of V. We shall consider the coefficients of the representation matrices $c_V(\xi,\eta)$ (resp. $c_E(\alpha,\beta)$) with respect to $\{v(\xi)\}$ (resp. $\{e(\alpha)\}$). The following two facts are constantly used in the argument below. (1) For $l\in U(\mathfrak{l})$, $c_V((\varkappa,\beta),(\iota,\alpha))(l)=0$ unless $\iota=\varkappa$, and $c_V((\iota,\beta),(\iota,\alpha)(l)=c_E(\beta,\alpha)$ (l). (2) Because of the simplicity of V, $(c_V(\xi,\eta)(k))_{\xi,\eta\in B(V)}$ runs all over the matrices of size dim V, when k runs over $U(\mathfrak{k})$.

Lemma 1.9. *For* $c_{\alpha\beta}=c_V(\alpha,\beta)$, (1) $\tau(c_{\alpha\beta}c^*_{\gamma\delta})=0$ *unless* $(\alpha,\beta)=(\gamma,\delta)$, *and* (2) $\tau(c_{\alpha\beta}c^*_{\alpha\beta})=(\dim V)^{-1}$.

Proof. Considering $U(\mathfrak{k})^*=U(\mathfrak{k})^*_R$, we get $\tau(k(c_{\alpha\beta}c^*_{\gamma\delta}))=\tau((kc_{\alpha\beta})\cdot c^*_{\gamma\delta}+c_{\alpha\beta}\cdot(kc^*_{\gamma\delta}))=\Sigma_\iota\tau(c_{\alpha\iota}c_{\iota\beta}(k)c^*_{\gamma\delta})+\Sigma_\varkappa\tau(c_{\alpha\beta}c^*_{\gamma\varkappa}c^*_{\varkappa\delta}(k))=\Sigma_\iota\tau(c_{\alpha\iota}c^*_{\gamma\delta})c_{\iota\beta}(k)-\Sigma_\varkappa\tau(c_{\alpha\beta}c^*_{\gamma\varkappa})c_{\delta\varkappa}(k)$ for $k\in\mathfrak{k}$. Since $c_{\alpha\beta}$'s are linearly independent in $U(\mathfrak{k})^*$, $\tau(c_{\alpha\iota}c^*_{\gamma\delta})=0$ unless $\iota=\delta$, and $\tau(c_{\alpha\delta}c^*_{\gamma\delta})=\tau(c_{\alpha\beta}c^*_{\gamma\beta})$. Combining this with the similar result obtained by replacing right with left, we get (1) and also we can show that $\tau(c_{\alpha\beta}c^*_{\alpha\beta})$ is independent of (α,β). Since $k\ \Sigma_{\alpha,\beta}c_{\alpha\beta}c^*_{\alpha\beta}=0$ $(k\in\mathfrak{k})$ in $U(\mathfrak{k})^*_R$, i.e., $\Sigma c_{\alpha\beta}c^*_{\alpha\beta}\in U(\mathfrak{k})^*_0$, we get $\tau(\Sigma_{\alpha,\beta}c_{\alpha\beta}c^*_{\alpha\beta})=\Sigma_{\alpha,\beta}c_{\alpha\beta}(1)c^*_{\alpha\beta}(1)=\dim V$. Thus we get (2).

1.10. For $(\iota,\xi)\in B(E,V)\times B(V)$, put

$$y(\iota,\xi):=\sum_{\alpha\in B(E)}c_V((\iota,\alpha),\xi)e(\alpha),\qquad\text{and}$$

$$y^*(\iota,\xi):=\sum_{\alpha\in B(E)}c^*_V((\iota,\alpha),\xi)e^*(\alpha).$$

Then $\{y(\iota,\xi)\}$ (resp $\{y^*(\iota,\xi)\}$) gives a linear basis of $Y(E)_V$ (resp. $Y(E^*)_{V^*}$). By (1.9), we have

$$\text{(1)}\qquad \begin{aligned}&\langle y^*(\iota,\xi),\ y(\varkappa,\eta)\rangle=0 \text{ unless } (\iota,\xi)=(\varkappa,\eta),\qquad\text{and}\\ &\langle y^*(\iota,\xi),\ y(\iota,\xi)\rangle=\dim E/\dim V.\end{aligned}$$

For $\Phi\in\mathrm{Hom}_\mathfrak{k}\ (Y(E^*),Y(F^*))$, let $\Psi=T(\Phi)\in\mathrm{Hom}_\mathfrak{k}(Y(F),Y(E))$, and define $\Phi_V(\varkappa,\eta\,;\,\iota,\xi)$, $\Psi_V(\iota,\xi\,;\,\varkappa,\eta)\in K$ by

$$\text{(2)}\qquad \Phi y^*(\iota,\xi)=\sum_{\substack{\varkappa\in B(F,V)\\ \eta\in B(V)}}y^*(\varkappa,\eta)\Phi_V(\varkappa,\eta\,;\,\iota,\xi),\qquad\text{and}$$

$$\text{(3)}\qquad \Psi y(\varkappa,\eta)=\sum_{\substack{\iota\in B(E,V)\\ \xi\in B(V)}}y(\iota,\xi)\Psi_V(\iota,\xi\,;\,\varkappa,\eta)$$

for $\iota\in B(E,V)$, $\varkappa\in B(F,V)$ and $\xi,\eta\in B(V)$. By (1)

$$\text{(4)}\qquad \Psi_V(\iota,\xi\,;\,\varkappa,\eta)=\frac{\dim F}{\dim E}\Phi_V(\varkappa,\eta\,;\,\iota,\xi).$$

Compare the coefficients of $c_V(\eta,\xi)$'s of $\Phi(ky^*(\iota,\xi))=k(\Phi y^*(\iota,\xi))$ using

$$ky^*(\iota,\xi)=\sum_{\zeta\in B(V)} y^*(\iota,\zeta)c_V(\zeta,\xi)^*(k)$$

for $k\in U(\mathfrak{k})$ and $(\iota,\xi)\in B(E,V)\times B(V)$. Then we can show that $\Phi_V(\varkappa,\eta\,;\,\iota,\xi)=0$ unless $\xi=\eta$, and $\Phi_V(\varkappa,\xi\,;\,\iota,\xi)$ is independent of ξ, which we shall denote by $\Phi_V(\varkappa,\iota)$ omitting ξ. Define $\Psi_V(\iota,\varkappa)$ in the same way. Then

$$(2') \qquad \Phi y^*(\iota,\xi)=\sum_{\varkappa\in B(F,V)} y^*(\varkappa,\xi)\Phi_V(\varkappa,\iota),$$

$$(3') \qquad \Psi y(\varkappa,\eta)=\sum_{\iota\in B(E,V)} y(\iota,\eta)\Psi_V(\iota,\varkappa), \qquad \text{and}$$

$$(4') \qquad \Psi_V(\iota,\varkappa)=\frac{\dim F}{\dim E}\Phi_V(\varkappa,\iota).$$

The following lemma concludes the proof of (1.7).

Lemma 1.11. *Let E and F be simple $U(\mathfrak{l})$-modules, $\Phi\in\mathrm{Hom}_{\mathfrak{k}}(Y(E^*),Y(F^*))$ and $\Psi=T(\Phi)\in\mathrm{Hom}_{\mathfrak{k}}(Y(F),Y(E))$. The following conditions are equivalent.*

(1) $\Phi\in\mathrm{Hom}^{\#}_{\mathfrak{k}}(Y(E^*),Y(F^*))$.

(2) *For any $f\in F$, there exists a finite set $\{e_i\in E,k_i\in U(\mathfrak{k})\}$ such that for any $V\in\hat{\mathfrak{k}}$, $\beta\in B(F)$, $\iota\in B(E,V)$, and $\varkappa\in B(F,V)$, we have*

$$\langle f^*(\beta),f\rangle\Phi_V(\varkappa,\iota)=\sum_{\substack{i\\ \alpha\in B(E)}} c_V((\varkappa,\beta),(\iota,\alpha))(k_i)\langle e^*(\alpha),e_i\rangle.$$

(3) *There exists $k_0\in U(\mathfrak{k})$ such that for any $V\in\hat{\mathfrak{k}}$, $\alpha\in$* B*(E), $\beta\in B(F)$, $\iota\in B(E,V)$, and $\varkappa\in B(F,V)$, we have*

$$\Phi_V(\varkappa,\iota)=c_V((\varkappa,\beta),\,(\iota,\alpha))(k_0).$$

(4) *For any $e^*\in E^*$, there exists a finite set $\{f_j^*\in F^*,k_j'\in U(\mathfrak{k})\}$ such that for any $V\in\hat{\mathfrak{k}}$, $\alpha\in B(E)$, $\iota\in B(E,V)$, and $\varkappa\in B(F,V)$, we have*

$$\langle e(\alpha),e^*\rangle\frac{\dim F}{\dim E}\Phi_V(\varkappa,\iota)=\sum_{\substack{j\\ \beta\in B(F)}} c_V((\varkappa,\beta),\,(\iota,\alpha))(k_j')\langle f(\beta),f_j^*\rangle.$$

(5) $\Psi\in\mathrm{Hom}^{\#}_{\mathfrak{k}}(Y(F),Y(E))$.

Proof. (1)$\Rightarrow$(2) Let us write down the left hand side of the condition (#) for Φ (cf. (1.6)):

$$\langle(\Phi y^*(\iota,\xi))(k),f\rangle$$

$$=\sum_{\varkappa\in B(F,V)}\langle y^*(\varkappa,\xi)(k)\Phi_V(\varkappa,\iota),f\rangle$$

$$= \sum_{\substack{\varkappa \in B(F,V) \\ \beta \in B(F)}} c^*((\varkappa,\beta),\xi)(k)\langle f^*(\beta),f\rangle \Phi_V(\varkappa,\iota).$$

Next let us write down the right hand side:

$$\sum_{i\in I} \langle y^*(\iota,\xi)(k_i k),e_i\rangle$$

$$= \sum_{\substack{i\in I \\ \alpha\in B(E)}} c^*((\iota,\alpha),\ \xi)(k_i k)\langle e^*(\alpha),e_i\rangle$$

$$= \sum_{\substack{i\in I \\ \alpha\in B(E)}} c^*((\iota,\alpha),\ (\varkappa,\beta))(k_i)c^*((\varkappa,\beta),\xi)(k)\langle e^*(\alpha),e_i\rangle.$$

Since $c^*(\xi,\eta)(k)=c(\eta,\xi)(k^\top)$, comparing the coefficients of $c^*((\varkappa,\beta),\xi)$, we get (2). To prove (2)$\Rightarrow$(1), read the above proof backward. We can prove (4)$\Rightarrow$(5) in the same way, using (1.11, (4′)).

(3)$\Rightarrow$(2) For $l\in U(\mathfrak{l})$ and $e_0\in E$,

$$\sum_{\alpha\in B(E)} c_V((\varkappa,\beta),(\iota,\alpha))(lk_0)\langle c^*(\alpha),e_0\rangle$$

$$= \sum_{\substack{\alpha\in B(E) \\ \gamma\in B(F)}} c_V((\varkappa,\beta),(\varkappa,\gamma))(l)c_V((\varkappa,\gamma),\ (\iota,\alpha))(k_0)\langle c^*(\alpha),e_0\rangle$$

$$= \sum_{\alpha,\gamma} c_F(\beta,\gamma)(l)\Phi_V(\varkappa,\iota)\langle e^*(\alpha),e_0\rangle.$$

Taking l and e_0 so that $\Sigma_\gamma c_F(\beta,\gamma)(l)=\langle f^*(\beta),f\rangle$ for any $\beta\in B(F)$, and $\Sigma_\alpha\langle e^*(\alpha), e_0\rangle=1$, we get (2).

(2)$\Rightarrow$(3) Since E is a simple $\mathfrak{l}$-module, there exist $l_i\in U(\mathfrak{l})$ such that $c_E(\gamma,\alpha)(l_i) =\langle e^*(\gamma),e_i\rangle$ for any $\alpha,\gamma\in B(E)$. Then the right hand side of the equality in (2) is

$$\sum_{\substack{i \\ \gamma\in B(E)}} c_V((\varkappa,\beta),\ (\iota,\gamma))(k_i)c_V((\iota,\gamma),\ (\iota,\alpha))(l_i)=c_V((\varkappa,\beta),\ (\iota,\alpha))(k)$$

for any $\alpha\in B(E)$, where $k:=\Sigma_i k_i l_i$. For any $l\in U(\mathfrak{l})$,

$$c_V((\varkappa,\beta),\ (\iota,\alpha))(lk)$$

$$= \sum_{\gamma\in B(F)} c_V((\varkappa,\beta),\ (\varkappa,\gamma))(l)c_V((\varkappa,\gamma),\ (\iota,\alpha))(k)$$

$$= \sum_{\gamma\in B(F)} c_F(\beta,\gamma)(l)\langle f^*(\gamma),f\rangle\Phi_V(\varkappa,\iota).$$

Take $f\in F$ and $l\in U(\mathfrak{l})$ so that $\Sigma_\gamma c_F(\beta,\gamma)(l)\langle f^*(\gamma)f\rangle=1$ for any $\beta\in B(F)$, and put $k_0=lk$. Then we get (3). We can prove (3)$\Leftrightarrow$(4) in the same way.

§2

The purpose of this section is to prove (2.8), which is used later in (3.7).

2.1. Let M be a K-vector space. Consider the discrete topology in K and the finite-open topology in $M^*:=\mathrm{Hom}(M,K)$.

Lemma. (1) *The totality M^{**} of the continuous linear functionals on M^* can be naturally identified with M.* (2) *If M has a countable linear basis, M^* has a countable open basis for the neighbourhoods of* 0.

Proof. (1) We have a natural injection $\omega: M\to M^{**}$. Let us prove its surjectivity. Let $\{m_i\}$ be a linear basis of M and define $m_j^*\in M^*$ so that $\langle m_i,m_j^*\rangle=1$ if $i=j$, otherwise $=0$. Then $M=\oplus_i Km_i$ and $M^*=\Pi_i Km_i^*$. Given $\mu\in M^{**}$, put $\mu_i=\mu(m_i^*)$, and assume that $\mu_{i_p}\neq 0$ for infinitely many i_p's ($p=1,2,\cdots$). Then $\Sigma_{p=1}^k \mu_{i_p}^{-1}m_{i_p}^* =: n_k^*$ converges to $\Sigma_{p=1}^\infty \mu_{i_p}^{-1}m_{i_p}^*$, but $\mu(n_k^*)=k$ is not convergent. This contradicts the continuity of μ. Hence $m=\Sigma_i\mu_i m_i$ is a finite sum, and $\langle m, m_i^*\rangle=\langle\mu,m_i^*\rangle$ for any i, i.e., $\omega(m)=\mu$. (2) If $\{m_1,m_2,\cdots\}$ is a linear basis of M, then $\{m_1,\cdots,m_p\}^\perp\subset M^*(p=1,2,\cdots)$ form an open basis for the neighbourhoods of 0.

2.2. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{q}$ a parabolic subalgebra, and $\mathfrak{m}$ a Levi subalgebra of $\mathfrak{q}$. Let $\mathscr{A}_\mathfrak{q}$ be the category of finite dimensional $\mathfrak{q}$-modules which are $\mathfrak{m}$-semisimple. For $E\in\mathscr{A}_\mathfrak{q}$, put

$$M(E)=\mathrm{ind}(E|\mathfrak{q}\to\mathfrak{g}):=U(\mathfrak{g})\otimes_{U(\mathfrak{q})}E, \qquad \text{and}$$

$$M'(E)=\mathrm{coind}(E|\mathfrak{q}\to\mathfrak{g}):=\mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}),E).$$

Cf. [6, 5.1 and 5.5]. We always consider the discrete topology in E and the finite-open topology in $M'(E)$. For an element $m^*\in M(E)^*=\mathrm{Hom}(U(\mathfrak{g})\otimes_{U(\mathfrak{q})}E, K)$, define the element $m'\in\mathrm{Hom}(U(\mathfrak{g}),E^*)$ by $\langle m'(u),e\rangle=m^*(u^\top\otimes e)$ ($u\in U(\mathfrak{g}),e\in E$). Then $m'\in M'(E^*)$ and $m^*\to m'$ gives an isomorphism $M(E)^*\to M'(E)^*$ including topology ([6, 5.5.4]). Thus by (2.1), we get the following assertions.

Lemma 2.3. *For E, $F\in\mathscr{A}_\mathfrak{q}$,* $\mathrm{Hom}_\mathfrak{g}(M(E),M(F))$ *is naturally identified with* $\mathrm{Hom}_\mathfrak{g}^c(M'(F^*),M'(E^*))$, *where the latter is the set of continuous homomorphisms.*

Lemma 2.4. *The $\mathfrak{g}$-submodules of $M(E)$ are naturally in one to one correspondence with the closed $\mathfrak{g}$-submodules of $M'(E^*)\simeq M(E)^*$.*

2.5. Let $\mathfrak{a}$ be the center of $\mathfrak{m}$. Any $\lambda\in\mathfrak{a}^*$ can be uniquely extended to a Lie algebra character of $\mathfrak{q}$, which we shall denote by the same letter λ, or by $\lambda_\mathfrak{q}$ if the specification is necessary. We also denote the corresponding one dimensional $\mathfrak{q}$-module by the same letter. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$, and put $\mathfrak{l}=\mathfrak{k}\cap\mathfrak{q}$. Let $E\in\mathscr{A}_\mathfrak{q}$ and consider the following conditions.

(1) $\mathfrak{k}$ is reductive in $\mathfrak{g}$ and $\mathfrak{l}$ is reductive in $\mathfrak{k}$.

(2) $\mathfrak{g}=\mathfrak{k}+\mathfrak{q}$.

(3) For any composition factor $\bar{E}$ of E, there exists a finite dimensional simple $\mathfrak{k}$-module V such that $\mathrm{Hom}_{\mathfrak{l}}(V,\bar{E})\neq 0$.

(4) For any composition factor $\bar{E}$ of E, there exists $\lambda\in\mathfrak{a}^*$ such that $\lambda_{\mathfrak{q}}|\mathfrak{l}=0$ and $M(\bar{E}^*\otimes\lambda_{\mathfrak{q}})$ is simple.

Lemma 2.6. *If* (1)-(4) *are satisfied, then* $M'(E)_{(\mathfrak{k})}$ *is a dense* $\mathfrak{g}$*-submodule of* $M'(E)$.

Proof. By [6, 1.7.9], $M'(E)_{(\mathfrak{k})}$ is a $\mathfrak{g}$-submodule. Since

(2.6.1) $$M'(E)_{(\mathfrak{k})}=\mathrm{coind}(E|\mathfrak{q}\to\mathfrak{g})_{(\mathfrak{k})}=\mathrm{coind}(E|\mathfrak{l}\to\mathfrak{k})_{(\mathfrak{k})} \text{ by (2) and [6, 5.5.8]}$$

$$=Y(E)=(U(\mathfrak{k})'\otimes E)^{\mathfrak{l}}=\bigoplus_{V\in\hat{\mathfrak{k}}}(U(\mathfrak{k})^*_V\otimes E)^{\mathfrak{l}},$$

the functor $\mathscr{A}_{\mathfrak{q}}\ni E\to M'(E)_{(\mathfrak{k})}$ is exact. Hence we can reduce the proof to the case where $E\in\hat{\mathfrak{q}}$. By (3) and (2.6.1), $M'(E)_{(\mathfrak{k})}\neq 0$. As $\mathfrak{k}$-modules, we can naturally identify

$$M'(E)=\mathrm{coind}(E|\mathfrak{q}\to\mathfrak{g})=\mathrm{coind}(E|\mathfrak{l}\to\mathfrak{k})$$

$$=\mathrm{coind}(E\otimes(-\lambda)|\mathfrak{l}\to\mathfrak{k})=\mathrm{coind}(E\otimes(-\lambda)|\mathfrak{q}\to\mathfrak{g})=M(E^*\otimes\lambda)^*$$

including topology, for λ as in (4). In particular the $\mathfrak{k}$-module $M'(E)_{(\mathfrak{k})}$ ($\neq 0$) is identified with $M(E^*\otimes\lambda)^*_{(\mathfrak{k})}$, whose closure is $M(E^*\otimes\lambda)^*=M'(E)$ by (4) and (2. 4).

2.7. For $E\in\mathscr{A}_{\mathfrak{q}}$, put $X(E)=\mathrm{coind}(E|\mathfrak{q}\to\mathfrak{g})_{(\mathfrak{k})}=M'(E)_{(\mathfrak{k})}$. Since we can naturally identify $X(E)|\mathfrak{k}$ with $Y(E)=\mathrm{coind}(E|\mathfrak{l}\to\mathfrak{k})_{(\mathfrak{k})}$, we have a natural mapping

$$\mathrm{Hom}^{\#}_{\mathfrak{k}}(Y(F),Y(E))\to\mathrm{Hom}_{\mathfrak{k}}(X(F),X(E))\ (E,F\in\mathscr{A}_{\mathfrak{q}}),$$

whose image we shall denote by $\mathrm{Hom}^{\#}_{\mathfrak{k}}(X(F),X(E))$. Put $\mathrm{Hom}^{\#}_{\mathfrak{g}}=\mathrm{Hom}^{\#}_{\mathfrak{k}}\cap\mathrm{Hom}_{\mathfrak{g}}$.

Lemma 2.8. *Let* $E,F\in\mathscr{A}_{\mathfrak{q}}$, *and assume the conditions* (1)-(4) *of* (2.5) *for E and F. Then we have a natural isomorphism* (= *restriction*)

$$R:\ \mathrm{Home}^{c}_{\mathfrak{g}}(M'(F),M'(E))\to\mathrm{Hom}^{\#}_{\mathfrak{g}}(X(F),X(E)).$$

Proof. The restriction of $\psi'\in\mathrm{Hom}^{c}_{\mathfrak{g}}(M'(F),M'(E))$ to $X(F)=M'(F)_{(\mathfrak{k})}$ gives a $\mathfrak{g}$-homomorphism $X(F)\to X(E)$, which we shall denote by $R(\psi')=\Psi$. By (2.6), $\psi'\to\Psi$ is injective.

Let us determine the image of R. Let $\psi: M(E^*)\to M(F^*)$ be the $\mathfrak{g}$-homomorphism corresponding to $\psi': M'(F)\to M'(E)$ (cf. (2.3)). Let e^* be any element of E^*. By (2.5, (2)),

$$\psi(1\otimes e^*)=\sum_i k_i^{\top}\otimes f_i^* \text{ (finite sum)}$$

with some $k_i\in U(\mathfrak{k})$ and $f_i^*\in F^*$. Let $\psi^*: M(F^*)^*\to M(E^*)^*$ be the dual of ψ. If $m^*\in M(F^*)^*$ corresponds to $m'\in M'(F)$ (cf. (2.2)), then

$$\langle(\psi' m')(k),e^*\rangle=\langle\psi^* m^*,k^{\top}\otimes e^*\rangle=\langle m^*,\psi(k^{\top}\otimes e^*)\rangle$$

$$=\langle m^*,\sum_i k^{\top}k_i^{\top}\otimes f_i^*\rangle=\sum_i\langle m'(k_i k),f_i^*\rangle$$

for any $k\in U(\mathfrak{k})$. Thus the condition (#) of (1.6) is satisfied, and hence the image of R is contained in $\mathrm{Hom}_{\mathfrak{g}}^{\#}(X(F),X(E))$.

Given $\Psi\in\mathrm{Hom}_{\mathfrak{g}}^{\#}(X(F),X(E))$, let us show that Ψ can be extended to a continuous $\mathfrak{g}$-homomorphism $M'(F)\to M'(E)$. For $m'\in M'(F)=M(F^*)^*$, we can find a sequence $\{x_1,x_2,\cdots\}$ in $X(F)$ converging to m', by (2.1,(2)) and (2.6). Since $\langle(\Psi x_p)(k),e^*\rangle=\Sigma_i\langle x_p(k_i k),f_i^*\rangle$, $\lim_{p\to\infty}(\Psi x_p)(k)$ is convergent for any $k\in U(\mathfrak{k})$. Define this limit value to be $(\psi m')(k)$. Then we can show that $\psi\in\mathrm{Hom}_{\mathfrak{g}}^{c}(M'(F),M'(E))$ and $R(\psi)=\Psi$.

§3

The purpose of this section is to prove the duality theorem (3.9). In (3.1)-(3.6), modifying the argument of [6, 9.6.9], we construct an invariant pairing between certain Harish-Chandra modules over $\mathfrak{g}\times\mathfrak{g}$. Without to say, this pairing is an algebraic counterpart of the L^2-inner product on a homogeneous space. Using this pairing and also the results of the previous sections, we prove in (3.7) the duality theorem in the special case where the Lie algebra is of the form $\mathfrak{g}\times\mathfrak{g}$. The general case (3.9) follows from this special case by a simple trick (3.8).

3.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra containing it. Let $\mathfrak{p}$ be a standard parabolic subalgebra, $\mathfrak{l}$ its Levi subalgebra containing $\mathfrak{h}$, and $\mathfrak{p}_-$ the parabolic subalgebra such that $\mathfrak{p}\cap\mathfrak{p}_-=\mathfrak{l}$.

Let $\rho\in\mathfrak{h}^*$ be the half of the sum of the positive roots, W the Weyl group, and w the longest element of the Weyl subgroup corresponding to $\mathfrak{p}$. Then $2\varepsilon:=w\rho+\rho\in\mathfrak{h}^*$ can be extended to a Lie algebra character of $\mathfrak{l}$, and can be regarded as a character of $\mathfrak{p}$ by the projection $\mathfrak{p}\to\mathfrak{l}$. Using [11, 1.17], we can show that $\mathrm{ind}(-2\varepsilon|\mathfrak{p}\to\mathfrak{g})$ is a simple $\mathfrak{g}$-module, whose annihilator we shall denote by J_P. Then J_P is a primitive ideal with the trivial central character.

Let G be the simply connected semisimple algebraic group whose Lie algebra is $\mathfrak{g}$, and, P and L the connected algebraic subgroups of G corresponding to $\mathfrak{p}$ and $\mathfrak{l}$, respectively. Let $D(G/P)$ be the ring of differential operators on G/P, and I_P the kernel of the natural algebra homomorphism $U(\mathfrak{g})\to D(G/P)$. Then

(1) $\qquad U(\mathfrak{g})/J_P=U(\mathfrak{g})I_P \qquad$ by [2, 3.7]

$$=D(G/P) \qquad \text{by [2, 3.8].}$$

Furthermore, as $\mathfrak{g}$-modules

$$D(G/P) \simeq \operatorname{gr} D(G/P) \tag{2}$$

$$=K[T^*(G/P)] \qquad \text{by [2, 1.4]}$$

$$=K[G\times^P \mathfrak{p}^{\perp}] \qquad \text{by [2, 2.4],}$$

where $K[-]$ denotes the regular function ring, $T^*(-)$ the cotangent bundle, and $\mathfrak{p}^{\perp}$ the orthogonal complement in $\mathfrak{g}^*$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form. Let y be a generic element of the center of $\mathfrak{l}$, and $Y := G\times^P (Ky+\mathfrak{p}^{\perp})$. Since the function $Ky+\mathfrak{p}^{\perp}\to Ky\simeq K$ is P-invariant, it induces $f: Y\to K$. Since the natural morphism $Y\to G/P\times K$ is smooth (a fibre bundle with fibre $\mathfrak{p}^{\perp}$), $f: Y\to K$ is also smooth. By the proof of [3, A1],

$$K[G\times^P \mathfrak{p}^{\perp}]=K[f^{-1}(0)]=K[Y]/fK[Y]. \tag{3}$$

(Read infra p. 101, *l*. 14, noting that the assumption '$G_x=P_x$' is not used there. Cf. (3.2) below.) By the proof of [3, 7.6, Behauptung (1)],

$$K[Y]/fK[Y]\simeq K[Y]/(f-1)K[Y] \tag{4}$$

as $\mathfrak{g}$-modules. (Read p. 97,*ll*. 4-7.) Put $K^{\times}=K\backslash\{0\}$ and $Y^{\times}=f^{-1}K^{\times}$. Then we can show that $f: Y^{\times}\to K^{\times}$ is isomorphic to the trivial bundle $f^{-1}(0)\times K^{\times}\to K^{\times}$. Hence

$$K[Y]/(f-1)K[Y]=K[Y^{\times}]/(f-1)K[Y^{\times}]=K[f^{-1}(1)]. \tag{5}$$

Since $\mathfrak{g}_y=\mathfrak{l}$ and G_y is known to be connected [14, 8.5], $G_y=L$. Here $\mathfrak{g}_y$ (resp. G_y) denotes the centralizer of y in $\mathfrak{g}$ (resp. G). Let U be the unipotent radical of P. Then $U_y=1$ and hence $U\cdot y\subset y+\mathfrak{p}^{\perp}$ is dense. By the proposition of [15, 2.5], $U\cdot y$ is closed, and hence $y+\mathfrak{p}^{\perp}=U\cdot y=P\cdot y\simeq P/L$. Thus

$$f^{-1}(1)=G\times^P(y+\mathfrak{p}^{\perp})=G\times^P(P/L)=G/L. \tag{6}$$

Let $\hat{G}$ be the set of isomorphism classes of irreducible rational representations of G. Identify $\hat{G}$ with $\hat{\mathfrak{g}}$ in a natural way. For $\sigma\in\hat{\mathfrak{g}}=\hat{G}$,

$$\operatorname{mtp}_{\mathfrak{g}}(\sigma,K[G/L])=\operatorname{mtp}_G(\sigma,K[G/L])=\operatorname{mtp}_L(0,\sigma|L)=\operatorname{mtp}_{\mathfrak{l}}(0,\sigma|\mathfrak{l}) \tag{7}$$

by an algebraic Frobenius reciprocity theorem. Here $\operatorname{mtp}_{\mathfrak{g}}(\sigma,-)$ etc. denote the multiplicity of $\sigma\in\hat{\mathfrak{g}}$ etc.

Remark 3.2. At the end of the proof of [3, A1], the vanishing theorem of Grauert-Riemenschneider [8, 2.3] is used in the following form. *Let X and $\hat{X}$ be algebraic varieties over a field of characteristic zero, $\pi: \hat{X}\to X$ a proper birational morphism, and $\mathcal{K}$ the sheaf of absolutely regular highest differential forms on $\hat{X}$. Then $R^j\pi_*\mathcal{K}=0$ for $j>0$.* (By the argument of [13, p. 236, *ll*. 14-23], we may assume $\hat{X}$ a smooth projective variety, and X a (normal) projective variety. Now we reproduce the argument of [8, 2.3] in a slightly modified fashion. Let $\mathscr{L}$ be an ample line bundle on X. Put $\hat{\mathscr{K}}(n)=\hat{\mathscr{K}}\otimes\pi^*\mathscr{L}^n$. Consider the spectral sequence $E_2^{ij}(n)$: $=H^i(X,R^j\pi_*\hat{\mathscr{K}}(n))\Rightarrow H^{i+j}(\hat{X},\hat{\mathscr{K}}(n))$. Assume that $R^j\pi_*\hat{\mathscr{K}}=0$ $(0<j<q)$ and $R^q\pi_*\hat{\mathscr{K}}\neq 0$, for some $q>0$. Then $E_2^{ij}(n)=0$ $(0<j<q)$, and by [9, (III, 2.2.2)], $E_2^{q0}(n)=E_2^{q+1,0}(n)=0$ and $E_2^{0q}(n)\neq 0$ *if* n is sufficiently large. But then $0\neq E_2^{0q}(n)=\cdots=E_\infty^{0q}(n)=H^q(\hat{X},\hat{\mathscr{K}}(n))$, which contradicts the vanishing theorem of Kodaira. See [5] for an algebraic proof of the vanishing theorem of Kodaira.)

3.3. Put $\mathfrak{k}=\{(x,x)\in\mathfrak{g}\times\mathfrak{g}\}$ and $\mathfrak{l}_\mathfrak{k}=\{(l,l)\in\mathfrak{l}\times\mathfrak{l}\}$. Identify $\hat{\mathfrak{l}}$ with $\hat{\mathfrak{l}}_\mathfrak{k}$, $\mathfrak{P}$ and $\mathfrak{P}_-$ in a natural way. We also identify $\hat{\mathfrak{k}}$ with $\hat{\mathfrak{g}}$. For $E\in\mathscr{A}_\mathfrak{p}$ and $F\in\mathscr{A}_{\mathfrak{p}-}$, put

$$M(E,F)=\mathrm{ind}(E\otimes F|\mathfrak{p}\times\mathfrak{p}_-\to\mathfrak{g}\times\mathfrak{g}),$$

$$M'(E,F)=\mathrm{coind}(E\otimes F|\mathfrak{p}\times\mathfrak{p}_-\to\mathfrak{g}\times\mathfrak{g}),$$

$$X(E,F)=M'(E,F)_{(\mathfrak{k})}, \text{ and}$$

$$Y(E,F)=\mathrm{coind}(E\otimes F|\mathfrak{l}_\mathfrak{k}\to\mathfrak{k})_{(\mathfrak{k})}.$$

Lemma 3.4. *Consider $U(\mathfrak{g})$ as a $\mathfrak{g}\times\mathfrak{g}$-module by $(x,x')u=xu-ux'$ for $x,x'\in\mathfrak{g}$, $u\in U(\mathfrak{g})$. Then*

$$\mathrm{mtp}_\mathfrak{k}(\sigma,U(\mathfrak{g})/J_P)=\mathrm{mtp}_\mathfrak{l}(0,\sigma|\mathfrak{l})=\mathrm{mtp}_\mathfrak{k}(\sigma,X(2\varepsilon,-2\varepsilon))$$

for any $\sigma\in\hat{\mathfrak{k}}$.

Proof. The first equality follows from (3.1, (1)-(7)). The second equality is proved as follows:

$$\mathrm{mtp}_\mathfrak{k}(\sigma,X(2\varepsilon,-2\varepsilon))$$

$$=\mathrm{mtp}_\mathfrak{k}(\sigma,\mathrm{coind}((2\varepsilon)\otimes(-2\varepsilon)|\mathfrak{p}\times\mathfrak{p}_-\to\mathfrak{g}\times\mathfrak{g}))$$

$$=\mathrm{mtp}_\mathfrak{k}(\sigma,\mathrm{coind}(0|\mathfrak{l}_\mathfrak{k}\to\mathfrak{k})) \qquad \text{by [6, 5.5.8]}$$

$$=\mathrm{mtp}_\mathfrak{g}(\sigma,\mathrm{coind}(0|\mathfrak{l}\to\mathfrak{g}))$$

$$=\mathrm{mtp}_\mathfrak{l}(0,\ \sigma|\mathfrak{l}) \qquad \text{by [6, 5.5.7].}$$

By (3.4), we can follow the same argument as the proof of [6, 9.6.6] to get the following assertion.

Lemma 3.5. *As a $\mathfrak{g}\times\mathfrak{g}$-module, $X(2\varepsilon,-2\varepsilon)$ is naturally isomorphic to $U(\mathfrak{g})/J_P$.*

3.6. Since J_P is contained in the largest primitive ideal of $U(\mathfrak{g})$ with the trivial central character $\mathfrak{g}U(\mathfrak{g})$, we get $\mathfrak{g}\times\mathfrak{g}$-homomorphisms

$$X(E,F)\times X(E^*\otimes(2\varepsilon),\ F^*\otimes(-2\varepsilon))\to X(E\otimes E^*\otimes(2\varepsilon),\ F\otimes F^*\otimes(-2\varepsilon))$$
$$\to X(2\varepsilon,-2\varepsilon)=U(\mathfrak{g})/J_P\to U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}=K.$$

See [6, 5.6.7] for the first arrow. Thus we get a $\mathfrak{g}\times\mathfrak{g}$-invariant pairing $\langle\ ,\ \rangle=\langle\ ,\ \rangle_X$ of $X(E,F)$ and $X(E^*\otimes(2\varepsilon),\ F^*\otimes(-2\varepsilon))$. If we restrict the $\mathfrak{g}\times\mathfrak{g}$-action to $\mathfrak{k}$, then $X(E,F)$ (resp. $X(E^*\otimes(2\varepsilon),\ F^*\otimes(-2\varepsilon))$) are naturally identified with $Y(E,F)$ (resp. $Y(E^*,F^*)$), and the above $\mathfrak{g}\times\mathfrak{g}$-homomorphisms become those of (1.4,(1)). Hence $\langle\ ,\ \rangle_X$ is identified with $\langle\ ,\ \rangle_Y$ given in (1.4), and we can apply the results of §1 to the duality with respect to $\langle\ ,\ \rangle_X$.

Lemma 3.7. *Let $E_1,\ F_1\in\mathscr{A}_{\mathfrak{p}}$ and $E_2,\ F_2\in\mathscr{A}_{\mathfrak{p}_-}$. Assume that for any weights $\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2$ of $E_1,\ E_2,\ F_1,\ F_2$, respectively, $\lambda_1+\lambda_2$ and $\mu_1+\mu_2$ are integral weights. Then we can naturally identify*

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(E_1,E_2),M(F_1,F_2)),$$

$$\mathrm{Hom}^{c}_{\mathfrak{g}\times\mathfrak{g}}(M'(F_1^*,F_2^*),M'(E_1^*,E_2^*)),$$

$$\mathrm{Hom}^{\#}_{\mathfrak{g}\times\mathfrak{g}}(X(F_1^*,F_2^*),X(E_1^*,E_2^*)),$$

$$\mathrm{Hom}^{\#}_{\mathfrak{g}\times\mathfrak{g}}(X(E_1\otimes(2\varepsilon),E_2\otimes(-2\varepsilon)),X(F_1\otimes(2\varepsilon),F_2\otimes(-2\varepsilon))),$$

$$\mathrm{Hom}^{c}_{\mathfrak{g}\times\mathfrak{g}}(M'(E_1\otimes(2\varepsilon),E_2\otimes(-2\varepsilon)),M'(F_1\otimes(2\varepsilon),F_2\otimes(-2\varepsilon))),\qquad \textit{and}$$

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(F_1^*\otimes(-2\varepsilon),F_2^*\otimes(2\varepsilon)),M(E_1^*\otimes(-2\varepsilon),E_2^*\otimes(2\varepsilon))).$$

Proof. Let us show that the assumptions of (1.7) and (2.8) are satisfied. First, consider the condition (2.5,(3)). Let $\bar{E}_1,\ \bar{E}_2,\ \bar{F}_1$, and $\bar{F}_2$ be composition factors of $E_1,\ E_2,\ F_1$, and F_2, respectively, and $\lambda_1,\ \lambda_2,\ \mu_1$, and μ_2 be the respective highest weights. Let V be the finite dimensional simple $\mathfrak{k}$-module which has the extremal weight $\lambda_1+\lambda_2$. Then $\mathrm{Hom}_{\mathfrak{k}}(V,\bar{E}_1\otimes\bar{E}_2)\neq 0$. In this way, we can show that $E_1\otimes E_2$, $E_1^*\otimes E_2^*$, $F_1\otimes F_2$ and $F_1^*\otimes F_2^*$ satisfy (2.5,(3)). Using [11,1.17], we can show that (2.5,(4)) is always satisfied. The remaining conditions are obvious.

Lemma 3.8. (1) *Let $E\in\mathfrak{P}$ and $F\in\mathscr{A}_{\mathfrak{p}_-}$(resp. $E\in\mathscr{A}_{\mathfrak{p}}$ and $F\in\mathfrak{P}_-$), put $M(E)=\mathrm{ind}(E|\mathfrak{p}\to\mathfrak{g})$ and $M_-(F)=\mathrm{ind}(F|\mathfrak{p}_-\to\mathfrak{g})$, and consider $M(E)$ (resp. $M_-(F)$) as a $\mathfrak{g}\times 0$-module (resp. $0\times\mathfrak{g}$-module). Then as as $\mathfrak{g}$-modules, we have natural identifications*

$$\mathrm{Hom}_{\mathfrak{g}\times 0}(M(E),M(E,F))=M_-(F),$$

$$(\textit{resp.}\ \mathrm{Hom}_{0\times\mathfrak{g}}(M_-(F),M(E,F))=M(E)).$$

(2) *For $E_1,\ E_2\in\mathscr{A}_{\mathfrak{p}}$ and $F\in\hat{\mathfrak{P}}_-$,*

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(E_1,F),M(E_2,F))=\mathrm{Hom}_{\mathfrak{g}}(M(E_1),M(E_2)).$$

(3) *For* $E \in \mathfrak{P}$ *and* $F_1, F_2 \in \mathscr{A}_{\mathfrak{p}_-}$,

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(E,F_1),M(E,F_2))=\mathrm{Hom}_{\mathfrak{g}}(M_-(F_1),M_-(F_2)).$$

Proof. (1) Since $M(E,F)=M(E)\otimes M_-(F)$, $m\in M(E)$ gives a $0\times\mathfrak{g}$-homomorphism ϕ_m: $M_-(F)\to M(E,F)$, $m'\to m\otimes m'$. Conversely, assume that a $0\times\mathfrak{g}$-homomorphism ϕ : $M_-(F)\to M(E,F)$ is given and $F\in\mathfrak{P}_-$. Take a highest weight vector f of F. Let $\phi(1\otimes f)=\Sigma_i m_i\otimes m'_i(m_i\in M(E),m'_i\in M_-(F))$ with linearly independent $\{m_i\}$. Considering the $0\times\mathfrak{g}$-action, we can show that every m'_i is proportional to $1\otimes f$. Hence we have $\phi(1\otimes f)=m_\phi\otimes(1\otimes f)$ with a uniquely determined $m_\phi\in M(E)$. Then $\phi(m')=m_\phi\otimes m'$ for any $m'\in M_-(F)$. By $\phi\to m_\phi$ and $m\to\phi_m$, we get the second identification. The first identification can be obtained in the same way.

(2) Naturally, $\phi\in\mathrm{Hom}_{\mathfrak{g}}(M(E_1),M(E_2))$ induces a $\mathfrak{g}\times\mathfrak{g}$-homomorphism $\phi\otimes 1$ between $M(E_i,F)=M(E_i)\otimes M_-(F)$ $(i=1,2)$. Conversely $\psi\in\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(E_1,F),M(E_2,F))$ induces a $\mathfrak{g}$-homomorphism ψ' : $M(E_1)\to M(E_2)$ by (1). By these correspondences, we get the desired identification. (3) can be proved in the same way.

Theorem 3.9. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(M(E),M(F))\simeq\mathrm{Hom}_{\mathfrak{g}}(M(F^*\otimes(-2\varepsilon)),M(E^*\otimes(-2\varepsilon)))$$

for any $E, F\in\mathscr{A}_{\mathfrak{p}}$.

Proof. For $\lambda\in\mathfrak{h}^*$, let $E_{\lambda+Q}$ be the sum of weight spaces $E_{\lambda'}$ such that $\lambda-\lambda'$ belong to the root lattice Q. Then $E_{\lambda+Q}\in\mathscr{A}_{\mathfrak{p}}$, $E=\bigoplus_{\lambda\in\mathfrak{h}^*/Q}E_{\lambda+Q}$ etc., and $\mathrm{Hom}_{\mathfrak{g}}(M(E),M(F))=\bigoplus_{\lambda\in\mathfrak{h}^*/Q}\mathrm{Hom}_{\mathfrak{g}}(M(E_{\lambda+Q}),M(F_{\lambda+Q}))$. Hence we may assume from the beginning that $E=E_{\lambda+Q}$ and $F=F_{\lambda+Q}$. Take $E_0\in\mathfrak{P}_-=\hat{\mathfrak{l}}$ whose highest weight belongs to $\lambda+Q$. Then, by (3.7),

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(E,E_0^*),M(F,E_0^*))\\ &=\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(M(F^*\otimes(-2\varepsilon)),E_0\otimes(2\varepsilon)),M(E^*\otimes(-2\varepsilon),E_0\otimes(2\varepsilon))).\end{aligned}$$

Thus we get the desired result by (3.8).

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The author learned another generalization from M. Duflo [7], in which $\mathfrak{g}$ and $\mathfrak{p}$ are allowed to be almost arbitrary, but Harish-Chandra modules do not appear.

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