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Busemann functions and positive eigenfunctions of Laplacian on noncompact symmetric spaces

By

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Introduction

Let X be a complete simply connected manifold of nonpositive sectional curvature. We can associate each geodesic ray γ in X with the following function $b(\gamma)$:

$$b(\gamma)(x)=\lim_{t\to\infty}\{d(x,\ \gamma(t))-t\} \qquad \text{for } x\in X, \tag{0.1}$$

where d is the distance on X. This is called the Busemann function associated with γ (which was defined in [7]) and is an important object in the study of nonpositively curved manifolds. It is a C^2 convex function and the inverse images $b(\gamma)^{-1}(t)$ ($t\in \mathbf{R}$) are called the horospheres ([13,17]). By investigating such objects, many results, for example, concerning co-finite discrete groups Γ of isometries of X and the structure of the ends of the quotient spaces $\Gamma\backslash X$ were obtained (e.g. [3,10]).

In this paper we point out that the Busemann function has other aspects which do not appear in its geometric definition in the case of symmetric spaces of non-compact type.

Let us consider the case where X has constant sectional curvature -1. In this case, the functions $e^{-b(\gamma)(x)}$ are minimal positive harmonic functions as pointed out in [2]. We can show this fact by direct computation. On the other hand, the author computed the Busemann functions on the symmetric space $SO(n)\backslash SL(n,\mathbf{R})$ in ([15, 16]). The result is as follows. Let $P(n,\mathbf{R})$ be the set of all positive definite symmetric matrices with determinant 1. If we identify $SO(n)\backslash SL(n,\mathbf{R})$ with $P(n,\mathbf{R})$ in the usual manner, the Busemann function $b(\gamma)$ associated with the geodesic ray

$$\gamma(t)=\mathrm{diag}(e^{2t\alpha_1/\|\alpha\|},e^{2t\alpha_2/\|\alpha\|},\cdots,e^{2t\alpha_n/\|\alpha\|})$$

is

$$b(\gamma)(x)=\frac{n}{\|\alpha\|}\log\Bigl(\prod_{i=1}^{n-1}\triangle_i(x)^{\alpha_{i+1}-\alpha_i}\Bigr) \qquad \text{for } x\in P(n,\mathbf{R}), \tag{0.2}$$

where $\alpha=\mathrm{diag}(\alpha_1,\cdots,\alpha_n)$ is an element of the Lie algebra of $SL(n,\mathbf{R})$ such that $\alpha_1<\alpha_2<\cdots<\alpha_n$, $\|\alpha\|$ its norm with respect to the metric induced from the Killing form,

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and $\triangle_i(x)$ the $(i\times i)$-minor determinant of x in the top left corner. What does the product $\prod_{i=1}^{n-1}\triangle_i(x)^{\alpha_{i+1}-\alpha_i}$ of minor determinants in (0.2) mean? Roughly speaking, it is also a minimal positive eigenfunction of Laplace-Beltrami operator Δ on $SO(n)\backslash SL(n,\mathbf{R})$ as we show in the sequel.

Let G be a connected semi-simple Lie group having finite center and no compact factors. Let K be a maximal compact subgroup of G and $X=G/K$ the associated symmetric space of noncompact type. We denote by x_0 the coset of identity element $e\in G$. We suppose that the metric of X is induced from some constant multiple of the Killing form B of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of K, and $\langle,\rangle$ the inner product on $\mathfrak{p}$ induced from the Riemannian metric on the tangent space $T_{x_0}(X)$ of X at x_0. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by A the analytic subgroup corresponding to $\mathfrak{a}$. Let Σ be the system of roots of the pair $(\mathfrak{g},\mathfrak{a})$. For each root $\theta\in\Sigma$, we choose the unique element H_θ of $\mathfrak{a}$ such that $\langle H_\theta,H\rangle=\theta(H)$ for all $H\in\mathfrak{a}$. Let Σ^+ be the system of positive roots determined by some ordering on $\mathfrak{a}$. We put

$$\mathfrak{a}^+=\{\alpha\in\mathfrak{a}\,|\,\theta(\alpha)\geqq 0 \text{ for all } \theta\in\Sigma^+\},$$

and

$$\rho=\frac{1}{2}\sum_{\theta\in\Sigma^+}H_\theta,$$

where in the sum every root occurs a number of times equal to its multiplicity. Let $\Delta=\mathrm{div}\circ\mathrm{grad}$ be the Laplace-Beltrami operator on X. Two geodesic rays γ_1, γ_2 in X are said to be asymptotic if $d(\gamma_1(t),\gamma_2(t))$ is uniformly bounded on $[0,\infty)$ (see §1.2).

Theorem A (Theorem 2.5). *Let $f: X\longrightarrow\mathbf{R}$ be the function defined by*

$$f(x)=e^{Cb(\gamma)(x)} \qquad \textit{for } x\in X,$$

where $\gamma:[0,\infty)\longrightarrow X$ is a geodesic ray and C is an arbitrary real number. Then f is an eigenfunction of Δ and the eigenvalue is given as follows: Let $\gamma':[0,\infty)\longrightarrow X$ be the geodesic ray emanating from x_0 which is asymptotic to γ and is written as $\gamma'(t)=k\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$, where $\alpha\in\mathfrak{a}^+-\{0\}$ and $k\in K$. Then the eigenvalue is

$$C\left(C+2\langle\rho,\frac{\alpha}{\|\alpha\|}\rangle\right), \tag{0.3}$$

where $\|\alpha\|=\sqrt{\langle\alpha,\alpha\rangle}$.

When $C>0$, the function $f=e^{Cb(k\gamma)}$ is something like a distance function from the point $k\gamma(\infty)$ at infinity (for precise definition of the points at infinity, see §1.2). So informally, on symmetric spaces of noncompact type, the inverse of the distance function from each point at infinity is harmonic.

Consider the equation

(0.4) $$\Delta f = cf$$

for an arbitrary fixed number $c \geqq -\|\rho\|^2$. The minimal positive solutions of this equation was studied extensively by Karpelevič ([18]). Let $X(\infty)$ be the boundary of the Eberline-O'Neil compactification of X and

(0.5) $$\gamma_0(t) = \left(\exp t \frac{\rho}{\|\rho\|}\right) \cdot x_0 \qquad \text{for } t \geqq 0.$$

Karpelevič called the G-orbit Ξ of $\xi_0 = \gamma_0(\infty) \in X(\infty)$ "the skeleton of the boundary of X" (see §2 for more precise description). And he constructed a family of functions $p(\cdot,\xi,\lambda): X \longrightarrow \mathbf{R}$ parametrized by $(\xi,\lambda) \in \Xi \times \mathfrak{a}$, which are positive solutions of $\Delta f = (\|\lambda\|^2 - \|\rho\|^2) f$, as follows. Let

(0.6) $$p(x,\xi_0,\lambda) = e^{\langle \rho+\lambda, H\rangle} \qquad \text{for } x = ne^H \cdot x_0,$$

where $G = NAK$ is the Iwasawa decomposition. Since K acts transitively on Ξ, for any $\xi = k\xi_0 \in \Xi$, $k \in K$, the function $p(x,\xi,\lambda)$ is defined by

(0.7) $$p(x,\xi,\lambda) = p(k^{-1}x,\xi_0,\lambda) \qquad \text{for } x \in X.$$

Theorem B (Theorem 2.6). *Let $k \in K$, $\xi = k\xi_0 \in \Xi$, $\mu \in \mathfrak{a}^+$. Then we have*

$$p(x,\xi,\mu) = e^{-\|\rho+\mu\| b(k\gamma)(x)} \qquad \textit{for all } x \in X,$$

where γ is the geodesic ray defined by

$$\gamma(t) = \left(\exp t \frac{\rho+\mu}{\|\rho+\mu\|}\right) \cdot x_0 \qquad \textit{for } t \geqq 0.$$

By combining this with the Karpelevič's result, we can describe the set $\mathcal{M}_c$ of all minimal positive solutions $f(x)$ of the equation (0.4) such that $f(x_0) = 1$ as follows.

Corollary C (Corollary 2.8).

$$\mathcal{M}_c = \{e^{-\|\rho+\mu\| b(k\gamma_{\rho+\mu})} \mid k \in K, \mu \in \mathfrak{a}_c^+\},$$

where $\mathfrak{a}_c^+ = \{\alpha \in \mathfrak{a}^+ \mid \|\alpha\|^2 = c + \|\rho\|^2\}$ and $\gamma_{\rho+\mu}(t) = \left(\exp t \frac{\rho+\mu}{\|\rho+\mu\|}\right) \cdot x_0$.

Let $\Phi: \mathcal{M}_c \longrightarrow X(\infty)$ be the map which sends each $e^{Cb(k\gamma)} \in \mathcal{M}_c$ to the point $k\gamma(\infty) \in X(\infty)$. We regard $X(\infty)$ as a geometric realization $|T|$ of the spherical Tits building of G (see §4 for more details) and obtain the following (see §1 for the definition of the Martin and the sphere topology).

Theorem D (Theorem 4.1). *The map Φ is a homeomorphism from $\mathcal{M}_c$ with the Martin topology to its image $\Phi(\mathcal{M}_c)$ with the induced topology from the sphere topology on $X(\infty)$. If $l = \text{rank } X \geqq 2$, there exists an open neighborhood W (with respect to the sphere topology) of the $(l-2)$-skeleton of $|T|$ such that $\Phi(\mathcal{M}_c) = X(\infty) - W$.*

In this way Busemann function $b(\gamma)$ leads to other analytically or algebraically defined functions when we consider its exponential functions $e^{Cb(\gamma)}$.

There are closely related descriptions in the book "Compactifications of symmetric spaces (Progress in Math. 156, Birkhäuser (1998)) written by Y. Guivarch, L. Ji, and J.C. Taylor. So, such relations have been known to the authors of this book and the persons concerned. Our paper was first written in 1996, and is independent of their work. The point of view in this paper is different from theirs.

This paper is organized as follows. In §1 we fix notation and recall basic definitions. In §2 we show that the exponential function of any Busemann function is an eigenfunction of the Laplacian Δ and clarify its relation to the Karpelevič's functions $p(x,\xi,\lambda)$. In §3 and §4, we give some applications of Theorems A, B. In §3, we consider the case where the rank of X is equal to 1 and describe the Poisson kernels in terms of the Busemann functions. In §4 we return to symmetric spaces of general rank and try to compare the minimal Martin boundary $\mathcal{M}_c$ with the Tits geometry of $X(\infty)$ through the map Φ. We also compute positive eigenfunctions on some symmetric spaces by using Theorem A and explain how to compute them on general symmetric spaces in §5.

§1. Preliminaries

1.1. Let G be a connected semi-simple Lie group having finite center and no compact factors. Let K be a maximal compact subgroup of G and $X=G/K$ the associated symmetric space of noncompact type. We suppose that X is m-dimensional, and denote by x_0 the coset of the identity element $e\in G$. We suppose that the metric of X is induced from a constant multiple of the Killing form B of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. We denote by $\langle , \rangle$ the inner product on $\mathfrak{p}$ induced from the Riemannian metric on the tangent space $T_{x_0}(X)$ of X at x_0. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and denote by A the corresponding analytic subgroup of G. Let Σ be the root system of the pair $(\mathfrak{g},\mathfrak{a})$. For each root $\theta\in\Sigma$, we choose the unique element H_θ of $\mathfrak{a}$ such that

$$\langle H_\theta, H\rangle=\theta(H) \qquad \text{for all } H\in\mathfrak{a},$$

and denote by $\mathfrak{g}^\theta$ the root space $\{Y\in\mathfrak{g}\,|\,[H,Y]=\theta(H)Y \text{ for all } H\in\mathfrak{a}\}$. We introduce a lexicographic order on $\mathfrak{a}$ and denote by Σ^+, Υ the system of positive, simple roots with respect to this order. We denote by l the cardinality of Υ, which is equal to the rank of the symmetric space X. Let

$$\mathfrak{a}^+=\{\alpha\in\mathfrak{a}\,|\,\theta(\alpha)\geqq 0 \text{ for all } \theta\in\Upsilon\}$$

the closure of the Weyl chamber $Int\ \mathfrak{a}^+=\{\alpha\in\mathfrak{a}\,|\,\theta(\alpha)>0 \text{ for all } \theta\in\Upsilon\}$. We put

$$\rho=\frac{1}{2}\sum_{\theta\in\Sigma^+}\mathrm{H}_\theta,$$

where every root occurs a number of times equal to its multiplicity in this sum. Let

$\mathfrak{n}=\sum_{\theta\in\Sigma^+}\mathfrak{g}^\theta$ be the Lie subalgebra of $\mathfrak{g}$ and N the analytic subgroup corresponding to $\mathfrak{n}$. Let M be the centralizer of A in K. The group $P=MAN$ is a minimal parabolic subgroup of G.

For each subset Θ of Υ, we denote by $\langle\Theta\rangle$ the set of roots which are linear combinations of elements in Θ, and put $\langle\Theta\rangle^+=\langle\Theta\rangle\cap\Sigma^+$. We denote by $\mathfrak{a}_\Theta$ (resp. $\mathfrak{n}_\Theta$) the Lie subalgebra of $\mathfrak{a}$ (resp. $\mathfrak{n}$) defined by

$$\mathfrak{a}_\Theta=\bigcap_{\theta\in\Theta}\ker\theta \quad (\text{resp. } \mathfrak{n}_\Theta=\sum_{\theta\in\Sigma^+-\langle\Theta\rangle^+}\mathfrak{g}^\theta),$$

and A_Θ (resp. N_Θ) the analytic subgroup of G corresponding to $\mathfrak{a}_\Theta$ (resp. $\mathfrak{n}_\Theta$). Let L_Θ be the centralizer of $\mathfrak{a}_\Theta$ in G and $P_\Theta=L_\Theta N_\Theta$. Note that $P_\phi=P$ and $P_\Upsilon=G$. P_Θ's are called the standard parabolic subgroups of G, and each proper parabolic subgroup Q is conjugate by some element of G to one of P_Θ's with $\Theta\neq\Upsilon$ (cf. [5,11]).

1.2. We recall the definition of the Eberlein-O'Neil compactification (cf. [3, 13]). This compactification can be defined for any complete, simply connected Riemannian manifold of nonpositive sectional curvature. But here X is a symmetric space of noncompact type as in §1.1. By the Cartan-Hadamard theorem, every unit speed geodesic $\gamma:[0,\infty)\longrightarrow X$ is a ray: $d(\gamma(t),\gamma(s))=|t-s|, t,s\geqq 0$. Two geodesic rays γ_1, γ_2 are called asymptotic if $d(\gamma_1(t),\gamma_2(t))$ is uniformly bounded on $[0,\infty)$. Being asymptotic is an equivalence relation. One define the sphere at infinity $X(\infty)$ of X to be the set of asymptote classes of geodesic rays in X. The equivalence class represented by a geodesic γ is denoted by $\gamma(\infty)$. A natural topology, the cone topology, on $\bar{X}=X\cup X(\infty)$ is defined as follows: For $v\in T_{x_0}(X)$, $\delta>0$, and $R>0$, let

$$C_{x_0}(v,\delta)=\{x\in X\,|\,\angle_{x_0}(v,T_{\overline{x_0x}})<\delta\},$$

$$T_{x_0}(v,\delta,R)=C_{x_0}(v,\delta)-\bar{B}_{x_0}(R),$$

where $T_{\overline{x_0x}}$ is the initial velocity of the geodesic ray through x emanating from x_0, and $\bar{B}_{x_0}(R)=\{x\in X\,|\,d(x_0,x)\leqq R\}$. Then the domains $T_{x_0}(v,\delta,R)$ together with the geodesic balls $B_x(r)$, $x\in X$ form a local basis for the cone topology. The induced topology on $X(\infty)$ is also called the "sphere topology". The set $ST_{x_0}(X)$ of all unit tangent vectors at x_0 is naturally identified with $X(\infty)$ by assigning each vector $v\in ST_{x_0}(X)$ the equivalence class of the ray $\gamma(t)=(\exp tv)\cdot x_0$. This map also gives a homeomorphism between the $(m-1)$-dimensional sphere $ST_{x_0}(X)$ and $X(\infty)$ equipped with the sphere topology. The cone topology is independent of the choice of the base point.

1.3. Let us consider the following equation on X:

$$\Delta f=cf, \tag{1.1}$$

where $\Delta=\operatorname{div}\circ\operatorname{grad}$ is the Laplace-Beltrami operator on X. If $c<-\|\rho\|^2$, (1.1) has no positive solution ([18, Theorem 17.1.1]). So we suppose $c\geqq-\|\rho\|^2$. We recall that a positive solution $f(x)$ of (1.1) is called minimal if every positive solution of this equation, not exceeding $f(x)$, differs from $f(x)$ only by a positive factor. The set $\mathcal{M}_c$ of minimal solutions $f(x)$ of (1.1) such that $f(x_0)=1$ is called the minimal Martin boundary ([19]). We can introduce a distance d_c on $\mathcal{M}_c$ by

$$d_c(f,f')=\int_{B_{x_0}(1)}\frac{|f(x)-f'(x)|}{1+|f(x)-f'(x)|}\qquad \text{for } f,f'\in\mathcal{M}_c.$$

We call the induced topology on $\mathcal{M}_c$ the Martin topology.

§2. Busemann functions

We associate each vector $\alpha\in\mathfrak{a}^+-\{0\}$ with the Busemann function $b(\gamma)$ with respect to the geodesic γ with initial velocity $\alpha/\|\alpha\|$. Then X is foliated by the horospheres $b(\gamma)^{-1}(t)$, $t\in\mathbf{R}$ ([13]). So we can take a natural (global) coordinate $(y_1,\cdots,y_m)$ such that y_1 is the signed distance from x_0 along γ and $(y_2,\cdots,y_m)$ corresponds to the coordinate of the horosphere $b(\gamma)^{-1}(-y_1)$. We express the Laplace-Beltrami operator Δ in terms of this coordinate and consider the relation between Busemann functions on X and positive eigenfunctions of Δ.

Lemma 2.1. *Let $\gamma_i:[0,\infty]\longrightarrow X$ $(i=1,2,\cdots)$ be a sequence of unit speed geodesics emanating from x_0 which converges to a unit speed geodesic γ. Then the sequence of Busemann functions $b(\gamma_i)$ converges to $b(\gamma)$ uniformly on every compact subset of X.*

Proof. It suffices to prove that the sequence $\{b(\gamma_i)\}$ converges to $b(\gamma)$ uniformly on the geodesic ball $B_{x_0}(R)$ centered at x_0 for any positive number R. The following argument is an improvement of the one in Lemma 2-3 of [16].

We fix an arbitrary point $x\in B_{x_0}(R)$. Let

$$b_s(\gamma)(x)=d(x,\gamma(s))-s\qquad \text{for } s>0.$$

For each positive integer j, we put $\delta_j=\angle_{\gamma(j)}(x_0,x)$ and $l_j=d(x,\gamma(j))$. Since the sectional curvatures of X are nonpositive, from the Rauch comparison theorem ([8]), we have

$$\begin{aligned}0\leqq b_j(\gamma)(x)-b_{j+s}(\gamma)(x)&=d(x,\gamma(j))-d(x,\gamma(j+s))+s\\ &\leqq l_j-\sqrt{l_j^2+s^2+2l_js\cos\delta_j}+s.\end{aligned}$$

Hence,

$$0\leqq b_j(\gamma)(x)-b(\gamma)(x)\leqq l_j(1-\cos\delta_j).$$

Again by the comparison theorem, we have $\cos\delta_j\geqq\dfrac{j^2+l_j^2-l_0^2}{2jl_j}$, where $l_0=d(x_0,x)$. So we obtain

$$0 \leqq b_j(\gamma)(x) - b(\gamma)(x) \leqq l_j\left(1 - \frac{j^2 + l_j^2 - l_0^2}{2jl_j}\right)$$

(2.1)

$$= \frac{l_0^2 - (l_j - j)^2}{2j} \leqq \frac{l_0^2}{2j} = \frac{1}{2j}\{d(x_0, x)\}^2 \leqq \frac{R^2}{2j}.$$

Similarly, for each positive integer j, we put

$$b_j(\gamma_i)(x) = d(x, \gamma_i(j)) - j.$$

Then we have

$$0 \leqq b_j(\gamma_i)(x) - b(\gamma_i)(x) \leqq \frac{R^2}{2j}. \tag{2.2}$$

For any $\varepsilon > 0$, we take and fix one positive integer j_0 such that $j_0 > \frac{2R^2}{\varepsilon}$. Then, from (2.1), (2.2), we have

$$\begin{cases} 0 \leqq b_{j_0}(\gamma_i)(x) - b(\gamma_i)(x) < \frac{\varepsilon}{4}, \\ 0 \leqq b_{j_0}(\gamma)(x) - h(\gamma)(x) < \frac{\varepsilon}{4}. \end{cases} \tag{2.3}$$

For this j_0, we can take a positive integer I so as the following holds.

$$\text{If } i \geqq I, \text{ then } d(\gamma_i(j_0), \gamma(j_0)) < \frac{\varepsilon}{4}. \tag{2.4}$$

Note that

$$|b_{j_0}(\gamma_i)(x) - b_{j_0}(\gamma)(x)| = |d(x, \gamma_i(j_0)) - d(x, \gamma(j_0))| \leqq d(\gamma_i(j_0), \gamma(j_0)).$$

So, from (2.3) and (2.4), we obtain

$$\begin{aligned} |b(\gamma_i)(x) - b(\gamma)(x)| &\leqq |b_{j_0}(\gamma_i)(x) - b(\gamma_i)(x)| + |b_{j_0}(\gamma_i)(x) - b_{j_0}(\gamma)(x)| \\ &\quad + |b_{j_0}(\gamma)(x) - b(\gamma)(x)| < \frac{3\varepsilon}{4}. \end{aligned}$$

Since x is an arbitrary point of $B_{x_0}(R)$, the convergence is uniform on $B_{x_0}(R)$.

Lemma 2.2. *Let* $\gamma(t) = \left(\exp t\frac{\alpha}{\|\alpha\|}\right) \cdot x_0$ *be a unit speed geodesic, where* $\alpha \in \mathfrak{a} - \{0\}$. *If* $\alpha \in \mathfrak{a}^+$, *then the Busemann function* $b(\gamma)$ *is invariant under the action of* N *on* X.

Proof. Lemma 2.1 implies that it suffices to show the following equality in the case where α lies in the Weyl chamber

$$Int\ \mathfrak{a}^+ = \{\alpha \in \mathfrak{a} \mid \theta(\alpha) > 0 \text{ for all } \theta \in \Upsilon\}.$$

$$(2.5) \qquad b(\gamma)(h\cdot x)=b(\gamma)(x) \qquad \text{for all } x\in X,\ h\in N$$

We note that any element h of the nilpotent Lie group N can be written as

$$h=e^{Y};\ Y=\sum_{\theta\in\Sigma^+} Y_\theta;\ Y_\theta\in\mathfrak{g}^\theta.$$

Let $\beta=\alpha/\|\alpha\|$. Since $Ad(e^{-t\beta})Y_\theta=e^{-t\theta(\beta)}Y_\theta$ and $\theta(\beta)>0$, we have

$$\lim_{t\to\infty} Ad(e^{-t\beta})Y=0 \text{ and } \lim_{t\to\infty} e^{-t\beta}h^{-1}e^{t\beta}=\lim_{t\to\infty}\exp(-Ad(e^{-t\beta})Y)=e.$$

Hence

$$\lim_{t\to\infty} d(\gamma(t),h^{-1}\gamma(t))=\lim_{t\to\infty} d(x_0,e^{-t\beta}h^{-1}e^{t\beta})=0$$

and

$$|b(\gamma)(h\cdot x)-b(\gamma)(x)|=|\lim_{t\to\infty}\{d(h\cdot x,\gamma(t))-d(x,\gamma(t))\}|$$

$$=|\lim_{t\to\infty}\{d(x,h^{-1}\gamma(t))-d(x,\gamma(t))\}|\leqq\lim_{t\to\infty} d(\gamma(t),h^{-1}\gamma(t))=0.$$

This implies the equality (2.5).

Lemma 2.3. *Let* $\alpha\in\mathfrak{a}-\{0\}$ *and* $\gamma(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$. *The restriction of* $b(\gamma)$ *to the submanifold* $A\cdot x_0$ *is given by*

$$b(\gamma)(e^H\cdot x_0)=-\langle H,\frac{\alpha}{\|\alpha\|}\rangle$$

for $H\in\mathfrak{a}$.

Proof. We remark that $A\cdot x_0$ is a totally geodesic submanifold of X and isometric to the Euclidean space $\mathbf{R}^l$. We take an orthonormal basis $\mathbf{v}_1,\cdots,\mathbf{v}_l$ of $\mathfrak{a}$ such that $\mathbf{v}_1=\alpha/\|\alpha\|$. And let $(t_1,\cdots,t_l)$ be the coordinate of $\mathfrak{a}$ with respect to this basis. We can regard this coordinate as a global coordinate of the submanifold $A\cdot x_0$ under the diffeomorphism $\mathfrak{a}\xrightarrow{\cong}A\cdot x_0$. Then, for $H=\sum_{i=1}^l t_i\mathbf{v}_i$, we obtain

$$d(e^H\cdot x_0,\gamma(s))-s=d_{\mathbf{R}^l}((t_1,\cdots,t_l),(s,0,\cdots,0))-s$$

$$=\left\{(t_1-s)^2+\sum_{i=2}^l(t_i)^2\right\}^{1/2}-s=\frac{\sum_{i=1}^l(t_i)^2-2t_1\cdot s}{\sqrt{(t_1-s)^2+\sum_{i=2}^l(t_i)^2}}.$$

Hence

$$b(\gamma)(e^H\cdot x_0)=-t_1=-\langle H,\mathbf{v}_1\rangle=-\langle H,\frac{\alpha}{\|\alpha\|}\rangle.$$

Let $G=NAK$ be the Iwasawa decomposition. Then $X=NA\cdot x_0$ is diffeomorphic to NA. If the initial velocity v of geodesic ray γ belongs to $\mathfrak{a}^+$, from Lemmas 2.2, 2.3, we have

$$b(\gamma)(ne^H\cdot x_0)=-\langle H,v\rangle \qquad \text{for } n\in N,\ H\in\mathfrak{a}.$$

Since $b(h\gamma)(x)=b(\gamma)(h^{-1}\cdot x)$ for all $h\in G$, $x\in X$, *every Busemann function is C^∞ on symmetric spaces of noncompact type. (*This fact should be known. But we cannot find its proof in literature.) Let $\mathfrak{a}^\perp$ be the orthogonal complement of the $\mathbf{R}$-span of α in $\mathfrak{a}$. We denote by $A^\perp$ the analytic subgroup of G corresponding to $\mathfrak{a}^\perp$. From lemmas 2.2, 2.3, we have

Corollary 2.4. *Under the assumptions of Lemma 2.2, $b(\gamma)$ is $NA^\perp$-invariant. In particular,*

$$b(\gamma)^{-1}(-s)=NA^\perp e^{s\alpha/\|\alpha\|}\cdot x_0.$$

In what follows we take a certain global coordinate of X. Let $\alpha\in\mathfrak{a}^+-\{0\}$. The exponential map $\exp:\mathfrak{n}\longrightarrow N$ is a diffeomorphism. We identify $\mathfrak{n}$ with $\mathbf{R}^{m-l}$ and denote by φ_1 the diffeomorphism $N\longrightarrow\mathbf{R}^{m-l}$ induced from the exponential map. We take an orthonormal basis $\{\mathbf{v}_1,\cdots,\mathbf{v}_l\}$ of $\mathfrak{a}$ such that $\mathbf{v}_1=\alpha/\|\alpha\|$. We define a diffeomorphism φ_2: $\mathbf{R}^{m-l}\times\mathbf{R}^l\longrightarrow X$ by

$$\varphi_2((y_1,\cdots,y_{m-l}),(t_1,\cdots,t_l))=\varphi_1^{-1}(y_1,\cdots,y_{m-l})\cdot\exp\left(\sum_{i=1}^{l} t_i\mathbf{v}_i\right)\cdot x_0$$

for $(y_1,\cdots,y_{m-l})\in\mathbf{R}^{m-l}$, $(t_1,\cdots,t_l)\in\mathbf{R}^l$, and put $\varphi=\varphi_2^{-1}$. The ambiguity of the choice of $\mathbf{v}_2,\cdots,\mathbf{v}_l$ does not affect the proceeding computation. So we call $\varphi: X\longrightarrow\mathbf{R}^{m-l}\times\mathbf{R}^l$ the global coordinate of X determined by α. (When α is in the Weyl chamber, this corresponds to the "orispherical coordinate" in [18] and the calculation (2.8) corresponds to [18,(9.9.2)].)

Let

$$\sum_{i,j=1}^{m} g_{ij}dy_i\otimes dy_j$$

be the metric tensor of X in terms of the above coordinate. In this expression, we have written as $y_{m-l+i}=t_i$ $(i=1,\cdots,l)$ for the sake of convenience. Let us compute

$$\sqrt{g}=\sqrt{\det(g_{ij})_{1\leq i,j\leq m}}.$$

First, we define a function $\psi_N:\mathbf{R}^{m-l}\longrightarrow\mathbf{R}$ so that the volume form (with respect to the induced metric from X) of the submanifold $N\cdot x_0$ of X is given by

$$\psi_N(y_1,\cdots,y_{m-l})dy_1\wedge\cdots\wedge dy_{m-l}.$$

That is,

$$\psi_N(y_1,\cdots,y_{m-l})=\sqrt{\det(g_{ij}(x'))_{1\leq i,j\leq m-l}},$$

where $x'=\varphi^{-1}((y_1,\cdots,y_{m-l}),(0,\cdots,0))$. Next, we compute

$$\sqrt{\det(g_{ij}(x))_{1\leq i,j\leq m-l}}$$

at the point $x=\varphi^{-1}((y_1,\cdots,y_{m-l}),(t_1,\cdots,t_l))$. We put $H=\sum_{i=1}^{l}t_i\mathbf{v}_i$, $a=e^H$.

We remark that for $Y_\theta \in \mathfrak{g}^\theta$, we have

$$d(e^{tY_\theta}a\cdot x_0, a\cdot x_0)=d(a^{-1}e^{tY_\theta}a\cdot x_0, x_0)$$

(2.6)

$$=d(\exp(te^{-adH}(Y_\theta))\cdot x_0, x_0)=d(\exp(te^{-\theta(H)}Y_\theta)\cdot x_0, x_0).$$

Let τ be the Cartan involution of $\mathfrak{g}$: i.e. $\tau|_{\mathfrak{k}}=id._{\mathfrak{k}}$, $\tau|_{\mathfrak{p}}=-id._{\mathfrak{p}}$. We decompose $Y\in \mathfrak{n}$ as

$$Y=Y_1+Y_2;\ Y_1=\frac{1}{2}(Y+\tau Y)\in\mathfrak{k},\ Y_2=\frac{1}{2}(Y-\tau Y)\in\mathfrak{p}.$$

Then

$$e^{tY}\cdot x_0=\exp\left(tY_2+\frac{t^2}{2}[Y_1, Y_2]+O(t^2)\right)\cdot x_0,$$

and the initial velocity of the curve $t\longmapsto e^{tY}\cdot x_0$ is $Y_2=\frac{1}{2}(Y-\tau Y)$. Hence, from (2.6), we have

$$\sqrt{\det(g_{ij}(x))_{1\leq i,j\leq m-l}}=\psi_N(y_1,\cdots,y_{m-l})\prod_{\theta\in\Sigma^+}e^{-\theta(H)}$$

(2.7)

$$=\psi_N(y_1,\cdots,y_{m-l})\prod_{\theta\in\Sigma^+}e^{-\langle H_\theta,H\rangle}=\psi_N(y_1,\cdots,y_{m-l})e^{-2\langle\rho,H\rangle},$$

where $e^{-\theta(H)}$ and $e^{-\langle H_\theta,H\rangle}$ appear in the product the same times as the multiplicity of the root θ. Let $\rho=\sum_{i=1}^{l}\rho_i\mathbf{v}_i$. Then $\langle\rho,H\rangle=\sum_{i=1}^{l}\rho_i t_i$. We define a function $\psi_{A^\perp}$: $\mathbf{R}^{l-1}\longrightarrow\mathbf{R}$ by

$$\psi_{A^\perp}(t_2,\cdots,t_l)=\exp\left(-2\sum_{i=2}^{l}\rho_i t_i\right).$$

We remark that $\left(\frac{\partial}{\partial t_i}\right)_x$ and $\left(\frac{\partial}{\partial y_j}\right)_x$ are orthogonal to each other for $i=1,\cdots,l$ and $j=1,\cdots,m-l$. From (2.7), we have at the point x,

(2.8) $$\sqrt{g}=\psi_N(y_1,\cdots,y_{m-l})\psi_{A^\perp}(t_2,\cdots,t_l)\mathrm{e}^{-2\rho_1 t_1}.$$

We now relate the Busemann functions to eigenfunctions of Laplacian.

Theorem 2.5. *Let f : $X\longrightarrow\mathbf{R}$ be the function defined by*

$$f(x)=\exp(Cb(\gamma)(x))\qquad \text{for } x\in X,$$

where γ: $[0,\infty)\longrightarrow X$ is a unit speed geodesic and C is an arbitrary real number. Then f is an eigenfunction of Δ. The eigenvalue is given as follows: Let γ': $[0, \infty)\longrightarrow X$ be the unit speed geodesic emanating from x_0 which is asymptotic to γ and is written as $\gamma'(t)=k\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$, where $\alpha\in\mathfrak{a}^+-\{0\}$ and $k\in K$. Then the *eigenvalue is* $C\left(C+2\langle\rho,\frac{\alpha}{\|\alpha\|}\rangle\right)$.

Proof. Since $b(\gamma)=b(\gamma')+c$ for some real number c ([3]), we have $e^{Cb(\gamma)}=e^{cC}$ $e^{Cb(\gamma')}$. So we can assume that $\gamma=\gamma'$. And it suffices to show the equation

$$\Delta f=C\left(C+2\langle\rho,\frac{\alpha}{\|\alpha\|}\rangle\right)f \tag{2.9}$$

in the case where $k=e$, because $b(k\gamma)(x)=b(\gamma)(k^{-1}x)$ for any $k\in K$. We use the (global) coordinate determined by α. The Laplace-Beltrami operator Δ is expressed in terms of $(y_1,\cdots,y_m)$ by

$$\Delta=\frac{1}{\sqrt{g}}\sum_{i,j=1}^{m}\frac{\partial}{\partial y_i}\left(\sqrt{g}\,g^{ij}\frac{\partial}{\partial y_j}\right),$$

where $(g^{ij})=(g_{ij})^{-1}$ and $y_{m-l+i}=t_i$ for $i=1,\cdots,l$. From Lemma 2.3 and Corollary 2. 4, we have

$$f((y_1,\cdots,y_{m-l}),(t_1,\cdots,t_l))=e^{-Ct_1}. \tag{2.10}$$

Notice that

$$g\left(\frac{\partial}{\partial t_i},\frac{\partial}{\partial y_j}\right)=0 \qquad \text{for } i=1,\cdots,l\ ;\ j=1,\cdots,m-l,$$

$$g\left(\frac{\partial}{\partial t_i},\frac{\partial}{\partial t_j}\right)=0 \qquad \text{for } i\neq j,$$

and $g\left(\frac{\partial}{\partial t_i},\frac{\partial}{\partial t_i}\right)=1$ for $i=1,\cdots,l$. Hence, from (2.8) and (2.10), we obtain

$$\Delta f=\frac{1}{\sqrt{g}}\frac{\partial}{\partial t_1}\left(\sqrt{g}\frac{\partial f}{\partial t_1}\right)=\frac{1}{\sqrt{g}}\frac{\partial}{\partial t_1}(-\sqrt{g}\,Ce^{-Ct_1})$$

$$=-C\frac{1}{\sqrt{g}}\frac{\partial}{\partial t_1}\{\psi_N(y_1,\cdots,y_{m-l})\psi_{A^\perp}(t_1,\cdots,t_l)e^{-(C+2\rho_1)t_1}\}$$

$$=C(C+2\rho_1)e^{-Ct_1}=C\left(C+2\langle\rho,\frac{\alpha}{\|\alpha\|}\rangle\right)f.$$

Let us consider the relation between the Karpelevič's functions $p(x,\xi,\lambda)$ ([18,§ 17]) and the functions $e^{Cb(k\gamma)}$. Notice that $\rho\in Int\ \mathfrak{a}^+$. And let

$$\gamma_0(t)=\left(\exp t\frac{\rho}{\|\rho\|}\right)\cdot x_0 \qquad \text{for } t\geqq 0. \tag{2.11}$$

Karpelevič called the G-orbit Ξ of $\xi_0=\gamma_0(\infty)\in X(\infty)$ "the skeleton of the boundary of X" ([18,§14]). (More precisely, he called each set consisting of all mutually asymptotic geodesics a finite bundle. And for $\alpha\in Int\ \mathfrak{a}^+$, he denoted by ξ_0 the point on the boundary of the Karpelevič compactification determined by the finite bundle containing the geodesic ray $\gamma(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$. The point ξ_0 is independent of the choice of $\alpha\in Int\ \mathfrak{a}^+$. He defined Ξ to be the G-orbit of this point ξ_0.) And he constructed a family of functions $X\longrightarrow\mathbf{R}$ parametrized by $(\xi,\lambda)\in\Xi\times\mathfrak{a}$ as follows. Let

$$p(x,\xi_0,\lambda)=e^{\langle\rho+\lambda,H\rangle}\qquad\text{for } x=ne^H\cdot x_0.$$

Here we remark that each $x\in X$ is uniquely expressed as $x=ne^H\cdot x_0$; $n\in N$, $H\in\mathfrak{a}$. Since K acts transitively on Ξ, any element $\xi\in\Xi$ can be expressed as $\xi=k\xi_0$ by using some element k of K. Let

$$p(x,\xi,\lambda)=p(k^{-1}x,\xi_0,\lambda)\qquad\text{for } x\in X.$$

This definition does not depend on the choice of k.

Theorem 2.6. *Let $k\in K$, $\xi=k\xi_0\in\Xi$, $\mu\in\mathfrak{a}^+$. Then we have*

$$p(x,\xi,\mu)=\exp(-\|\rho+\mu\|b(k\gamma)(x))$$

for all $x\in X$, where γ is the geodesic ray defined by

$$\gamma(t)=\left(\exp t\frac{\rho+\mu}{\|\rho+\mu\|}\right)\cdot x_0\qquad\textit{for } t\geqq 0.$$

Proof. Since

$$\exp(-\|\rho+\mu\|b(k\gamma)(x))=\exp(-\|\rho+\mu\|b(\gamma)(k^{-1}x)),$$

and $p(x,\xi,\mu)=p(k^{-1}x,\xi_0,\mu)$, we may only consider the case where $k=e$, $\xi=\xi_0$. Since $\rho+\mu\in\mathfrak{a}^+$, from Lemma 2.2, the function $f(x)=\exp(-\|\rho+\mu\|b(\gamma)(x))$ is N-invariant. And from its definition, $p(x,\xi_0,\mu)$ is also N-invariant. Therefore, it suffices to show that the two functions coincide on $A\cdot x_0$. From Lemma 2.3, the restriction of f to $A\cdot x_0$ is given by

$$f(e^H\cdot x_0)=\exp\left\{-\|\rho+\mu\|\cdot\left(-\langle H,\frac{\rho+\mu}{\|\rho+\mu\|}\rangle\right)\right\}=e^{\langle\rho+\mu,H\rangle}$$

for all $H\in\mathfrak{a}$. Hence, $p(x,\xi_0,\mu)=f(x)$ on $A\cdot x_0$.

Karpelevič showed the following in (the proof of) Theorem 17.2.1 of [18].

Theorem 2.7 (Karpelevič). *For $c\geqq-\|\rho\|^2$, let*

$$\mathfrak{a}_c^+=\{\alpha\in\mathfrak{a}^+\,|\,\|\alpha\|^2=c+\|\rho\|^2\}.$$

Then the set $\mathcal{M}_c$ of all minimal positive solutions $f(x)$ of equation $\Delta f=cf$ such that $f(x_0)=1$ coincides with the set $\{p(x,\xi,\mu)\,|\,\xi\in\Xi,\mu\in\mathfrak{a}_c^+\}$.

For each $\mu\in\mathfrak{a}_c^+$, let

$$\gamma_{\rho+\mu}(t)=\left(\exp t\frac{\rho+\mu}{\|\rho+\mu\|}\right)\cdot x_0$$

be the geodesic with initial velocity $\frac{\rho+\mu}{\|\rho+\mu\|}$. By combining this with Theorem B, we have the following.

Corollary 2.8. *For any number c such that $c\geqq-\|\rho\|^2$, the set $\mathcal{M}_c$ of all minimal positive solutions $f(x)$ of equation $\Delta f=cf$ such that $f(x_0)=1$ is given by*

$$\mathcal{M}_c=\{e^{-\|\rho+\mu\|b(k\gamma_{\rho+\mu})(x)}|k\in K,\mu\in\mathfrak{a}_c^+\}.$$

§3. Poisson kernels in the rank 1 case

In this section we consider the case where the rank of X is equal to 1. The sectional curvatures of X are bounded between two negative constants. The vector $\alpha/\|\alpha\|$ in (0.3) is unique and coincides with $\rho/\|\rho\|$. So, the functions $e^{-(\|\rho\|+\sqrt{\|\rho\|^2+c})b(k\gamma)}$ are positive solutions of (0.4).

We first recall the definition of Poisson kernel.

Definition 3.1 ([21]). *A Poisson kernel f normalized at x_0 for $q\in X(\infty)$ is a positive harmonic function on X such that $f(x_0)=1$ and f extends continuously to the zero function on $X(\infty)-\{q\}$.*

Proposition 3.2 ([2, Corollary 5.3], see also [21]). *There exists a unique Poisson kernel for every $q\in X(\infty)$.*

By using the Rauch comparison theorem, we have the following.

Lemma 3.3. *Suppose that the sectional curvatures K_X of X satisfy $K_X\leqq-a^2<0$ $(a>0)$. Let $\alpha\in\mathfrak{p}=T_{x_0}(X)$ and $\gamma(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$. Let $v\in T_{x_0}(X)$, $\delta>0$ satisfy $\angle_{x_0}(v,\alpha)=\delta_0>\delta$. Then, for any positive number C, we have*

$$e^{-Cb(\gamma)(x)}\leqq\left(\frac{1-\cos(\delta_0-\delta)}{2}\right)^{-C/a}\cdot e^{-Cd(x_0,x)} \tag{3.1}$$

for all $x\in C_{x_0}(v,\delta)-\{x_0\}$.

Proof. Let γ' be the unit speed geodesic joining x_0 and $x\in C_{x_0}(v,\delta)-\{x_0\}$ such that $\gamma'(0)=x_0$, $\gamma'(s)=x$. We denote by $\dot\gamma'(0)$ the initial velocity of γ'. Let $\angle_{x_0}(\alpha,\dot\gamma'(0))=\omega$. Then $\omega\geqq\delta_0-\delta>0$.

Let us consider the following geodesic triangle $\triangle(z_1,z_2,z_3)$ in the complete simply connected Riemannian manifold M_{-a^2} with constant sectional curvature $-a^2$:

$$d_{M_{-a^2}}(z_1,z_2)=t,\ d_{M_{-a^2}}(z_1,z_3)=s,\ \angle_{z_1}(z_2,z_3)=\omega.$$

By the Rauch comparison theorem, we have

$$d(x,\gamma(t))\geqq d_{M_{-a^2}}(z_2,z_3).$$

Let

$$\psi=\psi(s,t,\omega)=\cosh as\cdot\cosh at-\sinh as\cdot\sinh at\cdot\cos\omega.$$

By the cosine formula in M_{-a^2}, we have

$$\cosh(a\cdot d_{M_{-a^2}}(z_2,z_3))=\psi(s,t,\omega).$$

Hence,

$$e^{a\cdot d(x,\gamma(t))}+e^{-a\cdot d(x,\gamma(t))}\geqq 2\psi.$$

We are interested in the value $\lim_{t\to\infty}\{d(x,\gamma(t))-t\}$. So we may assume that $t\gg s$. Since $\omega>0$, we have

$$\psi-\sqrt{\psi^2-1}\leqq e^{a(s-t)}<1<e^{a\cdot d(x,\gamma(t))}.$$

Hence,

$$e^{a\cdot d(x,\gamma(t))}\geqq\psi+\sqrt{\psi^2-1},$$

and

$$d(x,\gamma(t))\geqq\frac{1}{a}\log(\psi+\sqrt{\psi^2-1}).$$

We obtain

$$\begin{aligned} b(\gamma)(x)&=\lim_{t\to\infty}\{d(x,\gamma(t))-t\} \\ &\geqq\lim_{t\to\infty}\left\{\frac{1}{a}\log(\psi+\sqrt{\psi^2-1})-t\right\}=\frac{1}{a}\lim_{t\to\infty}\log\frac{\psi+\sqrt{\psi^2-1}}{e^{at}}. \end{aligned}\tag{3.2}$$

Since $a>0$,

$$\lim_{t\to\infty}\frac{\psi}{e^{at}}=\frac{1}{2}\cosh as-\frac{1}{2}\sinh as\cdot\cos\omega>0.$$

Therefore,

$$\lim_{t\to\infty}\frac{\psi+\sqrt{\psi^2-1}}{e^{at}}=\cosh as-\sinh as\cdot\cos\omega$$

$$=\frac{1}{2}(1-\cos\omega)e^{as}+\frac{1}{2}(1+\cos\omega)e^{-as}\tag{3.3}$$

$$\geqq \frac{1}{2}(1-\cos\omega)e^{as} \geqq \frac{1}{2}(1-\cos(\delta_0-\delta))e^{-as}.$$

From (3.2) and (3.3), we have

$$b(\gamma)(x) \geqq \frac{1}{a}\log\left(\frac{1-\cos(\delta_0-\delta)}{2}\right)+s.$$

This implies the inequality (3.1).

From this and Theorems A, B, Proposition 3.2, we have the following.

Proposition 3.4. *For any* $k\in K$,

$$p(x,k\xi_0,\rho)=e^{-2\|\rho\| b(k\gamma_0)(x)}$$

is the unique Poisson kernel normalized at x_0 *for* $k\gamma_0(\infty)$, *where*

$$\gamma_0(t)=\left(\exp t\frac{\rho}{\|\rho\|}\right)\cdot x_0$$

as in (0.5), (2.11).

Remark. (1) The fact that $e^{-2\|\rho\| b(k\gamma_0)(x)}$ is the Poisson kernel should be already known, since it is written (without proof) in [4] that $e^{-hb(\gamma)}$, where h is the entropy, is the Poisson kernel for $\gamma(\infty)$.
(2) For $c\neq 0$, $c\geqq -\|\rho\|^2$, it also follows from the above argument that the function $P_c=e^{-(\|\rho\|+\sqrt{\|\rho\|^2+c})b(k\gamma)}$ is something like Poisson kernel in the following sense: (a) $(\Delta-c)P_c=0$, (b) $P_c(x_0)=1$, (c) P_c extends continuously to the zero function on $X(\infty)-\{k\gamma(\infty)\}$.

§4. Minimal Martin boundary and sphere at infinity

We recall that the sphere $X(\infty)$ at infinity of X can be considered (as a set) a geometric realization $|T|$ of the spherical Tits building T of G ([3,11]). For each $\alpha\in\mathfrak{a}^+-\{0\}$, let

$$\gamma_\alpha(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$$

be the unit speed geodesic with initial velocity $\alpha/\|\alpha\|$. We put

$$\bar{C}_0=\{\gamma_\alpha(\infty)|\alpha\in\mathfrak{a}^+\}.$$

Since $\mathfrak{a}$ is decomposed as the union of the closures of Weyl chambers, $\{\gamma_\alpha(\infty)|\alpha\in\mathfrak{a}-\{0\}\}$, the intersection of the flat $A\cdot x_0$ with $X(\infty)$, is decomposed as the union of some translates $k\bar{C}_0(k\in K)$ of $\bar{C}_0$. And $X(\infty)$ is decomposed as follows.

$$X(\infty)=\bigcup_{h\in G}h\bar{C}_0=\bigcup_{k\in K}k\bar{C}_0.$$

Let $\Upsilon=\{\theta_1,\cdots,\theta_l\}$ be the system of fundamental roots of $\mathfrak{a}$ as in §1.1. For each subset Θ of Υ, we put

$$\bar{C}_0(\Theta)=\{\gamma_\alpha(\infty)|\alpha\in\mathfrak{a}^+-\{0\},\theta(\alpha)=0 \text{ for all } \theta\in\Upsilon-\Theta\}.$$

Then $\bar{C}_0$ becomes an $(l-1)$-dimensional simplex with l vertices $\bar{C}_0(\{\theta_1\}),\cdots,\bar{C}_0(\{\theta_l\})$. Each $\bar{C}_0(\Theta)$ $(\Theta\subset\Upsilon)$ is a $(\#\Theta-1)$-dimensional boundary face. The set $\mathscr{K}$ of all simplices $h\bar{C}_0(\Theta)$ of $X(\infty)$ becomes a simplicial complex.

On the other hand, the spherical Tits building T of G is, by definition, the set of all (identity components of) parabolic subgroups $Q\subsetneq G$ equipped with the following partial ordering $<$: $Q<Q'$ iff $Q\supset Q'$ (see [6,22]). If ξ is an interior point (in the sense of simplex) of $h\bar{C}_0(\Theta)$, where $h\in G$, then the isotropy subgroup G_ξ of ξ is $hP_\Theta h^{-1}$. So, the set $\mathscr{K}$ is naturally identified with the set of all (proper) parabolic subgroups of G. For $h_1,h_2\in G$, and $\Theta_1,\Theta_2\subset\Upsilon$, we have $h_1\bar{C}_0(\Theta_1)\subset h_2\bar{C}_0(\Theta_2)$ iff $h_1P_{\Theta_1}h_1^{-1}\supset h_2P_{\Theta_2}h_2^{-1}$. Therefore, in the above identification, the inclusion in $\mathscr{K}$ is compatible with the relation $<$: $h_1\bar{C}_0(\Theta_1)\subset h_2\bar{C}_0(\Theta_2)$ iff $h_1P_{\Theta_1}h_1^{-1}<h_2P_{\Theta_2}h_2^{-1}$. Thus, $X(\infty)$ can be regarded as a geometric realization $|T|$ of T. We denote by $|T|^{l-2}$ the $(l-2)$-skeleton of $|T|$ for $l\geqq 2$.

Let $c\geqq-\|\rho\|^2$. We recall that

$$\mathscr{M}_c=\{p(x,\xi,\mu)|\xi\in\Xi,\mu\in\mathfrak{a}_c^+\}=\left\{e^{-\|\rho+\mu\|b(k\gamma_{\rho+\mu})}|\mu\in\mathfrak{a}_c^+,k\in K\right\}.$$

We define a map $\Phi:\ \mathscr{M}_c\longrightarrow X(\infty)$ by

$$\Phi(e^{-\|\rho+\mu\|b(k\gamma_{\rho+\mu})})=k\cdot\gamma_{\rho+\mu}(\infty)\qquad \text{for } \mu\in\mathfrak{a}_c^+,\ k\in K.$$

In the rank 1 case, it follows from Proposition 3.4 and the succeeding Remark that our map Φ coincides with the homeomorphisms $\mathscr{M}_c\longrightarrow X(\infty)$ constructed in [1, 2] by Anderson-Schoen and Ancona. In the higher rank case, Φ is a somewhat different kind of map from the ones in [1,2].

Theorem 4.1. *The map Φ is a homeomorphism from $\mathscr{M}_c$ with the Martin topology to its image $\Phi(\mathscr{M}_c)$ with the induced topology from the sphere topology on $X(\infty)$. If rank $X=1$, then $\Phi(\mathscr{M}_c)=X(\infty)$. If $l=$rank $X\geqq 2$, there exists an open neighborhood W (with respect to the sphere topology) of $|T|^{l-2}$, such that $\Phi(\mathscr{M}_c)=X(\infty)-W$.*

Remark. (1) The neighborhood W depends on the value c. If c becomes larger, then W becomes smaller (see Figure). Let us identify $\mathscr{M}_c$ with the image of Φ. Then $\mathscr{M}_{-\|\rho\|^2}=\Xi$, $\mathscr{M}_c\subset\mathscr{M}_{c'}$ if $c<c'$, and

$$X(\infty)=\overline{(\bigcup_{c\geqq-\|\rho\|^2}\mathscr{M}_c)},$$

where the closure in the last equality is relative to the sphere topology.

(2) In the case $c=0$, we also have an ideal boundary of G, namely the maximal boundary $B(G)$ of G defined by Furstenberg ([15]). The maximal boundary of G is identified with $G/P=\Xi$. Thus by embedding $B(G)$ and $\mathscr{M}_0$ into $X(\infty)$, we have

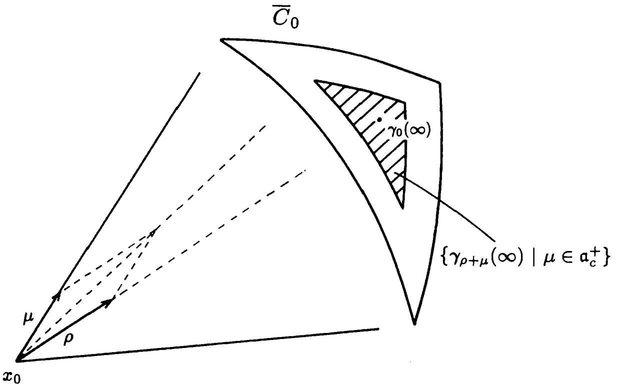

Figure. The case $l = rank X = 3$. The intersection of Ξ and $\bar{C}_0$ consists of a single point $\gamma_0(\infty)$ and $\mathcal{M}_c \cap \bar{C}_0 = \{\gamma_{\rho+\mu}(\infty) | \mu \in \mathfrak{a}_c^+\}$.

an inclusion relation as follows.

$$\begin{pmatrix} \text{maximal} \\ \text{boundary } \Xi \end{pmatrix} \subset \begin{pmatrix} \text{minimal Martin} \\ \text{boundary } \mathcal{M}_0 \end{pmatrix} \subset \begin{pmatrix} \text{geometric} \\ \text{boundary } X(\infty) \end{pmatrix}$$

Proof. Let

$$\mathcal{A}_c = \{\gamma_{\rho+\mu}(\infty) | \mu \in \mathfrak{a}_c^+\},$$

where $\mathfrak{a}_c^+ = \{\alpha \in \mathfrak{a}^+ | \|\alpha\|^2 = c + \|\rho\|^2\}$ as in §2. Since $\mathcal{A}_c$ is compact and $\Phi(\mathcal{M}_c) = K\mathcal{A}_c$, $\Phi(\mathcal{M}_c)$ is a compact subset of $X(\infty)$. We notice that $\mathcal{M}_c$ is Hausdorff and that $\Phi(\mathcal{M}_c)$ is metrizable (by using the angle metric). So we show that Φ^{-1} is continuous.

It suffices to prove the following: If a sequence $\{p_i\}_{i=1,2,\cdots}$ of points of $\Phi(\mathcal{M}_c)$ converges to $p \in \Phi(\mathcal{M}_c)$, then

$$\lim_{i\to\infty} d_c(\Phi^{-1}(p_i), \Phi^{-1}(p)) = 0.$$

Let

$$\Phi^{-1}(p_i)(x) = e^{-\|\rho+\mu_i\| b(k_i\gamma_{\rho+\mu_i})(x)}\ ;\ \mu_i \in \mathfrak{a}^+ - \{0\}\ ;\ k_i \in K$$

for each i, and

$$\Phi^{-1}(p)(x) = e^{-\|\rho+\mu\| b(k\gamma_{\rho+\mu})(x)}\ ;\ \mu \in \mathfrak{a}^+ - \{0\}\ ;\ k \in K.$$

Then,

$$p_i = k_i \cdot \gamma_{\rho+\mu_i}(\infty),\ p = k \cdot \gamma_{\rho+\mu}(\infty).$$

Since $\lim_{i\to\infty} p_i = p$, the sequence $\{k_i\gamma_{\rho+\mu_i}\}$ of geodesics converges to $k\gamma_{\rho+\mu}$. Let $a_i = \|\rho+\mu_i\|$, $a = \|\rho+\mu\|$. We write b_i, b instead of $b(\gamma_{\rho+\mu_i})$, $b(\gamma_{\rho+\mu})$, respectively. From Lemma 2.1, for any positive ε, there exists a positive integer I such that the following holds. If $i \geqq I$, then

$$(4.1) \qquad a|b_i(x) - b(x)| < \log(1+\varepsilon) \qquad \text{for all } x \in B_{x_0}(1),$$

$$|a_i - a| < \log(1+\varepsilon). \tag{4.2}$$

Lemma 4.2. *Let*

$$\psi_{1,i}(x) = |1 - e^{a(b(x) - b_i(x))}|, \quad \psi_{2,i}(x) = |e^{(a-a_i)b_i(x)} - 1|.$$

Then, for $i \geqq I$, we have

$$\psi_{1,i}(x) < \varepsilon, \quad \psi_{2,i}(x) < \varepsilon \qquad \text{on } B_{x_0}(1).$$

Proof. We only show the inequality for $\psi_{2,i}$. (We can show the inequality for $\psi_{1,i}$ in a similar way.)

We remark that $\|\text{grad } b_i\| \equiv 1$ ([3]). Since $b_i(x_0) = 0$, we have $|b_i(x)| \leqq 1$ on $B_{x_0}(1)$.

CASE 1. Suppose that $a_i \geqq a$. If $b_i(x) \geqq 0$, from (4.2), we have

$$\psi_{2,i}(x) = 1 - e^{(a-a_i)b_i(x)} \leqq 1 - e^{a-a_i} < 1 - e^{-\log(1+\varepsilon)} = 1 - \frac{1}{1+\varepsilon} < \varepsilon.$$

Similarly, if $b_i(x) < 0$,

$$\psi_{2,i}(x) = e^{(a-a_i)b_i(x)} - 1 \leqq e^{a_i - a} - 1 < e^{\log(1+\varepsilon)} - 1 = \varepsilon.$$

CASE 2. Suppose that $a_i < a$. If $b_i(x) \geqq 0$, then

$$\psi_{2,i}(x) = e^{(a-a_i)b_i(x)} - 1 \leqq e^{a-a_i} - 1 < e^{\log(1+\varepsilon)} - 1 = \varepsilon.$$

If $b(x) < 0$,

$$\psi_{2,i}(x) = 1 - e^{(a-a_i)b_i(x)} \leqq 1 - e^{a_i - a} < 1 - e^{-\log(1+\varepsilon)} = 1 - \frac{1}{1+\varepsilon} < \varepsilon.$$

From the above lemma and that $|b_i(x)| \leqq 1$, $|b(x)| \leqq 1$ on $B_{x_0}(1)$, we obtain

$$|e^{-ab_i(x)} - e^{-ab(x)}| = e^{-ab(x)} \cdot \psi_{1,i}(x) \leqq e^a \cdot \psi_{1,i}(x) < \varepsilon e^a,$$

$$|e^{-a_i b_i(x)} - e^{-ab_i(x)}| = e^{-ab_i(x)} \cdot \psi_{2,i}(x) \leqq e^a \cdot \psi_{2,i}(x) < \varepsilon e^a.$$

We have

$$|\Phi^{-1}(p_i)(x) - \Phi^{-1}(p)(x)| = |e^{-a_i b_i(x)} - e^{-ab(x)}| < 2\varepsilon e^a$$

uniformly on $B_{x_0}(1)$. Hence,

$$d_c(\Phi^{-1}(p_i), \Phi^{-1}(p)) \leqq \int_{B_{x_0}(1)} |\Phi^{-1}(p_i)(x) - \Phi^{-1}(p)(x)|$$

$$\leqq 2e^a \operatorname{Vol}(B_{x_0}(1)) \cdot \varepsilon.$$

This shows that $\lim_{i\to\infty} d_c(\Phi^{-1}(p_i), \Phi^{-1}(p)) = 0$.

§5. Computing positive eigenfunctions

5.1. We compute positive eigenfunctions of Laplacian Δ in the case $X = SU(n)\backslash SL(n,\mathbf{C})$ and compare it with the result of Dynkin ([9]). We must make a few modifications (besides the obvious modifications) on the results in §2, since the symmetric space is regarded as the set of right cosets in [9]. That is, we need to change

$$\gamma(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$$

in Lemma 2.2, Theorem 2.5 into

$$\gamma(t)=x_0\cdot\exp\left(-t\frac{\alpha}{\|\alpha\|}\right),$$

and

$$\gamma(t)=\left(\exp t\frac{\rho+\mu}{\|\rho+\mu\|}\right)\cdot x_0$$

in Theorem 2.6 into

$$\gamma(t)=x_0\cdot\exp\left(-t\frac{\rho+\mu}{\|\rho+\mu\|}\right).$$

When we treat the set of left cosets $SL(n,\mathbf{C})/SU(n)$, the isotropy subgroup $G_{\gamma(\infty)}$ of $\gamma(\infty)$ is N iff the initial velocity of γ belongs to $\mathfrak{a}^+$. But when we treat the set of right cosets, $G_{\gamma(\infty)}=N$ iff the initial velocity of γ belongs to $-\mathfrak{a}^+=\{-\alpha|\alpha\in\mathfrak{a}^+\}$. We denote by Lemma 2.2', Theorems 2.5', 2.6' respectively the correspondents obtained by this modifications. We also remark that in [9], the metric of $SU(n)\backslash SL(n,\mathbf{C})$ is induced from the $1/n$ of the Killing form.

Let $G=SL(n,\mathbf{C})$ and $K=SU(n)$. We can identify $K\backslash G$ with the set $P(n,\mathbf{C})$ of all $n\times n$ positive hermitian matrices with determinant 1 ; i.e. G acts transitively on $P(n,\mathbf{C})$ by $x\cdot h={}^t\bar{h}xh$ for $x\in P(n,\mathbf{C})$, $h\in G$, and the isotropy subgroup of $I_n=\text{diag}(1,\cdots,1)$ is K. (We use bars over complex matrices to denote their complex conjugates. And we denote by th the transpose of the matrix h.) Let

$$\mathfrak{a}=\{\alpha=\text{diag}(\alpha_1,\cdots,\alpha_n)|\text{trace }\alpha=0,\alpha_i\in\mathbf{R}\text{ for all }i\}.$$

For each i, we define a linear map $e_i:\mathfrak{a}\longrightarrow\mathbf{R}$ by

$$e_i(\alpha)=\alpha_i \qquad \text{for }\alpha=\text{diag}(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}.$$

The system Σ of roots (resp. the system Σ^+ of positive roots) is given by

$$\Sigma=\{e_i-e_j|i\neq j,1\leqq i,j\leqq n\}$$

$$(\text{resp. }\Sigma^+=\{e_i-e_j|1\leqq i<j\leqq n\}).$$

The root space $\mathfrak{g}^{e_i-e_j}$ corresponding to e_i-e_j is given by $\mathfrak{g}^{e_i-e_j}=\mathbf{C}E_{ij}$, where E_{ij} is the $n\times n$ matrices with i-j entry 1, all other entries being 0. So, every root has multiplicity 2. The metric is given by

$$\langle Y,Y'\rangle=4Re(\text{trace}(YY')) \qquad \text{for }Y,Y'\in\mathfrak{p}.$$

So, the norm of $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}$ is

$$\|\alpha\|=2\left\{\sum_{i=1}^{n}(\alpha_i)^2\right\}^{1/2}.$$

We also have the following.

$$H_{e_i-e_j}=\frac{1}{4}(E_{ii}-E_{jj}),$$

$$\rho=\frac{1}{2}\sum_{i<j}H_{e_i-e_j}=\frac{1}{2}\operatorname{diag}(c_1,\cdots,c_n),$$

$$c_i=\frac{n-2i+1}{2};\quad i=1,\cdots,n.$$

For

$$\alpha\in\mathfrak{a}^+=\{\operatorname{diag}(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}\,|\,\alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_n\},$$

let

$$\gamma(t)=x_0\cdot\exp\left(-t\frac{\alpha}{\|\alpha\|}\right)\qquad\text{for } t\geqq 0.$$

For $\beta=\operatorname{diag}(\beta_1,\cdots,\beta_n)\in\mathfrak{a}$, we put

$$d=\operatorname{diag}(d_1,\cdots,d_n)=x_0\cdot e^{\beta};\ \text{i.e.}\quad d_i=e^{2\beta_i}\ \text{for } i=1,\cdots,n.$$

Then, from Lemma 2.3, we have

$$b(\gamma)(x_0\cdot e^{\beta})=\langle\beta,\frac{\alpha}{\|\alpha\|}\rangle=\frac{\sum_{i=1}^{n}\alpha_i\log d_i}{\sqrt{\sum_{i=1}^{n}(\alpha_i)^2}}.\tag{5.1}$$

We rewrite the right hand side of (5.1) in an N-invariant form. For $x\in P(n,\mathbf{C})$, let $\triangle_i(x)$ the $(i\times i)$-minor determinant in the upper left corner. Then

$$d_i=\frac{\triangle_{i+1}(d)}{\triangle_i(d)}.\tag{5.2}$$

And for each i, we have

$$\triangle_i(d\cdot h)=\triangle_i(d)\qquad\text{for all } h\in N.$$

From (5.1),

$$b(\gamma)(d)=\frac{1}{\sqrt{\sum_{i=1}^{n}(\alpha_i)^2}}\left\{\sum_{i=1}^{n-1}\alpha_i\log d_i-\alpha_n(\log d_1+\cdots+\log d_{n-1})\right\}$$

$$=\frac{1}{\sqrt{\sum_{i=1}^{n}(\alpha_i)^2}}\sum_{i=1}^{n-1}(\alpha_i-\alpha_n)\log d_i=\frac{1}{\sqrt{\sum_{i=1}^{n}(\alpha_i)^2}}\log\left(\prod_{i=1}^{n-1}d_i^{\alpha_i-\alpha_n}\right).$$

Here, from (5.2), we have

$$\prod_{i=1}^{n-1}(d_i)^{\alpha_i-\alpha_n}$$

$$=\triangle_1(d)^{\alpha_1-\alpha_n}\cdot\left(\frac{\triangle_2(d)}{\triangle_1(d)}\right)^{\alpha_2-\alpha_n}\cdot\left(\frac{\triangle_3(d)}{\triangle_2(d)}\right)^{\alpha_3-\alpha_n}\cdots\cdot\left(\frac{\triangle_{n-1}(d)}{\triangle_{n-2}(d)}\right)^{\alpha_{n-1}-\alpha_n}$$

$$=\prod_{i=1}^{n-1}\triangle_i(d)^{\alpha_i-\alpha_{i+1}}.$$

Hence

$$b(\gamma)(d)=\frac{1}{\sqrt{\sum_{i=1}^{n}(\alpha_i)^2}}\log\left\{\prod_{i=1}^{n-1}\triangle_i(d)^{\alpha_i-\alpha_{i+1}}\right\}.$$

Therefore, from Lemma 2.2',

(5.3) $$b(\gamma)(x)=\frac{1}{\sqrt{\sum_{i=1}^{n}(\alpha_i)^2}}\log\left\{\prod_{i=1}^{n-1}\triangle_i(x)^{\alpha_i-\alpha_{i+1}}\right\}$$

for all $x\in P(n,\mathbf{C})$.

In particular, if

$$\alpha=\rho+\mu,\mu=\mathrm{diag}\,(\mu_1,\cdots,\mu_n)\in\mathfrak{a}^+,$$

then

$$\alpha_{i+1}-\alpha_i=\left(\mu_{i+1}+\frac{1}{2}c_{i+1}\right)-\left(\mu_i+\frac{1}{2}c_i\right)=\mu_{i+1}-\mu_i-\frac{1}{2}.$$

Hence, from (5.3) and Theorems 2.5,' 2.6',

$$p(x,\xi_0,\mu)=e^{-\|\rho+\mu\|b(\gamma)(x)}=\exp\left\{2\log\left(\prod_{i=1}^{n-1}\triangle_i(x)^{\mu_{i+1}-\mu_i-\frac{1}{2}}\right)\right\}$$

is a positive eigenfunction of Laplacian with eigenvalue $\|\mu\|^2-\|\rho\|^2$. In Dynkin's paper, the coordinate of $\alpha\in\mathfrak{a}^+$ is twice our coordinate $(\alpha_1,\cdots,\alpha_n)$. So, let $\gamma_i=2\mu_i$; $i=1,\cdots,n$. Then we have

$$p(x,\xi_0,\mu)=\prod_{i=1}^{n-1}\triangle_i(x)^{\gamma_{i+1}-\gamma_i+1}.$$

This coincides with the formula (39) in [9, Theorem 3].

5.2. We can compute the Busemann functions on $X=SL(n,\mathbf{R})/SO(n)$ in a similar way in **5.1.** We will use this result in the remainder of this section.

Let $G=SL(n,\mathbf{R})$ and $K=SO(n)$. We identify G/K with the set $P(n,\mathbf{R})$ of all $n\times n$ positive definite, symmetric matrices with determinant 1 ; i.e. G acts transitively on $P(n,\mathbf{R})$ by $h\cdot x=hx^th$ for $x\in P(n,\mathbf{R})$, $h\in G$, and the isotropy subgroup of I_n is K. Let

$$\mathfrak{a}=\{\alpha=\mathrm{diag}\,(\alpha_1,\cdots,\alpha_n)|\mathrm{trace}\ \alpha=0, \alpha_i\in\mathbf{R}\ \text{for all}\ i\}.$$

The system of positive roots is

$$\Sigma^+=\{e_i-e_j|1\leqq i<j\leqq n\},$$

and

$$\mathfrak{a}^+=\{\mathrm{diag}\,(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}|\alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_n\}.$$

The metric is given by

$$\langle Y, Y'\rangle=2n\cdot\mathrm{trace}\,(YY') \qquad \text{for } Y, Y'\in\mathfrak{p}.$$

So, the norm of $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}$ is

$$\|\alpha\|=\left\{2n\sum_{i=1}^{n}(\alpha_i)^2\right\}^{1/2}.$$

For $\alpha\in\mathfrak{a}-\{0\}$, we put

$$\gamma_\alpha(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0 \qquad \text{for } t\geqq 0.$$

Lemma 5.1. *Let* $\alpha=\mathrm{diag}\,(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}-\{0\}$.
(1) *If* $\alpha\in\mathfrak{a}^+$, *then*

$$b(\gamma_\alpha)(x)=-\frac{n}{\|\alpha\|}\log\left\{\prod_{i=1}^{n-1}\square_{n-i}(x)^{\alpha_{i+1}-\alpha_i}\right\},$$

where $\square_j(x)$ *is the* $(j\times j)$*-minor determinant in the lower right hand corner.*
(2) *If* $\alpha\notin\mathfrak{a}^+$, *take a permutation* σ *of* n *letters such that* $\alpha_{\sigma(1)}\geqq\cdots\geqq\alpha_{\sigma(n)}$. Then

$$b(\gamma_\alpha)(x)=-\frac{n}{\|\alpha\|}\log\left\{\prod_{i=1}^{n-1}\square_{n-i}(\sigma\cdot x)^{\alpha_{\sigma(i+1)}-\alpha_{\sigma(i)}}\right\},$$

where $\sigma\cdot x=(x_{\sigma(i)\sigma(j)})$ *for* $x=(x_{ij})\in P(n,\mathbf{R})$.

5.3. We compute some of the positive eigenfunctions of Laplacian on the upper half plane, the n product of upper half planes, and Siegel upper half spaces. Let us first consider the following situation.

The symmetric space X is embedded totally geodesically in another symmetric space X^* of noncompact type via a map $F: X\longrightarrow X^*$. We denote by g, g^* the metric of X, X^* respectively. There exists a positive constant C such that $F^*g^*=Cg$, where F^*g^* is the pulled back metric on X by F from the metric g^*. In this case we can reduce the computation of the Busemann functions on X to the one on X^* as follows.

Lemma 5.2. *Let* $F: X\longrightarrow X^*$, g, g^* *be as above. Let* $\gamma: [0,\infty)\longrightarrow X$ *be a unit speed geodesic, and* γ^* *the unit speed geodesic on* X^* *defined by* $\gamma^*(t)=$

$F(\gamma(t/\sqrt{C}))$ for $t \geqq 0$. Then we have

$$(5.4) \qquad b(\gamma)(x)=\frac{1}{\sqrt{C}} b(\gamma^*)(F(x)) \qquad \textit{for all } t \geqq 0.$$

We recall that every symmetric space $X=G/K$ of noncompact type can be embedded totally geodesically, isometrically into $P(m,\mathbf{R})$, where m is some positive integer, provided that one multiplies the metric on each irreducible de Rham factor of X by a suitable positive constant.

Proposition 5.3 ([20],[12, Appendix, Proposition (19)]). *Let $\iota: G \longrightarrow SL(m, \mathbf{R})$ be a faithful (mod center of G) representation such that*
(5.5) the image of G is self-adjoint: i.e. ${}^t\{\iota(G)\}=\iota(G)$,
(5.6) $\iota(K) \subset SO(m)$.
Then the map $F: X=G/K \longrightarrow P(m,\mathbf{R})=SL(m,\mathbf{R})/SO(m)$ given by

$$F(hK)=\iota(h)\,{}^t\{\iota(h)\} \qquad \textit{for all } h \in G$$

is an isometric, totally geodesic embedding after multiplying the metric on each irreducible de Rham factor of X by a suitable positive constant.

We can always construct such a representation $\iota: G \longrightarrow SL(m,\mathbf{R})$ satisfying (5.5), (5.6) by using the adjoint representation (, in which case $m=\dim G$). Therefore, in principle, we can compute the (minimal) positive eigenfunctions of Laplacian on any symmetric space of noncompact type explicitly by using Corollary 2.8, Lemmas 5.1, 5.2. But in each case which we treat in the remainder of this section, there exists a more natural representation $\iota: G \longrightarrow SL(m,\mathbf{R})$ than the adjoint representation.

a) upper half plane. Let

$$\mathcal{H}=\{z=x+\sqrt{-1}y \in \mathbf{C} \mid y>0\}$$

be the upper half plane with constant sectional curvature -1. Let $G=SL(2,\mathbf{R})$ and $K=SO(2)$. The group G acts from the left on $\mathcal{H}$ by fractional linear transformation. We identify G/K with $P(2,\mathbf{R})$ as in §5.2. Then the point in $P(2,\mathbf{R})$ corresponding to $z=x+\sqrt{-1}y \in \mathcal{H}$ is $\begin{pmatrix} y+x^2/y & x/y \\ x/y & 1/y \end{pmatrix}$. The metric is induced from the half of the Killing form of the Lie algebra $\mathfrak{g}$ of G; i.e.

$$\langle Y, Y' \rangle = 2 \operatorname{trace}(YY') \qquad \text{for } Y, Y' \in \mathfrak{p}.$$

Let

$$\gamma(t)=\begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} \cdot x_0 = \sqrt{-1} \cdot e^t \qquad \text{for } t \geqq 0.$$

The initial velocity of γ is 2ρ, where

$$\rho=\frac{1}{4}\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \qquad \text{and } \|\rho\|=\frac{1}{2}.$$

Then, we have

$$b(\gamma)(x+\sqrt{-1}y)=-\log y.$$

Let f_s be a function defined by

$$f_s(z)=e^{-2s\|\rho\|b(\gamma)(z)},$$

where s is a real number. Then we have

$$f_s(z)=(Im\ z)^s,$$

and from Theorem 2.5, $\Delta f_s=s(s-1)f_s$. We also have

$$f_s(h\cdot z)=\frac{f_s(z)}{|cz+d|^{2s}} \qquad \text{for } h=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G.$$

If we map $\mathcal{H}$ onto the interior of the unit disk by the Cayley transformation, the above result turns as follows. Let

$$\bar{\gamma}(t)=e^{\sqrt{-1}\delta}\cdot\frac{e^t-1}{e^t+1}$$

be the unit speed geodesic and $f_s(w)=e^{-2s\|\rho\|b(\bar{\gamma})(w)}$. Then

$$f_s(re^{\sqrt{-1}\omega})=\left\{\frac{1-r^2}{1+r^2-2r\cos(\delta-\omega)}\right\}^s.$$

b) the product of n upper half planes. Let $G=(SL(2,\mathbf{R}))^n$ and $K=(SO(2))^n$. The group G acts transitively from the left on $\mathcal{H}^n$ as follows.

$$h\cdot z=(h_1\cdot z_1,\cdots,h_n\cdot z_n)$$

$$\text{for } h=(h_1,\cdots,h_n)\in G \text{ and } z=(z_1,\cdots,z_n)\in\mathcal{H}^n.$$

Then the isotropy subgroup of $z_0=(\sqrt{-1},\cdots,\sqrt{-1})$ is K. We identify $\mathcal{H}^n$ with G/K by using this action. Notice that G can be regarded as a subgroup of $SL(2n,\mathbf{R})$ by identifying each element $h=(h_1,\cdots,h_n)$ of G with $\operatorname{diag}(h_1,\cdots,h_n)\in SL(2n,\mathbf{R})$. The metric of $\mathcal{H}^n$ is given by

$$\langle Y,Y'\rangle=2\,\mathrm{trace}\,(YY') \qquad \text{for } Y,Y'\in\mathfrak{p}.$$

Let

$$\gamma(t)=(e^{t/\sqrt{n}}\cdot\sqrt{-1},\cdots,e^{t/\sqrt{n}}\cdot\sqrt{-1}) \qquad \text{for } t\geqq 0.$$

The initial velocity of γ is $\frac{\rho}{\|\rho\|}$, where

$$\rho=\frac{1}{4}\operatorname{diag}(1,-1,\cdots,1,-1),$$

and $\|\rho\|=\sqrt{n}/2$. We calculate $b(\gamma)$ by embedding $\mathcal{H}^n$ into $P(2n,\mathbf{R})=SL(2n,\mathbf{R})/SO(2n)$. We define a totally geodesic embedding $F:\mathcal{H}^n\longrightarrow P(2n,\mathbf{R})$ by

$$F(h\cdot z_0)=h^t h \qquad \text{for all } h\in G.$$

We denote by g,g^* the metric of $\mathcal{H}^n,P(2n,\mathbf{R})$ respectively. Then we have $F^*g^*=2n\cdot g$. Let

$$\alpha^*=\operatorname{diag}(\alpha_1^*,\cdots,\alpha_{2n}^*)=\frac{1}{2\sqrt{2n}}\operatorname{diag}(1,-1,\cdots,1,-1),$$

and

$$\gamma^*(t)=(\exp t\alpha^*)\cdot x_0^*$$

be a unit speed geodesic in $P(2n,\mathbf{R})$, where x_0^* is the coset of the identity element of $SL(2n,\mathbf{R})$. Let σ be the permutation of $(2n)$-letters defined as follows;

$$\sigma=\begin{pmatrix} 1 & \cdots & i & \cdots & n & n+1 & \cdots & n+j & \cdots & 2n \\ 1 & \cdots & 2i-1 & \cdots & 2n-1 & 2 & \cdots & 2j & \cdots & 2n \end{pmatrix}.$$

Then we have

$$\alpha_{\sigma(1)}^*\geqq\cdots\geqq\alpha_{\sigma(2n)}^*.$$

The only number i such that $\alpha_{\sigma(i+1)}^*-\alpha_{\sigma(i)}^*\neq 0$ is n, and then $\alpha_{\sigma(n+1)}^*-\alpha_{\sigma(n)}^*=\frac{1}{n\sqrt{2}}$. We have

$$\sigma\cdot F(z)=\begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

$$A=\operatorname{diag}\left(y_1+\frac{x_1^2}{y_1},\cdots,y_n+\frac{x_n^2}{y_n}\right),$$

$$B=C=\operatorname{diag}\left(\frac{x_1}{y_1},\cdots,\frac{x_n}{y_n}\right),$$

$$\mathrm{D}=\operatorname{diag}\left(\frac{1}{y_1},\cdots,\frac{1}{y_n}\right),$$

where $z=(z_1,\cdots,z_n)$ and $z_i=x_i+\sqrt{-1}y_i$ for each i. Hence,

$$\square_n(\sigma\cdot F(z))=\frac{1}{y_1\cdot\cdots\cdot y_n}.$$

From Lemma 5.1 and (5.4), we obtain

$$b(\gamma)(z)=-\frac{1}{\sqrt{n}}\log(y_1\cdot\cdots\cdot y_n).$$

Let

$$f_s(z)=e^{-2s\|\rho\|b(\gamma)(z)}$$

be a positive eigenfunction of Δ. From Theorem 2.5, the corresponding eigenvalue is $s(s-1)n$. For $z=(z_1,\cdots,z_n)\in\mathbf{C}^n$, we denote by $\mathcal{N}(z)$ the product $z_1\cdot\cdots\cdot z_n$. Then, we have

$$f_s(z)=(y_1\cdot\cdots\cdot y_n)^s=\left(\prod_{i=1}^{n} Im\, z_i\right)^s=\mathcal{N}(Im\, z)^s$$

for $z\in\mathcal{H}^n$.

Let $\mathbf{k}$ be a totally real algebraic number field of degree n with ring of integers $\mathcal{O}_{\mathbf{k}}$. We denote by $\phi_1,\cdots,\phi_n$ the distinct field embeddings $\mathbf{k}\longrightarrow R$, where ϕ_1 is just the inclusion Then we can embed $SL(2,\mathcal{O}_{\mathbf{k}})$ in G by

$$h\longmapsto(\phi_1(h),\cdots,\phi_n(h))\qquad \text{for } h\in SL(2,\mathcal{O}_{\mathbf{k}}),$$

where $\phi_i(h)$ is the matrix obtained from h by conjugating each entry of h by ϕ_i. The group $SL(2,\mathcal{O}_{\mathbf{k}})$ acts on $\mathcal{H}^n$ through this embedding. We denote by $cz+d$ the point

$$(\phi_1(c)z_1+\phi_1(d),\cdots,\phi_n(c)z_n+\phi_n(d))$$

of $\mathbf{C}^n$ for $z=(z_1,\cdots,z_n)\in\mathcal{H}^n$ and $c,d\in\mathcal{O}_{\mathbf{k}}$. Then we have

$$f_s(h\cdot z)=\frac{f_s(z)}{|\mathcal{N}(cz+d)|^{2s}}$$

for $z=x+\sqrt{-1}y$ and $h=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathcal{O}_{\mathbf{k}})$.

c) Siegel upper half plane. Let I_n be the unit matrix of order n, and $J=\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}$. Let

$$G=Sp(n,\mathbf{R})=\{h\in SL(2n,\mathbf{R})|{}^t hJ_nh=J_n\},$$

and $K=SO(2n)\cap Sp(n,\mathbf{R})$. We denote by $\mathcal{H}_n$ the Siegel upper half plane of degree n, the set of all $n\times n$ complex symmetric matrices with positive definite imaginary part. The group G acts transitively from the left on $\mathcal{H}_n$ by

$$h\cdot z=(Az+B)(Cz+D)^{-1}\qquad \text{for } h=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G,\ z\in\mathcal{H}_n.$$

The isotropy group of $z_0=\sqrt{-1}I_n$ is K. By using this action, we identify $\mathcal{H}_n$ with G/K. Let

$$\mathfrak{a}=\{\mathrm{diag}\,(a_1,\cdots,a_n,-a_1,\cdots,-a_n)\in\mathfrak{gl}(2n,\mathbf{R})\}.$$

The system of positive roots is

$$\Sigma^+=\{2e_i \mid 1\leqq i\leqq n\}\cup\{e_i\pm e_j \mid 1\leqq i<j\leqq n\}.$$

The metric is given by

$$\langle Y,Y'\rangle=(n+1)\,\mathrm{trace}\,(YY') \qquad \text{for } Y,Y'\in\mathfrak{p}.$$

So,

$$\rho=\mathrm{diag}\,(\rho_1,\cdots,\rho_n,-\rho_1,\cdots,-\rho_n)\,;\,\rho_i=\frac{n-i+1}{2(n+1)} \qquad \text{for each } i,$$

and

$$\|\rho\|^2=\frac{1}{12}n(2n+1).$$

Let $\alpha=\frac{1}{2}\begin{pmatrix} I_n & \\ & -I_n\end{pmatrix}$ and $\gamma(t)=\left(\exp t\frac{\alpha}{\|\alpha\|}\right)\cdot x_0$. We have

(5.7) $$\|\alpha\|^2=\frac{1}{2}n(n+1), \qquad \langle\rho,\alpha\rangle=\frac{1}{4}n(n+1).$$

We compute the positive eigenfunction

$$f_s(z)=e^{-s\|\alpha\|b(\gamma)(z)}$$

of Δ. We define a totally geodesic embedding $F:\mathcal{H}_n\longrightarrow P(2n,\mathbf{R})$ by

$$F(h\cdot z_0)=h^t h \qquad \text{for all } h\in G.$$

We denote by g,g^* the metric of $\mathcal{H}_n$, $P(2n,\mathbf{R})$ respectively. Then we have $F^*g^*=\frac{4n}{n+1}g$. Let

$$\alpha^*=\mathrm{diag}\,(\alpha_1^*,\cdots,\alpha_{2n}^*)=\frac{1}{2\sqrt{2n}}\begin{pmatrix} I_n & \\ & -I_n\end{pmatrix}$$

and $\gamma^*(t)=(\exp t\alpha^*)\cdot x_0^*$, where x_0^* is the coset of the identity element of $SL(2n,\mathbf{R})$. Then $\gamma^*(t)$ is a unit speed geodesic in $P(2n,\mathbf{R})$. We notice that $\alpha_1^*\geqq\cdots\geqq\alpha_{2n}^*$. The only number i such that $\alpha_{i+1}^*-\alpha_i^*\neq 0$ is n, and then $\alpha_{n+1}^*-\alpha_n^*=-\frac{1}{\sqrt{2n}}$. For $h=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G$, we have

$$F(h\cdot z_0)=h^t h=\begin{pmatrix} A^tA+B^tB & A^tC+B^tD\\ C^tA+D^tB & C^tC+D^tD\end{pmatrix}.$$

Hence, from Lemma 5.1 and (5.4),

$$b(\gamma)(h\cdot z_0)=\sqrt{\frac{n+1}{4n}}b(\gamma^*)(F(h\cdot z_0))$$

$$=\sqrt{\frac{n+1}{2n}}\log\{\det({C'}C+{D'}D)\}.$$

From (5.7),

$$f_s(h\cdot z_0)=\{\det({C'}C+{D'}D)\}^{-\frac{s(n+1)}{2}}.$$

We remark that

$$Im(h\cdot z)=(\bar{z}'C+{}'D)^{-1}(Im\, z)(Cz+D)^{-1}$$

for $h=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in G$, $z\in\mathcal{H}_n$. If $z=z_0$, then

$$Im(h\cdot z_0)=({C'}C+{D'}D)^{-1}.$$

Therefore, we have

$$f_s(z)=\{\det(Im\, z)\}^{\frac{s(n+1)}{2}} \qquad \text{for all } z\in\mathcal{H}_n.$$

From Theorem 2.5 and (5.7) this function satisfies the equation

$$\Delta f_s=\frac{1}{2}s(s-1)n(n+1)f_s.$$

We also have

$$f_s(h\cdot z)=\frac{f_s(z)}{|\det(Cz+D)|^{s(n+1)}}$$

for $z\in\mathcal{H}_n$, $h=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in G$.

Let Γ be a torsion-free subgroup of finite index of **a**) $SL(2,\mathbf{Z})$ or **b**) $SL(2,\mathcal{O}_{\mathbf{k}})$ or **c**) $Sp(n,\mathbf{Z})$. We put $\Gamma_\infty=\Gamma\cap N$. Let

$$\varphi_s(z)=\sum_{h\in\Gamma_\infty\backslash\Gamma} f_s(h\cdot z) \qquad \text{for } z\in X,$$

where the sum runs over a complete set of representatives $h\in\Gamma$ for the quotient $\Gamma_\infty\backslash\Gamma$. Then

$$E_s(z)=\frac{\varphi_s(z)}{f_s(z)}$$

is the series obtained from the so-called Eisenstein series by replacing each term by its absolute value.

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