J. Math. Kyoto Univ. (JMKYAZ)
40-4 (2000), 777–790

Rational equivalence and phantom map out of a loop space

By

Kouyemon IRIYE

Abstract

McGibbon asked if for a connected finite complex X there is a rational equivalence from the loop space of X to a product of spheres and loop spaces of spheres. We will show that the answer is yes if it has only a finite number of nonzero rational homotopy groups or if spaces are localised at a prime. We will also give a clear picture of phantom maps out of the iterated loop space of a finite complex.

1. Introduction

In this paper, all spaces are assumed to have basepoints and all maps and homotopies are assumed to preserve them. A phantom map out of a CW-complex X is a map whose restriction to each n skeleton $sk_n(X)$ is homotopic to the constant map. One of the basic problems in the study of phantom maps out of loop spaces is the following problem raised by McGibbon ([9], Question 4).

Question. *Does there exist a finite complex X and an essential phantom map from ΩX to a target of finite type?*

By Theorem 8.7 of [9], there exists a rational equivalence

$$\Omega X \to \prod_{\alpha} S^{2n_\alpha - 1} \times \prod_{\beta} \Omega S^{2n_\beta - 1}$$

if and only if there are no essential phantom maps $\Omega X \to Y$ for any finite type target Y, where a space is referred to as a *finite type target* if each of its homotopy groups is finitely generated. The existence of such a rational equivalence is known for X which has the rational homotopy type of a suspension (Corollary 3.4 and Theorem 8.7 of [9]) and for a homogeneous space [6]. In this paper we will extend these results to a rationally elliptic space, where a space X is said to be *rationally elliptic* if $\sum_{n>1} \dim \pi_n(X) \otimes \mathbb{Q}$ is finite.

1991 *Mathematics Subject Classification(s).* Primary 55P99, 55P35.
The author is partially suported by Grant-in-Aid for Scientific Research.
Communicated by Prof. A. Kono, November 5, 1999

Theorem 1. *Let X be a simply connected, rationally elliptic, finite complex. Then there exists a rational equivalence*

$$\Omega X \to \prod_{\alpha} S^{2n_\alpha - 1} \times \prod_{\beta} \Omega S^{2n_\beta - 1}. \qquad (*)$$

As for the other implication of the existence of such a map $(*)$ to the homotopy theory, see Section 8 of [9] or Introduction of [6]. This theorem is also an extension of a work of McGibbon and Wilkerson [11], where they proved a localised version of the above theorem. In fact, if spaces are localised we can say more. To state our next result we need a definition. A CW-complex X is said to be *pseudo-finite* if it satisfies the following two conditions:

(1) Each of its homotopy groups is finitely generated except for the fundamental group.

(2) It has a finite spherical cone-length, that is, there is a finite series of subcomplexes $X_0 = * \subset X_1 \subset X_2 \subset \cdots \subset X_{s-1} \subset X_s = X$ such that each subcomplex X_i is obtained from X_{i-1} by attaching (possibly infinitely many) cells. In particular, X has a finite Lusternik-Schnirelmann category.

We will also use a p-local version of this concept.

Theorem 2. *Let X be a connected, pseudo-finite* CW*-complex, p be a prime and $k \geq 1$. Then there exists a p-local map*

$$\Omega_0^k X \to \prod_{\alpha} S^{2n_\alpha - 1} \times \prod_{\beta} \Omega S^{2n_\beta - 1}$$

which is a rational equivalence.

Here by $\Omega_0^k X$ we denote the basepoint component of the k-fold loop space of X. Now we have a clear picture of phantom maps out of the iterated loop space of a finite complex, which is an extension of Proposition 8.2 of [9].

Corollary 3. *Let X be a connected, pseudo-finite* CW*-complex and $k \geq 2$.*

(1) *If $\Omega_0^k X_{(p)}$ is not contractible for a prime p, then the universal phantom map out of $\Omega_0^k X$ is essential at the prime p.*

(2) *There exist essential phantom maps from $\Omega_0^k X$ into finite type targets if and only if*

$$\pi_q(\Omega_0^k X) \otimes \mathbb{Q} \neq 0 \qquad \text{for some } q \geq 2.$$

These targets can be taken to be spheres.

(3) *However, for each prime p there are no essential phantom maps from* $\Omega_0^k X$ *to nilpotent p-local finite type targets.*

Theorem 2 has another corollary which may be useful to consider McGibbon's question.

Corollary 4. *Let* X, X' *be simply connected finite complexes. If they are rationally homotopy equivalent to each other, then there is an essential phantom map from* ΩX *to a finite type target if and only if so is from* $\Omega X'$.

Acknowledgements

A part of this paper was written while the author stayed at the University of Aberdeen and he would like to express his gratitude to its hospitality. Thanks are also due to John Hubbuck, Mamoru Mimura, Akira Kono, Hiroo Shiga, Toshihiro Yamaguchi and the referee for their comments on earlier versions of this paper.

2. Preliminary lemmas

In this section, we prove lemmas needed later. The main results will be proven in the next section.

Lemma 2.1. *Let* X *be a pseudo-finite* CW*-complex and* $\widetilde{X}$ *be its universal covering space. Then* $\widetilde{X}$ *is also pseudo-finite.*

Proof. Since the first condition is trivial for $\widetilde{X}$, we prove that $\widetilde{X}$ has a CW-structure with a finite spherical cone-length. Let $p : \widetilde{X} \to X$ be the covering projection. It is well-known, see e.g., p.53 of [15], that $\widetilde{X}$ has a CW-structure so that, for each cell of e in X, each piece of $p^{-1}(e)$ is a cell of $\widetilde{X}$. Since X is pseudo-finite, there is a finite series of subcomplexes $X_0 = * \subset X_1 \subset X_2 \subset \cdots \subset X_{s-1} \subset X_s = X$ such that each subcomplex X_i is obtained from X_{i-1} by attaching cells, which is equivalent to say that the boundary of each cell of X_i is in X_{i-1}. Then $\widetilde{X}_i = p^{-1}(X_i)$ is a subcomplex of $\widetilde{X}$ and the following series of subcomplexes has the desired property.

$$* \subset \widetilde{X}_0 \subset \widetilde{X}_1 \subset \widetilde{X}_2 \subset \cdots \subset \widetilde{X}_{s-1} \subset \widetilde{X}_s = \widetilde{X}$$ □

Let X be a connected, pseudo-finite CW-complex and $\widetilde{X}$ be its universal covering space. Since $\Omega_0 X$ is homeomorphic to $\Omega\widetilde{X}$ and $\widetilde{X}$ is also pseudo-finite by Lemma 2.1, we may assume that any connected pseudo-finite CW-complex is simply connected to consider our problems.

Lemma 2.2. *For a simply connected* CW*-complex* X*, there exist a simply connected* CW*-complex* Y *and a rational equivalence* $f : X \to Y$ *which satisfy the following conditions:*

(1) *If X is rationally m-connected, then the m-skeleton of Y is the basepoint.*

(2) *If X is of finite type (resp. pseudo-finite or finite dimensional, or finite), then so is Y.*

Proof. The proof uses induction on m, the rational connectivity of X. We can prove the first induction step (i.e., $m = 1$) by the same argument as Lemma 5.5.1 of [2]. In this case we can choose Y and f so that f is homotopy equivalent.

Next we assume that X is rationally m-connected, where $m > 1$, and that the $m-1$-skeleton of X is the basepoint by the induction assumption. Then again by the same argument as Lemma 5.5.1 of [2] it is easy to see that there is a subcomplex X', which is a Moore space of type $(H_m(X;\mathbb{Z}), m)$, containing the entire m-skeleton, $sk_m(X)$. Let $Y = X/X'$ and $f : X \to X/X' = Y$ be the canonical collapsing map. Since $H_m(X;\mathbb{Z})$ is a torsion group, $f_* : H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$ is isomorphic. Then f must be a rational equivalence since X and Y are simply connected. Needless to say, the m-skeleton of Y is the basepoint. Now it is easy to prove that Y satisfies the condition (2). □

Finally we prove the following key lemma to prove our main results.

Lemma 2.3. (1) *Let X be a simply connected finite complex which is rationally $m-1$-connected and $\pi_m(X)\otimes\mathbb{Q} \neq 0$.*

If $m = 2n+1$ is odd, then there is a simply connected finite complex Y with a rational equivalence $\varphi : X \to Y$ and maps $q : Y \to S^{2n+1}$ and $f : S^{2n+1} \to Y$ whose composite $qf : S^{2n+1} \to S^{2n+1}$ is homotopic to the identity.

If $m = 2n$ is even and $\alpha \in \pi_{2n}(X)$ is an element with infinite order, then there is a map $f : X \to BU(n)$ such that $f_*(\alpha) \in \pi_{2n}(BU(n))$ has infinite order.

(2) *Let p be a prime and X a simply connected, p-local pseudo-finite* CW-*complex which is rationally $m-1$-connected and $\pi_m(X)\otimes\mathbb{Q} \neq 0$.*

If $m = 2n+1$ is odd, then there is a simply connected, p-local pseudo-finite CW-*complex Y with a rational equivalence $X \to Y$ and maps $q : Y \to S^{2n+1}_{(p)}$ and $f : S^{2n+1}_{(p)} \to Y$ whose composite $qf : S^{2n+1}_{(p)} \to S^{2n+1}_{(p)}$ is homotopic to the identity. Moreover, the homotopy fibre of the map q has the homotopy type of a simply connected, p-local pseudo-finite* CW-*complex.*

If $m = 2n$ is even and $\alpha \in \pi_{2n}(X)$ is an element with infinite order, then there is a map $f : X \to BU(n)_{(p)}$ such that $f_*(\alpha) \in \pi_{2n}(BU(n)_{(p)})$ has infinite order.

Proof. Case I. $m = 2n+1$ is odd in (1) and (2). Here we will prove only the p-local case since the integral case can be proven quite similarly. Thus in this case we assume that all spaces are localised at the prime p.

By Lemma 2.2 we assume that $2n$-skeleton of X is the basepoint. Since X is pseudo-finite, there is a finite series of subcomplexes $X'_0 = * \subset X'_1 \subset X'_2 \subset$

$\cdots \subset X'_{s-1} \subset X'_s = X$ such that each subcomplex X'_i is obtained from X'_{i-1} by attaching cells. By taking the union of $sk_{2n+2}(X)$ and X'_i we have a new finite series of subcomplexes $X_0 = sk_{2n+2}(X) \subset X_1 \subset X_2 \subset \cdots \subset X_{s-1} \subset X_s = X$ with the same property. Using induction on i we will construct simply connected CW-complexes $\{Y_i^j\}_{i=0,1,2,\cdots}^{j=1,2,\cdots}$ together with maps $\varphi_i : X_i \to Y_i^1$, $\varphi_i^j : Y_i^j \to Y_i^{j+1}$, $f_i^j : S^{2n+1} \to Y_i^j$ and $q_i^j : Y_i^j \to S^{2n+1}$ which satisfy the following conditions:

(1) The cofibre spaces of the maps $\varphi_i : X_i \to Y_i^1$ and $\varphi_i^j : Y_i^j \to Y_i^{j+1}$ for $j = 1, 2, \cdots$ are all finite torsion, where a space is said to be finite torsion if each of its integral homology groups is a finite torsion group. In particular, $\varphi_i : X_i \to Y_i^1$ and $\varphi_i^j : Y_i^j \to Y_i^{j+1}$ for $j = 1, 2, \cdots$ are rational equivalences,

(2) $q_i^j f_i^j \simeq \mathrm{id}_{S^{2n+1}}$ for $j = 1, 2, \cdots$, and

(3) the following diagram is commutative up to homotopy, where $p : S^{2n+1} \to S^{2n+1}$ is a degree p map.

$$\begin{array}{ccccc}
 & & X_i & & \\
 & & \downarrow{\scriptstyle \varphi_i} & & \\
S^{2n+1} & \xrightarrow{f_i^1} & Y_i^1 & \xrightarrow{q_i^1} & S^{2n+1} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \varphi_i^1} & & \downarrow{\scriptstyle p} \\
S^{2n+1} & \xrightarrow{f_i^2} & Y_i^2 & \xrightarrow{q_i^2} & S^{2n+1} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \varphi_i^2} & & \downarrow{\scriptstyle p}
\end{array}$$

For $i = 0$ the construction is clear. $X_0 = sk_{2n+2}(X)$ is a wedge of $\vee S^{2n+2}$ and a Moore space of type $(H_{2n+1}(X;\mathbb{Z}), 2n+1)$ since $2n$-skeleton of X is the basepoint. By assumption that $\pi_{2n+1}(X) \otimes \mathbb{Q} \cong H_{2n+1}(X;\mathbb{Q}) \neq 0$, there is a direct summand $\mathbb{Z}_{(p)}$ in $H_{2n+1}(X;\mathbb{Z})$. Then there are maps $f' : S^{2n+1} \to X_0$ and $q' : X_0 \to S^{2n+1}$ such that $q'f' : S^{2n+1} \to S^{2n+1}$ is homotopic to the identity. Thus, $Y_0^j = X_0$, $\varphi_0 = \mathrm{id}$, $f_0^j = f'$, $q_0^j = q'$ and $\varphi_0^j = p$ for all $j \geq 1$ satisfy the all conditions.

In the induction step, assume that we have constructed the spaces and maps for i. Let $\psi_i : \vee_\alpha S^{n_\alpha} \to X_i$ be the attaching map for X_{i+1}. Since $n_\alpha > 2n+1$ for each α, the map $q_i^1 \varphi_i \psi_i : \vee_\alpha S^{n_\alpha} \to S^{2n+1}$ is rationally null homotopic. Then the composite $q_i^k \varphi_i^k \cdots \varphi_i^1 \varphi_i \psi_i \simeq p^k q_i^1 \varphi_i \psi_i : \vee_\alpha S^{n_\alpha} \to S^{2n+1}$ must be null homotopic for a sufficiently large k. Because spaces are assumed to be p-local and the exponent of the p-torsion in the homotopy groups of a sphere is finite by James [8] and Toda [14]. Thus we assume that $q_i^k \varphi_i^k \cdots \varphi_i^1 \varphi_i \psi_i$ is null homotopic. Then we put

$$Y_{i+1}^{\ell} = Y_i^{k+\ell} \cup_{\varphi_i^{k+\ell} \cdots \varphi_i^1 \varphi_i \psi_i} \vee_\alpha e^{n_\alpha + 1}.$$

Now it is easy to construct all maps with the property (2) and (3). Since there is the following commutative diagram of cofibration, the cofibre space of the map $\varphi_{i+1}^1 : X_{i+1} \to Y_{i+1}^1$ is clearly finite torsion.

$$\begin{array}{ccccc} \vee_\alpha S^{n_\alpha} & \xrightarrow{\psi_i} & X_i & \longrightarrow & X_{i+1} \\ \| & & \downarrow{\scriptstyle \varphi_i^{k+1}\cdots\varphi_i^1\varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}^1} \\ \vee_\alpha S^{n_\alpha} & \xrightarrow{\varphi_i^{k+1}\cdots\varphi_i^1\varphi_i\psi_i} & Y_i^{k+1} & \longrightarrow & Y_{i+1}^1 \end{array}$$

Similarly so are the cofibre spaces of the maps $\varphi_{i+1}^j : Y_{i+1}^j \to Y_{i+1}^{j+1}$ for $j = 1, 2, \cdots$.

Finally we put $Y = Y_s^1$, $\varphi = \varphi_s : X = X_s \to Y = Y_s^1$, $f = f_s^1 : S^{2n+1} \to Y = Y_s^1$ and $q = q_s^1 : Y = Y_s^1 \to S^{2n+1}$. By construction, the cofibre of the map $\varphi : X \to Y$ is finite torsion and X is of finite type over the ring $\mathbb{Z}_{(p)}$. Therefore Y is also of finite type. Clearly f is a rational equivalence, $qf : S^{2n+1} \to S^{2n+1}$ is homotopic to the identity and Y is a simply connected, pseudo-finite CW-complex. This completes the first assertion.

By F we denote the homotopy fibre of the map q. In the fibre sequence

$$\Omega S^{2n+1} \to F \to Y \to S^{2n+1},$$

the map $\Omega S^{2n+1} \to F$ is null homotopic since the fibration $q : Y \to S^{2n+1}$ has the cross section f. By construction there is a finite series of subcomplexes $Y_1 = S^{2n+1} \subset Y_2 \subset \cdots \subset Y_{s-1} \subset Y_s = Y$ such that each subcomplex Y_i is obtained from Y_{i-1} by attaching cells. We may also assume that the map $f : S^{2n+1} \to Y$ is just the inclusion $S^{2n+1} = Y_1 \to Y$. By restricting the fibration $F \to Y$ to each subcomplex Y_i we obtain the fibration

$$\Omega S^{2n+1} \to F_i \to Y_i.$$

Here we remark that this fibration is induced from the principal path fibration $\Omega S^{2n+1} \to \mathrm{P}S^{2n+1} \to S^{2n+1}$ by restricting the map $q : Y \to S^{2n+1}$ to Y_i. By induction on i we will show that each F_i has the homotopy type of a p-local pseudo-finite CW-complex. Since F_1 is contractible, this is clear. Now we assume that we have proved that F_{i-1} has the homotopy type of a p-local pseudo-finite CW-complex for $i > 1$. Let $Y_i = Y_{i-1} \cup \vee_\alpha e^{n_\alpha+1}$. The fibration restricted to each cell $e^{n_\alpha+1}$ is fibre homotopy equivalent to the trivial fibration. Thus we have a fibration

$$F_{i-1} \cup \vee_\alpha e^{n_\alpha+1} \times \Omega S^{2n+1} \to Y_i$$

which is fibre homotopy equivalent to the fibration $F_i \to Y_i$. Thus we have

$$\begin{aligned} F_i &\simeq F_{i-1} \cup \vee_\alpha e^{n_\alpha+1} \times \Omega S^{2n+1}, \\ &\simeq F_{i-1} \cup \vee_\alpha C(S^{n_\alpha} \wedge \Omega S_+^{2n+1}), \\ &\simeq F_{i-1} \cup \vee_\alpha C(S^{n_\alpha} \vee S^{2n+n_\alpha} \vee S^{4n+n_\alpha} \vee \cdots), \\ &\simeq F_{i-1} \cup \vee_\alpha (e^{n_\alpha+1} \vee e^{2n+n_\alpha+1} \vee e^{4n+n_\alpha+1} \vee \cdots). \end{aligned}$$

In the second homotopy equivalence we used the following fact. Since the inclusion $\Omega S^{2n+1} \to e^{n_\alpha+1} \times \Omega S^{2n+1}$ is null homotopic in F_i, there is an embedding $C\Omega S^{2n+1}$ into $F_{i-1} \cup \vee_\alpha e^{n_\alpha+1} \times \Omega S^{2n+1}$ for each α. Needless to say, collapsing these cones to a point does not change its homotopy type. Clearly F is simply connected.

Case II. $m = 2n$ is even in (1). Since X is simply connected and rationally $(2n-1)$-connected, the Hurewicz image $H(\alpha)$ has also infinite order in $H_{2n}(X;\mathbb{Z})$. Take a cohomology class $[g] \in H^{2n}(X;\mathbb{Z})$ such that $\langle [g], H(\alpha)\rangle \neq 0$.

We consider the lifting problem

$$\begin{array}{ccc} & & BU(n) \\ & \nearrow & \downarrow c_n \\ X & \xrightarrow{\;g\;} & K(\mathbb{Z},2n) \end{array}$$

where c_n is the n-th Chern class. Since $BU(n)_{(0)} \simeq \prod_{k\geq 2}^{n} K(\mathbb{Q},2k)$, there is a lift

$$\begin{array}{ccccc} & & & & BU(n)_{(0)} \\ & & & \nearrow & \downarrow c_n \\ X & \longrightarrow & K(\mathbb{Z},2n) & \longrightarrow & K(\mathbb{Q},2n). \end{array}$$

Let N be a positive integer larger than $\dim X$ and $2n$. The Postnikov approximation $BU(n)^{(N)}$ of $BU(n)$ through dimension N is 0-universal by Theorem 1.2 of [16], see also Proposition 4.1 of [10]. That is, its rationalisation $BU(n)^{(N)}_{(0)} \simeq BU(n)_{(0)}$ can be constructed as the infinite mapping telescope using a family of self maps. Then a lift $X \to BU(n)_{(0)}$ factors through $BU(n)^{(N)}$. Since X is a finite complex whose dimension is less than N, it in fact factors through $BU(n)$. Thus we have solved the lifting problem

$$\begin{array}{ccc} & & BU(n) \\ & \overset{g'}{\nearrow} & \downarrow c_n \\ X & \xrightarrow{\;k\cdot g\;} & K(\mathbb{Z},2n) \end{array}$$

for some non-zero integer k. Clearly the map $f = g'$ has the desired property.

Case III. $m = 2n$ is even in (2). By J_qX we denote the q-th filtration of the James construction $JX \simeq \Omega\Sigma X$. Here again we assume that all spaces are localised at the prime p.

First we construct a map $f' : X \to J_\ell S^{2n}$ for sufficiently large ℓ such that $f'_*(\alpha) \in \pi_{2n}(J_\ell S^{2n}) \cong \mathbb{Z}_{(p)}$ is non-zero by the similar method to in the proof of the odd case because $J_{p^k-1}S^{2n}$ has finite homotopy exponent for the prime p. This fact follows from the existence of the following p-local fibration [14]:

$$J_{p^k-1}S^{2n} \to \Omega S^{2n+1} \to \Omega S^{2np^k+1}.$$

Then we compose the map f' with a map $J_\ell S^{2n} \to BU(n)$ which is essential on the bottom cell. The existence of such a map was proved in Case II.

More explicitly we prove this as follows. By Lemma 2.2 we assume that $2n-1$-skeleton of X is the basepoint. Since X is pseudo-finite, there is a finite series of subcomplexes $X_0 = sk_{2n}(X) = \vee S^{2n} \subset X_1 \subset X_2 \subset \cdots \subset X_{s-1} \subset X_s = X$ such that each subcomplex X_i is obtained from X_{i-1} by attaching cells. By $\alpha : S^{2n} \to X_0 = sk_{2n}(X) \subset X$ we also denote a cellular map which represents a homotopy class $\alpha \in \pi_{2n}(X)$. There is a map $f_0 : X_0 \to S^{2n} \subset J_{p-1}S^{2n}$ such that $f_0\alpha : S^{2n} \to J_{p-1}S^{2n}$ is essential. Inductively we will construct maps $f_i : X_i \to J_{p^{i+1}-1}S^{2n}$ such that $f_{i*}(\alpha) \in \pi_{2n}(J_{p^{i+1}-1}S^{2n}) \cong \mathbb{Z}_{(p)}$ is essential. Let $\psi_i : \vee_j S^{n_j} \to X_i$ be the attaching map for X_{i+1} and $\varphi_i : J_{p^i-1}S^{2n} \to J_{p^{i+1}-1}S^{2n}$ be the inclusion map. Although the composite $f_i\psi_i : \vee_j S^{n_j} \to J_{p^{i+1}-1}S^{2n}$ may be rationally essential, $\varphi_{i+1}f_i\psi_i : \vee_j S^{n_j} \to J_{p^{i+1}-1}S^{2n} \to J_{p^{i+2}-1}S^{2n}$ must be rationally null homotopic. Then for sufficiently large t, $p^t\varphi_{i+1}f_i\psi_i : \vee_j S^{n_j} \to J_{p^{i+2}-1}S^{2n}$ is null homotopic. Thus we can extend the map $p^t\varphi_{i+1}f_i : X_i \to J_{p^{i+2}-1}S^{2n}$ to $f_{i+1} : X_{i+1} \to J_{p^{i+2}-1}S^{2n}$. Then let $f' = f_s : X = X_s \to J_{p^{s+1}-1}S^{2n}$. Take a map $g : J_{p^{s+1}-1}S^{2n} \to BU(n)$ which is essential on the bottom cell. The composite $f = gf' : X \to BU(n)$ has a desired property and we complete the proof. □

3. Proofs of main results

In this section we first prove Theorem 2 and its corollaries, then we prove Theorem 1 by using these results and a result of McGibbon and Wilkerson.

Proof of Theorem 2. In this proof we assume that all spaces are simply connected as was remarked in Section 2 and localised at the prime p.

First we prove the theorem for the case $k = 1$. To prove this it is sufficient to prove the following theorem.

Theorem 3.1. *Let X be a simply connected, p-local pseudo-finite* CW*-complex, $\alpha : S^N \to X$ be a map whose homotopy class $[\alpha] \in \pi_N(X)$ has infinite order and $\tilde{\alpha}$ denote the adjoint map to α.*

If N is odd (resp. even), then there exists a map $\beta : \Omega X \to \Omega S^N$ (resp. $\beta : \Omega X \to S^{N-1}$) such that the composite $\beta\tilde{\alpha} : S^{N-1} \to \Omega X \to \Omega S^N$ (resp. $\beta\tilde{\alpha} : S^{N-1} \to \Omega X \to S^{N-1}$) is rationally essential.

Proof. The proof uses induction on the rational homotopy rank of X in dimensions less than N, that is, rank $= \Sigma_{i<N} \dim \pi_i(X) \otimes \mathbb{Q}$. We assume that X is a simply connected, pseudo-finite CW-complex which is rationally $m-1$-connected and $\pi_m(X) \otimes \mathbb{Q} \neq 0$, where we may assume $m \leq N$.

In the first induction step (i.e., rank = 0) of the induction argument, N must be m. We use a space and maps constructed in Lemma 2.3. If N is odd, then we can take $\beta = \Omega(q\varphi)$. If $N = 2n$ is even, then the composite

$\beta = \pi\Omega f : \Omega X \to U(n) \to U(n)/U(n-1) = S^{2n-1}$ has the required property, where $\pi : U(n) \to U(n)/U(n-1)$ is the canonical quotient map.

Now we proceed on the induction step.

If $m = 2n+1$ is odd, then by Lemma 2.3 there is a simply connected, pseudo-finite CW-complex Y with a rational equivalence $\varphi : X \to Y$ and a map $q : Y \to S^{2n+1}$. Since q has a cross section $f : S^{2n+1} \to Y$, ΩY is homotopy equivalent to $\Omega F \times \Omega S^{2n+1}$. Consider the map

$$h : \Omega X \stackrel{\Omega\varphi}{\to} \Omega Y \simeq \Omega F \times \Omega S^{2n+1} \to \Omega F$$

$h_*(\tilde{\alpha}) \in \pi_{N-1}(\Omega F) \cong \pi_N(F)$ has infinite order since h induces an isomorphism $h_* : \pi_{N-1}(\Omega X) \otimes \mathbb{Q} \to \pi_{N-1}(\Omega F) \otimes \mathbb{Q}$. Then $\Sigma_{i<N}\ \dim \pi_i(F) \otimes \mathbb{Q}$ is one less than that of X. Since F is pseudo-finite by Lemma 2.3, if N is odd (resp. even) then by induction hypothesis there exists a map $\beta' : \Omega F \to \Omega S^N$ (resp. $\beta' : \Omega F \to S^{N-1}$) with the required property. $\beta = \beta' h$ satisfies the condition.

Now we consider the case $m = 2n$ is even. For $m > 2$ let $\alpha' : S^m \to X$ be any map whose homotopy class $[\alpha'] \in \pi_m(X)$ has infinite order. Then there is a map $f : X \to BU(n)$ such that $f\alpha' : S^{2n} \to BU(n)$ is rationally essential by Lemma 2.3. For $m = 2$ let $\alpha' : S^2 \to X$ be a map whose homotopy class generates a direct summand of order infinite in $\pi_2(X)$ and $f : X \to BU(1) = K(\mathbb{Z}_{(p)}, 2)$ be a map such that the composite $f\alpha' : S^2 \to X \to K(\mathbb{Z}_{(p)}, 2)$ is a generator of $\pi_2(K(\mathbb{Z}_{(p)}, 2)) \cong \mathbb{Z}_{(p)}$. Consider the pullback diagram of fibrations:

$$\begin{array}{ccc} S^{2n-1} & = & S^{2n-1} \\ \downarrow{\scriptstyle j} & & \downarrow \\ Y & \longrightarrow & BU(n-1) \\ \downarrow{\scriptstyle r} & & \downarrow \\ X & \stackrel{f}{\longrightarrow} & BU(n) \end{array}$$

Clearly Y is pseudo-finite. Apply $\pi_*(\ )$ to this diagram and observe that

(1) Y is simply connected even if $n = 1$,

(2) $\Sigma_{i<N}\ \dim \pi_i(Y) \otimes \mathbb{Q}$ is one less than that of X,

(3) $[j] \in \pi_{2n-1}(Y)$ has finite order, and

(4) $r_* : \pi_N(Y) \otimes \mathbb{Q} \to \pi_N(X) \otimes \mathbb{Q}$ is isomorphic.

If $[j]$ is of order k ($k = 1$ if j is null homotopic), then j can be extented to a map $j' : M^{2n-1}(k) = S^{2n-1} \cup_k e^{2n} \to Y$. Put $Y' = Y \cup_{j'} CM^{2n-1}(k)$ and let $h : Y \to Y'$ be the canonical inclusion map. Since $hj : S^{2n-1} \to Y'$ is null homotopic by the construction of Y', we can construct the following homotopy commutative diagram of fibrations.

$$
\begin{array}{ccccccccc}
\Omega Y & \xrightarrow{\Omega r} & \Omega X & \longrightarrow & S^{2n-1} & \xrightarrow{j} & Y & \xrightarrow{r} & X \\
\Big\downarrow{\scriptstyle \Omega h} & & \Big\downarrow{\scriptstyle \bar{h}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle h} & & \\
\Omega Y' & = & \Omega Y' & \longrightarrow & * & \longrightarrow & Y' & = & Y'
\end{array}
$$

Since $h : Y \to Y'$ is a rational equivalence, the pair Y' and $\bar{h}_*(\tilde{\alpha}) \in \pi_{N-1}(\Omega Y') \cong \pi_N(Y')$ satisfy the induction assumption. Thus we complete the proof by the induction hypothesis just as in the odd case.

Let $k > 1$ and

$$\Omega_0 X \to \prod_\alpha S^{2n_\alpha - 1} \times \prod_\beta \Omega S^{2n_\beta - 1}$$

be a rational equivalence. Loop this map $k-1$ times and we obtain another rational equivalence

$$\Omega_0^k X \to \prod_\alpha \Omega_0^{k-1} S^{2n_\alpha - 1} \times \prod_\beta \Omega_0^k S^{2n_\beta - 1}.$$

Since there are p-local maps $\Omega S^{2n+1} \to S^{2n-1}$ with degree p on the bottom cell by Cohen, Moore, and Neisendorfer [3, 4, 13], a desired map will be given as a composite of these rational equivalences. □

The following lemma is an extension of a result of Halperin (Lemma 5 of [5]) and is necessary to prove Corollary 3. For a minimal model $\Lambda(x_1, x_2, \cdots)$, where $\deg x_{i-1} \leq \deg x_i$ for each $i \geq 2$, by $\Lambda(x_n, x_{n+1}, \cdots)$ we denote the minimal model $\Lambda(x_1, x_2, \cdots)/(x_1, \cdots, x_{n-1})$.

Lemma 3.2. *Let X be a simply connected, pseudo-finite* CW*-complex and $\Lambda(x_1, x_2, \cdots)$ be its minimal model, where $\deg x_{i-1} \leq \deg x_i$ for each $i \geq 2$. Let $[x_n] \in H^*(\Lambda(x_n, x_{n+1}, \cdots))$ be the class represented by x_n. Then $[x_n]$ is nilpotent for each n.*

Proof. In the proofs of Lemma 2.3 and Theorem 3.1 it was proved that every n-connected cover of X is rationally pseudo-finite. Since in the cohomology ring of a pseudo-finite CW-complex every cohomology class is nilpotent, the assertion of the lemma is clear. □

Proof of Corollary 3. (1) is a corollary of Theorem 7.5 of [7].

(2) By $\mathrm{Ph}(X, Y)$ we denote the set of all homotopy classes of phantom maps from X to Y. If

$$\pi_q(\Omega_0^k X) \otimes \mathbb{Q} = 0 \qquad \text{for all } q \geq 2,$$

then there is a rational equivalence $\Omega_0^k X \to \prod S^1$. By Theorem 8.7 of [9] $\mathrm{Ph}(\Omega_0^k X, Y) = *$ for every finite type target Y.

Next we assume that

$$\pi_q(\Omega_0^k X) \otimes \mathbb{Q} \cong \pi_{q+k}(X) \otimes \mathbb{Q} \neq 0 \qquad \text{for some } q \geq 2.$$

We note that in this case $\pi_n(X) \otimes \mathbb{Q} \neq 0$ for some odd $n \geq k+2$. Otherwise the minimal model of the space X is given by $\Lambda(x_1, x_2, \cdots)$, where $\deg x_{i-1} \leq \deg x_i$ for each $i \geq 2$ and $\deg x_i$ is even if it is greater than or equal to $k+2$. Let $\Lambda(x_m, x_{m+1}, \cdots)$ be the minimal model obtained from $\Lambda(x_1, x_2, \cdots)$ by killing the generators of degree less than $k+2$. Then for the degree reason, $dx_i = 0$ in $\Lambda(x_m, x_{m+1}, \cdots)$ for all $i \geq m$. Thus the class $[x_m]$ has the infinite height in the cohomology. This contradicts Lemma 3.2.

Thus we can choose a map $S^n \to X$ for some odd integer $n \geq k+2$ whose homotopy class in $\pi_n(X)$ has infinite order. Then its k-fold loop map $\Omega^k S^n \to \Omega_0^k X$ induces a monomorphism between the rational homology groups by Theorem 2. By Theorem 7.3 of [9] this map induces an epimorphism of pointed sets

$$\mathrm{Ph}(\Omega_0^k X, S^m) \to \mathrm{Ph}(\Omega^k S^n, S^m).$$

By Proposition 8.2 of [9] for a suitable m the set $\mathrm{Ph}(\Omega^k S^n, S^m) \neq *$. Then $\mathrm{Ph}(\Omega_0^k X, S^m) \neq *$.

(3) is a corollary of Theorem 2 and [9], Theorem 8.7. □

To prove Corollary 4 first we prove

Proposition 3.3. *Let X, X' be simply connected finite complexes. If there is a rational equivalence $f : X \to X'$, then $\mathrm{Ph}(\Omega X, Y) = *$ if and only if $\mathrm{Ph}(\Omega X', Y) = *$, whenever Y is a finite type target.*

Proof. First we recall the notation about localisation. For a set $\mathbf{P}$ of primes we set

$$\mathbb{Q}_{\mathbf{P}} = \left\{ \frac{n}{m} \in \mathbb{Q}; m \text{ is coprime to all primes in } \mathbf{P} \right\}$$

and $\mathbb{Q}_{\mathbf{P}}$-localisation of a nilpotent space X will be denoted by $X_{\mathbf{P}}$. For example, if $\mathbf{P} = \{p\}$ then $X_{\mathbf{P}} = X_{(p)}$. By $\bar{\mathbf{P}}$ we denote the set of all primes which are *not* in $\mathbf{P}$.

To prove the proposition we use the tower approach, see Section 4 of [9]. $\mathrm{Ph}(\Omega X, Y)$ is isomorphic to the $\lim^1$ term of a tower $\{[\Sigma sk_n(\Omega X), Y]\}$. Each group of the tower is countable. This fact can be proven by induction on n using the following exact sequence.

$$\cdots \to \oplus\pi_{n+2}(Y) \to [\Sigma sk_{n+1}(\Omega X), Y] \to [\Sigma sk_n(\Omega X), Y] \to \cdots$$

Thus its $\lim^1$ term is trivial if and only if the tower satisfies the Mittag-Leffler condition by Theorem 4.4 of [9]. To complete the proof we need the following.

Lemma 3.4. *Let $G = \{G_1 \leftarrow G_2 \leftarrow G_3 \cdots\}$ be a tower of nilpotent groups and $\mathbf{P}_1, \cdots, \mathbf{P}_k$ be sets of primes such that the union of the sets is equal to the set of all primes. Then G satisfies Mittag-Leffler condition if and only if so does when localised at each $\mathbf{P}_i$ for $i = 1, \cdots, k$.*

Proof. Let $G_n^{(m)} = \mathrm{image}\{G_n \leftarrow G_m\}$ for $n \leq m$. Then G satisfies Mittag-Leffler condition if and only if for each n there exists an integer N, which depends on n, such that

$$G_n^{(N)} = G_n^{(m)} \quad \text{for all } m \geq N.$$

Let R be a subring of $\mathbb{Q}$. According to the notation of Bousfield-Kan the R-localisation (or Malcev completion) of a nilpotent group N is denoted by $R \otimes N$.

We assume that for each i the tower $\mathbb{Q}_{\mathbf{P}_i} \otimes G$ satisfies Mittag-Leffler condition. Then there exists an integer N, depending on n, such that for each i

$$(\mathbb{Q}_{\mathbf{P}_i} \otimes G_n)^{(N)} = (\mathbb{Q}_{\mathbf{P}_i} \otimes G_n)^{(m)} \quad \text{for all } m \geq N.$$

Since the Marcev completion is an exact functor [1], Ch. V, 2.4, this implies that

$$\mathbb{Q}_{\mathbf{P}_i} \otimes G_n^{(N)} = \mathbb{Q}_{\mathbf{P}_i} \otimes G_n^{(m)} \quad \text{for all } m \geq N.$$

By the definition of nilpotent groups and the exactness of the Marcev completion this implies that the tower satisfies Mittag-Leffler condition.

The opposite implication being clear, the lemma follows. □

Now we return to the proof of Proposition 3.3.

Let G (resp. G') be the tower $\{[\Sigma sk_n(\Omega X), Y]\}$ (resp. $\{[\Sigma sk_n(\Omega X'), Y]\}$). Since X and X' are simply connected, each group of these towers is nilpotent by Whitehead, [15], Ch. X, Section 3. Thus Lemma 3.4 can apply to these towers. It is easy to see that there is a finite set $\mathbf{P}$ of primes such that $f_{\bar{\mathbf{P}}} : X_{\bar{\mathbf{P}}} \to X'_{\bar{\mathbf{P}}}$ is homotopy equivalent. Thus the tower $\mathbb{Q}_{\bar{\mathbf{P}}} \otimes G$ satisfies Mittag-Leffler condition if and only if so does $\mathbb{Q}_{\bar{\mathbf{P}}} \otimes G'$. For each prime p in $\mathbf{P}$, $\mathbb{Z}_{(p)} \otimes G$ and $\mathbb{Z}_{(p)} \otimes G'$ satisfy Mittag-Leffler condition by Theorem 2. Therefore, by Lemma 3.4, G satisfies Mittag-Leffler condition if and only if so does G'. □

Proof of Corollary 4. We assume that there are no essential phantom maps from ΩX to a finite type target. Let $X = X_1 \to X_2 \to \cdots$ be a 0-sequence of X in the sence of Mimura-Nishida-Toda [12], i.e., its telescope construction gives the rationalisation of X. Then, by Proposition 3.3, for each i there are no essential phantom maps from ΩX_i to a finite type target. Since X' is finite and rationally homotopy equivalent to X, for some i there exists a rational equivalence $X' \to X_i$. Thus, by Proposition 3.3, there are no essential phantom maps from $\Omega X'$ to a finite type target. □

The proof of Corollary 4 implies also

Proposition 3.5. *Let X be a simply connected finite complex. Then there is an essential phantom map from ΩX to a finite type target if and only if there is a finite set $\mathbf{P}$ of primes such that there is an essential phantom map from $\Omega X_{\bar{\mathbf{P}}}$ to a finite type target.*

Finally we will finish the proof of Theorem 1.

Proof of Theorem 1. McGibbon and Wilkerson [11] proved that for a simply connected, rationally elliptic, finite complex X there is a finite set $\mathbf{P}$ of primes such that there is a $\bar{\mathbf{P}}$-equivalence

$$\Omega X \simeq_{\bar{\mathbf{P}}} \prod_{\alpha} S^{2n_\alpha - 1} \times \prod_{\beta} \Omega S^{2n_\beta - 1}.$$

Thus this result and Proposition 3.5 implies Theorem 1. □

Department of Applied Mathematics
Osaka Women's University
Sakai, Osaka 590-0035, JAPAN
e-mail: kiriye@appmath.osaka-wu.ac.jp

References

[1] A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localization, Springer Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin, 1972.

[2] G. Carlsson and R. J. Milgram, Stable homotopy and iterated loop spaces, Chapter 13 in The Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995.

[3] F. R. Cohen, Two-primary analogues of Selick's theorem and the Kahn-Priddy theorem for the 3-sphere, Topology, **23** (1984), 401–422.

[4] F. R. Cohen, J. C. Moore and J. A. Neisendorfer, Torsion in homotopy groups, Ann. of Math., **109** (1979), 121–168.

[5] S. Halperin, Finiteness in the minimal models of Sullivan, Trans. Amer. Math. Soc., **230** (1977), 173–199.

[6] K. Iriye, The space of loops on a homogeneous space and rational equivalence, Bull. London Math. Soc., **31** (1999), 484–488.

[7] K. Iriye, Universal phantom maps out of loop spaces, Proc. Roy. Soc. Edinburgh 130A (2000), 313–333.

[8] I. M. James, On the suspension sequence, Ann. of Math., **65** (1957), 74–107.

[9] C. A. McGibbon, Phantom maps, Chapter 25 in The Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995.

[10] C. A. McGibbon, On the localization genus of a space, Prog. in Math., **136** (1996), 285–306.

[11] C. A. McGibbon and C. W. Wilkerson, Loop spaces of finite complexes at large primes, Proc. Amer. Math. Soc., **96** (1986), 698–702.

[12] M. Mimura, G. Nishida and H. Toda, Localization of CW-complexes and its applications, J. Math. Soc. of Japan, **23** (1971), 593–624.

[13] J. A. Neisendorfer, 3-primary exponents, Math. Proc. Cambridge Phil. Soc., **90** (1981), 63–83.

[14] H. Toda, On the double suspension E^2, J. Inst. Polytech. Osaka City Univ. Ser. A, **7** (1956), 103–145.

[15] G. W. Whitehead, Elements of homotopy theory, Springer-Verlag, Berlin, 1978.

[16] A. Zabrodsky, p-equivalences and homotopy type, Springer Lecture Notes in Math., vol. 418, Springer-Verlag, Berlin, (1974), 161–171.