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# Plumbed homology 3-spheres bounding acyclic 4-manifolds

By

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**Abstract**

The main purpose of this paper is to give an explicit formula of a homology cobordism invariant of plumbed homology 3-spheres which was defined in a joint work with M. Furuta by using the Seiberg-Witten monopole equation on 4-dimensional V-manifolds [8]. This formula provides a sufficient condition for homology 3-spheres of plumbing type to have infinite order in the homology cobordism group.

## 1. Introduction

In a joint work with M. Furuta, we defined a homology cobordism invariant which is an integral lift of the Rohlin invariant [8]. The main purpose of this paper is to give an explicit formula of this invariant for plumbed homology 3-spheres. To compute this invariant, we generalized the notion of plumbing to the category of V-manifolds. For the definitions concerning V-manifolds, see [22]. Then we applied the Atiyah-Singer-Kawasaki V-index theorem [15] to obtain the explicit formula of this invariant. This formula provides a sufficient condition for plumbed homology 3-spheres to have infinite order in the homology cobordism group. Throughout this paper, we work in the category of smooth oriented (V-)manifolds.

M. Furuta [9] constructed a finite dimensional approximation of the Seiberg-Witten monopole equation and proved that any closed indefinite spin 4-manifold $X$ satisfies the $\frac{10}{8}$-inequality $\frac{5}{4}|\operatorname{sign} X|+2 \leq b_2(X)$. N. Saveliev [24] proved that a certain class of Seifert fibered homology 3-spheres have infinite order in the homology cobordism group by constructing spin 4-manifolds with boundaries in the framed link calculus which violate the $\frac{10}{8}$-inequality. In a joint work with M. Furuta, we used the V-manifold version of the $\frac{10}{8}$-inequality to define a homology cobordism invariant for some classes of integral homology 3-spheres [8]. For a triple $(\Sigma, X, c)$ consisting of a homology 3-sphere $\Sigma$, a 4-V-manifold $X$ with boundary $\Sigma$, and a V-spin$^c$-structure $c$ on $X$, we defined a $\mathbf{Z}$-valued invariant $w(\Sigma, X, c)$. Let $\mathcal{S}(k^+, k^-)$ be the set of homology 3-spheres

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$\Sigma$ such that there exists a spin 4-V-manifold $X$ satisfying $b_2^{\pm}(X) \leq k^{\pm}$. If we assume $k^+ + k^- \leq 2$ then $w(\Sigma, X, c)$ does not depend on the pair $(X, c)$ of a spin 4-V-manifold $X$ with boundary $\Sigma$ satisfying $b_2^{\pm}(X) \leq k^{\pm}$ and a V-*spin* structure $c$, and furthermore the map:

$$\mathcal{S}(k^+, k^-) \ni \Sigma \longmapsto w(\Sigma, X, c) \in \mathbf{Z}$$

gives a homology cobordism invariant. This invariant is an integral lift of the Rohlin invariant.

To apply this invariant to homology 3-spheres of plumbing type, we generalized the notion of plumbing to the V-manifold category. In this paper, we consider plumbing only among smooth points. It is possible to consider plumbing among V-singular points, but it requires some more complicated treatment, and so we describe only an example concerning Kirby problem 4.28 in Section 8. First we define a notion of Seifert graphs $\Gamma = (V, E, \omega)$ as follows. (1) $(V, E)$ is a connected *tree* graph consisting of a set of vertices $V$ and a set of edges $E$. (2) Each vertex $k \in V$ is assigned a Seifert invariant:

$$\omega(k) = \{b_k; (\alpha_{k1}, \beta_{k1}), \ldots, (\alpha_{kn_k}, \beta_{kn_k})\} \ \ (k \in V),$$

where $b_k$ are integers, and $(\alpha_{ki}, \beta_{ki})$ are coprime integers satisfying $1 \leq \beta_{ki} \leq \alpha_{ki} - 1$.

A plumbed 4-V-manifold $P(\Gamma)$ is constructed from a Seifert graph $\Gamma$ as follows. For each vertex $k \in V$, we take the Seifert fibered space with Seifert-invariant $\omega(k)$. This Seifert fibered space can be thought of as an $S^1$-V-bundle over a Riemannian V-sphere $Z_k$. We construct the associated disk V-bundle $DL_k$. If two vertices $k$, $k'$ are connected by an edge, then we choose a local trivialization of each disk V-bundle $DL_k|_{D_{kk'}} \cong D_{kk'} \times D^2$ respectively and glue them up by the map:

$$DL_k|_{D_{kk'}} \cong D_{kk'} \times D^2 \ni (z, w) \longmapsto (w, z) \in D_{k'k} \times D^2 \cong DL_{k'}|_{D_{k'k}}.$$

The plumbed 4-V-manifold $P(\Gamma)$ has singularities of the form of the cone on the lens space. If we denote by $I_\Gamma$ the rational intersection matrix of $P(\Gamma)$ then the $(k, k')$-entry of $I_\Gamma$ is:

$$(I_\Gamma)_{k,k'} = \begin{cases} e_k & k = k' \\ 1 & (k, k') \in E \\ 0 & \text{otherwise}, \end{cases}$$

where $e_k := b_k + \sum_{i=1}^{n_k} \beta_{ki}/\alpha_{ki}$. Let us denote the boundary of the plumbing $P(\Gamma)$ by $\Sigma(\Gamma)$. Then we see that $\Sigma(\Gamma)$ is a homology 3-sphere if and only if:

$$\text{(HS)} \quad \det I_\Gamma = \pm \frac{1}{\prod_{k \in V} \alpha_k}, \quad \alpha_k := \prod_i \alpha_{ki}.$$

Suppose $\Gamma$ satisfies (HS). Then $P(\Gamma)$ is V-spin if and only if:

$$\text{(SP)} \left\{ \begin{array}{l} \text{(1) one of the } \alpha_{ki}\text{'s is even, or} \\ \text{(2) all } \alpha_{ki}\text{'s are odd and } \alpha_k e_k \text{ is even} \\ \text{holds for each vertex } k \in V. \end{array} \right.$$

Note that if $P(\Gamma)$ has a V-spin structure then it is unique. Let $b^+(\Gamma)$ (resp. $b^-(\Gamma)$) be the number of positive (resp. negative) eigenvalues of the matrix $I_\Gamma$. Take any $\sharp V$-tuple of integers $\vec{m} = (m_1, \ldots, m_{\sharp V}) \in \mathbf{Z}^{\sharp V}$ which parametrizes V-spin$^c$-structures $c(\vec{m})$ on $P(\Gamma)$. Then we have the following explicit formula of the invariant of plumbed homology 3-spheres.

**Theorem 1.** *If $\Gamma$ satisfies (HS), then we have:*

$$\begin{aligned}
&w(\Sigma(\Gamma), P(\Gamma), c(\vec{m}))\\
&= \frac{1}{8}\Bigg[{}^t\vec{s} I_\Gamma \vec{s} - (b^+(\Gamma) - b^-(\Gamma))\\
&\quad - \sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{\alpha_{ki}}\sum_{l=1}^{\alpha_{ki}-1}\Bigg\{\cot\left(\frac{\pi l}{\alpha_{ki}}\right)\cot\left(\frac{\pi\beta_{ki} l}{\alpha_{ki}}\right)\\
&\quad +2\cos\left(\frac{\pi(1+\beta_{ki}+2m_k\beta_{ki})l}{\alpha_{ki}}\right)\operatorname{cosec}\left(\frac{\pi l}{\alpha_{ki}}\right)\operatorname{cosec}\left(\frac{\pi\beta_{ki} l}{\alpha_{ki}}\right)\Bigg\}\Bigg]\\
&=: w(\Gamma, \vec{m}),
\end{aligned}$$

*where* $\vec{s} = I_\Gamma^{-1}(\chi_k + e_k)_{k\in V} + 2\vec{m} \in \mathbf{Z}^{\sharp V}$, $\vec{s} = (s_k)_{k\in V}$, $\vec{m} = (m_k)_{k\in V}$

$$\chi_k = 2 - \sum_{i=1}^{n_k}\left(1 - \frac{1}{\alpha_{ki}}\right), \quad e_k = b_k + \sum_{i=1}^{n_k}\frac{\beta_{ki}}{\alpha_{ki}}.$$

Combining this formula with several properties of the invariant, we obtain the following theorems.

**Theorem 2.** *Suppose $\Gamma$ satisfies (HS), and $b^+(\Gamma) = 0$. If $w(\Gamma, \vec{m}) > 0$ for some $\vec{m} \in \mathbf{Z}^{\sharp V}$ then the connected sum of any number of copies of $\Sigma(\Gamma)$ cannot be the boundary of an acyclic 4-manifold.*

**Theorem 3.** *Suppose $\Gamma$ satisfies (HS) and (SP). If we put*

$$\vec{m}_{\rm sp} = -\frac{1}{2} I_\Gamma^{-1}(\chi_k + e_k)_{k\in V},$$

*then we have:*

1. *If $b^\pm(\Gamma) \le 1$, and $w(\Gamma, \vec{m}_{\rm sp}) \ne 0$ then the connected sum of any number of copies of $\Sigma(\Gamma)$ cannot be the boundary of an acyclic 4-manifold.*

2. *If $b^\pm(\Gamma) \le 2$, and $w(\Gamma, \vec{m}_{\rm sp}) \ne 0$ then the connected sum of any odd number of copies of $\Sigma(\Gamma)$ cannot be the boundary of an acyclic 4-manifold.*

Since $w(\Gamma, \vec{m}_{\rm sp})$ is an integral lift of $\mu(\Sigma(\Gamma))$, we have the next corollary which is related to the problem concerning the simplicial triangulability of closed topological manifold of dimension $\ge 5$, [18], [11], [19], [27].

**Corollary 1.** *Suppose $\Gamma$ satisfies (HS), (SP), and $b^\pm(\Gamma) \le 1$. If $\mu(\Sigma(\Gamma)) \ne 0$ then the connected sum of any number of copies of $\Sigma(\Gamma)$ cannot be the boundary of an acyclic 4-manifold.*

This paper is organized as follows. In Section 2, we review several facts concerning the invariant studied in [8]. In Section 3, to apply this invariant to homology 3-spheres of plumbing type, we naturally generalize the notion of the plumbing to the 4-V-manifold category. We give a sufficient condition for Seifert graphs to realize homology 3-spheres which belong to a class in which the invariant has a homology cobordism invariance. In Section 4, we describe the set of all V-spin$^c$-structures on the plumbed 4-V-manifold in term of the plumbing data. Section 5 is devoted to the index computation. The explicit formula of this invariant for plumbed homology 3-spheres is obtained by using the Atiyah-Singer-Kawasaki V-index theorem. In Section 6, we consider a resolution of singularities of plumbed 4-V-manifolds to apply the $\frac{10}{8}$-inequality, and we compare it with $w$-invariant. In the case of Brieskorn homology 3-spheres, we can consider also the Milnor fibers to which we may apply the $\frac{10}{8}$-inequality. In Section 7, we give explicit examples of plumbed homology 3-spheres which have infinite order in the homology cobordism group.

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## 2. An invariant of homology 3-spheres

First we review the definitions and several facts concerning $w$-invariant [8]. For a triple $(\Sigma, X, c)$ consisting of a homology 3-sphere $\Sigma$, a 4-V-manifold $X$ with boundary $\Sigma$ and a V-spin$^c$-structure $c$ on $X$, we define a $\mathbf{Z}$-valued invariant $w(\Sigma, X, c)$ as follows. Let $Y$ be a spin 4-manifold with boundary $-\Sigma$. Then we can patch them up and get the closed 4-V-manifold $X \cup_\Sigma Y$. Since $\Sigma$ is a homology 3-sphere, we can uniquely glue the spin$^c$-structures on $X$ and $Y$ along the boundary $\Sigma$ and get a spin$^c$-structure on $X \cup_\Sigma Y$.

**Definition 1.**

$$w(\Sigma, X, c) := \frac{1}{2}\mathrm{ind}_{\mathbf{R}}\mathcal{D}(X \cup_\Sigma Y) + \frac{1}{8}\mathrm{sign}\, Y.$$

Here $\mathcal{D}(X \cup_\Sigma Y)$ is the Dirac operator on the closed V-manifold $X \cup_\Sigma Y$ associated to the spin$^c$-structure $c$ on $X \cup_\Sigma Y$, $\mathrm{ind}_{\mathbf{R}} D$ is the *real* V-index of an elliptic operator $D$ over V-manifold defined as $\dim_{\mathbf{R}} \mathrm{Ker}_{\mathrm{V}}(D) - \dim_{\mathbf{R}} \mathrm{Coker}_{\mathrm{V}}(D)$, and $\mathrm{sign}\, Y$ is the signature of the intersection form on $\mathrm{H}^2(Y, \partial Y; \mathbf{R}) \cong \mathrm{H}^2(Y; \mathbf{R})$. Note that each term of the right hand side is an integer. The integer $w(\Sigma, X, c)$ is independent of the choice of $Y$ and its spin structure [8]. Next we define a certain class of homology 3-spheres, [8].

**Definition 2.** 1. Let $\mathcal{X}$ be the set of isomorphism classes of triples $(\Sigma, X, c)$ such that

(a) $X$ is a compact oriented *spin* 4-V-manifold with only isolated singularities in its interior,

(b) $\Sigma$ is the boundary of $X$ and we assume that $\Sigma$ is a homology 3-sphere, and

(c) $c$ is a *spin* structure on $X$.

2. $\mathcal{X}(k^+, k^-) = \{(\Sigma, X, c) \in \mathcal{X} | b_2^+(X) \leq k^+, b_2^-(X) \leq k^-\}$.

3. $\mathcal{S}(k^+, k^-)$ is the set of the isomorphism classes of homology 3-sphere $\Sigma$ such that $(\Sigma, X, c)$ is in the class $\mathcal{X}(k^+, k^-)$ for some $X$ and $c$.

Then we have the following theorem [8].

**Theorem 4.** *For $(k^+, k^-)$ satisfying $k^+ + k^- \leq 2$, the map*

$$w(k^+, k^-) : \mathcal{S}(k^+, k^-) \ni \Sigma \longmapsto w(\Sigma, X, c) \in \mathbf{Z}$$

*gives a homology cobordism invariant.*

## 3. V-plumbing $P(\Gamma)$

In this section, we extend the notion of the plumbing to the 4-V-manifold category, and state a necessary and sufficient condition for the boundary to be a homology 3-sphere. First we define the notion of Seifert graphs.

**Definition 3.** $\Gamma = (V, E, \omega)$ is a Seifert graph if and only if:

1. $(V, E)$ is a connected tree graph consisting of a set of vertices $V$ and a set of edges $E$.

2. $\omega$ is a map which assigns a Seifert invariant $\omega(k)$ to each vertex $k \in V$:

$$\omega(k) = \{b_k; (\alpha_{k1}, \beta_{k1}), \ldots, (\alpha_{kn_k}, \beta_{kn_k})\} \ \ (k \in V).$$

**Remark.** More general Seifert graphs can be defined as in the case of the usual plumbing graphs. Since we are only interested in integral homology 3-spheres, this definition is sufficient for our discussion.

Let $\Gamma$ be a Seifert graph. Then the 4-V-manifold $P(\Gamma)$ obtained by plumbing according to $\Gamma$ is defined as follows. For each vertex $k$, we construct a line V-bundle $L_k$ over a Riemannian V-sphere $Z_k$ with Seifert data $\omega(k)$,

$$L_k := (Z_k \setminus \cup_{i=0}^{n_k} \mathrm{Int}\, D_{ki}) \times \mathbf{C} \cup_{\{\varphi_{ki}\}} \bigcup_{i=0}^{n_k} \frac{\tilde{D}_{ki} \times \mathbf{C}}{\mathbf{Z}/\alpha_{ki}},$$

where $(D_{ki}, x_{ki})(\cong (D^2, 0))$ are small open neighborhoods around marked points $x_{ki}$ in $Z_k$, the action of $\mathbf{Z}/\alpha_{ki}$ on $\tilde{D}_{ki} \times \mathbf{C}$ is given by $\zeta_{ki}^l \cdot (z, w) := (\zeta_{ki}^l z, \zeta_{ki}^{\beta_{ki} l} w)$ for $\zeta_{ki}^l \in \mathbf{Z}/\alpha_{ki}$, $\zeta_{ki} = e^{2\pi\sqrt{-1}/\alpha_{ki}}$, and the map $\varphi_{ki}$ is defined by:

$$\varphi_{ki} : \frac{(\tilde{D}_{ki} \setminus \{0\}) \times \mathbf{C}}{\mathbf{Z}/\alpha_{ki}} \ni [z, w] \longmapsto (z^{\alpha_{ki}}, z^{-\beta_{ki}} w) \in (D_{ki} \setminus \{0\}) \times \mathbf{C}.$$

Here, we put $\alpha_{k0} = 1$, $\beta_{k0} = b_k$. Let $DL_k$ be the $D^2$-V-bundle associated to the line V-bundle $L_k$. For each vertex $k \in V$, we take for each vertex $k' \in V$ which are connected to $k$ by an edge a sufficiently small disks $D_{kk'}$ away from $D_{ki}$'s, and a trivialization $DL_k|_{D_{kk'}} \cong D_{kk'} \times D^2$ on each $D_{kk'}$. If two vertices $k$ and $k'$ are connected by an edge $e \in E$, we define an isomorphism $\sigma_e$ as follows.

$$\sigma_e : DL_k|_{D_{kk'}} \cong D_{kk'} \times D^2 \ni (z, w) \longmapsto (w, z) \in D_{k'k} \times D^2 \cong DL_{k'}|_{D_{k'k}}$$

Then $P(\Gamma)$ is defined by:

$$P(\Gamma) := \left( \coprod_{k \in V} DL_k \right) / \{\sigma_e\}_{e \in E}.$$

$P(\Gamma)$ is a 4-V-manifold with boundary and with isolated singularities in its interior. The neighborhood of each singularity has the form $\frac{\tilde{D}_{ki} \times D^2}{\mathbf{Z}/\alpha_{ki}}$ which is the cone on the lens space $L(\alpha_{ki}, \beta_{ki})$. Let $\Sigma(\Gamma)$ denote the boundary of $P(\Gamma)$.

Next we give a necessary and sufficient condition for the 3-manifold $\Sigma(\Gamma)$ to be a homology 3-sphere in terms of the data on the graph $\Gamma$. Now we compute the first homology of $\Sigma(\Gamma)$. Let $c_{ki}$ be the first homology class in $Z_k \setminus \cup_i \mathrm{Int}\, D_{ki}$ represented by the cycle $\partial D_{ki}$ and $h_k$ be the class represented by a fiber on the $S^1$-bundle $(Z_k \setminus \cup_i \mathrm{Int}\, D_{ki}) \times S^1$. A Mayer-Vietoris argument shows that the first homology group of $\Sigma(\Gamma) = \partial P(\Gamma)$ has the following form:

**Proposition 1.**

$$\begin{aligned} &\mathrm{H}_1(\Sigma(\Gamma); \mathbf{Z}) \\ &= \frac{\mathbf{Z}\left\langle c_{k1}, \ldots, c_{kn_k}, h_k,\ k \in V \right\rangle}{\left\langle \begin{array}{l} \sum_{i=1}^{n_k} c_{ki} + b_k h_k + \sum_{k' \in V(k)} h_{k'} = 0, \\ \hfill \alpha_{ki} c_{ki} - \beta_{ki} h_k = 0,\ i = 1, \ldots, n_k, \quad k \in V \end{array} \right\rangle}, \end{aligned}$$

*where $V(k)$ is the set of vertices which are connected to $k$ by an edge.*

Note that the rational intersection form on $P(\Gamma)$ can be defined by using cup product in the de Rham cohomology of smooth V-forms. Let $I_\Gamma$ be the intersection matrix of $P(\Gamma)$, then the $(k, k')$-entry of $I_\Gamma$ is:

$$(I_\Gamma)_{k,k'} = \begin{cases} e_k & k = k' \\ 1 & (k, k') \in E \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following theorem.

**Theorem 5.** $\Sigma(\Gamma)$ *is a homology* 3*-sphere if and only if:*

$$\text{(HS)}\quad \det I_\Gamma = \pm \frac{1}{\prod_{k\in V}\alpha_k}, \quad \alpha_k := \prod_i \alpha_{ki}.$$

*Proof.* If we denote by $R_\Gamma$ the relation matrix for $\mathrm{H}_1(\Sigma(\Gamma);\mathbf{Z})$ with respect to the basis $< c_{k1},\ldots,c_{kn_k}, h_k \;\; k\in V >$ then $\Sigma(\Gamma)$ is a homology 3-sphere if and only if $\det R_\Gamma = \pm 1$. Then the assertion is deduced from the following formula, which is derived from a direct computation of the determinant of $R_\Gamma$.

$$\det R_\Gamma = (-1)^{\sum_{k\in V} n_k}\left(\prod_{k\in V}\alpha_k\right)\det I_\Gamma. \qquad \square$$

We prove the following proposition which we use later.

**Proposition 2.** *The condition (HS) implies:*

1. $(\alpha_{k1},\ldots,\alpha_{kn_k})$ *are pairwise coprime integers for each* $k\in V$.
2. $(\alpha_{ki},\alpha_k e_k)$ *are coprime for each* $i=1,\ldots,n_k$ *and* $k\in V$.

*Proof.* Suppose that some pair $(\alpha_{k_0 i},\alpha_{k_0 j})$ $(k_0\in V)$ has the greatest common divisor $d\geq 2$, then $l_{k_0}=\alpha_{k_0}e_{k_0}$ also has the divisor $d$. $(\prod_{k\in V}\alpha_k)\det I_\Gamma = \pm 1$ is the summation of the terms of the following form

$$\left(\prod_{k\in V}\alpha_k\right)(\pm e_{k_1}\cdots e_{k_s}) = \pm\left(\prod_{k\in V\setminus\{k_1,\ldots,k_s\}}\alpha_k\right)\left(\prod_{k\in\{k_1,\ldots,k_s\}} l_k\right)$$

which contains the factor either $\alpha_{k_0}$ or $l_{k_0}$ and hence has the divisor $d$, a contradiction. The second assertion follows similarly. $\square$

## 4. V-spin$^c$-structures on $P(\Gamma)$

First we note that $P(\Gamma)$ has almost complex structures compatible with the structures of line V-bundles. So we fix one of them from now on. In this section, we describe the set of all V-spin$^c$-structures on $P(\Gamma)$ in terms of the plumbing data $\Gamma$. Note that the set of all V-spin$^c$-structures on $P(\Gamma)$ is the affine space over $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(P(\Gamma))$, where we denote by $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(X)$ the abelian group of all topological isomorphism classes of line V-bundles on a V-manifold $X$ (see [10]). For each vertex $k\in V$, we construct a line V-bundle $\tilde{L}_k$ on $P(\Gamma)$ satisfying:

$$c_1(\tilde{L}_k)[Z_{k'}] = \begin{cases} e_k & k=k' \\ 1 & (k,k')\in E. \\ 0 & \text{otherwise}\end{cases}$$

Let $L_k$ denote the line V-bundle over $Z_k$ whose associated $D^2$-V-bundle is $DL_k$, and $p_k : DL_k \to Z_k$ the projection. Then $\tilde{L}_k$ is obtained by gluing the pull back $p_k^* L_k$ and $(P(\Gamma) \setminus DL_k) \times \mathbf{C}$ over the solid tori $\partial DL_k \cap \overline{(P(\Gamma) \setminus DL_k)}$ so that the tautological section of $p_k^* L_k$ extends trivially to $P(\Gamma)$. Then we have the following proposition.

**Proposition 3.** *If* $\Gamma$ *satisfies (HS) then* $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(P(\Gamma))$ *is freely generated by* $\tilde{L}_k$*'s* ($k \in V$)*.*

*Proof.* Let $L_{0k}$ be the generator of $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(Z_k)$ with $c_1(L_{0k})[Z_k] = 1/\alpha_k$. Since the union $\bigcup_{k \in V} Z_k \subset P(\Gamma)$ of zero sections of $DL_k$'s is a V-deformation retract of $P(\Gamma)$ and $\Gamma$ is a tree,

$$\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(P(\Gamma)) \cong \mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}\left(\bigcup_{k \in V} Z_k\right) \cong \bigoplus_{k \in V} \mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(Z_k) \cong \bigoplus_{k \in V} \mathbf{Z}[L_{0k}].$$

For each $k \in V$, let $\tilde{L}_{0k}$ be the line V-bundle over $P(\Gamma)$ corresponding via the above isomorphism to the line V-bundle $L_{0k}$ over $Z_k$. Then $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(P(\Gamma))$ is freely generated by $\tilde{L}_{0k}$. $\tilde{L}_{0k}$ satisfies:

$$c_1(\tilde{L}_{0k})[Z_{k'}] = \frac{1}{\alpha_k} \delta_{kk'}$$

Comparing the Euler numbers,

$$\tilde{L}_k = \sum_{k' \in V} (I_\Gamma)_{kk'} \alpha_{k'} \tilde{L}_{0k'}.$$

Since $\Sigma(\Gamma)$ is a homology 3-sphere, $\det I_\Gamma = \pm 1 / \prod_{k \in V} \alpha_k$. Hence the matrix $I_\Gamma \cdot \mathrm{diag}(\alpha_k)$ has integral entries and determinant $\pm 1$. So it has the inverse with integral coefficients. $\square$

Note that we have fixed an almost complex structure on $P(\Gamma)$. Then we have the canonical V-spin$^c$-structure on $P(\Gamma)$ whose associated line V-bundle is the dual of the canonical line V-bundle $K^{-1}$. Let $S^+_{\mathrm{can}} \oplus S^-_{\mathrm{can}}$ be the spinor V-bundle associated to the canonical V-spin$^c$-structure. Then we have the following theorem.

**Theorem 6.** *Suppose* $\Gamma = (V, E, \omega)$ *satisfies (HS). Then there is a one-to-one correspondence between the set of all V-spin*$^c$*-structures on* $P(\Gamma)$ *and the lattice* $\mathbf{Z}^{\sharp V}$ *such that* $\vec{m} = (m_1, \ldots, m_{\sharp V}) \in \mathbf{Z}^{\sharp V}$ *corresponds to the V-spin*$^c$*-structure whose associated spinor V-bundle is*

$$(S^+_{\mathrm{can}} \oplus S^-_{\mathrm{can}}) \otimes \bigotimes_{k \in V} \tilde{L}_k^{m_k}.$$

**Remark.** The line V-bundle associated to the V-spin$^c$-structure on $P(\Gamma)$ which corresponds to $\vec{m} \in \mathbf{Z}^{\sharp V}$ is the determinant line V-bundle of $S^{\pm}_{\mathrm{can}} \otimes \bigotimes_{k \in V} \tilde{L}_k^{m_k}$:

$$K^{-1} \otimes \bigotimes_{k \in V} \tilde{L}_k^{2m_k},$$

where $K$ is the canonical line V-bundle of $P(\Gamma)$.

For V-spin structures, we have the following theorem.

**Theorem 7.** *Suppose $\Gamma$ satisfies the condition (HS). Then $P(\Gamma)$ has a V-spin structure if and only if:*

$$(SP) \left\{ \begin{array}{l} \textit{(1) one of the } \alpha_{ki} \textit{ 's is even, or} \\ \textit{(2) all } \alpha_{ki} \textit{ 's are odd and } \alpha_k e_k \textit{ is even} \\ \textit{for each vertex} k \in V. \end{array} \right.$$

*If $P(\Gamma)$ has a V-spin structure then it is unique.*

*Proof.* Note that there is a unique spin structure on contractible 4-manifolds, and its automorphism group is $\mathbf{Z}/2$. Therefore $P(\Gamma)$ is V-spin if and only if each $L_k \subset P(\Gamma)$ is V-spin since regions for gluing are contractible. The canonical line V-bundle $K_k$ over $L_k$ is

$$K_k = p_k^*(T_V Z_k \otimes L_k)^{-1},$$

where $T_V Z_k$ is the tangent V-bundle of $Z_k$. Let $L_{0k}$ be the generator of $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(Z_k)$. Then $T_V Z_k$ and $L_k$ has the form: $T_V Z_k = L_{0k}^{f_k}$, $L_k = L_{0k}^{l_k}$ for some integer $f_k$, $l_k$, respectively. The integers $f_k$ and $l_k$ are obtained by comparing the Euler numbers. The Euler number for $T_V Z_k$ is

$$c_1(T_V Z_k)[Z_k] = \chi_k := 2 - \sum_{i=1}^{n_k} \left(1 - \frac{1}{\alpha_{ki}}\right),$$

and we have $f_k = \alpha_k \chi_k$. Similarly, we obtain $l_k = \alpha_k e_k$. Hence the canonical V-bundle $K_k$ for $L_k$ is:

$$K_k = p_k^* L_{0k}^{-(f_k + l_k)}$$

and $K_k$ has a square root in $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(Z_k) \cong \mathbf{Z}$ if and only if $l_k + f_k$ is even. Therefore, each $L_k$ has a V-spin structure if and only if (1) one of the $\alpha_{ki}$'s is even or (2) all $\alpha_{ki}$'s are odd and $l_k = \alpha_k e_k$ is even. Note that $(\alpha_{ki}, l_k)$'s are coprime by Proposition 3. If $L_k$ has a V-spin structure then it is unique since the square root in $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{V}}(Z_k) \cong \mathbf{Z}$ is unique. Since $\Gamma$ is a tree, automorphisms of the spin structures on the gluing regions extend to $P(\Gamma)$, and hence the uniqueness follows. $\square$

Let $b^+(\Gamma)$ (resp. $b^-(\Gamma)$) be the number of positive (resp. negative) eigenvalues of the matrix $I_\Gamma$. Then Theorem 2 and 3 follows from Theorem 5, Theorem 6 in [8], respectively. By Theorem 5 and 7, we obtain the next corollary.

**Corollary 2.** *The boundary $\Sigma(\Gamma)$ of the plumbing $P(\Gamma)$ belongs to $\mathcal{S}(k^+, k^-)$ if $\Gamma$ satisfies the condition (HS), (SP), and $b^{\pm}(\Gamma) \leq k^{\pm}$.*

## 5. Computation of the V-index

In this section, we compute the index of the Dirac operator on $P(\Gamma) \cup_{\Sigma(\Gamma)} Y$ where $\Gamma$ is a Seifert graph with Seifert data $\omega(k) = \{b_k; (\vec{\alpha}_k, \vec{\beta}_k)\}$ satisfying (HS). We denote by $c(\vec{m})$ ($\vec{m} \in \mathbf{Z}^{\sharp V}$) the V-spin$^c$-structure on $P(\Gamma) \cup_{\Sigma(\Gamma)} Y$ which is the gluing of a spin structure on $Y$ and the V-spin$^c$-structure on $P(\Gamma)$ whose associated spinor V-bundle is $S^{\pm}_{\mathrm{can}} \otimes \bigotimes_{k \in V} \tilde{L}_k^{m_k}$ as in Theorem 6. Let $L_{\vec{m}}$ be the determinant line V-bundle associated to $c(\vec{m})$. Let $\mathcal{D}(P(\Gamma) \cup_{\Sigma(\Gamma)} Y)$ be the Dirac operator corresponding to the V-spin$^c$-structure $c(\vec{m})$. Then Theorem 1 follows from:

**Theorem 8.**

$$\begin{aligned}
&\mathrm{ind}_{\mathbf{R}} \mathcal{D}(P(\Gamma) \cup_{\Sigma(\Gamma)} Y) \\
&= \frac{1}{4} \Bigg[ {}^t \vec{s} I_\Gamma \vec{s} - (b^+(\Gamma) - b^-(\Gamma) + \mathrm{sign}\, Y) \\
&\quad - \sum_{k \in V} \sum_{i=1}^{n_k} \frac{1}{\alpha_{ki}} \sum_{l=1}^{\alpha_{ki}-1} \Bigg\{ \cot\left(\frac{\pi l}{\alpha_{ki}}\right) \cot\left(\frac{\pi \beta_{ki} l}{\alpha_{ki}}\right) \\
&\quad + 2 \cos\left(\frac{\pi(1 + \beta_{ki} + 2 m_k \beta_{ki}) l}{\alpha_{ki}}\right) \mathrm{cosec}\left(\frac{\pi l}{\alpha_{ki}}\right) \mathrm{cosec}\left(\frac{\pi \beta_{ki} l}{\alpha_{ki}}\right) \Bigg\} \Bigg],
\end{aligned}$$

*where* $\vec{s} = I_\Gamma^{-1} (\chi_k + e_k)_{k \in V} + 2\vec{m}$.

*Proof.* Let $X$ be the 4-V-manifold $P(\Gamma) \cup_{\Sigma(\Gamma)} Y$ with singular set $\Sigma X = \bigcup_{k \in V} \bigcup_{i=1}^{n_k} \Sigma_{ki}$, where $\Sigma_{ki}$ is the singular point of the cone on the lens space $L(\alpha_{ki}, \beta_{ki})$ whose multiplicity is $\alpha_{ki}$. By the Atiyah-Singer-Kawasaki V-index theorem [15]:

$$\begin{aligned}
&\mathrm{ind}_{\mathbf{C}} \mathcal{D}(X) \\
&= (-1)^{\dim X} \{ch([\sigma(\mathcal{D})]) td(T_V X \otimes \mathbf{C})\}[T_V X] \\
&\quad + \sum_{k \in V} \sum_{i=1}^{n_k} \frac{(-1)^{\dim \Sigma_{ki}}}{\alpha_{ki}} ch^{\Sigma_{ki}}([\sigma(\mathcal{D})]) \mathcal{J}^{\Sigma_{ki}}(X)[T_V \Sigma_{ki}].
\end{aligned}$$

Note that $\mathrm{ind}_{\mathbf{R}} = 2\, \mathrm{ind}_{\mathbf{C}}$.

1. The first term on the right hand side is:

$$\{ch([\sigma(\mathcal{D})]) td(T_V X \otimes \mathbf{C})\}[T_V X] = \frac{1}{8} \left( c_1(L_{\vec{m}})^2 - \frac{1}{3} p_1(T_V X) \right) [X].$$

2. On the second term on the right hand side, we have:

$$\begin{aligned} & ch^{\Sigma_{ki}}([\sigma(\mathcal{D})])\mathcal{J}^{\Sigma_{ki}}(X)[T_V\Sigma_{ki}] \\ &= \sum_{1\neq g\in \mathbf{Z}/\alpha_{ki}} \frac{ch_g(\tilde{j}^*_{ki}[\sigma(\tilde{\mathcal{D}}_{ki})])}{ch_g(\bigwedge_{-1}(N\tilde{\Sigma}_{ki}\otimes \mathbf{C}))} td(T\tilde{\Sigma}_{ki}\otimes \mathbf{C})[T\tilde{\Sigma}_{ki}], \end{aligned}$$

where $[\sigma(D)]$ is the symbol class of an elliptic operator $D$, $\tilde{\Sigma}_{ki} = \tilde{U}^g_{ki}$ is the fixed point of the action of $g \in \mathbf{Z}/\alpha_{ki}$ on the local uniformization $\tilde{U}_{ki}$ of the neighborhood $U_{ki}$ of the singular point $\Sigma_{ki}$ in $DL_k \subset P(\Gamma)$, $\tilde{j}_{ki}$ the inclusion $\{\tilde{\Sigma}_{ki}\} \hookrightarrow \tilde{U}_{ki}$, $N\tilde{\Sigma}_{ki}$ the normal bundle of $\tilde{\Sigma}_{ki}$ in $\tilde{U}_{ki}$, and $\tilde{\mathcal{D}}_{ki}$ is the $\mathbf{Z}/\alpha_{ki}$-invariant Dirac operator on $\tilde{U}_{ki}$ which is the lift of the Dirac operator $\mathcal{D}$ on $U_{ki}$. Note that the normal bundle $N\tilde{\Sigma}_{ki}$ is $\mathbf{Z}/\alpha_{ki}$-diffeomorphic to $\mathbf{C}\times\mathbf{C}$ with $\mathbf{Z}/\alpha_{ki}$-action given by

$$\zeta^l_{ki}\cdot(z,w) = (\zeta^l_{ki}z, \zeta^{\beta_{ki}l}_{ki}w)$$

for $g = \zeta^l_{ki} \in \mathbf{Z}/\alpha_{ki}$ with $\zeta_{ki} = e^{2\pi\sqrt{-1}/\alpha_{ki}}$, and the local uniformization of the positive and negative spinor V-bundle $S^+$, $S^-$ restricted to the point $\tilde{\Sigma}_{ki}$ is $\mathbf{Z}/\alpha_{ki}$-diffeomorphic to $\mathbf{C}\times\mathbf{C}$ with $\mathbf{Z}/\alpha_{ki}$-action

$$\zeta^l_{ki}\cdot(z,w) = (\xi_1^{l/2}\xi_2^{l/2}\zeta^{l/2}z, \xi_1^{-l/2}\xi_2^{-l/2}\zeta^{l/2}w),$$

$$\zeta^l_{ki}\cdot(z,w) = (\xi_1^{l/2}\xi_2^{-l/2}\zeta^{l/2}z, \xi_1^{-l/2}\xi_2^{l/2}\zeta^{l/2}w),$$

respectively, where

$$\xi_1^{1/2} = e^{\pi\sqrt{-1}/\alpha_{ki}}, \xi_2^{1/2} = e^{\pi\sqrt{-1}\beta_{ki}/\alpha_{ki}}, \zeta^{1/2} = e^{\pi\sqrt{-1}(1+\beta_{ki}+2m_k\beta_{ki})/\alpha_{ki}}.$$

Hence we have:

$$\begin{aligned} & \sum_{1\neq g\in \mathbf{Z}/\alpha_{ki}} \frac{ch_g(\tilde{j}^*_{ki}[\sigma(\tilde{\mathcal{D}}_{ki})])}{ch_g(\bigwedge_{-1}(N\tilde{\Sigma}_{ki}\otimes \mathbf{C}))} td(T\tilde{\Sigma}_{ki}\otimes \mathbf{C})[T\tilde{\Sigma}_{ki}] \\ &= \sum_{l=1}^{\alpha_{ki}-1} \frac{(\xi_1^{l/2}-\xi_1^{-l/2})(\xi_2^{l/2}-\xi_2^{-l/2})\zeta^{l/2}}{(1-\xi_1^l)(1-\xi_1^{-l})(1-\xi_2^l)(1-\xi_2^{-l})} \\ &= \sum_{l=1}^{\alpha_{ki}-1} \frac{\zeta^{l/2}}{(\xi_1^{l/2}-\xi_1^{-l/2})(\xi_2^{l/2}-\xi_2^{-l/2})} \\ &= -\sum_{l=1}^{\alpha_{ki}-1} \frac{1}{4}\cos\left(\frac{\pi(1+\beta_{ki}+2m_k\beta_{ki})l}{\alpha_{ki}}\right)\operatorname{cosec}\left(\frac{\pi l}{\alpha_{ki}}\right)\operatorname{cosec}\left(\frac{\pi\beta_{ki}l}{\alpha_{ki}}\right). \end{aligned}$$

3. The 1-st Pontrjagin number $p_1(T_VX)[X]$ is also computed by using the V-signature theorem of T. Kawasaki [14].

$$\operatorname{sign} X = \frac{1}{3}p_1(T_VX)[X] - \sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{\alpha_{ki}}\sum_{l=1}^{\alpha_{ki}-1}\cot\left(\frac{\pi l}{\alpha_{ki}}\right)\cot\left(\frac{\pi\beta_{ki}l}{\alpha_{ki}}\right)$$

4. By Proposition 3, the canonical line V-bundle $K$ for $P(\Gamma)$ can be written as $K = \bigotimes_{k\in V} \tilde{L}_k^{r_k}$ for some $r_k$'s in $\mathbf{Z}$. We have:

$$\begin{aligned} c_1(L_{\vec{m}})^2[X] &= c_1(L_{\vec{m}})^2[P(\Gamma)\cup_{\Sigma(\Gamma)} Y] \\ &= c_1\left(K^{-1}\otimes \bigotimes_{k\in V} \tilde{L}_k^{2m_k}\right)^2 [P(\Gamma),\Sigma(\Gamma)] = {}^t\vec{s} I_\Gamma \vec{s}\,, \end{aligned}$$

where $s_k = -r_k + 2m_k$.

5. By Novikov's addition formula:

$$\operatorname{sign} X = \operatorname{sign}(P(\Gamma)\cup_{\Sigma(\Gamma)} Y) = \operatorname{sign} I_\Gamma + \operatorname{sign} Y = b^+(\Gamma) - b^-(\Gamma) + \operatorname{sign} Y.$$

Then the theorem follows from the following lemma. □

**Lemma 1.** *$\vec{r} = \{r_k\}_{k\in V}$ is given by the following formula:*

$$\vec{r} = -I_\Gamma^{-1}(\chi_k + e_k)_{k\in V} \in \mathbf{Z}^{\sharp V}.$$

*Proof.* Now the canonical line V-bundle $K_k$ for $L_k$ is

$$K_k = p_k^* L_{0k}^{-(f_k+l_k)}.$$

Then we see:

$$\begin{aligned} c_1(K)[Z_k] &= c_1(K_k)[Z_k] \\ &= c_1(p_k^* L_{0k}^{-(f_k+l_k)})[Z_k] = -(f_k+l_k)c_1(p_k^* L_{0k})[Z_k] \\ &= -(f_k+l_k)\frac{1}{\alpha_k} = -(\chi_k + e_k). \end{aligned}$$

So the coefficient $r_k$'s are obtained by the following equations.

$$-(\chi_k+e_k) = c_1(K)[Z_k] = c_1\left(\bigotimes_{k\in V} \tilde{L}_{k'}^{r_{k'}}\right)[Z_k] = \sum_{k'\in V} r_{k'} c_1(\tilde{L}_{k'})[Z_k]$$

for each $k\in V$. If we put column vectors $\vec{t} = -(\chi_k+e_k)_{k\in V}$ in $\mathbf{Q}^{\sharp V}$, then the equation can be written by using the intersection matrix $I_\Gamma$ as

$$\vec{t} = I_\Gamma\,\vec{r}.$$

Since $\Gamma$ satisfies (HS), the determinant of $I_\Gamma$ is $\pm 1/\prod_{k\in V}\alpha_k$. It follows that the inverse matrix has integral entries, and we have:

$$\vec{r} = I_\Gamma^{-1}\vec{t} = I_\Gamma^{-1}\operatorname{diag}(\alpha_k)^{-1}\operatorname{diag}(\alpha_k)\vec{t} = (\operatorname{diag}(\alpha_k)I_\Gamma)^{-1}\operatorname{diag}(\alpha_k)\vec{t} \in \mathbf{Z}^{\sharp V} \quad □$$

## 6. Hirzebruch-Jung resolutions Milnor fibers and the $\frac{10}{8}$-Theorem

In this section, we consider the Hirzebruch-Jung resolution of singularities of the plumbed V-manifold $P(\Gamma)$ to apply the estimate of the $\frac{10}{8}$-theorem, and compare it with an application of $w$-invariant.

Let $\Gamma = (V, E, \omega)$ be a Seifert graph. The singularities in $P(\Gamma)$ are the cyclic quotient singularities: $C(\alpha, \beta) := \frac{\mathbf{C}\times\mathbf{C}}{\mathbf{Z}/\alpha}$, where $(\alpha, \beta)$ are coprime integers and the action of $\mathbf{Z}/\alpha$ on $\mathbf{C}\times\mathbf{C}$ is given by $\zeta_\alpha^l \cdot (z, w) = (\zeta_\alpha^l z, \zeta_\alpha^{\beta l} w)$ for $\zeta_\alpha^l \in \mathbf{Z}/\alpha$ with $\zeta_\alpha = e^{2\pi\sqrt{-1}/\alpha}$. The Hirzebruch-Jung resolutions of these singularities are obtained by the following method [12]. For each singularity $C(\alpha, \beta)$, we consider a continued fraction expansion of $\alpha/\beta$.

$$\frac{\alpha}{\beta} = [m_1, \ldots, m_s] = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\ddots - \cfrac{1}{m_s}}}$$

Then a resolution of the singularity $C(\alpha, \beta)$ is obtained by plumbing according to the following linear graph (Figure 6.1):

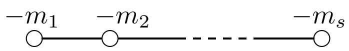


Figure 6.1

If $\Gamma$ is a graph with $\sharp V = 2, n_1 = 3, n_2 = 3$, for example, then a resolution of singularities in $P(\Gamma)$ is given by plumbing according to the following graph (Figure 6.2).

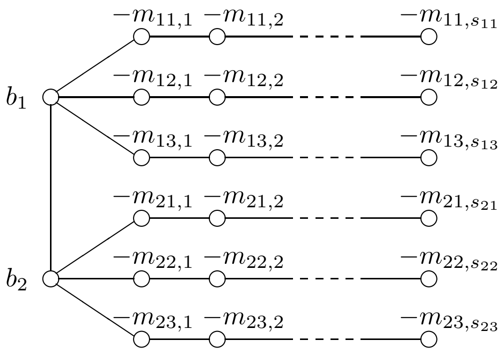


Figure 6.2

We denote this weighted graph by $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \tilde{\omega})$. Note that the boundary $\Sigma(\tilde{\Gamma})$ of the plumbing $P(\tilde{\Gamma})$ is diffeomorphic to $\Sigma(\Gamma)$.

**Remark.** Note that we do not assume $m_i \geqslant 2$, so our resolution is not necessarily comprex analytic. In genaral, resolution is not unique. In fact, $[m_1, \ldots, m_s]$ is not determined by $\alpha/\beta$. W. Neumann [20] proved that the oriented 3-manifolds $\Sigma(\Gamma_1)$ and $\Sigma(\Gamma_2)$ obtained by plumbing according to two graphs $\Gamma_1$ and $\Gamma_2$ are homeomorphic if and only if $\Gamma_1$ and $\Gamma_2$ are related by several fundamental operations of graphs.

Let $I_{\tilde{\Gamma}}$ be the intersection matrix for $P(\tilde{\Gamma})$. Then we have:

**Claim 1.** *Suppose $\Gamma$ satisfies (HS), (SP), and all $b_k$'s and $m_{ki,t}$'s in $\tilde{\Gamma}$ are even. If $\tilde{\Gamma}$ satisfies (1) $|\mathrm{sign}\, I_{\tilde{\Gamma}}| \neq \sharp\tilde{V}$ and $\frac{5}{4}|\mathrm{sign}\, I_{\tilde{\Gamma}}| > \sharp\tilde{V}$, or (2) $|\mathrm{sign}\, I_{\tilde{\Gamma}}| = \sharp V$ and $\sharp V \neq 0$ then $\Sigma(\tilde{\Gamma})$ has infinite order in the homology cobordism group.*

*Proof.* Since all $b_k$'s and $m_{ki,t}$'s are even, $P(\tilde{\Gamma})$ is spin. Suppose that for some integer $k$, the connected sum $k\Sigma(\tilde{\Gamma})$ is the boundary of an acyclic 4-manifold $Y$. Then we have the closed smooth spin 4-manifold $\tilde{X} := kP(\tilde{\Gamma}) \cup_{k\Sigma(\tilde{\Gamma})} -Y$. If $\tilde{\Gamma}$ satisfies the condition (1) then we get the following inequality: $\frac{5}{4}|\mathrm{sign}\, \tilde{X}| + 2 = k\frac{5}{4}|\mathrm{sign}\, I_{\tilde{\Gamma}}| + 2 > k\sharp\tilde{V} + 2 = b_2(\tilde{X}) + 2 > b_2(\tilde{X})$ which contradict the $\frac{10}{8}$-theorem [9]. On the other hand, suppose that $\tilde{\Gamma}$ satisfies the condition (2). Then we have a definite closed spin 4-manifold $\tilde{X} := kP(\tilde{\Gamma}) \cup_{k\Sigma(\tilde{\Gamma})} -Y$ with $b_2(\tilde{X}) > 0$ which contradict Donaldson's theorem [2], [3]. $\square$

If the number of vertices $\sharp V$ is one, the corresponding $\Sigma(\Gamma)$ is a Brieskorn homology 3-sphere. In this case, we can consider the Milnor fiber which is smooth spin 4-manifold with boundary $\Sigma(\Gamma)$. For simplicity, we consider only the case of Brieskorn homology 3-spheres with three singular fibers. For pairwise coprime integers $\alpha_1, \alpha_2, \alpha_3$, the Brieskorn homology 3-sphere $\Sigma(\alpha_1, \alpha_2, \alpha_3)$ is defined by:

$$\Sigma(\alpha_1, \alpha_2, \alpha_3) = \{(z_1, z_2, z_3) \in \mathbf{C}^3 | z_1^{\alpha_1} + z_2^{\alpha_2} + z_3^{\alpha_3} = 0\} \cup S^5$$

The Milnor fiber $V(\alpha_1, \alpha_2, \alpha_3)$ is defined by:

$$V(\alpha_1, \alpha_2, \alpha_3) = \{(z_1, z_2, z_3) \in \mathbf{C}^3 | z_1^{\alpha_1} + z_2^{\alpha_2} + z_3^{\alpha_3} = 1\} \cup D^6.$$

The Milnor fiber $V(\alpha_1, \alpha_2, \alpha_3)$ is a smooth spin 4-manifold with boundary diffeomorphic to $\Sigma(\alpha_1, \alpha_2, \alpha_3)$. The second Betti number of the fiber is called the Milnor number and calculated as

$$b_2(V(\alpha_1, \alpha_2, \alpha_3)) = (\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1).$$

The signature is calculated by Hirzebruch-Zagier [13], and it is in fact eight times the Casson invariant, see Fintushel-Stern [5], Fukuhara-Matsumoto-

Sakamoto [6] and Neumann-Wahl [21].

$$\operatorname{sign}(V(\alpha_1,\alpha_2,\alpha_3)) = -\frac{1}{N}\sum_{1+z^N=0}\left(\frac{1+z}{1-z}\prod_{i=1}^{3}\frac{1+z^{N/\alpha_i}}{1-z^{N/\alpha_i}}\right)$$

where $N=\alpha_1\alpha_2\alpha_3$. Then we have the following:

**Claim 2.** *If* $\frac{5}{4}|\operatorname{sign}(V(\alpha_1,\alpha_2,\alpha_3))| > b_2(V(\alpha_1,\alpha_2,\alpha_3))$ *then the Brieskorn homology 3-sphere* $\Sigma(\alpha_1,\alpha_2,\alpha_3)$ *has infinite order in the homology cobordism group.*

*Proof.* The claim follows from a direct application of the $\frac{10}{8}$-theorem to the closed spin 4-manifold $kV(\alpha_1,\alpha_2,\alpha_3)\cup_{k\Sigma(\alpha_1,\alpha_2,\alpha_3)} -Y$ for some integer $k>0$. $\square$

## 7. Examples

We give several examples of homology 3-spheres of plumbing type which have infinite order in the homology cobordism group of homology 3-spheres and we compare $w$-invariant with the application of Claim 1, Claim 2 by using computer. Here we consider only homology 3-spheres with vanishing Rohlin invariant. Let $\Gamma$ be a Seifert graph satisfying (HS) and (SP). We construct a resolution of singularities of 4-V-manifold $P(\Gamma)$ to get a smooth spin 4-manifold $P(\tilde{\Gamma})$ in the following way. For each vertex $k\in V$, we rewrite the Seifert invariants by the following steps. (Step 1) For every $i=1,\ldots,n_k$, if $\beta_{ki}$ and $\alpha_{ki}$ are odd then we substitute $\beta_{ki}\mapsto\beta_{ki}-\alpha_{ki}$, and $b_k\mapsto b_k+1$. (Step 2) If $b_k$ is odd then one of the $\alpha_{ki}$'s, $\alpha_{ki_0}$ say, is even by the condition (SP). In this case, we substitute $b_k\mapsto b_k+1$ and $\beta_{ki_0}\mapsto\beta_{ki_0}-\alpha_{ki_0}$. Note that these substitutions do not change the isomorphism class of the line V-bundle $L_k$'s and hence the diffeomorphism class of $P(\Gamma)$, and the condition (SP) is independent of these substitutions. Then all $b_k$'s are even and either $\alpha_{ki}$ or $\beta_{ki}$ is even for any $k,i$. For coprime integers $(\alpha,\beta)$ such that either $\alpha$ or $\beta$ is even, we can expand $\frac{\alpha}{\beta}=[m_1,\ldots,m_s]$ according to the following algorithm:

$$\begin{aligned}
\alpha &= \beta m_1 - q_1,\ \ |\beta|>|q_1|,\ \ |m_1|\geq 2,\ \ m_1 : \text{even}\\
\beta &= q_1 m_2 - q_2,\ \ |q_1|>|q_2|,\ \ |m_2|\geq 2,\ \ m_2 : \text{even}\\
q_1 &= q_2 m_3 - q_3,\ \ |q_2|>|q_3|,\ \ |m_3|\geq 2,\ \ m_3 : \text{even}\\
&\cdots\\
q_{s-2} &= q_{s-1}m_s,\ \ |m_s|\geq 2,\ \ m_s : \text{even}.
\end{aligned}$$

Then we have a resolution graph $\tilde{\Gamma}$ such that all weights are even and hence $P(\tilde{\Gamma})$ is spin. Note that $\Sigma(\tilde{\Gamma})=\partial P(\tilde{\Gamma})=\Sigma(\Gamma)$ is a homology 3-sphere.

1) Brieskorn homology 3-spheres $\Sigma(\alpha_1,\alpha_2,\alpha_3)$.

In the following table (Table 7.1), we give several examples of Brieskorn homology 3-spheres with the Rohlin invariants vanish which have infinite order in the homology cobordism group for which we can apply Theorem 3 and we cannot apply Theorem 2 and Claim 1, 2. Note that the disk V-bundle $L(\alpha_1, \alpha_2, \alpha_3)$ associated to the Seifert fibration $\Sigma(\alpha_1, \alpha_2, \alpha_3)$ has the negative definite intersection form. In the columns of resolutions and Milnor fibers, we write $\frac{10}{8}$sign on the left and the second Betti number on the right. $\max\{w\}$ means the maximum value of $w(\Sigma, X, c)$ considering $c$ as a variable.

| Brieskorn | $w$(spin) | $\max\{w\}$ | resolution | | Milnor fiber | |
|---|---|---|---|---|---|---|
| $\Sigma(3,5,16)$ | $-2$ | 0 | 20 | 20 | $-80$ | 120 |
| $\Sigma(5,13,18)$ | $-2$ | 0 | 20 | 20 | $-460$ | 816 |
| $\Sigma(7,8,19)$ | $-2$ | 0 | 20 | 24 | $-420$ | 756 |
| $\Sigma(2,9,19)$ | $-2$ | 0 | 20 | 22 | $-100$ | 144 |
| $\Sigma(11,12,19)$ | $-2$ | 0 | 20 | 26 | $-1020$ | 1980 |
| $\Sigma(13,16,19)$ | $-2$ | 0 | 20 | 26 | $-1620$ | 3240 |
| $\Sigma(10,17,19)$ | $-2$ | 0 | 20 | 28 | $-1320$ | 2592 |
| $\Sigma(9,11,20)$ | $-2$ | 0 | 20 | 26 | $-800$ | 1520 |
| $\Sigma(3,7,22)$ | $-2$ | 0 | 20 | 26 | $-160$ | 252 |
| $\Sigma(5,13,22)$ | $-2$ | 0 | 20 | 28 | $-560$ | 1008 |
| $\Sigma(15,19,22)$ | $-2$ | 0 | 20 | 28 | $-2580$ | 5292 |
| $\Sigma(4,17,23)$ | $-2$ | 0 | 20 | 28 | $-600$ | 1056 |
| $\Sigma(13,20,23)$ | $-2$ | 0 | 20 | 28 | $-2460$ | 5016 |
| $\Sigma(10,21,23)$ | $-2$ | 0 | 20 | 32 | $-1980$ | 3960 |
| $\Sigma(7,13,24)$ | $-2$ | 0 | 20 | 20 | $-880$ | 1656 |
| $\Sigma(2,17,25)$ | $-2$ | 0 | 20 | 22 | $-260$ | 384 |
| $\Sigma(17,22,25)$ | $-2$ | 0 | 20 | 34 | $-3860$ | 8064 |
| $\Sigma(19,22,25)$ | $-2$ | 0 | 20 | 28 | $-4320$ | 9072 |

Table 7.1

**Remarks.** 1. It seems that Claim 2 is not effective in our case. Among Brieskorn homology 3-spheres with $\alpha_i \leq 50$, this method detected only $\Sigma(2,3,5)$ (the Poincaré homology sphere) and $\Sigma(2,3,11)$.

2. It is remarkable that the detection using Theorem 2 has perfect agreement at least for the numerical computation in $\Sigma(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \leq 25$ with the detection using a theorem of T. Lawson [17] concerning the Fintushel-Stern invariant [4] which comes from the Donaldson theory.

2) Plumbed homology 3-spheres obtained by plumbing according to Seifert graphs $\Gamma = (V, E, \omega)$, $\sharp V = 2$, $n_1 = 3$, $n_2 = 3$.
The following table (Table 7.2) contains several examples of plumbed homology 3-spheres with the Rohlin invariants vanish which have infinite order in the homology cobordism group for which we can apply Theorem 3 and we cannot apply Theorem 2 and Claim 1, 2. In the column of resolutions, we write $\frac{10}{8}$sign on the left and the second Betti number on the right. "V-sig." in the table means the signature of plumbed 4-V-manifolds.

| Plumbed homology 3-spheres | $w$(spin) | V-sig. | resolution | |
|---|---|---|---|---|
| $\left\{\begin{matrix} -141,(2,1)(3,1)(5,1) \\ -1,(4,1)(5,3)(7,1) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 20 |
| $\left\{\begin{matrix} -2,(2,1)(3,2)(5,4) \\ -32,(4,3)(5,2)(7,6) \end{matrix}\right\}$ | 2 | 0 | $-20$ | 20 |
| $\left\{\begin{matrix} -141,(3,1)(4,1)(7,3) \\ -1,(4,1)(5,3)(7,1) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 20 |
| $\left\{\begin{matrix} -2,(3,2)(4,3)(7,4) \\ -86,(4,3)(5,2)(7,6) \end{matrix}\right\}$ | 2 | 0 | $-20$ | 20 |
| $\left\{\begin{matrix} -6,(3,2)(4,1)(5,1) \\ -1,(5,1)(6,1)(7,3) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 22 |
| $\left\{\begin{matrix} -3,(2,1)(3,1)(7,1) \\ -2,(3,2)(7,5)(8,1) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 22 |
| $\left\{\begin{matrix} -2,(2,1)(3,2)(7,5) \\ -9,(3,1)(7,1)(8,1) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 22 |
| $\left\{\begin{matrix} -8,(4,3)(5,3)(7,6) \\ -2,(3,2)(7,2)(8,7) \end{matrix}\right\}$ | 2 | 0 | $-20$ | 22 |
| $\left\{\begin{matrix} -10,(3,1)(5,1)(8,5) \\ -1,(3,1)(7,3)(8,1) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 22 |
| $\left\{\begin{matrix} -9,(2,1)(3,1)(5,3) \\ -1,(5,3)(7,1)(8,1) \end{matrix}\right\}$ | $-2$ | 0 | 20 | 22 |

Table 7.2

3) There are also many plumbed homology 3-spheres which could not be detected whether they have infinite order or not by using Theorem 2, 3 and Claim 1, 2. It is worth while to give several examples.

3-1) Brieskorn homology 3-spheres.
In the following table (Table 7.3), we list all Brieskorn homology 3-spheres $\Sigma(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \leq 10$ and with the Rohlin invariants vanish which we can not detect by using the realization of these homology 3-spheres as the boundary of plumbed 4-V-manifolds associated to the Seifert graph with one vertex. In the following list, the plumbed spin 4-V-manifold corresponding to the Brieskorn sphere $\Sigma(5,7,9)$ is not V-spin and hence we can not apply Theorem 3.

| Brieskorn | $w$(spin) | $\max\{w\}$ | resolution | | Milnor fiber | |
|---|---|---|---|---|---|---|
| $\Sigma(3,4,5)_{\mathrm{CH}}$ | 0 | 0 | 0 | 6 | $-20$ | 24 |
| $\Sigma(2,5,7)_{\mathrm{CH}}$ | 0 | 0 | 0 | 8 | $-20$ | 24 |
| $\Sigma(5,6,7)_{\mathrm{CH}}$ | 0 | 0 | 0 | 6 | $-80$ | 120 |
| $\Sigma(3,5,8)$ | 0 | 0 | 0 | 8 | $-40$ | 56 |
| $\Sigma(3,7,8)_{\mathrm{CH}}$ | 0 | 0 | 0 | 8 | $-60$ | 84 |
| $\Sigma(5,7,9)$ | — | 0 | — | — | $-120$ | 192 |
| $\Sigma(5,8,9)_{\mathrm{CH}}$ | 0 | 0 | 0 | 8 | $-140$ | 224 |
| $\Sigma(7,8,9)_{\mathrm{CH}}$ | 0 | 0 | 0 | 6 | $-200$ | 336 |

Table 7.3

**Remark.** Brieskorn homology spheres $\Sigma(\alpha_1, \alpha_2, \alpha_3)_{\mathrm{CH}}$ are in the list of A. Casson and J. Harer [1] which are homology cobordant to zero. The list in [1] includes Brieskorn homology spheres of the form: $\Sigma(p, ps \pm 1, ps \pm 2)$, $p \geq 3$ odd, $s \geq 1$, and $\Sigma(p, ps-1, ps+1)$, $p \geq 2$ even, $s \geq 1$ odd.

3-2) Plumbed homology 3-spheres obtained by plumbing according to Seifert graphs $\Gamma = (V, E, \omega)$, $\sharp V = 2$, $n_1 = 3$, $n_2 = 3$.
The following list (Table 7.4) contains several examples of plumbed homology 3-spheres with the Rohlin invariants vanish which we can not detect by using the realization of these homology spheres as the boundary of plumbed 4-V-manifolds associated to Seifert graphs. Note that $\max\{w\}$ are computed only for the negative definite plumbed 4-V-manifolds for which we can apply Theorem 2.

| Plumbed homology 3-spheres | $w$(spin) | $\max\{w\}$ | V-sig. | resolution | |
|---|---|---|---|---|---|
| $\left\{\begin{array}{c} -2,(2,1)(3,1)(5,1) \\ -3,(2,1)(3,2)(5,4) \end{array}\right\}$ | 0 | — | 0 | 0 | 16 |
| $\left\{\begin{array}{c} -2,(2,1)(3,2)(5,4) \\ -31,(2,1)(3,1)(5,1) \end{array}\right\}$ | 0 | — | 0 | 0 | 16 |
| $\left\{\begin{array}{c} -10,(2,1)(3,1)(5,3) \\ -2,(3,1)(4,3)(5,4) \end{array}\right\}$ | 0 | — | 0 | 0 | 14 |
| $\left\{\begin{array}{c} -1,(3,1)(4,1)(5,2) \\ -61,(3,1)(4,1)(5,2) \end{array}\right\}$ | 0 | 0 | $-2$ | 0 | 12 |
| $\left\{\begin{array}{c} -3,(3,2)(4,3)(5,3) \\ -2,(3,1)(4,1)(5,2) \end{array}\right\}$ | 0 | — | 0 | 0 | 16 |
| $\left\{\begin{array}{c} -1,(3,1)(4,1)(5,2) \\ -62,(3,2)(4,3)(5,3) \end{array}\right\}$ | 0 | — | 0 | 0 | 12 |

Table 7.4

3-3) Y. Matsumoto informed me about Problem 4.28 given by himself in Kirby problems [16]. Let $\Sigma$ be the homology 3-sphere represented by a framed link $L$ in $S^3$ consisting of two copies of trefoils $K_1$, $K_2$ with framing 0 and linking number $lk(K_1, K_2) = 1$ (Figure 7.1). The problem is whether $\Sigma$ is the boundary of an smooth acyclic 4-manifold or not. By the framed link calculus, we see that $\Sigma$ is obtained by plumbing according to the following graph $\Gamma$ (Figure 7.2). If we blow down linear arms in $P(\Gamma)$ then we get the 4-V-manifold $P(\hat{\Gamma})$ obtained by plumbing according to Seifert graph: $\hat{\Gamma} = (\hat{V}, \hat{E}, \hat{\omega})$, $\hat{V} = \{1, 2\}$, $\hat{E} = \{(1, 2)\}$, $\hat{\omega}(1) = \{-1, (2, 1), (3, 1), (35, 6)\}$, $\hat{\omega}(2) = \{-1, (2, 1), (3, 1), (35, 6)\}$, where we must plumb the singular point of type $(35, 6)$ in $\hat{\omega}(1)$ and that of type $(35, 6)$ in $\hat{\omega}(2)$. Note that $\Sigma(\hat{\Gamma}) = \Sigma(\Gamma) = \Sigma$. The signature of the intersection matrix $I_{\hat{\Gamma}}$ of $P(\hat{\Gamma})$ is zero, so we cannot apply Theorem 2. On the other hand, we see that $P(\hat{\Gamma})$ is V-spin and so we can compute $w$-invariant of spin structure, but $w(\Sigma(\hat{\Gamma}), P(\hat{\Gamma}), \vec{m}_{\mathrm{sp}}) = 0$, so we can not apply Theorem 3 also. Therefore we cannot detect whether $\Sigma$ is the boundary of an acyclic 4-manifold or not by our method.

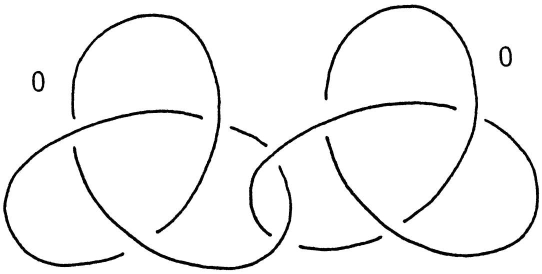


Figure 7.1

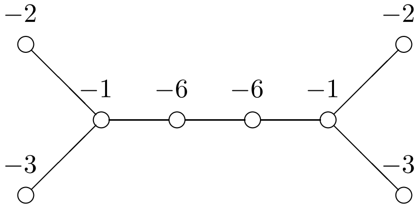


Figure 7.2

**Concluding remarks.** 1. We will check the agreement of the detection using Theorem 2 which follows from the Seiberg-Witten theory and the detection using the method of T. Lawson [17] which follows from the Donaldson theory by a numerical experimentation. This will give an observation of the conjecture concerning the equivalence of the Seiberg-Witten theory and the Donaldson theory.

2. Recently, we found that the following equation:

$$w(\Sigma(\Gamma), P(\Gamma), c(\vec{m}_{\rm sp})) = -\frac{1}{8}\mathrm{sign}\, P(\tilde{\Gamma})$$

holds for any $\Gamma$ satisfying (HS) and (SP), where $\tilde{\Gamma}$ is the plumbing graph with all weights even corresponding to a resolution of $P(\Gamma)$. The right hand side is in fact minus the invariant $\bar{\mu}(\Sigma(\tilde{\Gamma}))$ for plumbed homology 3-spheres which is an integral lift of the Rohlin invariant introduced by W. Neumann [19] and L. Siebenmann [27]. This means that the Neumann-Siebenmann invariant $\bar{\mu}$ has a homology cobordism invariance in the set of all homology 3-spheres which bound plumbed spin 4-V-manifolds with $b_2^{\pm} \leq 2$, [7]. On the other hand, N. Saveliev defined an invariant by using instanton Floer homology and proved that this invariant is equal to the Neumann-Siebenmann invariant for Seifert fibered homology 3-spheres [26], [23]. He checked vanishing of this invariant for homology 3-spheres which are known to bound acyclic 4-manifolds [25].

3. The arguments in this paper may be extended to the case of rational homology 3-spheres. We will treat them in the framework of spin cobordisms.

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