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Distances on topological self-similar sets and the kneading determinants

By

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Outline

In this paper we investigate the possible self-similar metrics on self-similar sets. Traditionally, a self-similar set is associated with a family of contractions on a metric space. One often finds two of these self-similar sets are homeomorphic to each other, for example, the unit interval and the Koch curve (Figure 1). These two self-similar sets have the same topological structure but the different 'metric structures'. Moreover, we will later see that there exist many metric structures on this 'topological' self-similar set (Example 1.16). Roughly speaking, our question is the following: What metric does a self-similar set admit?

Our notion of self-similar sets is slightly different from the classical one. We introduce the notion of *topological* self-similar sets, which is a generalization of self-similar sets. While a self-similar set is associated with a family of contractions on a metric space, a topological self-similar set is abstractly constructed from the shift space.

By definition, there are no a priori metric on a topological self-similar set K. Our first aim is to find a distance function which makes K self-similar, which is called a *self-similar metric*. We will construct a *self-similar pseudometric* on K, however, the existence of a self-similar metric depends on the topology of K. We will give an example of a topological self-similar set which admits no self-similar metric. We also discuss some sufficient conditions of the existence of a self-similar metric.

Secondly, we study a critical property of self-similar metrics. Suppose that a topological self-similar set K admits a self-similar metric. Then K together with the metric is a self-similar set associated with contractions. But there is some restriction, that is, the possible Lipschitz constants of the contractions are bounded below. We expect that the lower bound, which we call a *critical polyratio*, is an important characteristic of topological self-similar sets. Using

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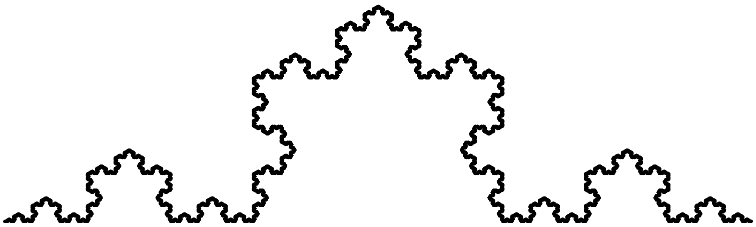

Figure 1: The Koch curve

an analogue of Milnor-Thurston's kneading theory, we will calculate of critical polyratios for a certain class of self-similar sets.

Introduction

The idea of self-similar sets has developed gradually. Classically, there are well-known self-similar figures including Cantor's ternary set and the Sierpinski Gasket. One can see that these figures are invariant sets of finitely many similitudes. Moran's result [15] is one of the earliest works from this point of view. This classical notion is refined through the works of Williams [20], Hutchinson [6] and Hata [4]. Their self-similar sets are constructed from finitely many contractions instead of similitudes (Definition 0.2). Afterward a purely topological definition (Definition 0.3) is given by the author [7] and Kigami [11]. One of the motivation of our study is to clarify the difference between these notions.

Definition 0.1. Let (X, d) be a metric space. A continuous mapping $F : X \to X$ is called a *contraction* with respect to the distance d if

$$\mathrm{Lip}_d(F) = \max_{x \neq y} \frac{d(F(x), F(y))}{d(x, y)} < 1.$$

The constant $\mathrm{Lip}_d(F)$ is called the *Lipschitz constant* of F, and it is also called the *contraction ratio* of F.

Definition 0.2. Let (X, d) be a complete metric space. Let $F_1, F_2, \ldots, F_N$ be contractions on X. Then there uniquely exists a nonempty compact set $K \subset X$ such that

$$K = F_1(K) \cup F_2(K) \cup \cdots \cup F_N(K).$$

We say that K is the *self-similar set* associated with $F_1, F_2, \ldots, F_N$.

By this definition, one can consider the self-similar set K as the attractor of the semigroup action generated by $F_1, F_2, \ldots, F_N$ (see [4] for detail). In fact, for any *word* $w = i_1 i_2 \ldots i_k \in \{1, 2, \ldots, N\}^k$, the composition $F_w = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_k}$ has a global attractive fixed point x_w, that is,

$$x_w = \lim_{n \to \infty} {F_w}^n(x) \tag{0.1}$$

for any $x \in X$. Then we have an expression

$$K = \text{closure}\left\{ x_w \mid w \in \bigcup_{k=1}^{\infty} \{1, 2, \ldots, N\}^k \right\}.$$

Thus a dense subset of K is 'coded' by the set of finite words. Moreover, let us see that self-similar sets have a stronger property called 'coding property,' the whole set K is coded by the set of infinite words. We denote by Σ_N the one-sided shift space with N symbols, i.e. the set of one-sided infinite sequences of $\{1, 2, \ldots, N\}$, which is identified with the mapping space $\{1, 2, \ldots, N\}^{\mathbb{N}} = \{\underline{a} : \mathbb{N} \to \{1, 2, \ldots, N\}\}$ and is equipped with the topology of the direct product of the finite set. Then for any $\underline{a} = i_1 i_2 \cdots \in \Sigma_N$, similarly to (0.1), we have a unique point $x_{\underline{a}} \in K$ such that

$$x_{\underline{a}} = \lim_{k \to \infty} F_{i_1 i_2 \ldots i_k}(x)$$

for any $x \in K$, and also an expression

$$K = \{x_{\underline{a}} \mid \underline{a} \in \Sigma_N\}.$$

This correspondence between $\underline{a}$ and $x_{\underline{a}}$ yields a continuous surjective 'coding map' $\pi : \Sigma \to K$ such that the diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\tau_i"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\ K \arrow[r, "F_i"'] & K \end{tikzcd}$$

commute for all $i = 1, 2, \ldots, N$, where $\tau_i(w_1 w_2 \ldots) = i w_1 w_2 \ldots$. In the light of the coding property, we propose a purely topological description of a self-similar sets as follows.

Definition 0.3. A compact Hausdorff topological space K is called a *topological self-similar set* if there exist continuous maps $F_1, F_2, \ldots, F_N : K \to K$ and a continuous surjection $\pi : \Sigma_N \to K$ such that the diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\tau_i"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\ K \arrow[r, "F_i"'] & K \end{tikzcd}$$

commutes for all i. We say that $(K, \{F_i\}_{i=1}^N)$, a topological self-similar set together with the set of continuous maps as above, is a *topological self-similar system*. We call π the *coding map* of $(K, \{F_i\}_{i=1}^N)$.

Clearly, a self-similar set associated with contractions $F_1, F_2, \ldots, F_N$ is a topological self-similar set. However it is not easy to see whether the converse is true or false.

Problem 1. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. (1) Is there a distance function $d(\cdot,\cdot)$ on K such that all F_i are contractions with respect to d? (Such a distance is called a *self-similar metric.*) (2) If the answer is negative, what kind of topological self-similar sets has a self-similar metric?

The first half of this paper is concerned with this problem. In Section 1 we construct a *standard pseudodistance* $D_\alpha(\cdot,\cdot)$ on K for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (0,1)^N$ which satisfies $D_\alpha(F_i(x), F_i(y)) \leq \alpha_i D_\alpha(x,y)$ for all i. We say that α is the *polyratio* of D_α. A standard pseudodistance is the basic tool throughout this paper. We will show that there exists a self-similar metric if and only if there is a polyratio such that the standard pseudodistance is a distance. Moreover, if the standard pseudodistance with polyratio $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a distance, then the standard pseudodistance with polyratio $(\alpha'_1, \alpha'_2, \ldots, \alpha'_N)$ such that $\alpha_i \leq \alpha'_i$ for all i is also a distance. This fact gives rise to the following problem:

Problem 2. Find *critical polyratios*, i.e. minimal polyratios such that the standard pseudodistances are distances if exists.

The set of critical polyratios is considered as a measure of the topological complexity of a topological self-similar set. We will see in Section 3 that it has a strong relation to the topological entropy. In Section 1 we also present a result on totally disconnected topological self-similar sets. For a self-similar set K associated with one-to-one contractions $F_1, F_2, \ldots, F_N$, it is known that the connectedness of K is restricted by the Lipschitz constants of the contractions: if $\sum_{i=1}^N \mathrm{Lip}(F_i) < 1$, then K is totally disconnected (see [20] and [4]). Our result is following: a topological self-similar set is totally disconnected if and only if the set of critical polyratios consists of only one point $(0,0,\ldots,0)$, i.e. any standard pseudodistance is a distance.

We also give a counterexample to Problem 1 in Section 1, that is, we will show that there exists a topological self-similar set without any self-similar metric. This example is constructed as follows. First we introduce the notion of the *critical set* of a topological self-similar system, which will play an important role in our study. As in the study of interval dynamics, we use the idea of *kneading invariants*, which is determined by the behavior of the critical set. We will see that a topological self-similar set is, in topological sense, a quotient space of Σ_N with respect to a equivalence relation 'generated' by the kneading invariant, moreover, under a certain condition, we can construct a topological self-similar system with a given kneading invariant. Specifically, we show that there exists a topological self-similar system which has the kneading invariant same as that of an irrational rotation on S^1. From the fact that an irrational rotation is volume-preserving, we see that this topological self-similar system has no self-similar metric.

In Section 2 we consider topological self-similar sets $(K, \{F_i\})$ satisfying a certain condition, which are often said to be 'finitely ramified.' Such a topological self-similar set has only finitely many critical points, and hence its 'dynamics' resembles to one-dimensional dynamics. Roughly speaking, in this context, it is natural to consider the dynamics of f, the 'inverse map' of $\{F_i\}$, which behaves as a piecewise monotone map on an interval or a rational map on Riemann sphere. With respect to a self-similar metric (if exists), f is an 'expanding map.' Thus the self-similarity is regarded as a kind of the hyperbolicity of the dynamics. We say that $(K, \{F_i\})$ is *non-recurrent* if the orbit of any critical point does not accumulate in the critical set. Such a condition often appears in the study of one-dimensional dynamics (see [18], [17] and [14], Chapter III, Section 6). For example, in [18], van Strien showed that a Misiurewicz map on an interval with some assumption is almost hyperbolic. We will prove that $(K, \{F_i\})$ has a self-similar metric if it is non-recurrent.

Problem 2 will be studied in Section 3. Under a certain situation, a topological self-similar set defines a dynamics on a topological tree. In such a case, if a critical polyratio $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfies $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_N$, then $-\log \alpha$ is the topological entropy. Thus we can consider the notion of critical ratios as a generalization of topological entropy. In [7], using matrices associated with directed graphs, the author calculated the critical polyratios of topological self-similar sets with the property called 'postcritically finite.' In this paper we will use a version of Milnor-Thurston's theory (see [16]) in order to study critical polyratios. Recall that in interval dynamics, the topological entropy is calculated from the asymptotic behavior of the lap number. In our case we will define a power series with coefficients corresponding to the lap numbers, and show that its radius of convergence is a critical polyratio. For the proof, Milnor-Thurston has used kneading determinants of one variable; we will use kneading determinants of N variables. (Kneading determinants of N variables are strongly related to dynamical zeta functions with locally constant weight. See [1].) We prove that the critical polyratios are zeros of the kneading determinant, and immediately we see that the set of the critical ratios is a real analytic set since the kneading determinant is an analytic function.

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Contents

1. General theory and examples

In this section we introduce the concept of standard pseudodistances on topological self-similar sets, which is the main tool throughout this paper. After that, several preparatory results, including a counterexample, are formulated.

1.1. Basic definitions

In this subsection we formalize our notation and give several examples. We also see the fundamental fact that a topological self-similar set is metrizable.

Notation and Definition 1.1.

(1) (The space of infinite sequences) We denotes by (Σ_N, σ) the one-sided symbolic dynamical system with N symbols ($N \geq 2$). Namely, $\Sigma_N = \{1, 2, \dots, N\}^{\mathbb{N}}$ is the space of infinite sequences of $\{1, 2, \dots N\}$. We write an element of Σ_N as $\underline{w} = w_1 w_2 \dots$. The shift map $\sigma : \Sigma_N \to \Sigma_N$ removes the leading symbol of a sequence, that is, $\sigma(w_1 w_2 \dots) = w_2 \dots$ for any $w_1 w_2 \dots \in \Sigma_N$. The shift map is an N-to-1 map, and we can naturally define the branches $\tau_1, \tau_2, \dots, \tau_N$ of σ^{-1} such that $\tau_i(w_1 w_2 \dots) = i w_1 w_2 \dots$ for $i = 1, 2, \dots, N$.

(2) (The space of words) The space of finite sequences of length n is denoted by

$$\mathcal{W}_n = \{1, 2, \dots, N\}^n = \{u_1 u_2 \dots u_n \,|\, u_k \in \{1, 2, \dots, N\}, k = 1, 2, \dots, n\}.$$

We write $\mathcal{W}_* = \bigcup_{n=0}^{\infty} \mathcal{W}_n$. An element of $\mathcal{W}_n$ is said to be a *word* of length (or *depth*) n. The set $\mathcal{W}_0$ consists of only one element, called the *empty word*, which we denote by $\emptyset$. The length of a word U is denoted by $|U|$. The mapping σ and τ_i are also applied on $\mathcal{W}_*$. More precisely, we set $\sigma(u_1 u_2 \dots u_n) = u_2 \dots u_n$ for $u_1 u_2 \dots u_n \in \bigcup_{n=1}^{\infty} \mathcal{W}_n$, $\sigma(\emptyset) = \emptyset$, and $\tau_i(u_1 u_2 \dots u_n) = i u_1 u_2 \dots u_n$ for $i = 1, 2, \dots, N$ and $u_1 u_2 \dots u_n \in \mathcal{W}_*$. A word U is called a *successor* of U' if $\sigma^k(U') = U$ for some k.

(3) (Basis) If $U = u_1 u_2 \dots u_n$ is a word, then τ_U is the composition $\tau_{u_1} \circ \tau_{u_2} \circ \dots \circ \tau_{u_n}$. For simplicity, we write UV instead of $\tau_U(V)$. For $\underline{u} = u_1 u_2 \dots$, we

write

$$[\underline{u}]_n = u_1 u_2 \dots u_n.$$

We also write

$$[U]_n = u_1 u_2 \dots u_n$$

if $U = u_1 u_2 \dots u_m \in \mathcal{W}_m$ and $m \geq n$. For a word $U = u_1 u_2 \dots u_n \in \mathcal{W}_*$, we write

$$\Sigma(U) = \tau_U(\Sigma_N) = \{\underline{u} \in \Sigma_N \,|\, [\underline{u}]_n = U\}.$$

Then $\{\Sigma(U) \,|\, U \in \mathcal{W}_*\}$ is a basis for the open sets of Σ_N.

(4) (Order) We define a partial order on $\mathcal{W}_*$,

$$U \prec U'$$

if $\Sigma(U) \subset \Sigma(U')$. Remark that

$$\Sigma(U) \cap \Sigma(U') \neq \emptyset \iff U \prec U' \text{ or } U' \prec U.$$

If $m > n$, we use the notation

$$\mathcal{W}_m(U) = \tau_U(\mathcal{W}_{m-n}) = \{V \in \mathcal{W}_m \,|\, [V]_n = U\} = \{V \in \mathcal{W}_m \,|\, V \prec U\}.$$

Notation 1.2. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system (Definition 0.3). For $U = w_1 w_2 \dots w_k \in W_k$, we write $K(U) = F_U(K)$, where $F_U = F_{w_1} \circ F_{w_2} \circ \dots F_{w_k}$. If $U = \emptyset$, then F_U denotes the identity. Remark that $K(U)$ is compact and $K(w_1 w_2 \dots w_{k-1}) \subset K(w_1 w_2 \dots w_k)$. We write

$$L_n(x) = \bigcup_{\substack{\pi^{-1}(x) \cap \Sigma(U) \neq \emptyset \\ U \in \mathcal{W}_n}} K(U).$$

Remark 1.3. Let K be a compact Hausdorff set, and let $F_1, F_2, \dots, F_N$ be continuous maps of K to itself. Then $(K, \{F_i\}_{i=1}^N)$ is a topological self-similar system if and only if $\bigcap_{n=0}^\infty K(u_1 u_2 \dots u_n)$ consists of only one point for any $u_1 u_2 \dots \in \Sigma_N$. In particular, if $(K, \{F_i\}_{i=1}^N)$ is a topological self-similar system, then the coding map is uniquely determined.

Indeed, if π is the coding map, then it has to satisfy $\pi(u_1 u_2 \dots) \in K(u_1 u_2 \dots u_n)$ for any $n \geq 0$. If $\bigcap_{n=0}^\infty K(u_1 u_2 \dots u_n)$ has more than one point, then π is not surjective. Conversely, suppose $\bigcap_{n=0}^\infty K(u_1 u_2 \dots u_n)$ has only one point for any $u_1 u_2 \dots \in \Sigma_N$. Then a surjective map $\pi : \Sigma_n \to K$ is defined by $\pi(u_1 u_2 \dots) \in \bigcap_{n=0}^\infty K(u_1 u_2 \dots u_n)$. If O is a neighborhood of $\pi(u_1 u_2 \dots)$, then there exists n such that $K(u_1 u_2 \dots u_n) \subset O$. Since $\pi^{-1}(K(u_1 u_2 \dots u_n))$ includes $\Sigma(u_1 u_2 \dots u_n)$, which is a neighborhood of $u_1 u_2 \dots$, we conclude that π is continuous.

Lemma 1.4. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system with coding map π. Then $\mathcal{N}(x) = \{L_n(x) \,|\, n = 0, 1, 2 \dots\}$ is a fundamental neighborhood system.*

Proof. Note that $X = \bigcup_{\substack{\pi^{-1}(x)\cap\Sigma(U)=\emptyset \\ U\in\mathcal{W}_n}} K(U)$ is compact. Since $x \notin X$, we conclude $K - X$ is a neighborhood of x in $L_n(x)$.

Conversely, let O be an open neighborhood of x. Then $\pi^{-1}(O)$ is also open. It is easy to see that

$$\pi^{-1}(O) = \bigcup_{\substack{\Sigma(U)\subset\pi^{-1}(O) \\ U\in\mathcal{W}_*}} \Sigma(U).$$

Since $\pi^{-1}(x)$ is compact, there exists a finite subset $\mathcal{U} \subset \{U \in \mathcal{W}_* \mid \Sigma(U) \subset \pi^{-1}(O)\}$ such that $\pi^{-1}(x) \subset \bigcup_{U\in\mathcal{U}} \Sigma(U)$. Therefore $L_n(x) \subset O$ for $n = \max_{U\in\mathcal{U}} |U|$. □

Theorem 1.5. *A topological self-similar set is metrizable.*

Proof. From Lemma 1.4, a topological self-similar set K satisfies the second countability axiom. Indeed,

$$\left\{ \operatorname{int} \bigcup_{U\in\mathcal{D}} K(U) \,\middle|\, \mathcal{D} \subset \mathcal{W}_n, n = 0, 1, 2, \ldots \right\}$$

is a basis for the open sets. A Hausdorff space together with the second countability axiom is metrizable (for example see [10]). □

Lemma 1.6. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system with coding map π. Let d be any distance on K which is compatible with the original topology. We denote, by* $\operatorname{diam} X$*, the diameter of $X \subset K$ with respect to the distance d. Then*

$$\lim_{n\to\infty} \max_{U\in\mathcal{W}_n} \operatorname{diam} K(U) = 0.$$

Proof. Suppose there exist a positive number $\varepsilon > 0$ and a sequence $U_1, U_2, \ldots$ such that $U_k \in \mathcal{W}_k$ and $\operatorname{diam} K(U_k) > \varepsilon$. Let us take a point $\underline{u_k} \in \Sigma(U_k)$. Since Σ_N is compact, we can assume that $\lim_{k\to\infty} \underline{u_k} = \underline{u}$. For each n there exists k such that $U_k \prec [\underline{u}]_n$. Thus

$$\operatorname{diam} K([\underline{u}]_n) \geq \operatorname{diam} K(U_k) > \varepsilon.$$

Since $\{L_n(x) \mid n = 1, 2, \ldots\}$ is a fundamental neighborhood system, the $\epsilon/3$-ball

$$B(\pi(\underline{u}), \epsilon/3) = \{y \mid D(\pi(\underline{u}), y) < \epsilon/3\}$$

includes $L_n(\pi(\underline{u}))$ for some n. Therefore

$$\operatorname{diam} L_n(\pi(\underline{u})) < 2\epsilon/3.$$

This contradicts the fact that $K([\underline{u}]_n) \subset L_n(\pi(\underline{u}))$. □

As we have seen in Introduction, a self-similar set associated with contractions is a topological self-similar set. The first problem discussed in this paper is the following.

Definition 1.7. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. A distance d on K which is compatible with the original topology of K is called a *self-similar metric* if $F_1, F_2, \dots, F_N$ are contractions with respect to the distance d.

Problem 1-(1). Does a self-similar system $(K, \{F_i\}_{i=1}^N)$ have any self-similar metric?

We will consider this problem in the following subsections. For the moment, we show several examples of self-similar sets, all of which are obtained from contractions.

Example 1.8. The first four examples are subsets of the unit interval $[0, 1]$; the last two examples are Julia sets of quadratic polynomials in the complex plane.

We use the symbols $\mathbf{1}, \mathbf{2}, \dots, \mathbf{N}$ instead of $1, 2, \dots, N$ in order to avoid confusion. If U is a word, we denote, by $\overline{U}$, the infinite periodic sequence $UU \cdots \in \Sigma_N$. For example, $\overline{\mathbf{12}} = \mathbf{121212}\dots$ and $\mathbf{1}\overline{\mathbf{2}} = \mathbf{1222}\dots$. Similarly, if j is a nonnegative integer, we write $U^j = \underbrace{UU \dots U}_{j \text{ times}}$. For example, $(\mathbf{12})^3 = \mathbf{121212}$, $\mathbf{12}^3 = \mathbf{1222}$ and $\mathbf{11}^0 = \mathbf{1}$.

(1) Let X be the unit interval $[0, 1]$, and we define maps on X by

$$F_1(x) = x/3, F_2(x) = (x+2)/3.$$

Then the self-similar set K associated with F_1 and F_2 is Cantor's ternary set. The coding map $\pi : \Sigma_2 \to K$ is written as

$$\pi(u_1 u_2 \dots) = \sum_{u_k = \mathbf{2}} 2 \cdot 3^{-k}.$$

It is easy to see that π is a homeomorphism. For $x \in K$, the inverse image $u_1 u_2 \dots \in \pi^{-1}(x)$ is obtained by

$$u_n = \begin{cases} \mathbf{1} & \text{if} \quad f^{n-1}(x) \in [0, 1/3] \\ \mathbf{2} & \text{if} \quad f^{n-1}(x) \in [2/3, 1] \end{cases},$$

where $f(x) = 3x$ if $0 \le x \le 1/3$, and $f(x) = 3x - 2$ if $1/3 \le x \le 1$. For example, $\pi(\mathbf{1}\overline{\mathbf{2}}) = 1/3$ and $\pi(\overline{\mathbf{12}}) = 1/4$.

(2) Let X be the unit interval $[0, 1]$, and we define maps on X by

$$F_1(x) = x/2, F_2(x) = (x+1)/2.$$

Then the self-similar set K associated with F_1 and F_2 is the unit interval itself. The coding map $\pi : \Sigma_2 \to K$ is written as

$$\pi(u_1 u_2 \dots) = \sum_{u_k = \mathbf{2}} 2^{-k}.$$

The coding map is not injective. Indeed, $\pi(\mathbf{1}\overline{\mathbf{2}}) = \pi(\mathbf{2}\overline{\mathbf{1}}) = 1/2$. Note that $1/2$ is the point where $K(\mathbf{1})$ and $K(\mathbf{2})$ intersect. In fact, if $\#\pi^{-1}(x) > 1$, then $\#\pi^{-1}(x) = 2$ and there exist a positive integer n and distinct words $U, V \in \mathcal{W}_n$ such that $\{x\} = K(U) \cap K(V)$. Moreover, x has the form $k \cdot 2^{-n}$ for some odd number k. This is verified by the fact that $K(U)$ is the interval with endpoints $F_U(0)$ and $F_U(1)$. It is easy to see that

$$\pi^{-1}(k \cdot 2^{-n}) = \{u_1 u_2 \dots u_{n-1}\mathbf{1}\overline{\mathbf{2}}, u_1 u_2 \dots u_{n-1}\mathbf{2}\overline{\mathbf{1}}\},$$

where

$$k = 1 + \sum_{\substack{u_j = \mathbf{2} \\ j=1,2,\dots,n-1}} 2^{n-j}.$$

Thus if two distinct words $U, V \in \mathcal{W}_*$ satisfy the condition $K(U) \cap K(V) \neq \emptyset$ and $\Sigma(U) \cap \Sigma(V) = \emptyset$, then $\{U, V\} = \{U'\mathbf{1}\mathbf{2}^i, U'\mathbf{2}\mathbf{1}^j\}$ for some nonnegative integers i, j and some $U' \in \mathcal{W}_*$.

(3) Let X be the unit interval $[0,1]$, and we define maps on X by

$$F_1(x) = (1-x)/2, F_2(x) = (x+1)/2.$$

Then the self-similar set K associated with F_1 and F_2 is the unit interval itself. The coding map $\pi : \Sigma_2 \to K$ is written as

$$\pi(u_1 u_2 \dots) = \sum_{k=1}^{\infty} \epsilon(k) 2^{-k},$$

where we set $n = \#\{j \mid u_j = \mathbf{1}, j = 1, 2, \dots, k-1\}$ and $\epsilon(k) = (-1)^n$.

(4) Let X be the unit interval $[0,1]$, and we define maps on X by

$$\begin{aligned} F_1(x) &= \begin{cases} x/2 & \text{if} \quad 0 \leq x \leq 1/3 \\ 1/6 & \text{if} \quad 1/3 < x \leq 2/3 \\ (x-1/3)/2 & \text{if} \quad 2/3 < x \leq 1 \end{cases}, \\ F_2(x) &= \begin{cases} x/2 + 2/3 & \text{if} \quad 0 \leq x \leq 1/3 \\ 5/6 & \text{if} \quad 1/3 < x \leq 2/3 \\ (x-1/3)/2 + 2/3 & \text{if} \quad 2/3 < x \leq 1 \end{cases}. \end{aligned}$$

Then the self-similar set K associated with F_1 and F_2 is the union of two intervals $[0,1/3]$ and $[2/3,1]$. The coding map $\pi : \Sigma_2 \to K$ is written as

$$\pi(u_1 u_2 \dots) = \begin{cases} \dfrac{2}{3} \displaystyle\sum_{\substack{u_k = \mathbf{2} \\ k \geq 2}} 2^{-k} & \text{if } u_1 = \mathbf{1} \\ \dfrac{2}{3} + \dfrac{2}{3} \displaystyle\sum_{\substack{u_k = \mathbf{2} \\ k \geq 2}} 2^{-k} & \text{if } u_1 = \mathbf{2} \end{cases}.$$

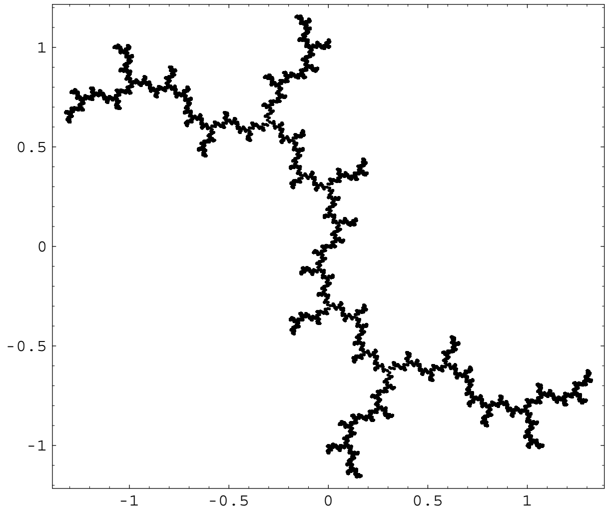

Figure 2: The Julia set of $f(z)_{\sqrt{-1}} = z^2 + \sqrt{-1}$

(5) Let K be the Julia set of the polynomial $f_{-2}(z) = z^2 - 2$. It is known that K is the interval $[-2, 2]$. The polynomial map f_{-2} has two inverse branches on K:

$$F_1(x) = -\sqrt{x+2}, F_2(x) = \sqrt{x+2}.$$

Then $(K, \{F_1, F_2\})$ is a self-similar system. Indeed, this is topologically conjugate to the third example above. That is to say, the two contractions in the third example are inverse branches of the map

$$g(x) = \begin{cases} 1 - 2x & \text{if } 0 \leq x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leq 1 \end{cases},$$

which is conjugate to the map f_{-2} by the homeomorphism $Q : [0, 1] \to [-2, 2]$ defined by

$$Q(x) = -2\cos(\pi x).$$

(6) Let K be the Julia set of the polynomial $f_{\sqrt{-1}}(z) = z^2 + \sqrt{-1}$. In this case the map $f_{\sqrt{-1}}$ also has two inverse branches F_1, F_2 on K, and $(K, \{F_1, F_2\})$ is a self-similar system (Figure 2). Indeed, there exists a 'metric' on a neighborhood of K for which f is expanding (see [3]). The metric can be written in the form $v(z)|dz|$, where v is continuous except at the postcritical set $\{\sqrt{-1}, -1 + \sqrt{-1}, -\sqrt{-1}\}$. Such a polynomial is said to be subhyperbolic. If all critical points of a given polynomial are not periodic but eventually periodic, then it is subhyperbolic, and then the Julia set is a topological self-similar set (see [8]).

1.2. Standard pseudodistances

As we will show an example later, a topological self-similar system does not always have a self-similar metric. However we can always construct a pseudodistance, which is a criterion of existence of a self-similar metric.

Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. We say that an ordered N-tuple $(\alpha_1, \alpha_2, \dots, \alpha_N)$ is a *polyratio* if all α_i are positive numbers less than one. We denote by $\mathbf{Ra}_N$ the set of polyratios. For a polyratio $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, we will construct a pseudodistance $D_\alpha(\cdot,\cdot)$ on K, which satisfies $D_\alpha(F_i(x), F_i(y)) \le \alpha_i D_\alpha(x, y)$ for any $i = 1, 2, \dots, N$. This is called the standard pseudodistance for α. If the pseudodistance D_α is a distance, then of course it is a solution to Problem 1-(1). The following fact, which will be proved later, is important: there exists a self-similar metric if and only if the pseudodistance for some polyratio is a distance.

Definition 1.9. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. We say that an ordered l-tuple $(U_1, U_2, \dots, U_l)$ is a *pre-chain* of $(K, \{F_i\}_{i=1}^N)$ if $U_j \in \mathcal{W}_*$ $(j = 1, 2, \dots, l)$ and $K(U_j) \cap K(U_{j+1}) \ne \emptyset$ $(j = 1, 2, \dots, l-1)$. A pre-chain $(U_1, U_2, \dots, U_l)$ is called a pre-chain of *depth* n if every U_i belongs to $\mathcal{W}_n$. We say that l is the *length* of the pre-chain.

Let $x, y \in K$. We say that $(U_1, U_2, \dots, U_l)$ is a pre-chain between x and y if $x \in K(U_1)$ and $y \in K(U_l)$. A pre-chain $(U_1, U_2, \dots, U_l)$ is called a *chain* if $\Sigma(U_j) \cap \Sigma(U_{j'}) = \emptyset$ for $j \ne j'$. We denote, by $G(x, y)$ (resp. $G'(x, y)$), the set of chains (resp. pre-chains) between x and y. The set of chains of depth n (resp. of depth at most n) between x and y is denoted by $\tilde{G}_n(x, y)$ (resp. $G_n(x, y)$). Since $K(\emptyset) = K$, the set $G(x, y)$ is not empty.

Definition 1.10. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a polyratio. We construct a pseudodistance $D(\cdot,\cdot) = D_\alpha(\cdot,\cdot)$ as follows. For a word $U = w_1 w_2 \dots w_n \in \mathcal{W}_n$, we write

$$A(U) = \alpha_{w_1} \alpha_{w_2} \cdots \alpha_{w_n}.$$

We set $A(U) = 1$ for $U = \emptyset$. For a pre-chain $\mathcal{C} = (U_1, U_2, \dots, U_l)$, we write

$$A(\mathcal{C}) = A(U_1) + A(U_2) + \cdots A(U_l).$$

We define

$$D(x, y) = \inf_{\mathcal{C} \in G(x,y)} A(\mathcal{C}) = \inf_{\mathcal{C} \in G'(x,y)} A(\mathcal{C}).$$

Remark that it is also described as $\lim_{n\to\infty} \min_{\mathcal{C} \in G_n(x,y)} A(\mathcal{C})$, since $\min_{\mathcal{C} \in G_n(x,y)} A(\mathcal{C})$ is decreasing as $n \to \infty$.

It is evident that if $(U_1, U_2, \dots, U_l) \in G'(x, y)$ and $(U'_1, U'_2, \dots, U'_{l'}) \in G'(y, z)$, then $(U_1, U_2, \dots, U_l, U'_1, U'_2, \dots, U'_{l'}) \in G'(x, z)$. Thus we have $D(x, y) + D(y, z) \ge D(x, z)$, and so the function D is a pseudodistance. The pseudodistance D is a distance if and only if $D(x, y) > 0$ for any distinct point x, y. We say that D is the *standard pseudodistance* of $(K, \{F_i\}_{i=1}^N)$ for polyratio α.

From the following proposition, D is compatible with the topology of K.

Proposition 1.11. *For any $\epsilon > 0$ there exists $n \geq 0$ such that $L_n(x) \subset B(x, \epsilon)$ for any x, where $B(x, \epsilon)$ is the ϵ-ball $\{y \mid D(x, y) < \epsilon\}$.*

Moreover, suppose that D is a distance. Then for any $n \geq 0$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset L_n(x)$.

Proof. If $x, y \in K(U)$ for some $U \in \mathcal{W}_n$, then $(U) \in G(x, y)$ and $D(x, y) \leq A(U) \leq (\max_i \alpha_i)^n$. Therefore $L_n(x) \subset B(x, \epsilon)$ for $n \geq \log \epsilon / \log(\max_i \alpha_i)$.

Suppose that D is a distance. Assume that there exists a sequence x_1, $x_2, \ldots$ outside $L_n(x)$ such that $\lim_{i\to\infty} D(x, x_i) = 0$. We may also assume that x_n converges to some $y \in K$ as $n \to \infty$ in the topology of K. Remark that $y \neq x$ since each x_i is not contained in a neighborhood $L_n(x)$. Then from the first assertion we have $\lim_{i\to\infty} D(x_i, y) = 0$. Thus $D(x, y) \leq D(x, x_i) + D(x_i, y) \to 0$. This is a contradiction. □

Proposition 1.12. *For each $i = 1, 2, \ldots, N$,*

$$D(F_i(x), F_i(y)) \leq \alpha_i D(x, y).$$

Proof. For any $\epsilon > 0$ there exists a chain $\mathcal{C} = (U_1, U_2, \ldots, U_l) \in G(x, y)$ satisfies

$$A(\mathcal{C}) < D(x, y) + \epsilon.$$

Then $G(F_i(x), F_i(y))$ contains $(iU_1, iU_2, \ldots, iU_l)$, and

$$\begin{aligned} D(F_i(x), F_i(y)) &\leq A(iU_1) + A(iU_2) + \cdots + A(iU_l) \\ &= \alpha_i(A(U_1) + A(U_2) + \cdots + A(U_l)) \\ &< \alpha_i(D(x, y) + \epsilon). \end{aligned}$$

□

Proposition 1.13. *Suppose that there exists a self-similar metric d. If we choose positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$ such that $\mathrm{Lip}_d(F_i) \leq \alpha_i < 1$ $(i = 1, 2, \ldots, N)$, then the standard pseudodistance $D = D_\alpha$ for the polyratio $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a distance.*

Proof. We set $M = \max_{x,y\in K} d(x, y)$. Let $\epsilon > 0$ be a positive number. Choose a chain $\mathcal{C} = (U_1, U_2, \ldots, U_l) \in G(x, y)$ such that

$$A(\mathcal{C}) < D(x, y) + \epsilon.$$

Let $x_i \in K(U_i) \cap K(U_{i+1})$ for $i = 1, 2, \ldots, l-1$. We take points a_i for $i = 0, 1, \ldots, l-1$ and b_i for $i = 1, 2, \ldots, l$ so that $F_{U_i}(a_{i-1}) = x_{i-1}$ and $F_{U_i}(b_i) = x_i$, where $x_0 = x, x_l = y$. Then

$$d(x_{i-1}, x_i) \leq A(U_i) d(a_{i-1}, b_i) \leq A(U_i) M.$$

Thus

$$\begin{aligned} d(x, y) &\leq d(x, x_1) + d(x_1, x_2) + \cdots + d(x_{l-1}, y) \\ &\leq (A(U_1) + A(U_2) + \ldots A(U_l)) M \\ &< (D(x, y) + \epsilon) M. \end{aligned}$$

Therefore $0 < d(x, y)/M \leq D(x, y)$. □

Corollary 1.14. *A topological self-similar system has a self-similar metric if and only if there exists* $0 < \alpha < 1$ *such that the standard pseudodistance* D *for the polyratio* $(\alpha, \alpha, \dots, \alpha)$ *is a distance.*

Definition 1.15. We say a polyratio $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a *metric polyratio* if D_α is a distance. A *critical polyratio* is an infimum of metric polyratios. We denote, by $\mathrm{CR} = \mathrm{CR}(K, \{F_i\}_{i=1}^N)$, the set of critical polyratios of $(K, \{F_i\}_{i=1}^N)$. Precisely, we say that $(\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbf{Ra}_N$ belongs to CR if

- if $0 < \alpha'_i < \alpha_i$ for $i = 1, 2, \dots, N$, then $(\alpha'_1, \alpha'_2, \dots, \alpha'_N)$ is not a metric polyratio,
- if $\alpha_i < \alpha'_i < 1$ for $i = 1, 2, \dots, N$, then $(\alpha'_1, \alpha'_2, \dots, \alpha'_N)$ is a metric polyratio.

The following cases are exceptional: If every polyratio is a metric polyratio, then we set $\mathrm{CR} = \{(0, 0, \dots, 0)\}$; if every polyratio is not a metric polyratio, then we set $\mathrm{CR} = \{(1, 1, \dots, 1)\}$.

To study CR is one of the aims in this paper. We will see in Section 3 the properties of CR for some class of topological self-similar systems. Here we give two examples for which we can easily describe CR.

Example 1.16. (1) Consider the self-similar system $(K, \{F_1, F_2\})$ in Example 1.8-(1). Then any (α_1, α_2) is a metric polyratio. Indeed, since the coding map π is a homeomorphism, $K(U) \cap K(V)$ is empty if $\Sigma(U) \cap \Sigma(V) = \emptyset$. Thus

$$G(x, y) = \{(U) \,|\, U \in \mathcal{W}_*, \pi^{-1}(x), \pi^{-1}(y) \in \Sigma(U)\}.$$

In other words, if $x = \pi(u_1 u_2 \dots u_n u_{n+1} \dots)$ and $y = \pi(u_1 u_2 \dots u_n u'_{n+1} \dots)$ with $u_{n+1} \neq u'_{n+1}$, then $G(x, y) = \{(u_1), (u_1 u_2), \dots, (u_1 u_2 \dots u_n)\}$. For example, $G(0, 1) = \{(\emptyset)\}$ and $G(2/9, 1/3) = \{(\mathbf{1}), (\mathbf{12})\}$. Therefore $D(x, y) = \alpha_{u_1} \alpha_{u_2} \cdots \alpha_{u_n} > 0$.

(2) Consider the self-similar system $(K, \{F_1, F_2\})$ in Example 1.8-(2). We will show that

$$\mathrm{CR} = \{(\alpha_1, \alpha_2) \,|\, \alpha_1 + \alpha_2 = 1, 0 < \alpha_1 < 1, 0 < \alpha_2 < 1\}.$$

This set is seen as the gray region in Figure 3. Suppose that $(\alpha_1, \alpha_2) \in \mathbf{Ra}_2$ satisfies $\alpha_1 + \alpha_2 < 1$. Let n be a positive integer, and let k be an integer such that $1 \le k \le 2^n$. Let $U_{n,k} = u_1 u_2 \dots u_n \in \mathcal{W}_*$ be the word defined by

$$k = 1 + \sum_{\substack{u_j = \mathbf{2} \\ j = 1, 2, \dots, n}} 2^{n-j}.$$

For example, $U_{2,1} = \mathbf{11}, U_{2,2} = \mathbf{12}, U_{2,3} = \mathbf{21}, U_{2,4} = \mathbf{22}$. It is clear that $\{U_{n,k} \,|\, 1 \le k \le 2^n\} = \mathcal{W}_n$. For any n, the 2^n-tuple $\mathcal{C}_n = (U_{n,1}, U_{n,2}, \dots, U_{n,2^n})$

is a chain between 0 and 1. Therefore

$$D(0,1) \leq A(\mathcal{C}_n) = \sum_{U \in \mathcal{W}_n} A(U) = (\alpha_1 + \alpha_2)^n \to 0 \ (n \to \infty),$$

and hence (α_1, α_2) is not a metric ratio.

Suppose that $(\alpha_1, \alpha_2) \in \mathbf{Ra}_2$ satisfies $\alpha_1 + \alpha_2 = 1$. Let $\mathcal{C} = (U_1, U_2, \ldots, U_l)$ be a chain between 0 and 1. Set $n(\mathcal{C}) = \max_i |U_i|$. If $|U_k| = n(\mathcal{C})$ and U_k has the form $U\mathbf{1}$, then $k \neq l$ and $U_{k+1} = U\mathbf{2}$. If $|U_k| = n(\mathcal{C})$ and U_k has the form $U\mathbf{2}$, then $k \neq 1$ and $U_{k-1} = U\mathbf{1}$. Remark that $K(U) = K(U_k) \cup K(U_{k+1})$ (or $K(U) = K(U_{k-1}) \cup K(U_k)$). Putting U instead of U_k, U_{k+1} (or U_{k-1}, U_k), we obtain a new chain $\mathcal{C}_1 = (U_1, U_2, \ldots, U_{k-1}, U, U_{k+2}, \ldots, U_l)$, which satisfies $A(\mathcal{C}) = A(\mathcal{C}_1)$ since $A(U\mathbf{1}) + A(U\mathbf{2}) = A(U)(\alpha_1 + \alpha_2) = A(U)$. This procedure gives us a sequence of chains $\mathcal{C}, \mathcal{C}_1, \ldots, \mathcal{C}_m$ such that $A(\mathcal{C}) = A(\mathcal{C}_1) = \cdots = A(\mathcal{C}_m)$ and $\mathcal{C}_m = (\emptyset)$. Thus $A(\mathcal{C}) = 1$ for any chain $\mathcal{C}$ in $G(0,1)$. Consequently, $D(0,1) = 1$. Moreover, $D(x,y) > 0$ for any distinct points $x, y \in K$. To prove this, we show that if $D(x,y) = 0$ for some distinct points in K, then $D(0,1) = 0$. Indeed, if $x < y$, then there exist integers n and k such that $x \leq k \cdot 2^{-n}, (k+1)2^{-n} \leq y$. Since a chain between x and y includes a chain between $k \cdot 2^{-n}$ and $(k+1)2^{-n}$, we have $D(k \cdot 2^{-n}, (k+1)2^{-n}) = 0$. There exists a word $U \in \mathcal{W}_*$ such that $K(U)$ is equal to the interval $[k \cdot 2^{-n}, (k+1)2^{-n}]$. For any ϵ, there exists a chain $\mathcal{C}$ between $k \cdot 2^{-n}$ and $(k+1)2^{-n}$ such that $A(\mathcal{C}) < \epsilon$. We can assume $\mathcal{C}$ has the form $(UU_1, UU_2, \ldots, UU_l)$. Clearly, $\mathcal{C}' = (U_1, U_2, \ldots, U_l)$ is a chain between 0 and 1. Thus $A(U)A(\mathcal{C}') < \epsilon$, and hence $D(0,1) = 0$. From this, it is follows that D is a distance.

Remark 1.17. Similar argument shows that the sets of critical polyratios for Example 1.8-(3), (4) and (5) are the same as that of Example 1.8-(2). The set of critical polyratios for Example 1.8-(6) is

$$\{(\alpha_1, \alpha_2) \,|\, \alpha_1\alpha_2 + \alpha_1^2\alpha_2 + \alpha_1^3 = 1, 0 < \alpha_1 < 1, 0 < \alpha_1 < 1\},$$

which will be shown by the argument in Section 3. See Figure 3.

In Section 3 we will see the relation of critical polyratios to topological entropies. Here we mention that the above calculation illustrate this relation. In Example 1.8-(5), the topological entropy of $(f_{-2}, [-2,2])$ is equal to $\log 2 = -\log 2^{-1}$; the intersection of the set of critical polyratio and the line $\alpha_1 = \alpha_2$ contains only one point $(2^{-1}, 2^{-1})$. In Example 1.8-(6), the topological entropy of $(f_{\sqrt{-1}}, T)$ is equal to $-\log \alpha$, where $T \subset K$ is the Hubbard tree (i.e. T is the minimal connected tree in K containing all postcritical points) and α is the positive root of the equation $t^2 + 2t^3 = 1$; a critical polyratio (α_1, α_2) satisfies the equation $\alpha_1\alpha_2 + \alpha_1^2\alpha_2 + \alpha_1^3 = 1$, which together with $\alpha_1 = \alpha_2 = t$ makes $t^2 + 2t^3 = 1$.

1.3. Kneading invariants

We introduce an important invariant of topological self-similar systems, which is called the kneading invariant. The notion of kneading invariants origi-

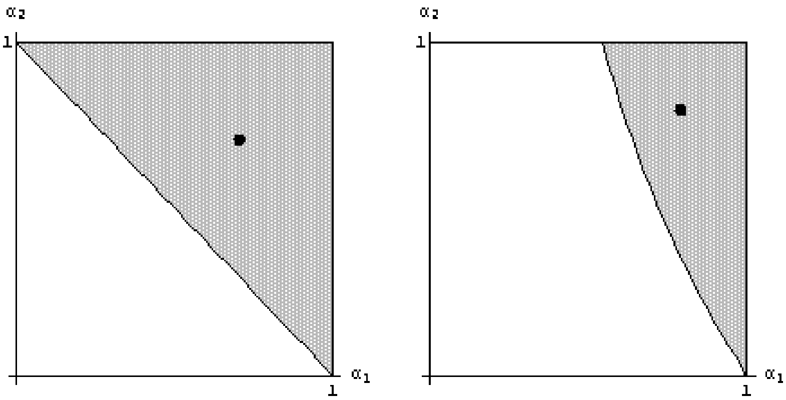

Figure 3: The sets of metric polyratios for Example 1.18-(2), (6)

nated in interval dynamics (see [16] and [2]). Recall that the kneading invariant of an interval map is obtained from the orbit of critical points, and it determines the combinatorial type of the dynamics. In this subsection we define kneading invariants $\mathcal{A} \subset 2^{\Sigma_N}$ from the coding of critical points, and we show that a topological self-similar set is homeomorphic to a quotient space of the shift space by an equivalence relation generated from its kneading invariant. Moreover, if $\mathcal{A} \subset 2^{\Sigma_N}$ is given with a suitable condition, we can construct a topological self-similar system whose kneading invariant is equal to $\mathcal{A}$.

Definition 1.18. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system with coding map π. The *critical set* of $(K, \{F_i\}_{i=1}^N)$ is the union of C_1 and C_2 defined by

$$C_1 = \bigcup_{\substack{1 \le i,j \le N \\ i \ne j}} (K(i) \cap K(j)),$$

$$C_2 = \bigcup_{1 \le i \le N} \{x \in K(i) \mid \# F_i^{-1}(x) \ge 2\}.$$

We denote the critical set by C. A point of C is called a *critical point*. The *kneading invariant* of $(K, \{F_i\}_{i=1}^N)$ is defined by

$$\mathcal{A} = \{\pi^{-1}(c) \mid c \in C\}.$$

Notation 1.19. For $x \in K$ we set

$$P^k(x) = \pi \sigma^k \pi^{-1}(x).$$

We also define

$$C(n) = \bigcup_{k=0}^{n} \bigcup_{V \in \mathcal{W}_k} F_V(C)$$

and

$$\tilde{C}(n) = \bigcup_{\substack{U,V \in \mathcal{W}_n \\ U \neq V}} (K(U) \cap K(V)).$$

Proposition 1.20. *Let $(K, \{F_i\}_{i=1}^N)$ and $(K', \{F_i'\}_{i=1}^N)$ be topological self-similar systems which are conjugate to each other. Namely, there exists a homeomorphism $h : K \to K'$ such that $F_i' \circ h = h \circ F_i$ for any $i = 1, 2, \ldots, N$. Then their kneading invarinats agree with each other.*

Proof. Let us denote by π the coding map of $(K, \{F_i\}_{i=1}^N)$. The assertion is obtained by the fact that $h \circ \pi$ is the coding map of $(K', \{F_i'\}_{i=1}^N)$. $\square$

Example 1.21. For each self-similar system in Example 1.8, the kneading invariant is as follows.

(1) By the fact that F_1 and F_2 are injective and that $K(1) \cap K(2) = \emptyset$, we have $C = \emptyset$. Consequently, $\mathcal{A} = \emptyset$.

(2) Since F_1 and F_2 are injective, the critical set is equal to $C_1 = K(1) \cap K(2) = \{1/2\}$. Thus $\mathcal{A} = \{\pi^{-1}(1/2)\} = \{\{\mathbf{1\overline{2}}, \mathbf{2\overline{1}}\}\}$.

(3) Similarly, the critical set is equal to $C_1 = K(1) \cap K(2) = \{1/2\}$. Consider the map $g(x) = |2x - 1|$, of which inverse branches are F_1 and F_2. Since $1/2$ is carried as $1/2 \to 1 \to 1$ by iteration of g, we see that the kneading invariant is $\mathcal{A} = \{\pi^{-1}(1/2)\} = \{\{\mathbf{11\overline{2}}, \mathbf{21\overline{2}}\}\}$.

(4) While $K(1) \cap K(2) = \emptyset$, the contractions are not injective. Thus the critical set is equal to $C_2 = \{1/6, 5/6\}$. The kneading invariant is $\mathcal{A} = \{\pi^{-1}(1/6), \pi^{-1}(5/6)\} = \{\{\mathbf{11\overline{2}}, \mathbf{12\overline{1}}\}, \{\mathbf{21\overline{2}}, \mathbf{22\overline{1}}\}\}$.

(5) The dynamics is conjugate to that of (3). The critical set is $C_1 = \{0\}$. The kneading invariant is $\mathcal{A} = \{\pi^{-1}(0)\} = \{\{\mathbf{11\overline{2}}, \mathbf{21\overline{2}}\}\}$.

(6) The critical set is $\{0\}$. Since the orbit of 0 for the map $f_{\sqrt{-1}}$ is $0 \to \sqrt{-1} \to -1 + \sqrt{-1} \to -\sqrt{-1} \to -1 + \sqrt{-1}$, the kneading invariant is $\mathcal{A} = \{\pi^{-1}(0)\} = \{\{\mathbf{11\overline{12}}, \mathbf{21\overline{12}}\}\}$.

Proposition 1.22. (1) If $\#P^1(x) \geq 2$, then $x \in C$.

(2) $P^k(x) = \{y \in K \mid F_V(y) = x \text{ for some } V \in \mathcal{W}_k\}$.

(3) If $x \notin C_1$, then there exists i such that $\pi^{-1}(x) = \tau_i \pi^{-1}(P^1(x))$.

Proof. (2) Suppose $y \in P^k(x)$. Then there exists $\underline{u} \in \pi^{-1}(x)$ such that $\pi\sigma^k(\underline{u}) = y$. Let $V = [\underline{u}]_k$. Then we have $F_V(y) = F_V \pi \sigma^k(\underline{u}) = \pi \tau_V \sigma^k(\underline{u}) = \pi(\underline{u}) = x$.

Conversely, suppose $F_V(y) = x$ for some word $V \in \mathcal{W}_k$. Let $\underline{w} \in \pi^{-1}(y)$. Then $\pi\tau_V(\underline{w}) \in \pi\tau_V\pi^{-1}(y) = F_V\pi\pi^{-1}(y) = \{x\}$. Thus $\tau_V(\underline{w}) \in \pi^{-1}(x)$ and $\pi\sigma^k(\tau_V(\underline{w})) = y$.

(1) Let $y_1 \neq y_2 \in P^1(x)$. By (2) we have $x = F_i(y_1) = F_j(y_2)$ for some i, j. If $i = j$, then $x \in C_2$. If $i \neq j$, then $x \in K(i) \cap K(j) \subset C_1$.

(3) Suppose $x \notin C_1$. Then there exists i such that $\pi^{-1}(x) \subset \Sigma(i)$. Thus $P^1(x) = F_i^{-1}(x)$. Consequently, $\pi\tau_i\pi^{-1}(P^1(x)) = F_i\pi\pi^{-1}(P^1(x)) = F_i(P^1(x)) = \{x\}$. Therefore $\tau_i\pi^{-1}(P^1(x)) \subset \pi^{-1}(x)$. Since $P^1(x) = \pi\sigma\pi^{-1}(x)$ and $\pi^{-1}(x) \subset \Sigma(i)$, we have

$$\tau_i\pi^{-1}(P^1(x)) = \tau_i\pi^{-1}\pi\sigma\pi^{-1}(x) \supset \tau_i\sigma\pi^{-1}(x) = \pi^{-1}(x). \qquad \square$$

Proposition 1.23. *Let $x \in K$. If $\#\pi^{-1}(x) \geq 2$, then there exist a critical point c and a word $U \in \mathcal{W}_*$ such that $\tau_U(\pi^{-1}(c)) = \pi^{-1}(x)$. In particular, $F_U(c) = x$.*

Proof. Suppose $\#\pi^{-1}(x) \geq 2$. Then there exists an integer $n \geq 0$ such that

$$\#\{i \in \{1, 2, \ldots, N\} \mid \sigma^n\pi^{-1}(x) \cap \Sigma(i) \neq \emptyset\} \geq 2.$$

Namely, there exist distinct symbols $i, j \in \{1, 2, \ldots, N\}$ such that $P^n(x) \cap K(i) \neq \emptyset$ and $P^n(x) \cap K(j) \neq \emptyset$. Let m be the smallest nonnegative integer such that $P^m(x) \cap C \neq \emptyset$. The integer m is well-defined. Indeed, if $P^k(x) \cap C = \emptyset$ for any k, then we have $\#P^k(x) = 1$ for any k from (1) of Proposition 1.22. The unique point $c \in P^n(x)$ is a critical point, since there exist distinct symbols i, j such that $c \in K(i) \cap K(j)$. This is a contradiction.

Now $\#P^k(x) = 1$ for $k = 1, 2, \ldots m$. In particular the critical point $c \in P^m(x)$ is unique. By (3) of Proposition 1.22, there exists $U \in \mathcal{W}_m$ such that $\pi^{-1}(x) = \tau_U\pi^{-1}(P^m(x))$. $\square$

Corollary 1.24. *Suppose that $K(U_1) \cap K(U_2) \neq \emptyset$ and $\Sigma(U_1) \cap \Sigma(U_2) = \emptyset$. Then $K(U_1) \cap K(U_2) \subset C(k)$, where $k = \min(|U_1|, |U_2|) - 1$. In particular, $\tilde{C}(n) \subset C(n-1)$.*

Proof. Let $x \in K(U_1) \cap K(U_2)$. Then there exists $\underline{u} \in \pi^{-1}(x) \cap \Sigma(U_1)$ and $\underline{v} \in \pi^{-1}(x) \cap \Sigma(U_2)$. By Proposition 1.23, there exists a critical point c and a word U such that $|U| \leq k$ and $x = F_U(c)$. $\square$

Since $\pi : \Sigma_N \to K$ is surjective, the self-similar set K is considered as a quotient space of Σ_N. Namely, K is homeomorphic to $\Sigma_N/\sim$, where we say $\underline{w} \sim \underline{u}$ if $\pi(\underline{w}) = \pi(\underline{u})$. Remark that an equivalence class of $\sim$ is written in the form $\pi^{-1}(x)$. By the previous proposition, all equivalence classes of $\sim$ are 'generated' by the kneading invariant $\mathcal{A}$, that is, $X \subset \Sigma_N$ is an equivalence class with $\#X > 1$ if and only if $X = \tau_U(A)$ for some $U \in \mathcal{W}_*$ and $A \in \mathcal{A}$. Thus the topology of a self-similar set is determined by the kneading invariant.

Definition 1.25. We call $(K, \{F_i\}_{i=1}^N)$ a *pre-self-similar system* if K is a compact topological space which satisfies all the condition of a topological self-similar set except the Hausdorff separation axiom. We call K a *pre-self-similar set.*

Lemma 1.28 gives a sufficient condition for a pre-self-similar set to be Hausdorff. See [7] for a necessary and sufficient condition.

Proposition 1.26. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. Then the kneading invariant $\mathcal{A}$ satisfies the following property:*

Let $U \in \mathcal{W}_*$ and let $A, B \in \mathcal{A}$. If $\tau_U(A) \cap B \neq \emptyset$, then $\tau_U(A) \subset B$; moreover, $\tau_U(A) = B$ if and only if $U = \emptyset$ and $A = B$.

Conversely, let $\mathcal{A}$ be a collection of subsets of Σ_N satisfies the property above and the additional condition that any member of $\mathcal{A}$ has more than one elements. Then there exists a pre-self-similar system $(K, \{F_i\}_{i=1}^N)$ with kneading invariant $\mathcal{A}$.

Proof. Let $U \in \mathcal{W}_*$ be a word, and A and B members of $\mathcal{A}$. Suppose $\tau_U(A) \cap B \neq \emptyset$. Let us denote, by c and c', the critical points such that $A = \pi^{-1}(c)$ and $B = \pi^{-1}(c')$. Note that $\pi(\tau_U(A) \cap B) \subset \pi\tau_U(A) \cap \pi(B) = \{F_U(c)\} \cap \{c'\}$. Thus $F_U(c) = c'$. Since $\pi\tau_U(A) = \{c'\}$, we have $\tau_U(A) \subset B$. The condition that $\tau_U(A) = B$ and $U \neq \emptyset$ implies the contradiction that c' is not a critical point. Indeed, $c' \notin C_1$, because $\pi^{-1}(c') = \tau_U(A) \subset \Sigma(u_1)$, where u_1 is the leading symbol of U. By (2) of Proposition 1.22, $P^1(c') = F_{u_1}^{-1}(c')$. By (3) of Proposition 1.22, $\pi^{-1}(c') = \tau_{u_1}\pi^{-1}F_{u_1}^{-1}(c')$, and so $\sigma(B) = \pi^{-1}F_{u_1}^{-1}(c')$. Thus

$$F_{u_1}^{-1}(c') = \pi\pi^{-1}F_{u_1}^{-1}(c') = \pi\sigma(B) = \pi\sigma\tau_U(A) = \pi\tau_{\sigma(U)}(A) = \{F_{\sigma(U)}(c)\}.$$

Therefore $\#F_{u_1}^{-1}(c') = 1$, and hence $c' \notin C_2$. Consequently, $\tau_U(A)$ is a proper subset of B if $U \neq \emptyset$.

Suppose $\mathcal{A}$ is given. We define a relation $\sim$ on Σ_N as $\underline{x} \sim \underline{y}$ if $\underline{x} = \underline{y}$ or there exist $U \in \mathcal{W}_*$ and $A \in \mathcal{A}$ such that $\underline{x}, \underline{y} \in \tau_U(A)$. By assumption, this relation is an equivalence relation. Indeed, suppose $\underline{x} \sim \underline{y}$ and $\underline{y} \sim \underline{z}$. Then there exist words $U, V \in \mathcal{W}_*$ and $A, B \in \mathcal{A}$ such that $\underline{x}, \underline{y} \in \tau_U(A)$ and $\underline{y}, \underline{z} \in \tau_V(B)$. Since $\tau_U(A) \cap \tau_V(B) \neq \emptyset$, we have $\Sigma(U) \cap \Sigma(V) \neq \emptyset$. We can assume $U \prec V$. Let $n = |V|$. Then $\tau_{\sigma^n(U)}(A) \cap B \neq \emptyset$, and hence $\tau_{\sigma^n(U)}(A) \subset B$. Therefore $\tau_U(A) \subset \tau_V(B)$, and so $\underline{x} \sim \underline{z}$. We have a quotient space $K = \Sigma_N/\sim$ and the natural surjection $\pi : \Sigma_N \to K$. If maps $F_1, F_2, \ldots, F_N : K \to K$ are defined as $F_i(x) = \pi\tau_i\pi^{-1}(x)$, then $F_i \circ \pi = \pi \circ \tau_i$. Their continuity is easily verified by this commutative diagram. Hence $(K, \{F_i\}_{i=1}^N)$ is a pre-self-similar system. It is clear that $\mathcal{A}$ is its kneading invariant. $\square$

Corollary 1.27. *Let $(K, \{F_i\}_{i=1}^N)$ and $(K', \{F_i'\}_{i=1}^N)$ be a topological self-similar systems. If their kneading invariants agree with each other, then $(K, \{F_i\}_{i=1}^N)$ and $(K', \{F_i'\}_{i=1}^N)$ are conjugate to each other.*

Proof. Let $\mathcal{A}$ be the kneading invariant of $(K, \{F_i\}_{i=1}^N)$. We can construct, from $\mathcal{A}$, a self-similar system $(K_{\mathcal{A}}, \{F_{\mathcal{A},i}\}_{i=1}^N)$ by the method in the last half of the proof of the previous proposition. It is easy to see that there exists a homeomorphism $h : K \to K_{\mathcal{A}}$ such that $F_{\mathcal{A},i} \circ h = h \circ F_i$. $\square$

1.4. Counterexample

We construct an example of a self-similar system without self-similar metric.

Consider an irrational rotation on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which is defined by $R(x) = x + \theta \mod 1$, where θ is an irrational number in $[0, 1]$. Divide the circle into two intervals: $J_1 = [0, 1/2]$ and $J_2 = [1/2, 1]$. For $x \in \mathbb{T}$, we define the itinerary $i(x) = \{w_1 w_2 \dots, u_1 u_2 \dots\} \subset \Sigma_2$ as follows.

$$\begin{aligned} w_k = u_k = i \qquad & \text{if } R^{k-1}(x) \in \operatorname{int} J_i, \\ w_k = 1, u_k = 2 \qquad & \text{if } R^{k-1}(x) = 0 \text{ or } 1/2. \end{aligned}$$

Since 0 and 1/2 are not periodic, we see that $\#\{i \,|\, w_i \neq u_i\} \leq 1$. For example, if $i(0) = \{w_1 w_2 \dots, u_1 u_2 \dots\}$, then $w_1 \neq u_1$ but $w_k = u_k$ for $k = 2, 3, \dots$. If $\# i(t) = 2$, then there exists $U \in \mathcal{W}_*$ such that either $i(t) = \tau_U(i(0))$ or $i(t) = \tau_U(i(1/2))$.

For $U = w_1 w_2 \dots w_n \in \mathcal{W}_*$, we write $J_U = \bigcap_{i=1}^n R^{-i+1}(J_{w_i})$. Then

$$J_U = \{x \in \mathbb{T} \,|\, i(x) \cap \Sigma(U) \neq \emptyset\}.$$

Since J_U is the intersection of semicircles and θ is irrational, we see that J_U is an interval or an empty set. If $x \neq y$, then $i(x) \cap i(y) = \emptyset$. Indeed, there exists $n \geq 0$ such that $R^n(x) \in \operatorname{int} J_1$ and $R^n(y) \in \operatorname{int} J_2$ since θ is irrational. Therefore, for $\underline{w} = w_1 w_2 \dots \in \Sigma_N$, the length of $J_{w_1 w_2 \dots w_n}$ tends to zero as n to infinity.

Since $R^k(0) \neq 1/2$ for any integer k, we have $i(R^k(0)) \cap i(1/2) = \emptyset$. Consequently, $\tau_U(i(0)) \cap i(1/2) = \emptyset$ for any $U \in \mathcal{W}_*$. By Proposition 1.26, there exists a pre-self-similar system $(K, \{F_1, F_2\})$ with kneading invariant $\mathcal{A} = \{i(0), i(1/2)\}$. We show that K is metrizable in Lemma 1.28 and that any standard pseudodistance is not a distance.

Lemma 1.28. *Let $(K, \{F_i\}_{i=1}^N)$ be a pre-self-similar system with coding map π. Suppose the critical set C is a finite set, and $\#\pi^{-1}(x)$ is a compact set for any $x \in K$. Then $(K, \{F_i\}_{i=1}^N)$ is a topological self-similar system.*

Proof. We will show that K is Hausdorff. Choose any two points $x, y \in K$. Then there exists n such that

$$\{U \in \mathcal{W}_n \,|\, \Sigma(U) \cap \pi^{-1}(x) \neq \emptyset\} \cap \{U \in \mathcal{W}_n \,|\, \Sigma(U) \cap \pi^{-1}(y) \neq \emptyset\} = \emptyset.$$

Then $y \notin L_n(x)$, $x \notin L_n(y)$, and $L_n(x) \cap L_n(y)$ contains at most finite points. Thus there exists $m \geq n$ such that $x, y \notin L_m(z)$ for any $z \in L_n(x) \cap L_n(y)$. We have $L_m(x) \cap L_m(y) = \emptyset$, and so K is Hausdorff. $\square$

Next we will show that the standard pseudodistance $D = D_{(\alpha,\alpha)}$ is not a distance for any $0 < \alpha < 1$. We can define $h : \mathbb{T} \to K$ by $h(t) = \pi(i(t))$, since either $i(t) = \tau_U(i(0))$ or $i(t) = \tau_U(i(1/2))$ for some word U if $\#i(t) = 2$. Note that if two words $U, V \in \mathcal{W}_*$ satisfies $a \in J_U \cap J_V$, then $h(a) \in K(U) \cap K(V)$. Now let us consider the pseudodistance between $c_1 = h(0)$ and $c_2 = h(1/2)$. For any $k > 0$, the intersection $\{R^{-k}(0), R^{-k}(1/2)\} \cap J_1$ is one point. For $U \in W_n$ the endpoints of the interval J_U is contained in $\bigcup_{k=0}^{n-1}\{R^{-k}(0), R^{-k}(1/2)\}$. Since $\bigcup_{k=0}^{n-1}\{R^{-k}(0), R^{-k}(1/2)\}$ has exactly $n+1$ elements in J_1, we have $\#\{U \in \mathcal{W}_n \mid J_U \subset J_1\} = n$. Let us denote, by $U_1, U_2, \ldots, U_n$, the members of $\{U \in \mathcal{W}_n \mid J_U \subset J_1\}$ to satisfy $J_{U_i} \cap J_{U_{i+1}} \neq \emptyset$ for $i = 1, 2, \ldots, n-1$ and $0 \in J_{U_1}$, $1/2 \in J_{U_k}$. Then $K_{U_i} \cap K_{U_{i+1}} \neq \emptyset$ for $i = 1, 2, \ldots, n-1$ and $c_1 \in K_{U_1}$, $c_2 \in K_{U_n}$. Therefore $(U_1, U_2, \ldots, U_n)$ is a chain between c_1 and c_2. Consequently,

$$D(c_1, c_2) \leq A(U_1) + A(U_2) + \cdots + A(U_k) = n\alpha^n \to 0.$$

Hence D is not a distance.

We have constructed an abstract topological self-similar system $(K, \{F_1, F_2\})$. In the last of this subsection, we give a possible candidate of a geometric realization of $(K, \{F_1, F_2\})$. See Figure 4. This is made by two maps f_1, f_2 of $\overline{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ to itself defined by $f_1(x, y) = R_\theta^{-1}(x, (y + 2\sqrt{1-x^2})/3)$ and $f_2(x, y) = R_\theta^{-1}(x, (y - 2\sqrt{1-x^2})/3)$, where R_θ is the θ-rotation $R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. We can recognize the figure to be an invariant set

$$K' = \bigcap_{k=0}^{\infty} \bigcup_{U \in \mathcal{W}_k} f_U(\overline{D}),$$

which includes the circle $S^1 = \partial D$. On S^1, the restrictions $f_1|S^1$ and $f_2|S^1$ form two inverse branches of the rotation R_θ. Thus f_1 and f_2 on S^1 are considered to be conjugate to F_1 and F_2 on $h(\mathbb{T})$. Although the figure looks like a self-similar set, the two maps f_1 and f_2 are not contractions. It is very likely that $(K', \{f_1|K', f_2|K'\})$ is a topological self-similar system which is conjugate to $(K, \{F_1, F_2\})$. However we do not succeed to verify it so far.

1.5. Connectedness of self-similar sets

In this subsection we discuss the connection between the self-similarity and the connectedness of topological self-similar sets. We show that the standard pseudodistance is positive between two points that belong to distinct connected components (or component for short). As a corollary, we have a sufficient conditions for a topological self-similar system to have a self-similar metric: the case where K is *totally disconnected*, namely, every connected component of K has only one point.

Let K be a self-similar set associated with contractions $F_1, F_2, \ldots, F_N$. It is known that if $\sum_{i=1}^{N} \mathrm{Lip}(F_i) < 1$, then K is totally disconnected (see [4] and [20]). From our viewpoint, it is natural to ask the following inverse

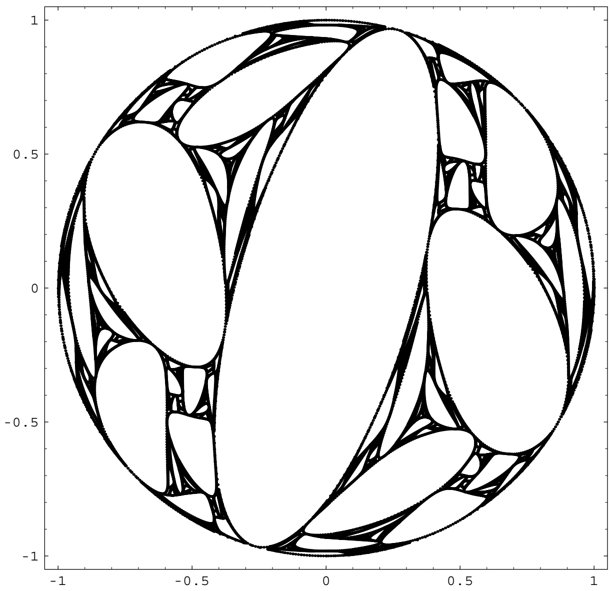

Figure 4: The invariant set K'

problem. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system with K totally disconnected. Does it have a self-similar metric d such that $\sum_{i=1}^N \mathrm{Lip}_d(F_i) < 1$? The following proposition gives an affirmative answer, moreover, that implies a stronger statement: K is totally disconnected if and only if any polyratio is a metric polyratio.

Proposition 1.29. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. Two points x and y in K are contained in two distinct components of K if and only if $\tilde{G}_n(x,y) = \emptyset$ for some n. Recall that $\tilde{G}_n(x,y)$ is the set of chains of depth n between x and y.*

Proof. Let $x, y \in K$ such that $\tilde{G}_n(x,y) = \emptyset$. Then there exist $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{W}_n$ such that $\mathcal{W}_n = \mathcal{E}_1 \cup \mathcal{E}_2$, $x \in K(\mathcal{E}_1)$, $y \in K(\mathcal{E}_2)$ and $K(\mathcal{E}_1) \cap K(\mathcal{E}_2) = \emptyset$, where $K(\mathcal{E}_i) = \bigcup_{U \in \mathcal{E}_i} K(U)$. Since each of $K(\mathcal{E}_i)$ is closed, each of $K(\mathcal{E}_i) = K \setminus K(\mathcal{E}_j)$ $(i \neq j)$ is open. Therefore any subset containing x and y is not connected.

Suppose that $\tilde{G}_n(x,y) \neq \emptyset$ for any n. Let us take $(U_1^n, U_2^n, \dots, U_{l_n}^n) \in \tilde{G}_n(x,y)$. We write $X_n = K(U_1^n) \cup K(U_2^n) \cup \cdots \cup K(U_{l_n}^n)$. We show that

$$X = \bigcap_{k=0}^{\infty} \overline{\bigcup_{n=k}^{\infty} X_n}$$

is connected. Assume that X is not connected. Then there exists a subset $Y_1 \subset X$ such that both of Y_1 and $Y_2 = X \setminus Y_1$ are closed and open in the relative topology of X. Since X is closed, so are Y_1 and Y_2. Consider a distance function d on K. Since Y_1 and Y_2 are compact, we have $d(Y_1, Y_2) = \inf_{y_1 \in Y_1, y_2 \in Y_2} d(y_1, y_2) = \epsilon > 0$. Let O_1 and O_2 be the $\epsilon/3$-neighborhoods of Y_1 and Y_2: $O_i = \{z \mid d(Y_i, z) \leq \epsilon/3\}$. Then $d(O_1, O_2) \geq \epsilon/3$. It is easy to see that there exists a positive integer m such that $X_n \subset O_1 \cup O_2$ for $n > m$. Since $(U_1^n, U_2^n, \ldots, U_{l_n}^n)$ is a chain, for any $n > m$ there exists $1 \leq i_n \leq l_n$ such that $K(U_{i_n}^n) \cap O_1 \neq \emptyset$ and $K(U_{i_n}^n) \cap O_2 \neq \emptyset$. Hence the diameter of $K(U_{i_n}^n)$ is equal to or bigger than $\epsilon/3$ for any $n > m$. This is a contradiction to Lemma 1.6. Therefore X is a connected set which contains x and y. $\square$

Corollary 1.30. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. If two points x and y are contained in distinct connected components of K, then $D(x, y) > 0$.*

Proof. There exists n such that $\tilde{G}_n(x, y) = \emptyset$. Therefore if $(U_1, U_2, \ldots, U_l) \in G(x, y)$, then at least one of U_i belongs to $\bigcup_{k=0}^{n-1} \mathcal{W}_k$. Thus $D(x, y) > (\min_i \alpha_i)^{n-1}$. $\square$

Corollary 1.31. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. Then K is totally disconnected, if and only if every polyratio is a metric polyratio, or equivalently* $\mathrm{CR} = \{(0, 0, \ldots, 0)\}$.

Proof. The sufficiency is an immediate consequence. Suppose that $X \subset K$ is a component containing two points x and y. By Proposition 1.29, $\tilde{G}_n(x, y) \neq \emptyset$ for any n. Since $\#\mathcal{W}_n = N^n$, we can take a chain $\mathcal{C}_n \in \tilde{G}_n(x, y)$ with length at most N^n. If we take a polyratio $((2N)^{-1}, (2N)^{-1}, \ldots, (2N)^{-1})$, then

$$A(\mathcal{C}_n) \leq N^n (2N)^{-n} = 2^{-n} \to 0.$$

Thus $D(x, y) = 0$. $\square$

1.6. Existence of self-similar metrics

As we have seen in Proposition 1.13 and Corollary 1.14, a condition of the existence of self-similar metrics is described in term of standard pseudodistances. In this subsection, we reduce this condition using critical sets C and pre-postcritical sets P under the assumption $C \neq \emptyset$.

Definition 1.32. Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system with kneading invariant $\mathcal{A}$. The *pre-postcritical set* is defined as

$$P = \bigcup_{k>0, A \in \mathcal{A}} \pi\sigma^k(A) = \bigcup_{k>0, c \in C} P^k(c).$$

The *postcritical set* is the closure of P. A point of P is called a *postcritical point.*

Theorem 1.33. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. Then α is a metric polyratio if and only if $D_\alpha(x, y) > 0$ for any distinct points $x, y \in \bigcup_{n=1}^\infty C(n)$.*

Theorem 1.34. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. Then $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a metric polyratio if and only if there exists a distance d on $C \cup P$ compatible with the original topology such that $d(x, y) \le \alpha_i d(x', y')$ for any $i \in \{1, 2, \ldots, N\}$, any $x, y \in (C \cup P) \cap K(i)$ and any $x' \in F_i^{-1}(x), y' \in F_i^{-1}(y)$, and such that $M = \sup_{x,y \in C \cup P} d(x, y) < \infty$.*

Proof of Theorem 1.33. Let $x, y \in K$ be distinct points. Assuming that $D(x, y) = 0$, we show a contradiction. Let n be an integer such that $L_n(x) \cap L_n(y) = \emptyset$. Then $M = \inf_{a,b} D(a, b)$ is positive, where the infimum is taken over all $a \in C(n) \cap L_n(x)$ and all $b \in C(n) \cap L_n(y)$. Let $0 < \epsilon < M$. Then there exists a chain $\mathcal{C} = (U_1, U_2, \ldots, U_l)$ between x and y such that $A(\mathcal{C}) < \epsilon$. Let $1 \le i_1 \le l$ be the minimal integer such that $K(U_{i_1}) \not\subset L_n(x)$. Then it is easy to see that $K(U_{i_1}) \cap C(n) \cap L_n(x) \ne \emptyset$. Similarly, there exists an integer $1 \le i_2 \le l$ such that $K(U_{i_2}) \cap C(n) \cap L_n(y) \ne \emptyset$. Therefore $(U_{i_1}, \ldots, U_{i_2})$ is a chain between $a \in C(n) \cap L_n(x)$ and $b \in C(n) \cap L_n(y)$. Hence we have a contradiction $M \le D(a, b) \le A(U_{i_1}) + \cdots + A(U_{i_2}) < A(\mathcal{C}) < \epsilon$. $\square$

Proof of Theorem 1.34. We define a function d_n on $(P \cup C(n)) \times (P \cup C(n))$ as follows. First we set $d_0 = d$. If d_{n-1} is defined, then for $x, y \in (P \cup C(n)) \cap K(i)$, we set

$$d_n^i(x, y) = \alpha_i \inf_{x' \in F_i^{-1}(x), y' \in F_i^{-1}(y)} d_{n-1}(x', y').$$

For $x, y \in P \cup C(n)$, we set

$$d_n(x, y) = \inf(d_n^{i_1}(x, x_1) + d_n^{i_2}(x_1, x_2) + \cdots + d_n^{i_l}(x_{l-1}, y)),$$

where the infimum is taken over all pre-chains $(i_1, i_2, \ldots, i_l)$ between x and y of depth one and all $x_j \in K(i_j) \cap K(i_{j+1}) \cap (P \cup C(n))$. If there does not exist such a chain, then we set $d_n(x, y) = \sup_{a,b} d_n(a, b)$, where the supremum is taken over all $a, b \in P \cup C(n)$ such that there exists a pre-chain of depth one between a and b. $\square$

Lemma 1.35. *For $n = 1, 2, \ldots$, we have*

(1) *d_n is a distance on $P \cup C(n)$ compatible with the original topology.*

(2) *For $x, y \in P \cup C(n-1)$, we have $d_{n-1}(x, y) \le d_n(x, y)$.*

(3) *$d_n(x, y) \le \alpha_i d_{n-1}(x', y')$ for any $i \in \{1, 2, \ldots, N\}$, any $x, y \in (P \cup C(n)) \cap K(i)$ and any $x' \in F_i^{-1}(x), y' \in F_i^{-1}(y)$.*

Proof. We prove the claims by induction. For convenience, we set $d_{-1} = d_0 = d$ and $C(-1) = \emptyset$. Then (1), (2) and (3) are satisfied for $n = 0$. Suppose that they are satisfied for $n = k - 1$.

Let $x, y \in P \cup C(k)$. Then for any $\epsilon > 0$ there exist pre-chain $\mathcal{C} = (i_1, i_2, \ldots, i_{l_1})$ between x and y of depth one and points $x_j \in K(i_j) \cap K(i_{j+1}) \cap (P \cup C(k))$ such that

$$\sum_{j=1}^{l-1} d_k^{i_j}(x_{j-1}, x_j) < d_k(x, y) + \epsilon, \tag{1.1}$$

where $x = x_0, y = y_l$. If $x_j \notin C$, then $i_j = i_{j+1}$ and $\#F_j^{-1}(x_j) = 1$. Thus we can assume that $x_1, x_2, \ldots, x_{l-1}$ are critical points. By the definition of d_k^i, there exist $x'_{j-1} \in F_{i_j}^{-1}(x_{j-1})$ and $x''_j \in F_{i_j}^{-1}(x_j)$ such that $\alpha_{i_j} d_{k-1}(x'_{j-1}, x''_j) < d_k^{i_j}(x_{j-1}, x_j) + \epsilon/l$. From this together with (1.1),

$$\sum_{j=1}^{l} d_k(x_{j-1}, x_j) \le \sum_{j=1}^{l} \alpha_{i_j} d_{k-1}(x'_{j-1}, x''_j) \le d_k(x, y) + 2\epsilon. \tag{1.2}$$

If $x, y \in P \cup C(k-1)$, then

$$d_{k-1}(x, y) \le \sum_{j=1}^{l} d_{k-1}(x_{j-1}, x_j) \le \sum_{j=1}^{l} d_k(x_{j-1}, x_j) \le d_k(x, y) + 2\epsilon.$$

Thus we have (2).

To prove that d_k is a distance, it is sufficient to show that $d_k(x, y) > 0$ if $x \neq y$. Here we prove a stronger fact: Let y be a point in $P \cup C(k)$ and let $a_1, a_2, \ldots$ be a sequence in $P \cup C(k)$. If $d_k(y, a_m) \to 0$ $(m \to \infty)$, then $a_m \to y$ $(m \to \infty)$ (i.e. d_k is equivalent to or stronger than the original topology). Without loss of generality, we assume that a_m converges to some point in K, say a. We show that a contradiction follows from $a \neq y$. Let O_1 and O_2 be open neighborhoods of a and y such that $\overline{O_1} \cap \overline{O_2} = \emptyset$. Then $Q = \inf\limits_{c, c'} d_{k-1}(c, c')$ is positive, where the infimum is taken over all $c \in C \cap O_1$ and all $c' \in C \cap O_2$. Let m be an integer such that $a_n \in O_1$ if $n \ge m$. We write $S = \{a\} \cup \{a_m, a_{m+1}, \ldots\}$. Since $F_i^{-1}(S)$ and $F_i^{-1}(y)$ are compact, we choose $\epsilon' > 0$ so small that

$$\{z \in P \cup C(k-1) \,|\, d_{k-1}(z, x') < \epsilon'/\alpha_i \text{ for some } x' \in F_i^{-1}(S)\} \subset F_i^{-1}(O_1),$$
$$\{z \in P \cup C(k-1) \,|\, d_{k-1}(z, y') < \epsilon'/\alpha_i \text{ for some } y' \in F_i^{-1}(y)\} \subset F_i^{-1}(O_2)$$

for any $i = 1, 2, \ldots, N$. We write $E(z, \epsilon, i) = \{c \in C \cap K(i) \,|\, d_k^i(c, z) < \epsilon\}$ if $z \in (P \cup C(k)) \cap K(i)$, and $E(z, \epsilon, i) = \emptyset$ otherwise. Then $\bigcup_{x \in S} E(x, \epsilon', i) \subset O_1, E(y, \epsilon', i) \subset O_2$ for any $i = 1, 2, \ldots, N$. We set $\epsilon = \min\{Q/4, \epsilon'/2\}$. Let $m' > m$ be an integer such that $d_k(a_{m'}, y) < \epsilon$. For $x = a_{m'}$ we have a pre-chain $\mathcal{C}$ and points x_i which satisfy (1.1). Since $d_k^{i_1}(x, x_1) < \epsilon', d_k^{i_l}(x_{l-1}, y) < \epsilon'$, we have $x_1 \in O_1, x_{l-1} \in O_2$. Therefore $l \ge 3$ and $x_1, x_{l-1} \in C$. From (1.2),

$$Q \le d_{k-1}(x_1, x_{l-1}) \le \sum_{j=2}^{l-1} d_{k-1}(x_{j-1}, x_j) \le d_k(x, y) + 2\epsilon < 3Q/4,$$

and hence we arrive at a contradiction. Thus d_k is a distance equivalent to or stronger than the original topology.

Let $x, y \in (P \cup C(k)) \cap K(i)$, and let $x' \in F_i^{-1}(x), y' \in F_i^{-1}(y)$. Then

$$d_k(x,y) \le d_k^i(x,y) \le \alpha_i d_{k-1}(x', y').$$

Hence (3) is verified.

Finally, we prove that the distance d_k is equivalent to or weaker than the original topology. Note that $M_k = \sup_{x,y \in P \cup C(k)} d_k(x,y)$ is finite. Therefore $\sup_{y \in (P \cup C(k)) \cap K(U)} d_k(x,y) \le A(U) M_k$ for any $x \in P \cup C(k)$ and any $U \in \mathcal{W}_*$. Thus for any $\epsilon > 0$, if we take n such that $(\max_i \alpha_i)^n M_k < \epsilon$, then $L_n(x) \cap (P \cup C(k)) \subset \{y \in P \cup C(k) \mid d_k(x,y) < \epsilon\}$. $\square$

Now we continue the proof of the theorem. By Theorem 1.33, it is sufficient to show that $D(x,y) > 0$ for any distinct $x, y \in C(n)$. Let $x, y \in C(n)$ be distinct points and let $\mathcal{C} = (U_1, U_2, \ldots, U_l)$ be a chain between x and y. Choose $x_i \in K(U_i) \cap K(U_{i+1})$ for $i = 1, 2, \ldots, l-1$. Then $x_i \in C(m)$, where $m = \max\{|U_1|, |U_2|, \ldots, |U_l|, n\}$. Let $x'_{i-1} \in F_{U_i}^{-1}(x_{i-1})$ and $x''_i \in F_{U_i}^{-1}(x_i)$ for $i = 1, 2, \ldots, l$, where $x = x_0$, $y = y_l$. Note that $x'_{i-1}, x''_i \in P \cup C$. We have

$$0 < d_n(x,y) \le d_m(x,y) \le \sum_{i=1}^{l} d_m(x_{i-1}, x_i) \le \sum_{i=1}^{l} A(U_i) d(x'_{i-1}, x''_i) \le A(\mathcal{C}) M.$$

Thus $0 < d_n(x,y)/M \le D(x,y)$.

Remark 1.36. A related topic is discussed by Kigami [12]. He states a necessary and sufficient condition for a p.c.f. self-similar set K to admit a strictly self-similar metric (i.e. a metric d satisfying $d(F_i(x), F_i(y)) = \alpha_i d(x,y)$) such that there exists a 'geodesic' between any two points in K.

Example 1.37. Consider the self-similar system $(K, \{F_1, F_2\})$ of Example 1.8-(6). Recall that it has the critical set $C = \{c\}$ and the postcritical set $P = \{p_1, p_2, p_3\}$ such that $F_1(p_1) = F_2(p_1) = c, F_1(p_2) = p_1, F_1(p_3) = p_2, F_2(p_2) = p_3$. Suppose $(\alpha_1, \alpha_2) \in \mathbf{Ra}_2$ is a polyratio such that $\alpha_1\alpha_2 + \alpha_1^2\alpha_2 + \alpha_1^3 = 1$. Set $d(c, p_1) = \alpha_1, d(c, p_2) = \alpha_1^2 + \alpha_1\alpha_2, d(c, p_3) = \alpha_2, d(p_1, p_2) = 1, d(p_1, p_3) = \alpha_1 + \alpha_2, d(p_2, p_3) = 1/\alpha_1$. Then d is a distance on $C \cup P$ which satisfies the condition of Theorem 1.34. Thus (α_1, α_2) is a metric polyratio.

2. Non-recurrent self-similar sets

In this section we study a sufficient condition for topological self-similar systems to have self-similar metrics. We consider topological self-similar systems $(K, \{F_i\}_{i=1}^N)$ satisfying the following conditions:

(1) The critical set $C = C_1 \cup C_2$ is a finite set.

(2) $F_i^{-1}(x)$ is a finite set for any $i \in \{1, 2, \ldots, N\}$ and any $x \in K(i)$.

Definition 2.1. A topological self-similar system satisfying the above conditions is said to be *finitely ramified*.

Remark 2.2. In another context, the word 'finitely ramified self-similar sets' has been used in slightly different formulations (see for example [13], [11]).

Definition 2.3. A topological self-similar system is said to be *non-recurrent* if the critical set C contains no cluster point of the pre-postcritical set P. That is to say, there is a neighborhood O of C such that $O \cap P \subset C$.

In this section we prove the following.

Theorem 2.4. *A non-recurrent finitely ramified topological self-similar system has a self-similar metric.*

We have seen the prototype of the proof in Example 1.16. In general, the proof is rather complicated. We will prepare several lemmas in the next subsection.

2.1. Lemmas

Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system.

Lemma 2.5. *Let $x_1, x_2, \ldots$ be a sequence in K which converges to x, and let $V_1, V_2, \ldots$ be a sequence of words. Then*

$$\lim_{i\to\infty} F_{V_i}(x) = y \iff \lim_{i\to\infty} F_{V_i}(x_i) = y.$$

Proof. For any $k \geq 0$ there exists i_0 such that $x_i \in L_k(x)$ if $i \geq i_0$. Clearly,

$$F_{V_i}(x_i) \in F_{V_i}(L_k(x)) \subset L_k(F_{V_i}(x)).$$

Hence there exists $U_k \in \mathcal{W}_k$ such that $F_{V_i}(x_i)$ and $F_{V_i}(x)$ belong to $K(U_k)$. By Lemma 1.6, the assertion is true. $\square$

Lemma 2.6. *Let c be a critical point, and let $x \in \bigcup_{k=1}^\infty P^k(c)$. Then the set*

$$X = \{y \,|\, y = F_V(x), c = F_U(y) \text{ for some } V, U \in \mathcal{W}_*\} \subset \bigcup_{k=1}^\infty P^k(c)$$

is finite.

Proof. Note that $F_i^{-1}(x)$ is finite for any i and any x. Hence we see that $P^k(c)$ is finite for each k. Since the critical set C is finite, there exists n such that $C \cap X \subset \bigcup_{k=1}^n P^k(c)$. Let $B_0 = \bigcup_{k=1}^n P^k(c) \cap X$ and $B_i = (P^{n+i}(c) - \bigcup_{k=0}^{i-1} B_k) \cap X$ for $i = 1, 2, \ldots$. Then $B_{i+1} \subset \bigcup_{y \in B_i} P^1(y)$. If

$i' > 0$ and $y \in B_{i'}$, then $\#P^1(y) = 1$ from (1) of Proposition 1.22. Thus $\#B_{i'+1} \leq \#B_{i'}$. Let i_0 be the integer such that $x \in B_{i_0}$, and let $i' = \max\{1, i_0\}$. For $y \in B_{i'}$, there exists $m > 0$ such that $P^m(y) \subset \bigcup_{i=0}^{i_0} B_i$. This implies that B_j is empty for some large j. Consequently, $X = \bigcup_{i=0}^{\infty} B_i = \bigcup_{i=0}^{j-1} B_i$ is a finite set. $\square$

Lemma 2.7. *Let c be a critical point, and let $\{x\} \cup \{a_1, a_2, \dots\}$ be an infinite subset of $\bigcup_{k=1}^{\infty} P^k(c)$. If they satisfies the following:*

(2.1) *There exist words $V_1, V_2, \dots$ such that $F_{V_i}(x) = F_{V_i}(a_i) = c$ for each i,*

then $C \cup \{x\}$ contains a cluster point of P.

Proof. By Lemma 2.6, $X = \{y \,|\, y = F_V(x), c = F_U(y)$ for some $V, U \in \mathcal{W}_*, \}$ is a finite set. The lengths of V_i are not bounded, since $P^k(c)$ is finite set for any k. Hence for any integer $l \geq 0$, there exist an infinite subset $A_l \subset \{a_1, a_2, \dots\}$ and a word $W(l)$ of length l such that $A_0 \supset A_1 \supset A_2 \supset \dots$ and that $W(l)$ is a successor of V_i if $a_i \in A_l$. Moreover we can assume that if $a_i \in A_l$ and if W is a successor of $W(l)$, then $F_W(a_i) \notin X$. Indeed, it is sufficient that we take $A_l - \bigcup_{k=0}^{l} \bigcup_{y \in X} P^k(y)$ instead of A_l.

Note that if $a_i \in A_l$ and if W is a successor of $W(l)$, then $F_W(a_i) \in P$ and $F_W(x) \in X$. Let us denote, by Y, the set of points $y \in X$ satisfying the following condition: There exist a sequence of integers $l(1) < l(2) < \cdots$ and a sequence of words $U_1, U_2, \dots$ such that

- U_k is a successor of $W(l(k))$ for every k,
- $y = F_{U_k}(x)$ for every k,
- $|U_k| \to \infty$ as $k \to \infty$.

Then $y \in Y$ is a cluster point of P. Indeed, let $a_{i(k)} \in A_{l(k)}$ for $k = 1, 2, \dots$. Then

$$F_{U_k}(a_{i(k)}) \in K(U_k) \subset L_{|U_k|}(y).$$

Since $F_{U_k}(a_{i(k)})$ does not belong to X, the point y is a cluster point of $\{F_{U_k}(a_{i(k)})\} \subset P$.

We will prove that either $x \in Y$ or $Y \cap C \neq \emptyset$. Suppose $Y \cap C = \emptyset$. We use the notation $x(l, t) = F_{\sigma^t(W(l))}(x)$ for each l and $0 \leq t \leq l$. If $\{x(l, t) \,|\, t = 0, 1, \dots, l\} \cap C \neq \emptyset$, we define

$$p(l) = \min\{t \,|\, x(l, t) \text{ is a critical point }\},$$

and set $p(l) = l + 1$ otherwise. Then $p(l)$ $(l = 0, 1, \dots)$ are unbounded. Indeed, otherwise, $|\sigma^{p(l)}(W(l))| = l - p(l)$ are unbounded, and so we can choose $l(1) < l(2) < \dots$ and $U_k = \sigma^{p(l(k))}(W(l(k)))$ to satisfy the above condition for some critical point, which have to belong to Y. Thus $p(l)$ $(l = 0, 1, \dots)$ are unbounded. Note that $x(l, t)$ is contained in $X - C$ for $t = 0, 1, \dots, p(l) - 1$. If $p(l) > \#(X - C)$, there exist $s(1), s(2) \in \{0, 1, \dots, p(l) - 1\}$ such that $s(1) < s(2)$ and $x(l, s(1)) = x(l, s(2)) = z$. Since z is not a critical point, we have $\#P^1(z) = 1$. Thus

$$x(l, s(1) + 1) = x(l, s(2) + 1).$$

We can also see that

$$x(l, s(1) + m) = x(l, s(2) + m)$$

for $m = 1, 2, \dots, p(l) - 1 - s(2)$. Consequently,

$$x(l, p(l) - s(2) + s(1) - 1) = x(l, p(l) - 1).$$

Moreover, if $p(l) \leq l$, then

$$x(l, p(l) - s(2) + s(1)) = x(l, p(l)).$$

This is a contradiction; because $x(l, p(l))$ is a critical point by definition, but $p(l) - s(2) + s(1) < p(l)$. Hence $p(l) = l + 1$, and then every $x(l, t)$ is not a critical point. Therefore we conclude that $(x(l, l), x(l, l-1), \dots, x(l, 0))$ is a periodic sequence containing x. Thus there exists $t(l)$ such that $0 \leq t(l) \leq \#(X - C)$ and $F_{U_l}(x) = x(l, t(l)) = x$, where $U_l = \sigma^{t(l)}(W(l))$. Since $|U_l| \to \infty$ as $l \to \infty$, we have $x \in Y$. $\square$

Lemma 2.8. *Let x be a point in K, and let $y_1, y_2, \dots$ be a sequence in P. Suppose there exists a word $U_i \in \mathcal{W}_*$ for each $i = 1, 2, \dots$ such that $F_{V_i}(x), F_{V_i}(y_i) \in K(V_i U_i)$ for some word V_i and $|U_i| \to \infty$ as $i \to \infty$. Moreover we suppose that for each i there exists a successor W_i of V_i such that $F_{W_i}(y_i) \in C$. Then either of the following properties is satisfied.*

- *$F_{V_i}(x) = F_{V_i}(y_i)$ for some i.*
- *$C \cup \{x\}$ contains a cluster point of P.*

Proof. Choose $a_i \in {F_{V_i}}^{-1}(F_{V_i}(x)) \cap K(U_i)$ and $b_i \in {F_{V_i}}^{-1}(F_{V_i}(y_i)) \cap K(U_i)$ for each i. There exist a successor V_i'' of V_i such that $F_{V_i''}(b_i) \in C$. Indeed, if $b_i = y_i$, then take $V_i'' = W_i$. If $b_i \neq y_i$, then there exists V_i'' such that $F_{V_i''}(b_i) = F_{V_i''}(y_i)$ and $F_{\sigma(V_i'')}(b_i) \neq F_{\sigma(V_i'')}(y_i)$. Thus $b_i \in P$. Without loss of generality, we can assume that $\bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} K(U_i)}$ consists of only one point, say z. Note that $\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = z$.

To prove the lemma we divide the situation into several cases.

(A) Suppose $x = z$. If $\{b_i\}_i$ is an infinite set, then x is a cluster point of P. If $\{b_i\}_i$ is finite, then $b_i = x$ and hence $F_{V_i}(x) = F_{V_i}(y_i)$ for large i.

(B) Suppose $x \neq z$. Then $x \neq a_i$ for large i. Hence there exists a successor V_i' of V_i such that $F_{V_i'}(x) = F_{V_i'}(a_i) \in C$ for large i. Since $\#C < \infty$, we may assume $F_{V_i'}(x) = F_{V_i'}(a_i) = c$ for each large i without loss of generality.

(B-I) If $\{a_i\}_i$ is an infinite set, then $\{x\} \cup \{a_i\}_i$ satisfies (2.1).
(B-II) Suppose that $\{a_i\}_i$ is a finite set. Then $a_i = z$ for large i. Note that $\lim_{i\to\infty} F_{V_i'}(b_i) = \lim_{i\to\infty} F_{V_i'}(z) = c$ by Lemma 2.5. We have four cases:

(1) $\{b_i\}_i$ is finite.

(2) $\{b_i\}_i$ is infinite.

- (a) The length of V_i'' is bigger than that of V_i' for infinitely many i.
  - (i) $\{F_{V_i'}(b_i)\}_i$ is finite.
  - (ii) $\{F_{V_i'}(b_i)\}_i$ is infinite.
- (b) The length of V_i'' is bigger than that of V_i' for at most finitely many i.

(1) If $\{b_i\}_i$ is a finite set, then $b_i = z$ for large i. Thus $F_{V_i}(y_i) = F_{V_i}(b_i) = F_{V_i}(z) = F_{V_i}(x)$.
(2) We assume that $\{b_i\}_i$ is infinite. We may assume that $F_{V_i''}(b_i) = c'$ for each i.

(2-a) If the length of V_i'' is bigger than that of V_i' for infinitely many i, then V_i' is a successor of V_i'', and $F_{V_i'}(b_i) \in P$ for such i.

(2-a-i) If $F_{V_i'}(b_i) = c$ for large i, then $F_{V_i'}(x) = F_{V_i'}(b_i)$. Thus $\{x\} \cup \{b_i\}_i$ satisfies (2.1).

(2-a-ii) In the case where $\{F_{V_i'}(b_i)\}_i$ is a infinite set, c is a cluster point of P.

(2-b) Suppose that the length of V_i'' is bigger than that of V_i' for at most finitely many i. Then $F_{V_i'''} \circ F_{V_i''}(z) = c$ for large i, where $V_i'''V_i'' = V_i'$. Thus $F_{V_i''}(z) \in X = \{a \mid a = F_V(z), c = F_U(a) \text{ for some } V, U \in \mathcal{W}_*\}$. On the other hand, from Lemma 2.5, we have $\lim_{i\to\infty} F_{V_i''}(z) = \lim_{i\to\infty} F_{V_i''}(b_i) = c'$. Since X is finite, $F_{V_i''}(z) = c'$ for large i. Consequently, $F_{V_i'}(b_i) = F_{V_i'''} \circ F_{V_i''}(b_i) = F_{V_i'''}(c') = F_{V_i'''} \circ F_{V_i''}(z) = c$. Thus $\{x\} \cup \{b_i\}_i$ satisfies (2.1). □

Definition 2.9. • Let x, y be two points in K. There exists the maximal integer $t = t(x, y)$ such that $x, y \in K(U)$ for some $U \in \mathcal{W}_t$. Such a word U is called a *bridge* between x and y.

• Let $W \in \mathcal{W}_p$ be a word, and let $a, b \in K$ be distinct points. We say (W, a, b) is a *p-mesh* if $a \in F_W(C)$ and there exists a word W' such that $W \prec W'$ and $b \in F_{W'}(C) \cap K(W)$. The number p is called the depth of the mesh.

• Let W be a word, and let $a, b \in K$ be distinct points. We say (W, a, b) is a *p-block* if $|W| \geq p$ and $a, b \in K(W) \cap C(p)$.

Proposition 2.10. *Let (W, a_1, a_2) be a p-block. Then there exists a word W_1 such that $W \prec W_1$, $|W_1| \leq p$ and either (W_1, a_1, a_2) or (W_1, a_2, a_1) is a $|W_1|$-mesh.*

Proof. Let (W, a_1, a_2) be a p-block. Note that $a_1, a_2 \in K(W)$. Let $p_i \leq p$ be the smallest integer such that $a_i \in C(p_i)$ $(i = 1, 2)$. Say $p_1 \geq p_2$. There uniquely exist critical points $x_i \in P^{p_i}(a_i)$ and words $W_i \in \mathcal{W}_{p_i}$ such that $F_{W_i}(x_i) = a_i$ $(i = 1, 2)$. By (3) of Proposition 1.22, $\pi^{-1}(a_i) = \tau_{W_i}\pi^{-1}(x_i)$. Since $\tau_{W_i}\pi^{-1}(x_i) \cap \Sigma(W) \neq \emptyset$, we have $W \prec W_i$. Consequently, $W_1 \prec W_2$. Since $a_2 \in K(W) \subset K(W_1)$, we conclude that (W_1, a_1, a_2) is a p_1-mesh. $\square$

Lemma 2.11. *Let $a, b \in K$. Let U be a bridge between a and b. If $(U_1, U_2, \ldots, U_l)$ is a chain between a and b with $l \geq 2$, then there exists $1 \leq j \leq l - 1$ such that $K(U_j) \cap K(U_{j+1}) \subset \tilde{C}(|U| + 1)$.*

Proof. We write $p = |U| + 1$. Since $K(U_j) \cap K(U_{j+1}) \subset \tilde{C}(\min(|U_j|, |U_{j+1}|))$, the assertion is true in the case where $|U_j| < p$ for some j. We assume $|U_j| \geq p$ for any j. Let $V = [U_1]_p$, and let $t > 1$ be the smallest integer such that $V' = [U_t]_p \neq V$, which is well-defined because $|U| < p$. Hence $K(U_{t-1}) \cap K(U_t) \subset K(V) \cap K(V') \subset \tilde{C}(p)$. $\square$

2.2. Proof

Now we start the proof of Theorem 2.4. The proof consists of several steps.

Proof of Theorem 2.4. Suppose that $(K, \{F_i\}_{i=1}^N)$ is non-recurrent and finitely ramified .

Step 1: In this step we show the following lemma, and then obtain a corollary.

Lemma 2.12. *There exists an integer n_1 such that $|W| \leq p + n_1$ for any p and any p-block (W, a, b).*

Proof. Let (W, a, b) be a p-mesh. Let us denote, by $k = k(W, a, b)$, the greatest number such that there exists a word $U \in \mathcal{W}_k$ with $a, b \in K(U)$ and $U \prec W$. We first show that $k - p$ are bounded. Otherwise, for each $i = 1, 2, \ldots$ there exist a mesh (W_i, a_i, b_i) of depth p_i such that

$$k(W_i, a_i, b_i) - p_i \to \infty \text{ as } i \to \infty.$$

Set $k_i = k(W_i, a_i, b_i)$. Then there exists a word $U_i \prec W_i$ such that $|U_i| = k_i$ and $a_i, b_i \in K(U_i)$.

Since (W_i, a_i, b_i) is a p_i-mesh, we have points x_i and y_i' which satisfy $x_i, y_i' \in C$, $F_{W_i}(x_i) = a_i$, $F_{W_i'}(y_i') = b_i$, where $W_i \prec W_i'$. Let us take a point y_i such that $y_i \in F_{W_i''}^{-1}(y_i')$, where $W_i = W_i' W_i''$. Since C is finite, we can assume $x_i = x$ for each i. The word $\tilde{U}_i = \sigma^{p_i}(U_i)$ has length $k_i - p_i$. Thus $|\tilde{U}_i| \to \infty$ as $i \to \infty$. The points $a_i = F_{W_i}(x), b_i = F_{W_i}(y_i)$ are contained in $K(U_i) = F_{W_i}(K(\tilde{U}_i))$. Moreover, $F_{W_i''}(y_i) = y_i' \in C$. Consequently, the point x and the sequence $y_1, y_2, \ldots$ together satisfy the condition of Lemma 2.8. Since the topological self-similar system is non-recurrent, we have $F_{W_i}(x) = F_{W_i}(y_i)$ for some i. But this is impossible, because $a_i \neq b_i$. Thus we have proved that $k(W, a, b) - p$ are bounded by some integer n_1.

Let (W, a, b) be a p-block. From Proposition 2.10, there exists W_1 such that $W \prec W_1$, $|W_1| \le p$ and (W_1, a, b) is a $|W_1|$-mesh. Since $a, b \in K(W)$, we have $|W| - |W_1| \le k(W_1, a, b) - |W_1| \le n_1$. Thus $|W| \le |W_1| + n_1 \le p + n_1$. □

In particular, we immediately obtain the following.

Corollary 2.13. *Let $a, b \in C(p)$, and let U be a word such that $a, b \in K(U)$. Then $|U| \le p + n_1$.*

Proof. If $|U| > p + n_1$, then (U, a, b) is a p-block. □

Step 2: We set $\alpha = 2^{-1/(n_1+1)}$. Our goal is to show that $D = D_{(\alpha,\alpha,\dots,\alpha)}$ is a distance on K. We will show that $D(a, b) \ge \alpha^{p+n_1}$ if $a, b \in C(p)$ and $a \ne b$ from Step 2 to Step 4. This completes the proof by Theorem 1.33.

Let $a, b \in C(p)$ with $a \ne b$. Let $\mathcal{C} = (U_1, U_2, \dots, U_l)$ be a chain between a and b. It is sufficient to show that $A(\mathcal{C}) \ge \alpha^{p+n_1}$. Let us take $a_0 = a \in K(U_1), a_1 \in K(U_1) \cap K(U_2), \dots, a_{l-1} \in K(U_{l-1}) \cap K(U_l), a_l = b \in K(U_l)$. We can assume that $a_0, a_1, \dots, a_l$ are disjoint. We take a chain $\mathcal{C}' = (U'_1, U'_2, \dots, U'_l)$ such that U'_j is a bridge between a_{j-1} and a_j. Then $A(\mathcal{C}') \le A(\mathcal{C})$. Let U be a bridge between a and b. Then $|U| \le p + n_1$. We construct pre-chains $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_r$ between a and b such that $\mathcal{C}_0 = (U)$ and $\mathcal{C}_r = \mathcal{C}'$. The i-th pre-chain is written as $\mathcal{C}_i = (U^i_1, U^i_2, \dots, U^i_{l_i})$. They are required to satisfy the following properties.

- For each $i = 0, 1, \dots, r$ there exists a non-decreasing onto mapping $h_i : \{1, 2, \dots, l\} \to \{1, 2, \dots, l_i\}$. We denote $h_i^{-1}(j) = \{s(i, j) + 1, s(i, j) + 2, \dots, s(i, j + 1)\}$. Then U^i_j is a bridge between $a_{s(i,j)}$ and $a_{s(i,j+1)}$.
- Set
$$E_i = \{a_{s(i,j)} \,|\, j = 1, 2, \dots, l_i\} \cup \{a_l\}.$$
Then $E_0 = \{a_0, a_l\} \subset E_1 \subset \cdots \subset E_r = \{a_0, a_1, \dots, a_l\}$.
- Let $0 \le i \le r - 1$, $1 \le j \le l_i$ and $s(i, j) < j_0 < s(i, j + 1)$. Then $j_0 = s(i+1, j')$ for some $j' \in \{1, 2, \dots, l_{i+1}\}$ if and only if $a_{j_0} \in C(|U^i_j|+1)$.

First we set a trivial mapping $h_0 : \{1, 2, \dots, l\} \to \{1\}$. Each chain $\mathcal{C}_i = (U^i_1, U^i_2, \dots, U^i_{l_i})$ and each non-decreasing mapping $h_i : \{1, 2, \dots, l\} \to \{1, 2, \dots, l_i\}$ are inductively determined as follows. Suppose $E_{i'}$, $\mathcal{C}_{i'}$ and $h_{i'}$ are determined for $i' \le i$.

(1) (Construction of E_{i+1}) Every element of $E_i = \{a_{s(i,j)} \,|\, j = 1, 2, \dots, l_i\} \cup \{a_l\}$ is an element of E_{i+1}. If $j_0 \notin E_i$, then $j_0 \in E_{i+1}$ if and only if $a_{j_0} \in C(|U^i_j|+1)$, where j is the integer such that $1 \le j \le l_i$ and $s(i, j) < j_0 < s(i, j + 1)$.

(2) (Construction of h_{i+1}) Let $l_{i+1} = \#E_{i+1} - 1$. Then we set integers $s(i + 1, 1) < s(i + 1, 2) < \cdots < s(i + 1, l_{i+1})$ such that $E_{i+1} - \{a_l\} = \{a_{s(i+1,1)}, a_{s(i+1,2)}, \dots, a_{s(i+1,l_{i+1})}\}$. The mapping $h_{i+1} : \{1, 2, \dots, l\} \to \{1, 2, \dots, l_{i+1}\}$ is defined by $h_{i+1}(j) = j'$ if $s(i + 1, j') < j \le s(i + 1, j' + 1)$.

(3) (Construction of $\mathcal{C}_{i+1}$) We choose an arbitrary bridge between $a_{s(i+1,j)}$ and $a_{s(i+1,j+1)}$, which we denote by U_j^{i+1}.

For $j \in \{1, 2, \dots, l_i\}$ we have the subchain $\mathcal{C}_{i,j} = (U'_{j'}, U'_{j'+1}, \dots, U'_{j''})$ of $\mathcal{C}'$, where $j' = s(i,j)+1$, $j'' = s(i,j+1)$. The chain $\mathcal{C}_{i,j}$ is a chain between $a_{s(i,j)}$ and $a_{s(i,j+1)}$. From Lemma 2.11, if $j''-j' \geq 1$, then there exists $1 \leq m \leq j''-j'$ such that $a_{j'+m-1} \in K(U'_{j'+m-1}) \cap K(U_{j'+m}) \subset C(|U_j^i|+1)$. This implies that E_i is a proper subset of E_{i+1} if $\#E_i < l+1$. Therefore there exists an integer r such that $\#E_r = l+1$, and then each U_j^r is a bridge between a_{j-1} and a_j for each j. Thus we have constructed a sequence of pre-chains $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_r$ as required.

Step 3: Let $1 \leq j \leq l$ and $0 \leq i \leq r-1$. We write $j_1 = h_i(j)$ and $j_2 = h_{i+1}(j)$. We show

$$|U_{j_2}^{i+1}| \leq |U_{j_1}^i| + n_1 + 1. \tag{2.2}$$

Lemma 2.14. *If $s(i, j_1+1) - s(i, j_1) \geq 2$, then*

$$|U_{j_0}^{i-1}| < |U_{j_1}^i|.$$

Proof. As we have seen above, $h_{i+1}^{-1}(j_2)$ is a proper subset of $h_i^{-1}(j_1)$. Thus either $a_{s(i+1,j_2)} \neq a_{s(i,j_1)}$ or $a_{s(i+1,j_2+1)} \neq a_{s(i,j_1+1)}$. Say $a_{s(i+1,j_2)} \neq a_{s(i,j_1)}$. Then the point $a_{s(i+1,j_2)}$ belongs to $C(|U_{j_1}^i|+1)$ but it does not belong to $C(|U_{j_0}^{i-1}|+1)$. Thus $|U_{j_0}^{i-1}| < |U_{j_1}^i|$. $\square$

Lemma 2.15. *If $s(i, j_1+1) - s(i, j_1) \geq 2$, then both of the point $a_{s(i+1,j_2)}$ and $a_{s(i+1,j_2+1)}$ belong to $C(|U_{j_1}^i|+1)$.*

Proof. Let i' be the minimal integer such that $s(i+1, j_2) = s(i', j')$ for some j'. Then $a_{s(i+1,j_2)} \in C(|U_{h_{i'-1}(j)}^{i'-1}|+1)$. Thus by Lemma 2.14, we obtain

$$a_{s(i+1,j_2)} \in C(|U_{h_{i'-1}(j)}^{i'-1}|+1) \subset C(|U_{h_{i'}(j)}^{i'}|+1) \subset \cdots \subset C(|U_{j_1}^i|+1).$$

Similarly, $a_{s(i+1,j_2+1)} \in C(|U_{j_1}^i|+1)$. $\square$

Proof of (2.2). If $s(i, j_1+1) - s(i, j_1) = 1$, then $U_{j_2}^{i+1}$ and $U_{j_1}^i$ are bridges between the same two points, and hence $|U_{j_2}^{i+1}| = |U_{j_1}^i|$. Suppose $s(i, j_1+1) - s(i, j_1) \geq 2$. By Lemma 2.15, $U_{j_2}^{i+1}$ is a bridge of two points in $C(|U_{j_1}^i|+1)$. Therefore we obtain $|U_{j_2}^{i+1}| \leq |U_{j_1}^i| + n_1 + 1$ from Corollary 2.13. $\square$

Step 4: We will show

$$A(\mathcal{C}_i) = \sum_{j=1}^{l_i} A(U_j^i) \geq \alpha^{p+n_1}$$

for all $i = 0, 1, \dots, r$. Since $A(\mathcal{C}_0) = A(U) \geq \alpha^{p+n_1}$, it is sufficient to show that

$$A(\mathcal{C}_i) \leq A(\mathcal{C}_{i+1})$$

for $i = 0, 1, \dots, r-1$. This inequality is reduced to

$$A(U^i_j) \leq \sum_{j'=j_1}^{j_2} A(U^{i+1}_{j'}),$$

where $j_1 = h_{i+1}(s(i,j)+1)$, $j_2 = h_{i+1}(s(i,j+1))$. If $s(i,j+1) - s(i,j) = 1$, then $j_1 = j_2$, and so $|U^i_j| = |U^{i+1}_{j_1}|$. If $s(i,j+1) - s(i,j) \geq 2$, then $j_1 < j_2$. By (2.2),

$$\sum_{j'=j_1}^{j_2} A(U^{i+1}_{j'}) \geq A(U^{i+1}_{j_1}) + A(U^{i+1}_{j_2}) \geq 2\alpha^{|U^i_j|+n_1+1} = \alpha^{|U^i_j|} = A(U^i_j).$$

This completes the proof of Theorem 2.4. □

When we consider only the case where all F_i are injective, the proof is notably shortened. Almost all the lemmas are unnecessary. In fact, the integer n_1 which is obtained in Step 1 is found to be

$$m = \min\{n \mid \text{ for all } c \in C, L_n(c) \cap (P \cup C - \{c\}) = \emptyset\} - 1.$$

Indeed, let (W, a, b) be a p-mesh. Then $x = F_W^{-1}(a) \in C$ and $y = F_W^{-1}(b) \in P \cup C$. Since F_W is injective, the points x and y are distinct. Recall the integer $k = k(W, a, b)$ which is defined in Step 1. Namely, there exists $W' \in \mathcal{W}_k$ such that $W' \prec W$ and $a, b \in K(W')$. Then $\sigma^p(W')$ is a word of length $k - p$ such that $x, y \in K(\sigma^p(W'))$. Hence $k - p \leq m$.

Consider the self-similar systems of Example 1.8-(2) and (6) again. They are non-recurrent finitely ramified self-similar systems. For the self-similar system of (2), we can take the integer n_1 to be equal to one. For the self-similar system of (6), we can take the integer n_1 to be equal to two. By the estimate in our proof above, we have $\alpha = 2^{-1/2}$ and $\alpha = 2^{-1/3}$ respectively. They are far from the critical ratios. In Figure 3, the ratios (α, α) are shown by black dots.

3. Critical Polyratios

In the previous section we have found a metric polyratio for non-recurrent cases. That estimate is, however, far from critical polyratios. The aim of this section is finding exact critical polyratios.

The standard pseudodistance D is determined from $G_n(x, y)$, the set of chains of depth at most n, as

$$D(x, y) = \lim_{n\to\infty} \min_{\mathcal{C}\in G_n(x,y)} A(\mathcal{C}).$$

That is not true for $\tilde{G}_n(x,y)$, the set of chains of depth n; in general,

$$\lim_{n\to\infty} \min_{\mathcal{C}\in\tilde{G}_n(x,y)} A(\mathcal{C})$$

can not form a pseudodistance. In this section, however, we mainly consider $\tilde{G}_n(x,y)$ instead of $G_n(x,y)$. In fact, the set $G_n(x,y)$ is so complicated. On the other hand, $\tilde{G}_n(x,y)$ is related to the lap number, which is familiar to us.

We imagine that the 'asymptotic behavior' of $\tilde{G}_n(x,y)$ is the same as that of $G_n(x,y)$, and hence that it defines the critical polyratio. As for a finitely ramified topological self-similar system, for a given simple path γ between two points, a chain $\mathcal{C}_n^\gamma = (U_1, U_2, \ldots, U_l)$ of depth n between the points is uniquely determined such that each $K(U_i)\cap\gamma$ includes an arc. We expect that if $\sum_{k=1}^{\infty} A(C_k^\gamma) = \infty$ for any simple path γ between x and y, then $D(x,y) > 0$. If it is established, then we think of $\sum_{k=1}^{\infty} A(C_k^\gamma)$ as a power series of variables $\alpha_1, \alpha_2, \ldots, \alpha_N$, proving its polyradius of convergence to be a critical amount.

In this section we put a restriction. We will assume that $(K, \{F_i\}_{i=1}^N)$ is a finitely ramified topological self-similar system which satisfies the following condition:

Condition A

(1) Each component of K is simply connected.

(2) There exists a minimal trees $T_1, T_2, \ldots T_m \subset K$ which satisfy the following: For any simple path γ in K there exist T_k and a positive integer p, n such that $T_k \subset \bigcup_{i=n}^{n+p-1} \eta^i(\gamma)$, where $\eta^i(\gamma)$ is the i-the image of γ, which we will define later.

In Subsection 3.2 we will introduce the notion of invariant trees in K. If T is an invariant tree, then a (piecewise-continuous) dynamics is defined on T. A minimal tree is defined as an invariant trees in K that satisfies a certain condition like topological transitivity.

We will introduce a power series $v(T)(X_1, X_2, \ldots, X_N)$ of N variables for a tree $T \subset K$. For given two points x, y in a component of K there uniquely exists a simple path γ between x and y. Then we will see the power series $v(x,y) = v(\gamma)$ satisfies $v_n(x,y)(\alpha_1, \alpha_2, \ldots, \alpha_N) = A(\mathcal{C}_n^\gamma)$, where $v_n(x,y)$ is the homogeneous part of degree n. We say $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ (ε_k's are non-negative) is a *polyradius of convergence* of $v(x,y)$ if the radius of convergence of the 1-variable power series $v(x,y)(\varepsilon_1 t, \varepsilon_2 t, \ldots, \varepsilon_N t)$ is equal to one.

Then it is easily seen that the polyradius of convergence of $v(T)$ gives a lower estimate of critical polyratios (Lemma 3.14). Moreover,

Theorem 3.1. *Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system satisfying Condition A. If $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a polyradius of convergence of $v(x,y)$ for any two points x, y in a component of K, then $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a critical polyratio.*

To prove this, we use the kneading determinant for a dynamics on a topological tree. The kneading determinants is a holomorphic function on the unit polydisc

$$\mathbf{D} = \{(X_1, X_2, \dots, X_N) \in \mathbf{C}^N \mid |X_i| < 1, i = 1, 2, \dots, N\}$$

with a zero point which equals a critical polyratio. That is a simple generalization of Milnor-Thurston's theory.

Precise formulations will be given in Subsection 3.2. Here we only give an example in advance as a guideline of discussion.

Example 3.2. Consider the self-similar system $(K, \{F_1, F_2\})$ of Example 1.8-(2). We have the piecewise-continuous dynamics $f = (f_1, f_2)$ on $K = [0, 1]$, which is the pair of continuous maps

$$\begin{cases} f_1(x) = 2x & \text{on } [0, 1/2], \\ f_2(x) = 2x - 1 & \text{on } [1/2, 1]. \end{cases}$$

We see that $f_1 = F_1^{-1}$ and $f_2 = F_2^{-1}$. Let $\gamma \subset K$ be a subinterval not a point. Then the n-th image of γ is defined by

$$\begin{aligned} &f^0(\gamma) = \gamma, \\ &f^n(\gamma) = f_1(f^{n-1}(\gamma)) \cup f_2(f^{n-1}(\gamma)). \end{aligned}$$

Then it is easy to see that K is minimal, that is, for any subinterval γ not a point, there exists n such that $f^n(\gamma) = K$. In other words, for any γ there exists $U \in \mathcal{W}_*$ such that $K(U) \subset \gamma$.

The power series $v(x, y)(X_1, X_2)$ of two variables X_1, X_2 is defined as follows. Let $x, y \in K = [0, 1]$ with $x < y$. Consider the interval $[x, y]$ between x and y. We set

$$v_n(x, y)(X_1, X_2) = \sum_{u_1 u_2 \dots u_n} X_{u_1} X_{u_2} \cdots X_{u_n},$$

where $u_1 u_2 \dots u_n$ runs through all words in $\mathcal{W}_n$ such that $[x, y] \cap K(u_1 u_2 \dots u_n)$ contains more than one points. Note that the set of such words forms a chain $\mathcal{C}_n$ of depth n. We set

$$v(x, y)(X_1, X_2) = \sum_{n=0}^{\infty} v_n(x, y)(X_1, X_2).$$

If $x = 0$ and $y = 1$, then

$$v_n(0, 1)(X_1, X_2) = \sum_{u_1 u_2 \dots u_n \in \mathcal{W}_n} X_{u_1} X_{u_2} \cdots X_{u_n} = (X_1 + X_2)^n.$$

Thus

$$v(0, 1)(X_1, X_2) = \sum_{n=0}^{\infty} (X_1 + X_2)^n = \frac{1}{1 - X_1 - X_2}.$$

Consequently, the series is convergent on $\{(X_1,X_2) \mid |1-X_1-X_2|<1\}$; it is not convergent if $X_1+X_2=1$.

Suppose $x=k2^{-n}$ and $y=(k+1)2^{-n}$, where n,k are nonnegative integers such that $0\le k\le 2^n-1$. Then it is easily seen that

$$\begin{aligned} v(x,y)(X_1,X_2) &= \sum_{V\neq U, U\prec V} X_V + X_U\sum_{n=0}^{\infty}(X_1+X_2)^n \\ &= \sum_{V\neq U, U\prec V} X_V + \frac{X_U}{1-X_1-X_2}, \end{aligned}$$

where $K(U)=[x,y]$ and $X_{u_1u_2\ldots u_n}=X_{u_1}X_{u_2}\cdots X_{u_n}$. Thus $v(x,y)(X_1,X_2)$ is convergent on $\{(X_1,X_2)\mid |1-X_1-X_2|<1\}$; it is not convergent if $X_1+X_2=1$. That is true for any $x,y\in K$; because $[x,y]$ is included in an interval of the form $[k2^{-n},(k+1)2^{-n}]$, and also it includes such an interval. In fact, $v(x,y)(X_1,X_2)$ is written in the form

$$v(x,y)(X_1,X_2) = \sum_{V:[x,y]\subsetneq K(V)} X_V + \frac{\sum_U X_U}{1-X_1-X_2},$$

where U runs through all words satisfying the properties that $K(U)\subset[x,y]$ and that if $U\prec V$ then $K(V)\not\subset[x,y]$. From the minimality of K, we see that $\sum_U X_U$ does not vanish. It is clear that $\sum_U X_U$ is convergent if $|X_1|<1$ and $|X_2|<1$. For this reason, we consider $H(x,y)(X_1,X_2)=(1-X_1-X_2)v(x,y)(X_1,X_2)$ as an analytic function on $\mathbf{D}=\{(X_1,X_2)\mid |X_1|<1,|X_2|<1\}$. Note that

$$v(F_i(x),F_i(y))(X_1,X_2)=1+X_iv(x,y)(X_1,X_2)$$

for $i=1,2$, and hence

$$H(F_i(x),F_i(y))(X_1,X_2)=1-X_1-X_2+X_iH(x,y)(X_1,X_2). \tag{3.1}$$

For a polyratio (α_1,α_2), we can see $v_n(x,y)(\alpha_1,\alpha_2)=A(\mathcal{C}_n)$. If the series $v(x,y)(\alpha_1,\alpha_2)$ is convergent, then $v_n(x,y)(\alpha_1,\alpha_2)\to 0$ as $n\to\infty$, and so $D(\alpha_1,\alpha_2)(x,y)=0$. Conversely, if $v(x,y)(\alpha_1,\alpha_2)$ is not convergent, then (α_1,α_2) is a metric polyratio. Although this have been proved in Example 1.16, we give another proof. Indeed, suppose $\alpha_1+\alpha_2=1$. Consider the function

$$d(x,y)=\lim_{t\to 1-}\frac{v(x,y)(\alpha_1t,\alpha_2t)}{v(0,1)(\alpha_1t,\alpha_2t)}=H(x,y)(\alpha_1,\alpha_2),$$

which takes a positive value for $x\neq y$. The function d is a distance on K compatible with the topology of $[0,1]$ because of the fact that if $x_1\le x_2\le x_3$ in K, then $d(x_1,x_3)\le d(x_1,x_2)+d(x_2,x_3)$, the fact that if $x_1\le x_2\le x_3\le x_4$, then $d(x_2,x_3)\le d(x_1,x_4)$, and the fact that $d(k2^{-n},(k+1)2^{-n})=A(U)$, where U is the word satisfying $K(U)=[k2^{-n},(k+1)2^{-n}]$. By (3.1), we can see that d is a self-similar metric with polyratio (α_1,α_2).

3.1. Preliminaries — dynamics of self-similar system

If a topological self-similar system $(K, \{F_i\}_{i=1}^N)$ satisfies

$$\# \bigcup_{i=1}^{N} F_i^{-1}(x) = 1$$

for any $x \in K$, then there exists a continuous map $g : K \to K$ such that F_i $(i = 1, 2, \dots, N)$ are the inverse branches of g, namely, the diagram

$$\begin{array}{ccc} \Sigma_N & \xrightarrow{\sigma} & \Sigma_N \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ K & \xrightarrow[g]{} & K \end{array}$$

commutes. Then we consider (g, K) as the dynamics of $(K, \{F_i\}_{i=1}^N)$.

The set

$$C' = \{x \mid \# \bigcup_{i=1}^{N} F_i^{-1}(x) > 1\} \subset C$$

is, however, not always empty. In general, the continuous map g is defined only on $K - C'$. For example, recall Example 3.2. Only the point $1/2 \in K$ satisfies $\# \bigcup_{i=1}^{N} F_i^{-1}(1/2) > 1$. We have a continuous map $g : K - \{1/2\} \to K$ which is defined by

$$g(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x - 1 & \text{if } 1/2 < x \le 1 \end{cases}.$$

If the dynamics is extended on the whole space K, then ambiguity appears at $1/2$. When we consider $1/2$ as a member of $[0, 1/2]$, the value of $g(1/2)$ is one; when we consider $1/2$ as a member of $[1/2, 1]$, the value of $g(1/2)$ is zero. To avoid the ambiguity, we write $g(1/2^-) = 1$, the left-hand limit, and $g(1/2^+) = 0$, the right-hand limit.

In general, the left(right)-hand limit at a discontinuity point $x \in C'$ is not well-defined, since there is no natural linear order on K. Thus we consider a point x in K together with a simple path $\gamma : [0, 1] \to K$ which passes through x. We will examine a dynamics working on the set of ordered pairs (x, γ).

Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system. Let γ be a simple path, and a a point in γ. By the symbol γ, we may denote not only the mapping $[0, 1] \to K$ but also the image of the mapping. (For example, we write $a \in \gamma$ instead of $a \in \gamma([0, 1])$.) Considering the topological self-similar system $(K, \{F_i\}_{i=1}^N)$ as a complex of dynamics on paths, we can treat it as some kind of interval dynamics.

Remark 3.3. Precisely, we consider equivalent classes of paths. We identify γ with γ', say $\gamma \simeq \gamma'$, if $\gamma \circ h = \gamma'$ for some orientation-preserving homeomorphism $h : [0, 1] \to [0, 1]$.

Notation 3.4. The set of simple paths in K is denoted by

$$Q_0 = \{\gamma : [0,1] \to K \,|\, \gamma \text{ is injective and continuous}\}.$$

We set

$$\Xi_0 = \{(a,\gamma) \in K \times Q_0 \,|\, \gamma \in Q_0, a \in \gamma\}.$$

Usually, an element of Ξ_0 will be referred by a symbol ξ.

In this section we will define many functions with argument $(a^\star, \gamma)$, where $\star$ is $+$, $-$ or empty. If $\xi = (a,\gamma)$, the argument is written as $\xi^\star$.

Definition 3.5. Let $\gamma \in Q_0$ be a simple path. We say $a \in \gamma$ is a *turning point* of γ if for any $\epsilon > 0$ there is no $i \in \{1, 2, \ldots, N\}$ such that $\gamma([\gamma^{-1}(a) - \epsilon, \gamma^{-1}(a) + \epsilon]) \subset K(i)$, in other words, if $\gamma \cap K(i)$ is not a neighborhood of a in γ for any symbol i. We say a is k-turning point of γ if for some $\epsilon > 0$ there is $U \in \mathcal{W}_k$ such that $\gamma([\gamma^{-1}(a) - \epsilon, \gamma^{-1}(a) + \epsilon]) \subset K(U)$, but if for any $\epsilon > 0$ there is no $U \in \mathcal{W}_{k+1}$ such that $\gamma([\gamma^{-1}(a) - \epsilon, \gamma^{-1}(a) + \epsilon]) \subset K(U)$. A turning point is a 0-turning point. We denote, by $\mathrm{Tur}_k(\gamma) \subset \gamma$, the set of k-turning points of γ. For convenience, we set $\mathrm{Tur}_{-1}(\gamma) = \emptyset$. We say that $(a,\gamma) \in \Xi_0$ is a k-turning point if a is a k-turning point of γ.

A turning point of γ is a critical point. Since the critical set is finite, even if $a \in \gamma$ is a turning point there exist $\epsilon > 0$ and $i, j \in \{1, 2, \ldots, N\}$ such that $\gamma([t_a - \epsilon, t_a]) \subset K(i)$ and $\gamma([t_a, t_a + \epsilon]) \subset K(j)$, where $\gamma(t_a) = a$. We use the notation

$$Y(a^-, \gamma) = Y(\xi^-) = i, \; Y(a^+, \gamma) = Y(\xi^+) = j,$$

where $\xi = (a,\gamma)$. If ξ is not a turning point, then $Y(\xi^-) = Y(\xi^+)$, so it is denoted by $Y(\xi) = Y(a,\gamma)$. We call $Y(\xi^\pm)$ the *address* of $\xi^\pm$.

Since the critical set is finite, $\#F^{-1}_{Y(a^\star,\gamma)}(a) = 1$ except for finitely many a. Thus

$$\begin{aligned} g(a^-, \gamma) = g(\xi^-) &= \lim_{\epsilon \to 0} F_i^{-1}(\gamma(t_a - \epsilon)) \\ g(a^+, \gamma) = g(\xi^+) &= \lim_{\epsilon \to 0} F_j^{-1}(\gamma(t_a + \epsilon)) \end{aligned}$$

are well-defined, where $\gamma(t_a) = a$ and $\gamma(t_a - \epsilon) \in K(i)$, $\gamma(t_a + \epsilon) \in K(j)$ for small ϵ. We simply write $g(\xi)$ if $g(\xi^-) = g(\xi^+)$. The point $g(a^\pm, \gamma)$ is considered as the image of $a^\pm$ by the 'map' $g(\cdot, \gamma)$.

Definition 3.6. We say $a \in \gamma$ is an *essential critical point* of γ, that is to say $\xi = (a,\gamma)$ is an essential critical point, if either ξ is a turning point or $g(\xi^-) \neq g(\xi^+)$. It is clear that an essential critical point is a critical point and that a turning point is an essential critical point. Then the number of essential critical points of γ is clearly finite.

Remark 3.7. If $C_2 = \bigcup_{i=1}^N \{x \in K \,|\, \#F_i^{-1}(x) \geq 2\}$ is empty, then (a,γ) is a turning point if and only if (a,γ) is an essential critical point.

The essential critical points of γ divide the path γ into finite sub-paths on which we can define a continuous map g. Precisely speaking, the unit interval

$[0,1]$ is divided to subintervals $I = I_1 \cup I_2 \cup \cdots \cup I_l$, where $I_k = [t_{k-1}, t_k]$ ($k = 1, 2, \ldots, l$), $t_0 = 0$, $t_l = 1$ and where $\gamma(t_k)$ ($k = 1, 2, \ldots, l-1$) are essential critical points. Then for any $k = 1, 2, \ldots, l$ there exists $i_k \in \{1, 2, \ldots, N\}$ such that $\gamma(I_k) \subset K(i_k)$. Moreover, $g(\gamma(t)^-, \gamma) = g(\gamma(t)^+, \gamma)$ for any $t \in \mathrm{int} I_k$. Consequently, a continuous map $g_k : \gamma(I_k) \to K$ is defined as $g_k(a) = g(a, \gamma)$. We use the notation

$$\mathbf{i}(a, \gamma) = \mathbf{i}(a^{\pm}, \gamma) = I_k \quad \text{ if } a \in \mathrm{int} I_k$$

and

$$\mathbf{i}(\gamma(t_{k-1})^+, \gamma) = \mathbf{i}(\gamma(t_k)^-, \gamma) = I_k.$$

For $a \in \gamma$, we take $h_{\mathbf{i}(a^{\pm}, \gamma)}$, an orientation-preserving homeomorphism of $[0,1]$ onto $\mathbf{i}(a^{\pm}, \gamma)$. Then we obtain a simple path

$$\eta(a^{\pm}, \gamma) = g \circ \gamma \circ h_{\mathbf{i}(a^{\pm}, \gamma)} : [0,1] \to K.$$

Notation 3.8. For $\xi = (a, \gamma)$, we define

$$\eta^0(\xi^{\pm}) = \gamma,\ g^0(\xi^{\pm}) = a,\ \mu^0(\xi^{\pm}) = \xi,\ \mathbf{I}_0(\xi^{\pm}) = \gamma([0,1]),$$

and we inductively define for $k = 1, 2, \ldots$

$$\begin{array}{rcl} \eta^k(\xi^{\pm}) & = & \eta(\mu^{k-1}(\xi^{\pm})^{\pm}), \\ g^k(\xi^{\pm}) & = & g(\mu^{k-1}(\xi^{\pm})^{\pm}), \\ \mu^k(\xi^{\pm}) & = & (g^k(\xi^{\pm}), \eta^k(\xi^{\pm})), \\ \mathbf{I}_k(\xi^{\pm}) & = & F_{Y(\xi^{\pm})}(\mathbf{I}_{k-1}(\mu^1(\xi^{\pm})^{\pm})), \\ Y_{k-1}(\xi^{\pm}) & = & Y(\mu^{k-1}(\xi^{\pm})^{\pm}). \end{array}$$

For $k = 0, 1, \ldots$, we write

$$\mathcal{Y}_k(\xi^{\pm}) = Y_0(\xi^{\pm}) Y_1(\xi^{\pm}) \ldots Y_k(\xi^{\pm}) \in \mathcal{W}_{k+1}.$$

If $k = -1$, we set

$$\mathcal{Y}_{-1}(\xi^{\pm}) = \emptyset \in \mathcal{W}_0.$$

If $\mathbf{I}_k(\xi^-) = \mathbf{I}_k(\xi^+)$, $\mathcal{Y}_k(\xi^-) = \mathcal{Y}_k(\xi^+)$, etc, then we also use the notation $I_k(\xi)$, $\mathcal{I}_{k-1}(\xi)$, etc, respectively.

By definition,

$$\mathbf{I}_1(\xi^{\pm}) = F_{Y(\xi^{\pm})}(\eta(\xi^{\pm})([0,1])) = \gamma(\mathbf{i}(\xi^{\pm})) \subset \gamma([0,1]) = \mathbf{I}_0(\xi^{\pm}).$$

If $k \geq 0$, then

$$\mathbf{I}_k(\xi^{\pm}) = F_{\mathcal{Y}_{k-1}(\xi^{\pm})}(\mathbf{I}_0(\mu^k(\xi^{\pm})^{\pm}))$$

and

$$\mathbf{I}_{k+1}(\xi^{\pm}) = F_{\mathcal{Y}_{k-1}(\xi^{\pm})}(\mathbf{I}_1(\mu^k(\xi^{\pm})^{\pm})).$$

Thus

$$\mathbf{I}_0(\xi^{\pm}) \supset \mathbf{I}_1(\xi^{\pm}) \supset \cdots .$$

By definition we know that $\{\mathbf{I}_1(a,\gamma) \mid a \in \gamma\}$ is a decomposition of $\mathbf{I}_0(\xi) = \gamma([0,1])$ into subarcs by essential critical points. Thus $\{\mathbf{I}_k(a,\gamma) \mid a \in \gamma\}$ is also a decomposition of $\gamma([0,1])$. It is evident that if $\xi = (a,\gamma)$,

$$a \in \mathbf{I}_k(\xi^{\pm}) \subset K(\mathcal{Y}_{k-1}(\xi^{\pm}))$$

Clearly,

$$\eta^{k+m}(\xi^{\pm}) = \eta^k(\mu^m(\xi^{\pm})^{\pm}), \tag{3.2}$$

$$g^{k+m}(\xi^{\pm}) = g^k(\mu^m(\xi^{\pm})^{\pm}), \tag{3.3}$$

$$Y_{k+m}(\xi^{\pm}) = Y_k(\mu^m(\xi^{\pm})^{\pm}), \tag{3.4}$$

and

$$\mathcal{Y}_{k+m}(\xi^{\pm}) = \mathcal{Y}_k(\xi^{\pm})\mathcal{Y}_{m-1}(\mu^{k+1}(\xi^{\pm})^{\pm}). \tag{3.5}$$

The following is clear.

Proposition 3.9. (1) Let $\xi \in \Xi_0$. The point ξ is a k-turning point if and only if

$$\mathcal{Y}_{k-1}(\xi^-) = \mathcal{Y}_{k-1}(\xi^+),\ \mathcal{Y}_k(\xi^-) \neq \mathcal{Y}_k(\xi^+).$$

The point ξ is an essential critical point if and only if

$$\mathbf{I}_1(\xi^-) \neq \mathbf{I}_1(\xi^+).$$

(2) For $a, b \in \gamma$ and $k = 1, 2, \ldots$

$$\begin{aligned}\mathbf{I}_k(a^\star,\gamma) = \mathbf{I}_k(b^\diamond,\gamma) \Longrightarrow\ & \mathbf{I}_{k-1}(a^\star,\gamma) = \mathbf{I}_{k-1}(b^\diamond,\gamma),\\ & \mathcal{Y}_{k-1}(a^\star,\gamma) = \mathcal{Y}_{k-1}(b^\diamond,\gamma),\ \eta^k(a^\star,\gamma) = \eta^k(b^\diamond,\gamma),\end{aligned}$$

where $\star, \diamond \in \{-,+\}$. Moreover, if $a = b$, we see that if $\mathbf{I}_k(a^-,\gamma) = \mathbf{I}_k(a^+,\gamma)$, then $g^k(a^-,\gamma) = g^k(a^+,\gamma)$.

Notation 3.10. If a is a k-turning point, then by Proposition 3.9 we can take the minimal integer $0 \leq s \leq k$ such that $\mathbf{I}_{s+1}(a^-,\gamma) \neq \mathbf{I}_{s+1}(a^+,\gamma)$. Then $g^s(a,\gamma)$ is an essential critical point of $\eta^s(a,\gamma)$. We write

$$s(a,\gamma) = s.$$

We set

$$\begin{aligned}B_k(\gamma) &= \{a \mid a \in \textstyle\bigcup_{m=0}^{\infty} \mathrm{Tur}_m(\gamma), k = s(a,\gamma)\}\\ &= \{a \in \gamma \mid \mathbf{I}_k(a^-,\gamma) = \mathbf{I}_k(a^+,\gamma), \mathbf{I}_{k+1}(a^-,\gamma) \neq \mathbf{I}_{k+1}(a^+,\gamma)\}.\end{aligned}$$

for $k = 0, 1, \ldots$. Then $B_k(\gamma)$ is a finite set, since $B_k(\gamma) \subset C(k) = \bigcup_{|U| \leq k} F_U(C)$. In fact, $\bigcup_{m=0}^{k} B_m(\gamma) \cup \{\gamma(0), \gamma(1)\}$ is the set of endpoints of arcs in the form $\mathbf{I}_{k+1}(a,\gamma)$.

Remark 3.11. If C_2 is empty, then $s(a,\gamma)=k$ for any k-turning point (a,γ). Thus $B_k(\gamma)$ is the set of k-turning points of γ.

Example 3.12. Consider the self-similar system $(K,\{F_1,F_2\})$ of Example 1.8-(3). Let $\gamma:[0,1]\to K$ be the simple path $\gamma(t)=t$ between 0 and 1. Note that $\gamma([0,1])=[0,1]=K$. The critical set $C=\{1/2\}$ is equal to the set of turning points of γ and equal to the set of essential critical points of γ. The interval γ is divided into two subintervals: $\gamma=[0,1/2]\cup[1/2,1]=K(1)\cup K(2)$. It is easy to see that

$$g(a,\gamma)=\begin{cases} -2a+1 & \text{if } a\in[0,1/2] \\ 2a-1 & \text{if } a\in[1/2,1] \end{cases},$$

and

$$Y(a,\gamma)=\begin{cases} \mathbf{1} & \text{if } a\in[0,1/2) \text{ or } a=1/2^- \\ \mathbf{2} & \text{if } a\in(1/2,1] \text{ or } 1=1/2^+ \end{cases}.$$

Since

$$\mathbf{i}(a,\gamma)=\begin{cases} [0,1/2] & \text{if } a\in[0,1/2) \text{ or } a=1/2^- \\ [1/2,1] & \text{if } a\in(1/2,1] \text{ or } 1=1/2^+ \end{cases},$$

taking homeomorphisms $h_{[0,1/2]}(t)=t/2$ and $h_{[1/2,1]}(t)=(t+1)/2$, we have

$$\eta(a,\gamma)(t)=\begin{cases} -t+1 & \text{if } a\in[0,1/2) \text{ or } a=1/2^- \\ t & \text{if } a\in(1/2,1] \text{ or } a=1/2^+ \end{cases}.$$

In the same way, we see that

$$\eta^k(a,\gamma)(t)=\begin{cases} -t+1 & \text{if } n \text{ is odd} \\ t & \text{if } n \text{ is even} \end{cases},$$

where $n=\#\{0\le l\le k-1\,|\,Y_l(a,\gamma)=\mathbf{1}\}$.

3.2. Main results

Let $(K,\{F_i\}_{i=1}^N)$ be a finitely ramified self-similar set such that every component of K is simply connected. Let x,y be points in a component of K. From Corollaries A.2 and A.5 there uniquely exists a simple path $\gamma_{x,y}$ between x and y. The set of at most $n-1$-turning points of $\gamma_{x,y}$, denoted by $\bigcup_{m=0}^n \mathrm{Tur}_{m-1}(\gamma_{x,y})$, divides the path $\gamma_{x,y}$ into subpaths $L_1,L_2,\dots,L_l$. We can define a set

$$\mathcal{L}(\gamma_{x,y},n)=\{(L_1,U_1),(L_2,U_2),\dots,(L_l,U_l)\},$$

where $(U_1,U_2,\dots,U_l)$ is a chain between x and y, $U_k\in\mathcal{W}_n$, $L_k\subset\gamma_{x,y}\cap K(U_k)$, and $\{L_k\}_k$ is a set of simple arcs, satisfying $\bigcup_k L_k=\gamma_{x,y}$, which are mutually disjoint but one point. Using that, we define a homogeneous polynomial

$$v_n(x,y)(X_1,X_2,\dots,X_N)=\sum_{i=1}^{l} X_{U_i},$$

where $X_{u_1u_2\ldots u_n} = X_{u_1}X_{u_2}\cdots X_{u_n}$. Note that $v_0(x,y)(X_1,X_2,\ldots,X_N)=1$. Now we define a formal power series

$$v(x,y)(X_1,X_2,\ldots,X_N)=\sum_{k=0}^{\infty} v_k(x,y)(X_1,X_2,\ldots,X_N).$$

Definition 3.13. A set T is called a *topological tree* if T is homeomorphic to a 1-dimensional simplicial complex each component of which is simply connected.

We also define a formal power series v for a topological tree T in K. A topological tree T is divided into subtrees $L_1, L_2, \ldots, L_l$ by $\bigcup_{m=0}^{n}\bigcup_{\gamma\subset T}\mathrm{Tur}_{m-1}(\gamma)$. We write

$$\mathcal{L}(T,n)=\{(L_1,U_1),(L_2,U_2),\ldots,(L_l,U_l)\},$$

where $U_k\in\mathcal{W}_n$, $L_k\subset K(U_k)\cap T$, and $\{L_k\}_k$ is a set of connected topological trees, satisfying $\bigcup_k L_k = T$, which are mutually disjoint but one point. Then we set

$$\begin{aligned} v_n(T)(X_1,X_2,\ldots,X_N) &= \sum_{i=1}^{l} X_{U_i},\\ v(T)(X_1,X_2,\ldots,X_N) &= \sum_{k=0}^{\infty} v_k(x,y)(X_1,X_2,\ldots,X_N). \end{aligned}$$

The following lemma is easy.

Lemma 3.14. *Let $(K,\{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system such that every component of K is simply connected. Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)\in\mathbf{Ra}_N$ be a polyratio such that the power series $v(x,y)(\alpha)$ converges for some $x\neq y\in K$. Then the standard pseudodistance D_α is not a distance.*

Proof. Since $v(x,y)(\alpha)$ converges, $D_\alpha(x,y)\le v_n(x,y)(\alpha)\to 0$ as $n\to\infty$. $\square$

Thus, if the pseudodistance D_α is a distance, then $v(x,y)(\alpha)$ is not convergent for any x,y. Our main theorem is the converse.

Notation 3.15. Let $(K,\{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system. For a simple path γ in K, we write

$$\eta^k(\gamma)=\bigcup_{a\in\gamma}\eta^k(a,\gamma)=\{g^k(a^\dagger,\gamma)\,|\,a\in\gamma,\dagger=-,+\}$$

for $k=1,2,\ldots$ If T is a topological tree, we use the notation

$$\eta^k(T)=\bigcup_{\gamma:\text{ simple path in }T}\eta^k(\gamma)$$

for $k=1,2,\ldots$.

Definition 3.16. A topological tree T in the topological self-similar set K is called an *invariant tree* if $\eta^1(\gamma) \subset T$ for any simple path γ in T. An invariant tree is said to be *minimal* if for any simple path γ in T, there exist p and n such that $\bigcup_{i=n}^{n+p-1} \eta^i(\gamma) = T$

Recall that the main theorem has been stated as follows.

Condition A

(1) Each component of K is simply connected.

(2) There exists a minimal trees $T_1, T_2, \dots T_m \subset K$ which satisfy the following: For any simple path γ in K there exist T_k and a positive integer p, n such that $T_k \subset \bigcup_{i=n}^{n+p-1} \eta^i(\gamma)$.

Theorem 3.1. *Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system satisfying Condition A. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a polyradius of convergence of $v(x, y)$ for any two points x, y in a component of K, then α is a critical polyratio.*

In more detail, we will prove:

Theorem 3.17. *Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system satisfying Condition A. Suppose that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfies the following. There exists a polyradius of convergence $(\alpha_1^j, \alpha_2^j, \dots, \alpha_N^j)$ of $v(T_j)$ for each $j = 1, 2, \dots, m$ such that $\alpha_i^j \le \alpha_i$ for any i and any j, where T_j is the minimal tree in Condition A. Then α is a metric polyratio.*

Remark 3.18. One of sufficient conditions for a finitely ramified topological self-similar system to satisfy Condition A is the following. We say that an invariant tree T is a *Hubbard tree* if the critical set C and the pre-postcritical set P are included in T. If non-recurrent finitely ramified topological self-similar system has a Hubbard tree, then it satisfies Condition A. This claim, which is not proved in this paper, will be discussed in another paper of the author [9]. In particular, if the pre-postcritical set P of a finitely ramified topological self-similar system is finite, then Condition A is fulfilled. All self-similar systems in Example 1.8 satisfy Condition A.

Now let us start the proof of Theorem 3.17.

To construct a self-similar metric on K, we consider a distance on a minimal tree as follows. Let T be a minimal tree in K, and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ be a polyradius of convergence of $v(T)$. We write $X = (X_1, X_2, \dots, X_N)$. We will define

$$d(x, y) = \lim_{X \to \alpha} \frac{v(x, y)(X)}{v(T)(X)}$$

for x, y in a component of T. In fact, in the next section we will prove the following.

Lemma 3.19. *Let T be a minimal tree in K, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ be a polyradius of convergence of $v(T)$. Then for any $x \neq y$ in T, the series $v(x,y)(\alpha)$ diverges, and the limit*

$$d(x,y) = \lim_{t \to 1-} \frac{v(x,y)(\alpha t)}{v(T)(\alpha t)}. \tag{3.6}$$

exists, where $\alpha t = (\alpha_1 t, \alpha_2 t, \ldots, \alpha_N t)$.

Note that if (3.6) converges for any x, y, then

$$d(T') = \lim_{t \to 1-} \frac{v(T')(\alpha t)}{v(T)(\alpha t)}$$

converges for any subtree $T' \subset T$. We continue the proof, assuming Lemma 3.19.

Proposition 3.20. *Under the above assumption, we have*

(1) *Let T_1, T_2 be subtrees of T. If $T_1 \subset T_2$, then*

$$d(T_1) \leq d(T_2).$$

In particular, $d(T') \leq 1$ for any subtree $T' \subset T$.

(2) *Let T_1, T_2 be subtrees of T such that $T_1 \cap T_2$ is at most one point. Then*

$$d(T_1) + d(T_2) = d(T_1 \cup T_2).$$

(3) *Let T' be a subtree of T. Then*

$$\min_i \alpha_i d(\eta^1(T')) \leq d(T').$$

Moreover, if $T' \subset K(i)$ for some $i \in \{1, 2, \ldots, N\}$, then

$$\alpha_i d(\eta^1(T')) = d(T')$$

Proof. (1) Let T_1 and T_2 be a subtree of T with $T_1 \subset T_2$. Then there exists an integer $n_0 \geq 0$ such that if $n > n_0$ then $L \cap T_1$ is either connected or empty for any $(L, U) \in \mathcal{L}(T_2, n)$. The mapping $h_n : \mathcal{L}(T_1, n) \to \mathcal{L}(T_2, n)$ defined by $h_n(L', U') = (L, U)$ if $U = U'$ and $L' \subset L$ is well-defined, and it is injective if $n > n_0$. Consequently,

$$v_n(T_1)(X_1, X_2, \ldots, X_N) \leq v_n(T_2)(X_1, X_2, \ldots, X_N)$$

for $0 < X_i < 1$ if $n > n_0$. Since $\sum_{k=n_0+1}^{\infty} v_k(T_1)(\alpha t) \leq \sum_{k=n_0+1}^{\infty} v_k(T_2)(\alpha t)$ for $0 < t < 1$, and since $\sum_{k=n_0+1}^{\infty} v_k(T)(\alpha t) \to \infty$ as $t \to 1-$, we have $d(T_1) \leq d(T_2)$. The second assertion is verified by $d(T) = 1$.

(2) Suppose $T_1 \cap T_2$ is at most one point. Let r be the number of branches at the intersection point $a \in T_1 \cap T_2$, that is, $S - \{a\}$ has r connected components, where S a small connected neighborhood of a in T. Then there exists an integer $n_0 \geq 0$ such that if $n > n_0$ then

$$(\mathcal{L}(T_1, n) \cup \mathcal{L}(T_2, n) \cup \mathcal{L}(T_1 \cup T_2, n)) - ((\mathcal{L}(T_1, n) \cup \mathcal{L}(T_2, n)) \cap \mathcal{L}(T_1 \cup T_2, n))$$

consists of at most $3r/2$ members. Therefore the difference between $v_n(T_1)(\alpha t) + v_n(T_2)(\alpha t)$ and $v_n(T_1 \cup T_2)(\alpha t)$ is bounded by $3r(\max_i \alpha_i)^n/2$ for $0 < t < 1$. Since $0 < \alpha_i < 1$, we have $\sum_{k=n_0+1}^{\infty} 3r(\max_i \alpha_i)^n/2$ is finite, and hence $d(T_1) + d(T_2) = d(T_1 \cup T_2)$.

(3) Let T' be a subtree in T. If $(L, U) \in \mathcal{L}(\eta^1(T'), k)$, then there exist $i \in \{1, 2, \ldots, N\}$ and $(L', \tau_i(U)) \in \mathcal{L}(T', k+1)$ such that $L' \subset F_i(L)$. Thus

$$\min_i \alpha_i t \, v_k(\eta^1(T'))(\alpha t) \leq v_{k+1}(T')(\alpha t)$$

for $k = 0, 1, \ldots$. Consequently,

$$\min_i \alpha_i a \frac{v(\eta^1(T'))(\alpha t)}{v(T)(\alpha t)} \leq \frac{v(T')(\alpha t) - v_0(T')(\alpha t)}{v(T)(\alpha t)}$$

if $a < t < 1$. Since $v_0(T')$ is bounded, we have $\min_i \alpha_i d(\eta^1(T')) \leq d(T')$.

Moreover, suppose $T' \subset K(i)$ for some $i \in \{1, 2, \ldots, N\}$. Then $(L, U) \in \mathcal{L}(\eta^1(T'), k)$ if and only if $(F_i(L), \tau_i(U)) \in \mathcal{L}(T', k+1)$. Consequently,

$$X_i v_k(\eta^1(T'))(X) = v_{k+1}(T')(X)$$

for $k = 1, 2, \ldots$. The last assertion can be proved similarly.

□

Proposition 3.21. *Under the above assumption, $d(\cdot, \cdot)$ is a distance on each component of T which is compatible with the topology of T.*

Proof. It is clear that $d(x, y) = d(y, x)$. In the case $x = y$, although we have not defined $d(x, x)$, it is natural and reasonable to set $d(x, x) = 0$.

Let x, y, z be points in a component of T. Then $\gamma_{x,z} = \overline{\gamma_{x,y} - H} \cup \overline{\gamma_{y,z} - H}$, where $H = \gamma_{x,y} \cap \gamma_{y,z}$. Since $\overline{\gamma_{x,y} - H} \cap \overline{\gamma_{y,z} - H}$ consists of at most one point, we have

$$d(x, z) \leq d(x, y) + d(y, z).$$

Assume that there exists $x \neq y$ such that $d(x, y) = 0$. Since T is minimal, there exists a positive integer p, n such that $\bigcup_{i=n}^{n+p-1} \eta^i(\gamma_{x,y}) = T$. From Proposition 3.20,

$$d(T) \leq \sum_{i=n}^{n+p-1} d(\eta^i(\gamma_{x,y})) \leq \sum_{i=n}^{n+p-1} (\min_j \alpha_j)^{-i} d(\gamma_{x,y}) = 0.$$

This is a contradiction to the fact $d(T) = 1$. Therefore $d(x, y) > 0$ if $x \neq y$.

Let $x \in T$, and let $\{y_1, y_2, \dots\}$ be a sequence in T such that $d(x, y_k) \to 0$ as $k \to \infty$. Then $\{y_k\}$ converges to x. Indeed, we can assume that there exists a simple path γ such that x is one of its endpoints and that $\{y_1, y_2, \dots\} \subset \gamma$. If $\gamma_{x,y_k} \subset \gamma_{x,y_{k'}}$, then $d(x, y_k) \leq d(x, y_{k'})$. Thus we can assume that $\gamma_{x,y_1} \supset \gamma_{x,y_2} \supset \cdots$. We conclude $\bigcap_{k=1}^{\infty} \gamma_{x,y_k} = \{x\}$ from the fact that $d(x, y) > 0$ if $x \neq y$.

Let $x \in T$, and let $\{y_1, y_2, \dots\}$ be a sequence in T which converges to x. We will show that $d(x, y_k) \to 0$ as $k \to \infty$. We can assume that $\gamma_{x,y_1} \supset \gamma_{x,y_2} \supset \cdots$ and $d(x, y_1) \geq d(x, y_2) \geq \cdots$. Assume there exists a positive number δ such that $d(x, y_k) > \delta$ for every k. Let n be a positive integer such that $(\max_i \alpha_i)^n < \delta$. Then there exists k such that $\gamma_{x,y_k} \cap C(n) = \emptyset$. From (3) of Proposition 3.20, we have

$$(\max_i \alpha_i)^n d(\eta^n(x, \gamma_{x,y_k})) \geq d(\gamma_{x,y_k}) > \delta.$$

Therefore

$$1 < (\max_i \alpha_i)^{-n} \delta < d(\eta^n(x, \gamma_{x,y_k})) \leq d(T) = 1,$$

and this is a contradiction. □

To sum up, we have proved the following proposition. Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system satisfying Condition A. Let T_j $(j = 1, 2, \dots, m)$ be the minimal trees in Condition A. We denote, by $T_j^1, T_j^2, \dots, T_j^{q_j}$, the component of T_j. Let $(\alpha_1^j, \alpha_2^j, \dots, \alpha_n^j)$ be a polyradius of convergence of $v(T_j)$.

Proposition 3.22. *There exists a function d on $\bigcup_{j=1}^m \bigcup_{r=1}^{q_j} (T_j^r \times T_j^r)$ which is a distance on each T_j^r such that for any j, if two points x, y belong to T_j^r and if $\gamma_{x,y}$ contains no essential critical point, then $\alpha_i^j d(g(x, \gamma_{x,y}), g(y, \gamma_{x,y})) = d(x, y)$ for some $i \in \{1, 2, \dots, N\}$.*

The next step is to show the following. Suppose that a polyratio $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfies $\alpha_i^j \leq \alpha_i$ for any i and any j.

Lemma 3.23. *There exists a positive integer β which satisfies the following:*

(1) Let $a, b \in T_k^r$ such that $\gamma_{a,b}$ contains an essential critical point. If $\mathcal{C} = (U_1, U_2, \dots, U_l)$ is a chain between a and b such that $K(U_{i-1}) \cap K(U_i) \cap T_k^r = \emptyset$ for $i = 2, 3, \dots, l$, then $\min_i |U_i| \leq \beta$.

(2) Let $a, b \in T_k^r$. If $\mathcal{C} = (U_1, U_2, \dots, U_l)$ is a chain between a and b such that $K(U_{i-1}) \cap K(U_i) \cap T_k^r = \emptyset$ for $i = 2, 3, \dots, l$, and $K(U_{i-1}) \cap K(U_i) \cap C \neq \emptyset$ for some i, then $\min_i |U_i| \leq \beta$.

Consequently, if two points $a, b \in T_k^r$ and a chain $\mathcal{C}$ satisfy either (1) or (2), then $d(a, b) \leq A(\mathcal{C})/(\min_i \alpha_i)^{\beta}$.

Proof. Let

$$n_0 = \max\{n \mid \text{for any } a, b \text{ as (1)}, a, b \in K(U) \text{ for some } U \in \mathcal{W}_n\}.$$

If $\mathcal{C}$ is a chain as (1) such that $l \geq 2$, then there exists i such that $K(U_{i-1}) \cap K(U_i) \cap (\tilde{C}(n_0+1) - T_k^r) \neq \emptyset$ by Lemma 2.11. Note that $\tilde{C}(n_0+1) - T_k^r$ is a finite set. Therefore, if $\min_i |U_i|$ is not bounded, then using the same argument as Proposition 1.29 we obtain a connected set X containing a, b and a point in $\tilde{C}(n_0+1) - T_k^r$. This is a contradiction.

Suppose two points $a, b \in T_k^r$ and a chain $\mathcal{C}$ satisfy either (1) or (2). Let U_i be a word such that $|U_i| \leq \beta$. Then $d(a,b) \leq 1 \leq A(U_i)/(\min_j \alpha_j)^\beta \leq A(\mathcal{C})/(\min_j \alpha_j)^\beta$. □

Proposition 3.24. *α is a metric polyratio.*

Proof. Let $x \neq y \in K$. If x, y are contained in distinct connected components, then $D(x,y) > 0$. Suppose that they are contained in the same components and $D(x,y) = 0$. Then for any $\epsilon > 0$, there exists a chain $\mathcal{C}_\epsilon = (U_1^\epsilon, U_2^\epsilon, \ldots, U_{l_\epsilon}^\epsilon)$ between x and y such that $A(\mathcal{C}_\epsilon) < \epsilon$. By the same discussion as Proposition 1.29, we see that $X = \bigcap_{\epsilon>0} \overline{\bigcup_{\epsilon'<\epsilon} \bigcup_{k=1}^{l_{\epsilon'}} K(U_k^{\epsilon'})}$ is connected. Thus $\gamma_{x,y} \subset X$, and so $D(a,b) = 0$ for any two points $a, b \in \gamma_{a,b}$. By (2) of Condition A, the path $\gamma_{x,y}$ includes a subpath γ such that $\gamma \in K(U)$ and $g^n(\gamma) \subset T_j$ for some n, some $U \in \mathcal{W}_n$ and some j. Therefore it is easy to see that there exists $x', y' \in T_j$ such that $D(x', y') = 0$.

It suffices to show that $D(x,y) > 0$ if $x \neq y \in T_j^r$. Let $\mathcal{C} = (U_1, U_2, \ldots, U_l)$ be a chain between x and y. We set

$$\{i_1, i_2, \ldots, i_t\} = \{i \mid K(U_{i-1}) \cap K(U_i) \cap T_j^r \neq \emptyset\},$$

$i_0 = 0, i_{t+1} = l$. Choose $x_0 = x, x_i \in K(U_i) \cap K(U_{i+1}), x_l = y$ such that $x_i \in T_j^r$ if $i \in \{i_1, i_2, \ldots, i_t\}$. Let $0 \leq k \leq t$. Then there exists $n \geq 0$ such that $\mathrm{int} g^j(\gamma_{x_{i_k}, x_{i_{k+1}}})$ contains no essential critical point $(j = 0, 1, \ldots, n-1)$, $P^j(x_i) \cap C = \emptyset$ $(j = 0, 1, \ldots, n-1, i_k < i < i_{k+1})$ and either $\mathrm{int} g^n(\gamma_{x_{i_k}, x_{i_{k+1}}})$ contains an essential critical point or $P^n(x_i) \cap C \neq \emptyset$ for some $i_k < i < i_{k+1}$. Then there exists a word $V_k \in \mathcal{W}_n$ such that $U_i \prec V_k$ $(i_k < i \leq i_{k+1})$. Thus by Proposition 3.22 and Lemma 3.23,

$$\begin{aligned} d(x_{i_k}, x_{i_{k+1}}) &\leq A(V_k) d(g^n(x_{i_k}, \gamma_{x_{i_k}, x_{i_{k+1}}}), g^n(x_{i_{k+1}}, \gamma_{x_{i_k}, x_{i_{k+1}}})) \\ &\leq A(V_k) \sum_{i=i_k+1}^{i_{k+1}} A(\sigma^n(U_i))/(\min_j \alpha_j)^\beta \\ &= \sum_{i=i_k+1}^{i_{k+1}} A(U_i)/(\min_j \alpha_j)^\beta. \end{aligned}$$

Therefore

$$0 < d(x,y) \leq \sum_{k=0}^{t} d(x_{i_k}, x_{i_{k+1}}) \leq \sum_{k=0}^{t} \sum_{i=i_k+1}^{i_{k+1}} A(U_i)/(\min_j \alpha_j)^\beta = A(\mathcal{C})/(\min_j \alpha_j)^\beta.$$

Hence $A(\mathcal{C}) > 0$. □

To complete the proof of Theorem 3.17, we have to only show Lemma 3.19. We consider the function

$$\frac{v(\gamma)(\alpha t)}{v(T)(\alpha t)}$$

as a function of complex variable. Then it will be proved to be holomorphic.

The proof will be done by using kneading determinants. In [16], Milnor and Thurston have introduced a holomorphic function of one variable, called a kneading determinant, which is defined by the kneading sequence of an interval dynamics. In our case we extend it as a function of several variables. Although our kneading determinant is more complicated than the original one, the proof is almost parallel to that of Milnor-Thurston. There is no essential difference.

An interval naturally has a linearly order, which makes the kneading theory on the interval successful, but a tree is not so. Our new idea to settle the difficulty is the following: Considering all subintervals in the tree, we can treat the tree dynamics as a system of interval dynamics. On every interval of the system a linearly order is independently defined.

Furthermore, we will prove

Theorem 3.25. *Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system satisfying Condition A. Let $T_1, T_2, \ldots, T_m$ be minimal trees which satisfy Condition A. Then there exist analytic functions $\Delta_{T_1}, \Delta_{T_2}, \ldots, \Delta_{T_m}$ on $\mathbf{Ra}_N = \{(X_1, X_2, \ldots, X_N) \in \mathbf{R}^N \,|\, 0 < X_i < 1\}$ such that the set of metric polyratios is equal to the set of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbf{Ra}_N$ which satisfies the condition that for each $i = 1, 2, \ldots, m$ there exists $\beta(i) = (\beta_1(i), \beta_2(i), \ldots, \beta_N(i))$ in $\mathbf{Ra}_N$ such that $\beta_k(i) \leq \alpha_k$ $(k = 1, 2, \ldots, N)$ and $\Delta_{T_i}(\beta(i)) = 0$.*

3.3. Kneading determinants

Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system. In this subsection we will prove Lemma 3.19.

3.3.1. Orientations

Notation 3.26. Let γ, γ' be simple paths in K. We say

$$\gamma < \gamma'$$

if the image of γ' includes that of γ, and ${\gamma'}^{-1} \circ \gamma : [0,1] \to [0,1]$ is orientation-preserving. For a simple path γ, we define a simple path $-\gamma : [0,1] \to K$ as

$$(-\gamma)(t) = \gamma(1-t).$$

If $\xi = (a, \gamma) \in \Xi_0$, we write

$$-\xi = (a, -\gamma),\ -(\xi^{\pm}) = (a^{\mp}, -\gamma),\ (-\xi)^{\pm} = (a^{\pm}, -\gamma).$$

The following is easy.

Proposition 3.27. (1) Let $\xi \in \Xi_0$. Then

$$g^k(\xi^{\pm}) = g^k(-(\xi^{\pm})),\ \eta^k(\xi^{\pm}) = -\eta^k(-(\xi^{\pm})),\ Y_k(\xi^{\pm}) = Y_k(-(\xi^{\pm})).$$

(2) Let γ, γ' be a simple path in K satisfying $\gamma < \gamma'$, and let $a \in \gamma$. Then

$$-\gamma < -\gamma'$$

and

$$g^k(a^{\pm}, \gamma) = g^k(a^{\pm}, \gamma'),\ \eta^k(a^{\pm}, \gamma) < \eta^k(a^{\pm}, \gamma'),\ Y_k(a^{\pm}, \gamma) = Y_k(a^{\pm}, \gamma').$$

Definition 3.28. Let T be an invariant tree of $(K, \{F_i\}_{i=1}^N)$. Note that we have a natural one-to-one correspondence between

$$\begin{aligned} Q_1 &= Q_1(T) \\ &= \{(x, y) \in T \times T \mid x \neq y,\ x \text{ and } y \text{ belongs to the same component of } T\} \end{aligned}$$

and

$$\{\text{a simple path in } T\}$$

by identifying (x, y) with $\gamma_{x,y}$. (Precisely, $(x, y) \in Q_1$ is identified with the equivalence class including $\gamma_{x,y}$.) For (x, y) and (x', y') in Q_1, we say $(x, y) < (x', y')$ if $\gamma_{x,y} < \gamma_{x',y'}$. For $(x, y) \in Q_1$, we denote $-(x, y) = (y, x)$.

First we define the finite set Q' to be

$$Q' = \{(x, y) \in Q_1 \mid\ x \text{ and } y \text{ are endpoints of } T\}.$$

There exists a mapping

$$\chi : Q_1 \to Q'$$

which satisfies the conditions that $\chi(y, x) = -\chi(x, y)$, $(x, y) < \chi(x, y)$, and the restriction $\chi|Q'$ is the identity. We fix such a function χ. Then we fix a function $o : Q' \to \{-1, 1\}$ satisfying $o(x, y) = -o(y, x)$, and we obtain the sets

$$Q = Q(T) = \{(x, y) \in Q_1 \mid o(\chi(x, y)) = 1\}$$

and

$$Q^* = Q^*(T) = \{(x, y) \in Q' \mid o(x, y) = 1\}.$$

The function o, said to be an *orientation* on T, is extended on Q by $o(x, y) = o(\chi(x, y))$.

We use the notation

$$\Xi = \{(a, \gamma) \in \Xi_0 \mid \gamma \in Q\}.$$

Example 3.29. (1) Let $(K, \{F_1, F_2\})$ be the self-similar system of Example 1.8-(2). The unit interval $K = [0, 1]$ is a minimal tree. Since K has two endpoints 0 and 1, we see that $Q' = \{\gamma, -\gamma\}$, where $\gamma : [0, 1] \to K$ is defined by $\gamma(t) = t$. Setting $o(\gamma) = 1, o(-\gamma) = -1$, we have $Q^* = \{\gamma\}$. The mapping $\chi : Q_1 \to Q'$ is necessarily defined by $\chi(l) = \gamma$ if $l < \gamma$, $\chi(l) = -\gamma$ if $l < -\gamma$.

(2) Let $(K, \{F_1, F_2\})$ be the self-similar system of Example 1.8-(6). Recall that it has the critical set $C = \{c\}$ and the postcritical set $P = \{p_1, p_2, p_3\}$ such that $F_1(p_1) = F_2(p_1) = c, F_1(p_2) = p_1, F_1(p_3) = p_2, F_2(p_2) = p_3$. There exists a minimal tree T in K. The tree T, which is Y-figured, has three endpoints p_1, p_2 and p_3. (Remark that T has a branch point p, and the critical point c is contained in the simple path γ' between p_3 and p. See Figure 5.) Thus Q' has six members, and Q^* has three members. Set $Q^* = \{\gamma_1, \gamma_2, \gamma_3\}$, where γ_1 is a simple path between p_3 and p_1; γ_2 between p_3 and p_2; γ_3 between p_1 and p_2. There are several possibilities for the mapping χ. We choose χ as follows. Let $l : [0, 1] \to T$ be a simple path. If $\#\{\gamma \in Q' \,|\, l < \gamma\} = 1$, then $\chi(l)$ is uniquely determined. We set $\chi(l) = \pm\gamma_1$ if $l < \pm\gamma_1$ and $l < \mp\gamma_3$; $\chi(l) = \pm\gamma_1$ if $l < \pm\gamma_1$ and $l < \pm\gamma_2$; $\chi(l) = \pm\gamma_2$ if $l < \pm\gamma_2$ and $l < \pm\gamma_3$.

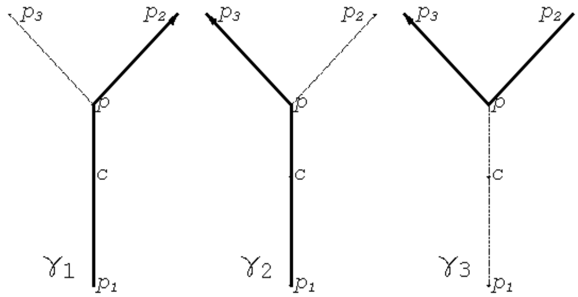

Figure 5: The minimal tree T and the curves $\gamma_1, \gamma_2, \gamma_3$

Definition 3.30. Let $\xi \in \Xi$. Then either $\eta(\xi^\star) \in Q$ or $-\eta(\xi^\star) \in Q$ for each $\star = -, +$. We write

$$\tilde{\eta}(\xi^\pm) = \begin{cases} \eta(\xi^\pm) & \text{if} \quad o(\eta(\xi^\pm)) = 1, \\ -\eta(\xi^\pm) & \text{if} \quad o(\eta(\xi^\pm)) = -1. \end{cases}$$

Namely,

$$\tilde{\eta}(\xi^\pm) = o(\eta(\xi^\pm))\, \eta(\xi^\pm). \tag{3.7}$$

For $\xi = (a, \gamma) \in \Xi$, we inductively define

$$\eta_*^0(\xi^\pm) = \chi(\gamma),\ e_0(\xi^\pm) = 1,\ \tilde{g}^0(\xi^\pm) = a^\pm,$$

and for $k = 1, 2, \ldots,$

$$\begin{array}{rcl} \eta_*^k(\xi^\pm) & = & \chi(\tilde{\eta}(\tilde{\mu}^{k-1}(\xi^\pm))), \\ e_k(\xi^\pm) & = & \displaystyle\prod_{m=0}^{k-1} o(\eta(\tilde{\mu}^m(\xi^\pm))) \in \{-1, 1\}, \\ \mu_*^k(\xi^\pm) & = & (g^k(\xi^\pm), \eta_*^k(\xi^\pm)), \\ \tilde{g}^k(\xi^\pm) & = & \begin{cases} g^k(\xi^\pm)^\pm & \text{if} \quad e_k(\xi^\pm) = 1, \\ g^k(\xi^\pm)^\mp & \text{if} \quad e_k(\xi^\pm) = -1, \end{cases} \\ \tilde{\mu}^k(\xi^\pm) & = & (\tilde{g}(\xi^\pm), \eta_*^k(\xi^\pm)). \end{array}$$

We say $\mu_*^k(\xi^\pm)$ is the k-th *successor* of $\xi^\pm$, $\eta_*^k(\xi^\pm)$ the *path component* of the k-th successor of $\xi^\pm$, and $e_k(\xi^\pm)$ the k-th *sign* of $\xi^\pm$.

From (3.2) and the definition of e_k,

$$\eta_*^{k+m}(\xi^\pm) = \eta_*^k(\tilde{\mu}^m(\xi^\pm)), \tag{3.8}$$

$$e_{k+m}(\xi^\pm) = e_m(\xi^\pm)\, e_k(\tilde{\mu}^m(\xi^\pm)). \tag{3.9}$$

Proposition 3.31. *Let $\xi \in \Xi$. Then*

$$\begin{array}{rl} \eta^k(\xi^\pm) < \eta_*^k(\xi^\pm) & \text{if } e_k(\xi^\pm) = 1, \\ -\eta^k(\xi^\pm) < \eta_*^k(\xi^\pm) & \text{if } e_k(\xi^\pm) = -1. \end{array}$$

Proof. We will prove the assertion by induction. Let $\xi = (a, \gamma)$. If $k = 0$, then $\eta^0(\xi^\pm) = \gamma$ and $\eta_*^0(\xi^\pm) = \chi(\gamma)$. We suppose the assertion is true when $k = n$. Then

$$e_n(\xi^\pm)\eta^n(\xi^\pm) < \eta_*^n(\xi^\pm).$$

By Proposition 3.27 and (3.7),

$$\begin{array}{rcl} e_{n+1}(\xi^\pm)\eta^{n+1}(\xi^\pm) & = & o(\eta(\tilde{\mu}^n(\xi^\pm)))e_n(\xi^\pm)\eta(\mu^n(\xi^\pm)^\pm) \\ & = & o(\eta(\tilde{\mu}^n(\xi^\pm)))\eta(\tilde{g}^n(\xi^\pm), e_n(\xi^\pm)\eta^n(\xi^\pm)) \\ & < & o(\eta(\tilde{\mu}^n(\xi^\pm)))\eta(\tilde{g}^n(\xi^\pm), \eta_*^n(\xi^\pm)) \\ & = & o(\eta(\tilde{\mu}^n(\xi^\pm)))^2\, \tilde{\eta}(\tilde{\mu}^n(\xi^\pm)) \\ & = & \tilde{\eta}(\tilde{\mu}^n(\xi^\pm)) \\ & < & \eta_*^{n+1}(\xi^\pm). \end{array}$$

This completes the proof. □

Corollary 3.32. *Let $\xi \in \Xi$. Then*

$$Y_{k+m}(\xi^\pm) = Y_m(\tilde{\mu}^k(\xi^\pm)) \quad \textit{and} \quad \mathcal{Y}_{k+m-1}(\xi^\pm) = \mathcal{Y}_{k-1}(\xi^\pm)\, \mathcal{Y}_{m-1}(\tilde{\mu}^k(\xi^\pm))$$

for $k = 0, 1, \ldots$ and $m = 0, 1, \ldots$.

The element of

$$\Pi^* = \{(\mathbf{I}_1(a, \gamma), \gamma) \mid \gamma \in Q^*, a \in \gamma\}$$

is called an *extended subinterval.* For $\xi \in \Xi$, we define for $k = 0, 1, \dots$

$$J_k(\xi^{\pm}) = (\mathbf{I}_1(\tilde{\mu}^k(\xi^{\pm})), \quad \eta_*^k(\xi^{\pm})) \in \Pi^*.$$

We say J_k is the *extended address* of the k-th successor of $\xi^{\pm}$. From (3.8) we have

$$J_{k+m}(\xi^{\pm}) = J_k(\tilde{\mu}^m(\xi^{\pm})). \tag{3.10}$$

If $\rho = (\mathbf{I}_1(a,\gamma),\gamma)$, then we write

$$I(\rho) = \mathbf{I}_1(a,\gamma), \ \gamma(\rho) = \gamma.$$

There uniquely exists $Y(\rho) \in \{1, 2, \dots, N\}$ such that $I(\rho) \subset K(Y(\rho))$. It is clear $Y(\rho) = Y(a, \gamma(\rho))$ for $a \in \mathrm{int} I(\rho)$. We write

$$\eta(\rho) = \eta(a,\gamma(\rho)), \ \eta_*(\rho) = \eta_*^1(a,\gamma(\rho)), \ e(\rho) = e_1(a,\gamma(\rho))$$

where $a \in \mathrm{int} I(\rho)$. This is independent of a.

Example 3.33. This is continued from Example 3.29.

(1) Consider the self-similar system $(K, \{F_1, F_2\})$ of Example 1.8-(2). If $0 \leq a < 1/2$, then $\mathbf{I}_1(a,\gamma) = [0, 1/2]$; if $1/2 < a \leq 1$, then $\mathbf{I}_1(a,\gamma) = [1/2, 1]$. Thus $\Pi^* = \{I_1, I_2\}$, where $I_1 = ([0, 1/2], \gamma), I_2 = ([1/2, 1], \gamma)$.

Let us calculate $e_k(c^{\pm},\gamma)$ and $J_k(c^{\pm},\gamma)$ for the critical point $c = 1/2$. We write $p_1 = 0$ and $p_2 = 1$. Since $\mu^k(c^-,\gamma) = (p_2,\gamma)$ for $k = 1, 2, \dots$ and $\mu^k(c^+,\gamma) = (p_1,\gamma)$ for $k = 1, 2, \dots$, we have $e_k(c^{\pm},\gamma) = 1$ for $k = 0, 1, \dots$ and

$$\begin{array}{ll} J_0(c^-,\gamma) = I_1, \quad J_k(c^-,\gamma) = I_2 & (k = 1, 2, \dots) \\ J_0(c^+,\gamma) = I_2, \quad J_k(c^+,\gamma) = I_1 & (k = 1, 2, \dots). \end{array}$$

For convenience, we write $e_\infty(c^{\pm},\gamma) = (\overline{1}) = (1, 1, \dots)$ and

$$J_\infty(c^-,\gamma) = (I_1, \overline{I_2}), \quad J_\infty(c^+,\gamma) = (I_2, \overline{I_1}).$$

(2) Consider the self-similar system $(K, \{F_1, F_2\})$ of Example 1.8-(6). We denote by L_1 the simple path between p_3 and c, by L_2 the simple path between c and p_1, by L_3 the simple path between c and p_2, by L_4 the simple path between p_1 and p_2. We consider L_i's as sets. Remark that L_4 is the image of γ_3. It is easy to see that $\Pi^* = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, where $\rho_1 = (L_1,\gamma_1), \rho_1 = (L_2,\gamma_1), \rho_1 = (L_1,\gamma_2), \rho_1 = (L_3,\gamma_2), \rho_1 = (L_4,\gamma_3)$.

Let us calculate e_k and J_k for the critical point c. Since

$$\begin{array}{lll} \mu^1(c^-,\gamma_1) = (p_1,-\gamma_3), & \mu^1(p_1,\gamma_3) = (p_2,-\gamma_2), & \mu^1(p_2,\gamma_2) = (p_3,-\gamma_1), \\ \mu^1(p_3,\gamma_1) = (p_2,-\gamma_3), & \mu^1(p_2,\gamma_3) = (p_3,-\gamma_2), & \mu^1(p_3,\gamma_2) = (p_2,-\gamma_3), \end{array}$$

we have

$$\begin{array}{rcl} e_\infty(c^-,\gamma_1) & = & (\overline{(1,-1)}) = (1,-1,1,-1,1,-1,1,\dots), \\ J_\infty(c^-,\gamma_1) & = & (\rho_1,\rho_5,\rho_4,\rho_1,\overline{(\rho_5,\rho_3)}). \end{array}$$

Similarly,

$$\begin{aligned} e_\infty(c^+,\gamma_1) &= (1,\overline{(1,-1)}), & J_\infty(c^+,\gamma_1) &= (\rho_2,\rho_5,\rho_4,\rho_1,\overline{(\rho_5,\rho_3)}), \\ e_\infty(c^-,\gamma_2) &= (\overline{(1,-1)}), & J_\infty(c^-,\gamma_2) &= (\rho_3,\rho_5,\rho_4,\rho_1,\overline{(\rho_5,\rho_3)}), \\ e_\infty(c^+,\gamma_2) &= (1,-1,\overline{(-1,1)}), & J_\infty(c^+,\gamma_2) &= (\rho_4,\rho_2,\overline{(\rho_5,\rho_3)}). \end{aligned}$$

3.3.2. Formal kneading matrices

Considering $\mathcal{W}_*$ as a monoid, we denote, by $\mathcal{R}_\infty$, the ring of formal infinite sums of $\mathcal{W}_*$ over $\mathbb{Z}$. Namely, $\mathcal{R}_\infty$ is the set of all functions $f:\mathcal{W}_*\to\mathbb{Z}$. For $f,f'\in\mathcal{R}_\infty$, the sum $f+f'$ is defined as $(f+f')(U)=f(U)+f'(U)$ and the product ff' is defined as $(ff')(U)=\sum_{VV'=U}f(V)f'(V')$. We may consider $\mathcal{W}_*$ as a subset of $\mathcal{R}_\infty$, that is, $U\in\mathcal{W}_*$ is considered as the mapping f_U which satisfies $f_U(U)=1$ and $f_U(V)=0$ if $U\neq V$. We set

$$\mathcal{R}_k=\{f\in\mathcal{R}_\infty\mid f(U)=0 \text{ if } |U|\neq k\}.$$

For $f\in\mathcal{R}_\infty$, we define $(f)_k\in\mathcal{R}_k$ as

$$(f)_k(U)=\begin{cases} f(U) & \text{if } |U|=k, \\ 0 & \text{otherwise.}\end{cases}$$

If $f_1,f_2,\ldots$ are elements of $\mathcal{R}$ such that $\#\{i\mid (f_i)_k\neq 0\}<\infty$ for each k, then $\sum_{i=1}^\infty f_i\in\mathcal{R}_\infty$ is naturally defined. Thus $f=\sum_{k=0}^\infty (f)_k$. If $a_U=f(U)$, then the element f is usually written in the form

$$f=\sum_{U\in\mathcal{W}_*}a_U U,$$

It is clear that $(f)_k=(f')_k$ for all k if and only if $f=f'$. Remark that the unit element for addition is 0 and the unit element for multiplication is identified with $\emptyset\in\mathcal{W}_0$:

$$0(U)=0 \text{ for any } U\in\mathcal{W}_*,$$

$$\emptyset(U)=\begin{cases} 1 & \text{if } U=\emptyset, \\ 0 & \text{if } U\neq\emptyset.\end{cases}$$

Definition 3.34. Let $\xi\in\Xi$ and $\rho\in\Pi^*$. For $k=0,1,\ldots$, we define an element of $\mathcal{R}_k$

$$\Theta_k^\rho(\xi^\pm)=\begin{cases} e_k(\xi^\pm)\,\mathcal{Y}_{k-1}(\xi^\pm) & \text{if } J_k(\xi^\pm)=\rho \\ 0 & \text{otherwise,}\end{cases}$$

where $\mathcal{Y}_{-1}(\xi^\pm)=\emptyset$, and we define a formal infinite sum

$$\Theta^\rho(\xi^\pm)=\sum_{k=0}^\infty \Theta_k^\rho(\xi^\pm).$$

Proposition 3.35. *Let $\xi \in \Xi$ and $\rho \in \Pi^*$. Then*

$$\Theta^{\rho}_{k+m-1}(\xi^{\pm}) = e_k(\xi^{\pm})\,\mathcal{Y}_{k-1}(\xi^{\pm})\,\Theta^{\rho}_{m-1}(\tilde{\mu}^k(\xi^{\pm})). \tag{3.11}$$

for $k = 0, 1, \dots$ and $m = 1, 2, \dots$.

Proof. From Corollary 3.32,

$$\begin{aligned}\Theta^{\rho}_{k+m-1}(\xi^{\pm}) &= \begin{cases} e_{k+m-1}(\xi^{\pm})\,\mathcal{Y}_{k+m-2}(\xi^{\pm}) & \text{if } J_{k+m-1}(\xi^{\pm}) = \rho \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} e_{k+m-1}(\xi^{\pm})\,\mathcal{Y}_{k-1}(\xi^{\pm})\,\mathcal{Y}_{m-2}(\tilde{\mu}^k(\xi^{\pm})) & \text{if } J_{k+m-1}(\xi^{\pm}) = \rho \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

On the other hand, by (3.10),

$$\Theta^{\rho}_{m-1}(\tilde{\mu}^k(\xi^{\pm})) \quad = \quad \begin{cases} e_{m-1}(\tilde{\mu}^k(\xi^{\pm}))\,\mathcal{Y}_{m-2}(\tilde{\mu}^k(\xi^{\pm})) & \text{if } J_{k+m-1}(\xi^{\pm}) = \rho \\ 0 & \text{otherwise.} \end{cases}$$

From (3.9), we have

$$e_{k+m-1}(\xi^{\pm}) = e_k(\xi^{\pm})\,e_{m-1}(\tilde{\mu}^k(\xi^{\pm})).$$

Thus

$$e_k(\xi^{\pm}) = e_{k+m-1}(\xi^{\pm})\,e_{m-1}(\tilde{\mu}^k(\xi^{\pm})),$$

and we obtain (3.11). □

The following is an immediate consequence.

Corollary 3.36. *Let $\xi \in \Xi$ and $\rho \in \Pi^*$. Then*

$$\Theta^{\rho}(\xi^{\pm}) = \sum_{j=0}^{k-1} \Theta^{\rho}_j(\xi^{\pm}) + e_k(\xi^{\pm})\,\mathcal{Y}_{k-1}(\xi^{\pm})\,\Theta^{\rho}(\tilde{\mu}^k(\xi^{\pm}))$$

for $k = 1, 2, \dots$.

Lemma 3.37. *Let $s : Q^* \to \mathbb{Z}$ be an arbitrary function. For $\rho \in \Pi^*$, we define m_s and n_s as*

$$m_s(\rho) = s(\gamma(\rho)) \text{ and } n_s(\rho) = s(\eta_*(\rho)).$$

Let $\xi \in \Xi$. Then for every $\rho \in \Pi^*$,

$$\sum_{\rho \in \Pi^*} \Theta^{\rho}(\xi^{\pm})\,(m_s(\rho)\emptyset - e(\rho)n_s(\rho)Y(\rho)) = s(\gamma)\emptyset.$$

Proof. Since $\emptyset \in \mathcal{W}_0$ and $Y(\rho) \in \mathcal{W}_1$, we have

$$\begin{aligned}&\big(\Theta^\rho(\xi^\pm)(m_s(\rho)\,\emptyset - e(\rho)\,n_s(\rho)\,Y(\rho))\big)_k\\&\quad= \begin{cases} m_s(\rho)\,\Theta^\rho_0(\xi^\pm) & \text{if } k=0\\ m_s(\rho)\,\Theta^\rho_k(\xi^\pm) - e(\rho)\,n_s(\rho)\,\Theta^\rho_{k-1}(\xi^\pm)\,Y(\rho) & \text{if } k \geq 1\end{cases}.\end{aligned}$$

Thus

$$\begin{aligned}\left(\sum_{\rho\in\Pi^*}\Theta^\rho(\xi^\pm)(m_s(\rho)\,\emptyset - e(\rho)\,n_s(\rho)\,Y(\rho))\right)_0 &= m_s(J_0(\xi^\pm))\,\Theta_0^{J_0(\xi^\pm)}(\xi^\pm)\\ &= s(\chi(\gamma))\emptyset.\end{aligned}$$

If $k \geq 1$,

$$\begin{aligned}&\left(\sum_{\rho\in\Pi^*}\Theta^\rho(\xi^\pm)(m_s(\rho)\,\emptyset - e(\rho)\,n_s(\rho)\,Y(\rho))\right)_k\\ &\qquad= m_s(J_k(\xi^\pm))\,\Theta_k^{J_k(\xi^\pm)}(\xi^\pm)\\ &\qquad\quad - e(J_{k-1}(\xi^\pm))n_s(J_{k-1}(\xi^\pm))\,\Theta_{k-1}^{J_{k-1}(\xi^\pm)}(\xi^\pm)\,Y(J_{k-1}(\xi^\pm))\\ &\qquad= e_k(\xi^\pm)\,s(\eta_*^k(\xi^\pm))\,\mathcal{Y}_{k-1}(\xi^\pm)\\ &\qquad\quad - e_1(\tilde{\mu}^{k-1}(\xi^\pm))s(\eta_*^k(\xi^\pm))\,e_{k-1}(\xi^\pm)\mathcal{Y}_{k-2}(\xi^\pm)\,Y_{k-1}(\xi^\pm)\\ &\qquad= 0.\end{aligned}$$

This complete the proof. $\square$

Definition 3.38. We set

$$C_e^* = C_e^*(T) = \{(c,\gamma) \in \Xi \mid \gamma \in Q^*, c \text{ is an essential critical point of } \gamma\}.$$

An element of C_e^* is referred by a symbol ϕ.

For $\phi = (c,\gamma) \in C_e^*$ and $\rho \in \Pi^*$, we define

$$M_{\phi\rho} = \Theta^\rho(c^+,\gamma) - \Theta^\rho(c^-,\gamma).$$

We say $(M_{\phi\rho})_{\phi\in C_e^*,\rho\in\Pi^*}$ is the *formal kneading matrix* of T.

Corollary 3.39. *Let $\phi \in C_e^*$ and $\gamma \in Q^*$. Then*

$$\sum_{\rho\in\Pi^*} M_{\phi\rho}(h^0_\gamma(\rho)\emptyset - e(\rho)h^1_\gamma(\rho)Y(\rho)) = 0,$$

where we set

$$h^0_\gamma(\rho) = \begin{cases} 1 & \textit{if } \gamma(\rho)=\gamma\\ 0 & \textit{if } \gamma(\rho)\neq\gamma\end{cases} \quad \textit{and} \quad h^1_\gamma(\rho) = \begin{cases} 1 & \textit{if } \eta^1_*(\rho)=\gamma\\ 0 & \textit{if } \eta^1_*(\rho)\neq\gamma\end{cases}.$$

Proof. When we consider the function $s : Q^* \to \mathbb{Z}$ defined by $s(\gamma') = \begin{cases} 1 & \text{if } \gamma' = \gamma \\ 0 & \text{if } \gamma' \neq \gamma \end{cases}$, the functions m_s and n_s defined in Lemma 3.37 are equal to h^0_γ and h^1_γ respectively. Thus if $\phi = (c, \delta)$, then

$$\sum_{\rho \in \Pi^*} M_{\phi\rho}(h^0(\rho)\emptyset - e(\rho)h^1(\rho)Y(\rho)) = s(\delta)\emptyset - s(\delta)\emptyset = 0.$$

□

Example 3.40. This is continued from Example 3.33. Let us calculate the formal kneading matrix $(M_{\phi\rho})_{\phi,\rho}$.

(1) Set $\phi = (c, \gamma)$. Then we have $C^*_e = \{\phi\}$. Since

$$\begin{aligned} &\Theta^{I_1}(\phi^-) = \emptyset, && \Theta^{I_1}(\phi^+) = \mathbf{2} + \mathbf{21} + \mathbf{21}^2 + \cdots, \\ &\Theta^{I_2}(\phi^-) = \mathbf{1} + \mathbf{12} + \mathbf{12}^2 + \cdots, && \Theta^{I_2}(\phi^+) = \emptyset, \end{aligned}$$

the formal kneading matrix is given by

$$M_{\phi I_1} = -\emptyset + \mathbf{2}\sum_{k=0}^{\infty} \mathbf{1}^k, \quad M_{\phi I_2} = \emptyset - \mathbf{1}\sum_{k=0}^{\infty} \mathbf{2}^k.$$

(2) We have $C^+_e = \{\phi_1, \phi_2\}$, where $\phi_1 = (c, \gamma_1), \phi_2 = (c, \gamma_2)$. Since

$$\begin{aligned} &\Theta^{\rho_1}(\phi_1^-) = \emptyset - \mathbf{211}, \quad \Theta^{\rho_1}(\phi_1^+) = \mathbf{111}, \\ &\Theta^{\rho_2}(\phi_1^-) = 0, \quad \Theta^{\rho_2}(\phi_1^+) = \emptyset, \\ &\Theta^{\rho_3}(\phi_1^-) = -\mathbf{21121} - \mathbf{2112121} - \cdots, \\ &\Theta^{\rho_3}(\phi_1^+) = \mathbf{11121} + \mathbf{1112121} + \cdots, \\ &\Theta^{\rho_4}(\phi_1^-) = \mathbf{21}, \quad \Theta^{\rho_4}(\phi_1^+) = -\mathbf{11}, \\ &\Theta^{\rho_5}(\phi_1^-) = -\mathbf{2} + \mathbf{2112} + \mathbf{211212} + \cdots, \\ &\Theta^{\rho_5}(\phi_1^+) = \mathbf{1} - \mathbf{1112} - \mathbf{111212} - \cdots, \end{aligned}$$

$$\begin{aligned} &\Theta^{\rho_1}(\phi_2^-) = -\mathbf{211}, \quad \Theta^{\rho_1}(\phi_2^+) = 0, \\ &\Theta^{\rho_2}(\phi_2^-) = 0, \quad \Theta^{\rho_2}(\phi_2^+) = -\mathbf{1}, \\ &\Theta^{\rho_3}(\phi_2^-) = \emptyset - \mathbf{21121} - \mathbf{2112121} - \cdots, \\ &\Theta^{\rho_3}(\phi_2^+) = \mathbf{111} + \mathbf{11121} + \cdots, \\ &\Theta^{\rho_4}(\phi_2^-) = \mathbf{21}, \quad \Theta^{\rho_4}(\phi_2^+) = \emptyset, \\ &\Theta^{\rho_5}(\phi_2^-) = -\mathbf{2} + \mathbf{2112} + \mathbf{211212} + \cdots, \\ &\Theta^{\rho_5}(\phi_2^+) = -\mathbf{11} - \mathbf{1112} - \cdots, \end{aligned}$$

the formal kneading matrix is given by

$$\begin{aligned} &M_{\phi_1\rho_1} = -\emptyset + (\mathbf{1} + \mathbf{2})\mathbf{11}, \quad M_{\phi_2\rho_1} = \mathbf{211}, \\ &M_{\phi_1\rho_2} = \emptyset, \quad M_{\phi_2\rho_2} = -\mathbf{1} \\ &M_{\phi_1\rho_3} = (\mathbf{1} + \mathbf{2})\mathbf{11}\textstyle\sum_{k=1}^{\infty}(\mathbf{21})^k, \\ &M_{\phi_2\rho_3} = -\emptyset + \mathbf{111} + (\mathbf{1} + \mathbf{2})\mathbf{11}\textstyle\sum_{k=1}^{\infty}(\mathbf{21})^k, \\ &M_{\phi_1\rho_4} = -(\mathbf{1} + \mathbf{2})\mathbf{1}, \quad M_{\phi_2\rho_4} = \emptyset - \mathbf{21}, \\ &M_{\phi_1\rho_5} = \mathbf{1} + \mathbf{2} - (\mathbf{1} + \mathbf{2})\mathbf{1}\textstyle\sum_{k=1}^{\infty}(\mathbf{12})^k, \\ &M_{\phi_2\rho_5} = \mathbf{2} - \mathbf{11} - (\mathbf{1} + \mathbf{2})\mathbf{1}\textstyle\sum_{k=1}^{\infty}(\mathbf{12})^k. \end{aligned}$$

Definition 3.41. For $\gamma \in Q$, $\phi \in C_e^*$ and $a \in B_k(\gamma)$ we set

$$Z^\phi(a,\gamma) = \begin{cases} \mathcal{Y}_{k-1}(a,\gamma) & \text{if } \mu_*^k(a,\gamma) = \phi \\ 0 & \text{otherwise} \end{cases}.$$

Let $\gamma \in Q_1$, $\gamma' \in Q$ and $\phi \in C_e^*$. Suppose $\gamma < \gamma'$. We define

$$\Lambda_k^\phi(\gamma,\gamma') = \sum_{a \in B_k(\gamma)} Z^\phi(a,\gamma') \in \mathcal{R}_k,$$

and

$$\Lambda^\phi(\gamma,\gamma') = \sum_{k=0}^{\infty} \Lambda_k^\phi(\gamma,\gamma') \in \mathcal{R}_\infty.$$

The following is the essential equality.

Proposition 3.42. *Let $\gamma \in Q_1, \gamma' \in Q$ and $\rho \in \Pi^*$. Suppose $\gamma = (x,y)$ and $\gamma < \gamma'$. Then*

$$\Theta^\rho(y^-,\gamma') - \Theta^\rho(x^+,\gamma') = \sum_{\phi \in C_e^*} \Lambda^\phi(\gamma,\gamma') M_{\phi\rho}.$$

Proof. For $\phi = (c,\delta) \in C_e^*$,

$$\begin{aligned}(\Lambda^\phi(\gamma,\gamma')M_{\phi\rho})_k &= \sum_{j=0}^{k} \Lambda_j^\phi(\gamma,\gamma')\,(M_{\phi\rho})_{k-j} \\ &= \sum_{j=0}^{k} \sum_{a \in B_j(\gamma)} Z^\phi(a,\gamma')\,(M_{\phi\rho})_{k-j} \\ &= \sum_{j=0}^{k} \sum_{\substack{a \in B_j(\gamma) \\ \eta_*^j(a,\gamma') = \delta \\ g^j(a,\gamma') = c}} \mathcal{Y}_{j-1}(a,\gamma')\,(M_{\phi\rho})_{k-j}.\end{aligned}$$

If

$$\mu_*^j(a,\gamma') = (c,\delta) = \phi,$$

then by definition

$$\tilde{g}^j(a^\pm,\gamma') = \begin{cases} c^\pm & \text{if } e_j(a,\gamma') = 1, \\ c^\mp & \text{if } e_j(a,\gamma') = -1. \end{cases}$$

By Proposition 3.35,

$$\begin{aligned}\Theta_k^\rho(a^+,\gamma') - \Theta_k^\rho(a^-,\gamma') &= e_j(a,\gamma')\,\mathcal{Y}_{j-1}(a,\gamma')\,(\Theta_{k-j}^\rho(\tilde{g}^j(a^+,\gamma'),\delta) \\ &\quad - \Theta_{k-j}^\rho(\tilde{g}^j(a^-,\gamma'),\delta)) \\ &= \mathcal{Y}_{j-1}(a,\gamma')\,(\Theta_{k-j}^\rho(c^+,\delta) - \Theta_{k-j}^\rho(c^-,\delta)) \\ &= \mathcal{Y}_{j-1}(a,\gamma')\,(M_{\phi\rho})_{k-j}.\end{aligned}$$

Thus

$$\left(\sum_{\phi\in C_e^*}\Lambda^{\phi}(\gamma,\gamma')M_{\phi\rho}\right)_k=\sum_a(\Theta_k^{\rho}(a^+,\gamma')-\Theta_k^{\rho}(a^-,\gamma')),$$

where the sum is over all $a\in\bigcup_{j=0}^{k}B_j(\gamma)$. Let us divide γ into finite arcs $I_1,I_2,\dots,I_l$ by $\bigcup_{j=0}^{k}B_j(\gamma)$. Then

$$\{I_i\}_{i=1}^{l}=\{\mathbf{I}_k(a,\gamma')\cap\gamma\,|\,a\in\gamma\}=\{\mathbf{I}_k(a,\gamma)\,|\,a\in\gamma\}.$$

Let us denote by a_i the unique point in $I_i\cap I_{i+1}$ $(i=1,2,\dots,l-1)$. Then $\{a_i\}_{i=1}^{l-1}=\bigcup_{j=1}^{k}B_j(\gamma)$. Consequently,

$$\begin{aligned}\left(\sum_{\phi\in C_e}\Lambda^{\phi}(\gamma,\gamma')\,M_{\phi\rho}\right)_k&=\sum_{i=1}^{l-1}(\Theta_k^{\rho}(a_i^+,\gamma')-\Theta_k^{\rho}(a_i^-,\gamma'))\\&=-\Theta_k^{\rho}(a_1^-,\gamma')+\sum_{i=1}^{l-2}(\Theta_k^{\rho}(a_i^+,\gamma')\\&\qquad-\Theta_k^{\rho}(a_{i+1}^-,\gamma'))+\Theta_k^{\rho}(a_{l-1}^+,\gamma')\\&=\Theta_k^{\rho}(a_l^-,\gamma')-\Theta_k^{\rho}(a_0^+,\gamma'),\end{aligned}$$

because $\mathbf{I}_k(a_{i-1}^+,\gamma')=\mathbf{I}_k(a_i^-,\gamma')$. Hence we obtain

$$\Theta^{\rho}(y^-,\gamma')-\Theta^{\rho}(x^+,\gamma')=\sum_{\phi\in C_e}\Lambda^{\phi}(\gamma,\gamma')M_{\phi\rho}.$$

This completes the proof. □

Definition 3.43. Let $\gamma\in Q_1$, and let k be a positive integer. Then the set $\bigcup_{m=0}^{k-1}\mathrm{Tur}_m(\gamma)$ divides the path γ into a finite arcs $I_1,I_2,\dots,I_l$ such that I_i neighbors I_{i+1} $(i=1,2,\dots,l-1)$. There exist $U_1,U_2,\dots,U_l\in\mathcal{W}_k$ such that $I_i\subset K(U_i)$. In the other word, $U_i=\mathcal{Y}_{k-1}(a,\gamma)$ if $a\in\mathrm{int}I_i$. This partition has been given in Subsection 3.2. Recall that the set $\mathcal{L}(\gamma,k)$ is given by

$$\{(I_1,U_1),(I_2,U_2),\dots,(I_l,U_l)\}.$$

We define

$$V_k(\gamma)=\sum_{i=1}^{l}U_i\in\mathcal{R}_k$$

and

$$V(\gamma)=\sum_{k=0}^{\infty}V_k(\gamma).$$

For a subtree $T'\subset T$, we similarly define $V(T')$.

For $\xi \in \Xi$, we define

$$\Omega(\xi^{\pm}) = \sum_{k=0}^{\infty} \mathcal{Y}_{k-1}(\xi^{\pm}).$$

For $\phi = (a, \gamma) \in C_e^*$, we denote, by $m = m(\phi)$, the minimal positive integer such that $\mathcal{Y}_m(c^+, \gamma) \neq \mathcal{Y}_m(c^-, \gamma)$. We define

$$\Psi(\phi) = \sum_{k=m(\phi)}^{\infty} (\mathcal{Y}_k(\phi^+) + \mathcal{Y}_k(\phi^-)).$$

Lemma 3.44. *Let $(x, y) = \gamma \in Q_1$, and $\gamma < \gamma' \in Q$. Then*

$$2V(\gamma) = \Omega(x^+, \gamma') + \Omega(y^-, \gamma') + \sum_{\phi \in C_e^*} \Lambda^{\phi}(\gamma, \gamma')\Psi(\phi).$$

Proof. Let

$$\mathcal{L}(\gamma, k) = \{(L_1, U_1), (L_2, U_2), \ldots, (L_l, U_l)\},$$

where the arc L_i neighbors the arc L_{i+1} $(i = 1, 2, \ldots, l-1)$. We denote $a_i \in L_i \cap L_{i+1}$ $(i = 1, 2, \ldots, l-1)$. Note that

$$\mathcal{Y}_{k-1}(a_{i-1}^+, \gamma') = \mathcal{Y}_{k-1}(a_i^-, \gamma') = U_i \ (i = 1, 2, \ldots, l),$$

where $a_0 = x, a_l = y$. By the definition of $\mathcal{L}(\gamma, k)$,

$$\{a_1, a_2, \ldots, a_{l-1}\} = \{a \in \gamma \,|\, \mathcal{Y}_{k-1}(a^-, \gamma') \neq \mathcal{Y}_{k-1}(a^+, \gamma')\} = \bigcup_{j=0}^{k-1} \mathrm{Tur}_j(\gamma).$$

We have defined $0 \leq s = s(a_i, \gamma') \leq k-1$ as the minimal integer such that $g^s(a_i, \gamma')$ is an essential critical point of $\eta^s(a_i, \gamma')$. Then

$$\begin{array}{rclcl} U_i & = & \mathcal{Y}_{k-1}(a_i^-, \gamma') & = & \mathcal{Y}_{s-1}(a_i, \gamma')\,\mathcal{Y}_{k-s-1}(\mu_*^s(a_i, \gamma')^-), \\ U_{i+1} & = & \mathcal{Y}_{k-1}(a_i^+, \gamma') & = & \mathcal{Y}_{s-1}(a_i, \gamma')\,\mathcal{Y}_{k-s-1}(\mu_*^s(a_i, \gamma')^+), \end{array}$$

where $\mu_*^s(a_i, \gamma') \in C_e^*$. Since $U_i \neq U_{i+1}$, we have $0 \leq m(\mu_*^s(a_i, \gamma')) \leq k-s-1$.

Conversely, let $a \in B_s(\gamma)$ such that $0 \leq m(\mu_*^s(a, \gamma')) \leq k-s-1$. Then $\mathcal{Y}_{k-1}(a^-, \gamma') \neq \mathcal{Y}_{k-1}(a^+, \gamma')$. Thus

$$\{a_1, a_2, \ldots, a_{l-1}\} = \bigcup_{s=0}^{k-1} \{a \in B_s(\gamma) \,|\, 0 \leq m(\mu_*^s(a, \gamma')) \leq k-s-1\}.$$

We denote this set by $E(\gamma, k)$, and we denote $E(\gamma, k, s) = E(\gamma, k) \cap B_s(\gamma)$. Note that $E(\gamma, 0)$ is empty.

We write $\phi_i = \mu_*^{s_i}(a_i, \gamma') \in C_e^*$, where $s_i = s(a_i, \gamma')$. Then

$$U_i + U_{i+1} = \mathcal{Y}_{s_i-1}(a_i, \gamma')\,(\mathcal{Y}_{k-s_i-1}(\phi_i^-) + \mathcal{Y}_{k-s_i-1}(\phi_i^+)).$$

Therefore

$$\begin{aligned}(2V(\gamma))_k &= \sum_{i=1}^{l-1}(U_i + U_{i+1}) + U_1 + U_l \\ &= \sum_{i=1}^{l-1} \mathcal{Y}_{s_i-1}(a_i, \gamma')\,(\mathcal{Y}_{k-s_i-1}(\phi_i^-) + \mathcal{Y}_{k-s_i-1}(\phi_i^+)) \\ &\quad + \mathcal{Y}_{k-1}(x^+, \gamma') + \mathcal{Y}_{k-1}(y^-, \gamma') \\ &= \sum_{i=1}^{l-1} Z^{\phi_i}(a_i, \gamma')\,(\Psi(\phi_i))_{k-s_i} + (\Omega(x^+, \gamma') + \Omega(y^-, \gamma'))_k.\end{aligned}$$

If we write $\phi(a) = \mu_*^s(a, \gamma')$ for $a \in B_s(\gamma)$, we have

$$\sum_{i=1}^{l-1} Z^{\phi_i}(a_i, \gamma')\,(\Psi(\phi_i))_{k-s_i} = \sum_{s=0}^{k-1} \sum_{a \in E(\gamma,k,s)} Z^{\phi(a)}(a, \gamma')(\Psi(\phi(a)))_{k-s}. \tag{3.12}$$

If $a \in B_s(\gamma) - E(\gamma, k, s)$, then $m(\phi(a)) \geq k - s$, and so $(\Psi(\phi(a)))_{k-s} = 0$. Therefore (3.12) is equal to

$$\sum_{s=0}^{k-1} \sum_{a \in B_s(\gamma)} Z^{\phi(a)}(a, \gamma')(\Psi(\phi(a)))_{k-s},$$

and hence it is equal to

$$\sum_{\phi \in C_e^*} \sum_{s=0}^{k-1} \sum_{a \in B_s(\gamma)} Z^{\phi}(a, \gamma')(\Psi(\phi))_{k-s},$$

because $Z^\phi(a, \gamma') = 0$ if $\phi \neq \phi(a)$. Consequently,

$$(2V(\gamma))_k = (\Omega(a^+, \gamma') + \Omega(y^-, \gamma'))_k + \sum_{\phi \in C_e^*} \sum_{s=0}^{k-1} (\Lambda^\phi(\gamma, \gamma'))_s (\Psi(\phi))_{k-s}.$$

This completes the proof. $\square$

Example 3.45. This is continued from Example 3.40.

(1) From Lemma 3.42, we have $\Theta^{I_i}(p_2^-, \gamma) - \Theta^{I_i}(p_1^+, \gamma) = \Lambda^\phi(\gamma, \gamma) M_{\phi I_i}$ for $i = 1, 2$. Considering $i = 1$, we obtain $-\sum_{k=1}^\infty \mathbf{1}^k = \Lambda^\phi(\gamma, \gamma)(-\emptyset + \mathbf{2}\sum_{k=0}^\infty \mathbf{1}^k)$. Consequently, $\Lambda^\phi(\gamma, \gamma) = \sum_{k=1}^\infty (\mathbf{1} + \mathbf{2})^k$. By Lemma 3.44, $V(\gamma) = \sum_{k=1}^\infty (\mathbf{1} + \mathbf{2})^k$.

(2) From Lemma 3.42,

$$\Theta^{\rho_i}(p^-,\gamma)-\Theta^{\rho_i}(q^+,\gamma)=\Lambda^{\phi_1}(\gamma,\gamma)M_{\phi_1\rho_i}+\Lambda^{\phi_2}(\gamma,\gamma)M_{\phi_2\rho_i}$$

for $\gamma=(q,p)\in Q_1$ and $i=1,2,3,4,5$. Since

$$\Theta^{\rho_1}(p_3^+,\gamma_1)=\emptyset,\Theta^{\rho_1}(p_1^-,\gamma_1)=0,\Theta^{\rho_2}(p_3^+,\gamma_1)=0,\Theta^{\rho_2}(p_1^-,\gamma_1)=\emptyset,$$

we have

$$\begin{aligned}-\emptyset&=\Lambda^{\phi_1}(\gamma_1,\gamma_1)(-\emptyset+\mathbf{1}^3+\mathbf{21}^2)+\Lambda^{\phi_2}(\gamma_1,\gamma_1)\mathbf{21}^2,\\ \emptyset&=\Lambda^{\phi_1}(\gamma_1,\gamma_1)-\Lambda^{\phi_2}(\gamma_1,\gamma_1)\mathbf{1}.\end{aligned}$$

Consequently,

$$\begin{array}{rcl}\Lambda^{\phi_2}(\gamma_1,\gamma_1)&=&(\mathbf{1}^2+\mathbf{21})\sum_{k=0}^{\infty}(\mathbf{21}+\mathbf{121}+\mathbf{1}^3)^k,\\ \Lambda^{\phi_1}(\gamma_1,\gamma_1)&=&\emptyset+\Lambda^{\phi_2}(\gamma_1,\gamma_1)\mathbf{1}.\end{array}$$

Since $\Psi(\phi_1)=\Psi(\phi_2)=(\mathbf{1}+\mathbf{2})(\emptyset+\mathbf{1}+\mathbf{11}+\mathbf{112}+\cdots)$,

$$\begin{aligned}V(\gamma_1)&=(\Omega(p_3^+,\gamma_1)+\Omega(p_1^-,\gamma_1)\\&\quad+\Lambda^{\phi_1}(\gamma_1,\gamma_1)\Psi(\phi_1)+\Lambda^{\phi_2}(\gamma_1,\gamma_1)\Psi(\phi_2))/2\\&=(\Omega(p_3^+,\gamma_1)+\Omega(p_1^-,\gamma_1)+(\emptyset+\Lambda^{\phi_1}(\gamma_1,\gamma_1)(\emptyset+\mathbf{1}))\Psi(\phi_1))/2\\&=\emptyset+\mathbf{1}+\mathbf{2}+\mathbf{1}^2+\mathbf{21}+\mathbf{1}^3+\mathbf{1}^2\mathbf{2}+\mathbf{21}^2+\mathbf{212}\\&\quad+\mathbf{1}^4+\mathbf{1}^3\mathbf{2}+\mathbf{1}^2\mathbf{21}+\mathbf{21}^3+\mathbf{21}^2\mathbf{2}+(\mathbf{21})^2+\cdots.\end{aligned}$$

Similarly,

$$\begin{aligned}V(\gamma_2)&=\emptyset+\mathbf{1}+\mathbf{2}+\mathbf{1}^2+\mathbf{12}+\mathbf{21}+\mathbf{1}^3+\mathbf{121}+\mathbf{21}^2+\mathbf{212}\\&\quad+\mathbf{1}^4+\mathbf{1}^3\mathbf{2}+\mathbf{121}^2+(\mathbf{12})^2+\mathbf{21}^3+\mathbf{21}^2\mathbf{2}+(\mathbf{21})^2+\cdots,\\ V(\gamma_3)&=\emptyset+\mathbf{1}+\mathbf{1}^2+\mathbf{12}+\mathbf{1}^3+\mathbf{1}^2\mathbf{2}+\mathbf{121}+\mathbf{1}^4+\mathbf{121}^2+\mathbf{1}^2\mathbf{21}\\&\quad+(\mathbf{12})^2+\cdots.\end{aligned}$$

Thus we have

$$\begin{aligned}V(T)&=\emptyset+\mathbf{1}+\mathbf{2}+\mathbf{1}^2+\mathbf{12}+\mathbf{21}+\mathbf{1}^3+\mathbf{1}^2\mathbf{2}+\mathbf{121}+\mathbf{21}^2+\mathbf{212}\\&\quad+\mathbf{1}^4+\mathbf{1}^3\mathbf{2}+\mathbf{1}^2\mathbf{21}+\mathbf{121}^2+(\mathbf{12})^2+\mathbf{21}^3+\mathbf{21}^2\mathbf{2}+(\mathbf{21})^2+\cdots.\end{aligned}$$

3.3.3. Kneading determinants

Let $X_1,X_2,\ldots,X_N$ be commutative variables. Consider the abelization

$$\beta:\mathcal{W}_*\to\langle X_1,X_2,\ldots,X_N\rangle,$$

where $\langle X_1,X_2,\ldots,X_N\rangle$ is the free commutative monoid generated by X_1, $X_2,\ldots,X_N$, and which is defined by

$$\beta(U)=X_U.$$

For $f \in \mathcal{R}_\infty$, we define a power series

$$\overline{f} = \sum_{U \in \mathcal{W}_*} f(U) X_U.$$

Then

$$f \mapsto \overline{f}$$

is considered as the abelization map from $\mathcal{R}_\infty$ to the formal power series ring $\mathbb{C}[[X]] = \mathbb{C}[[X_1, X_2, \ldots X_N]]$. For $\overline{f} \in \mathbb{C}[[X]]$, we denote, by $(\overline{f})_k$, its homogeneous part of degree k.

The power series

$$\theta^\rho(a^\pm, \gamma; X_1, X_2, \ldots, X_N) = \overline{\Theta^\rho(a^\pm, \gamma)}$$

is a holomorphic function on $\mathbf{D} = \{(X_1, X_2, \ldots, X_N) \,|\, |X_i| < 1\}$; because the absolute value of

$$\sum_{U \in \mathcal{W}_k} \Theta^\rho_k(a^\pm, \gamma)(U)$$

is 1, -1 or 0.

We set $\#Q^* = l$ and $\#\Pi^* = n$, then $\#C_e^* = n - l$. We define for $\rho \in \Pi^*, \phi \in C_e^*$,

$$R_{\phi\rho} = \overline{M_{\phi\rho}}.$$

Then $R = R(T) = (R_{\phi\rho})_{\phi \in C_e^*, \rho \in \Pi^*}$ is a $n - l \times n$-matrix in $\mathbb{C}[[X]]$, which is called the *kneading matrix* of T. We define for $\rho \in \Pi^*$ and $\gamma \in Q^*$,

$$H_{\rho\gamma} = h_\gamma(\gamma(\rho)) - e(\rho)\, h_\gamma(\eta_*(\rho))\, X_{Y(\rho)},$$

where

$$h_\gamma(\xi) = \begin{cases} 1 & \text{if } \gamma = \xi, \\ 0 & \text{if } \gamma \neq \xi. \end{cases}$$

Let us consider a $n \times l$-matrix

$$H = H(T) = (H_{\rho\gamma})_{\rho \in \Pi^*, \gamma \in Q^*}$$

in $\mathbb{C}[[X]]$. From Corollary 3.39, we have

$$RH = 0.$$

Definition 3.46. Let

$$G = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n-l\,1} & g_{n-l\,2} & \cdots & g_{n-l\,n} \end{pmatrix}, F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nl} \end{pmatrix}$$

be an $n - l \times n$-matrix and an $n \times l$-matrix. Suppose that $\{1, 2, \ldots, n\}$ is divided into $B = \{k(1), k(2), \ldots, k(l)\}$ and $B^c = \{1, 2, \ldots, n\} - B = \{k'(1), k'(2), \ldots,$

$k'(n-l)\}$. We assume that $k(1) < k(2) < \cdots < k(l)$ and $k'(1) < k'(2) < \cdots < k'(n-l)$. Then we write

$$G|\check{B} = \begin{pmatrix} g_{1k'(1)} & g_{1k'(2)} & \cdots & g_{1k'(n-l)} \\ g_{2k'(1)} & g_{2k'(2)} & \cdots & g_{2k'(n-l)} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n-l\,k'(1)} & g_{n-l\,k'(2)} & \cdots & g_{n-l\,k'(n-l)} \end{pmatrix},$$

$$F|B = \begin{pmatrix} f_{k(1)1} & f_{k(1)2} & \cdots & f_{k(1)l} \\ f_{k(2)1} & f_{k(2)2} & \cdots & f_{k(2)l} \\ \cdots & \cdots & \cdots & \cdots \\ f_{k(l)1} & f_{k(l)2} & \cdots & f_{k(l)l} \end{pmatrix}.$$

Lemma 3.47. *There exists a subset $B \subset \Pi^*$ such that $\# B = l$ and* $\det R|\check{B} \neq 0$.

Proof. Choose $B = \{\rho_\phi \mid \phi \in C_e^*\}$ such that $\rho_\phi = (\mathbf{I}_1(\phi^+), \gamma)$, where $\phi = (c, \gamma)$. Then the constant term of $R|\check{B}$ is equal to the unit matrix by permutations of the row vectors. Hence the constant term of $\det R|\check{B}$ is 1 or -1. □

We fix

$$B_0 = \{\rho_\gamma \mid \gamma \in Q^*\},$$

a subset of Π^* such that $\gamma(\rho_\gamma) = \gamma$.

Lemma 3.48. *The holomorphic function* $\det H|B_0$ *has no zero in* $\mathbf{D}$.

Proof. The matrix $H|B_0$ has the form $E + G$, where E is the unit matrix (by permutations of the row vectors) and each row vector of G has only one nonzero component $\pm X_k$. From this, it follows that $\det H|B_0 \neq 0$ at $X = (a_1, a_2, \ldots, a_N)$ if $|a_k| < 1$ for any k. □

The following is an immediate corollary of a known result (for example, see [5], Chapter VII, Section 3, Theorem I).

Lemma 3.49. *Let G be an $n-l \times n$-matrix, and F be a $n \times l$-matrix. Suppose that each component of these matrices is a holomorphic function on* $\mathbf{D}$, *and suppose that $GF = 0$. If B, B' are subsets of $\{1, 2, \ldots, n\}$ such that $\# B = \# B' = l$, then there exists* $\operatorname{sgn}(B, B') \in \{1, -1\}$ *such that*

$$\det G|\check{B} \det F|B' = \operatorname{sgn}(B, B') \det G|\check{B}' \det F|B.$$

Lemma 3.50. *Let $B \subset \Pi^*$ be a subset such that $\# B = l$. Then $R|\check{B} \neq 0$ if and only if $H|B \neq 0$.*

Proof. Suppose that $R|\check{B} = 0$ and $H|B \neq 0$. From Lemma 3.49, we see that $R|\check{B}' = 0$ for any $B' \subset \Pi^*$. This contradicts Lemma 3.47. The converse is also verified by Lemma 3.48. □

Definition 3.51. Let B be a subset of Π^* such that $\#B = l$ and $\det R|\check{B} \neq 0$. Remark that $\det R|\check{B}$ and $\det H|B$ are non-zero holomorphic functions on $\mathbf{D}$. Then the meromorphic function

$$\Delta = \Delta_T = \pm \frac{\det R|\check{B}}{\det H|B}$$

is said to be the *kneading determinant* of T, where we choose $+$ or $-$ so that $\Delta|_{X=(0,0,\dots,0)} = 1$. By Lemma 3.49, the kneading determinant is independent of B.

Lemma 3.52. *The kneading determinant Δ is holomorphic on $\mathbf{D}$.*

Proof. Consider the case $B = B_0$. □

We define for $\phi \in C_e^*, \gamma \in Q_1, \gamma' \in Q$ with $\gamma < \gamma'$

$$\lambda^\phi(\gamma, \gamma'; X_1, X_2, \dots, X_N) = \overline{\Lambda^\phi(\gamma, \gamma')}.$$

Then $\lambda^\phi(\gamma, \gamma')$ is a holomorphic function on

$$\{(X_1, X_2, \dots, X_N) \,|\, |X_i| < 1/N, i = 1, 2, \dots, N\};$$

because $\sum_{U \in \mathcal{W}_k} \Lambda^\phi(\gamma, \gamma')(U) \leq \#C(k) \leq N^k \#C$.

Lemma 3.53. *The function $\lambda^\phi(\gamma, \gamma')$ can be extended to a meromorphic function on $\mathbf{D}$. Moreover,*

$$\bigcup_{\phi \in C_e^*, \gamma \in Q_1, \gamma' \in Q : \gamma < \gamma'} \{X \in \mathbf{D} \,|\, \lambda^\phi(\gamma, \gamma'; X) = \infty\} = \{X \in \mathbf{D} \,|\, \Delta(X) = 0\}.$$

Proof. Let $\gamma = (x, y) \in Q_1$, and $\gamma < \gamma' \in Q$. From Lemma 3.42,

$$\theta^\rho(y^-, \gamma') - \theta^\rho(x^+, \gamma') = \sum_{\phi \in C_e^*} \lambda^\phi(\gamma, \gamma') R_{\phi\rho}$$

on $\{(X_1, X_2, \dots, X_n) \,|\, |X_i| < 1/N, i = 1, 2, \dots, N\}$.

Consider the subset $B = B_0$. There exists an $n - l \times n - l$-matrix

$$\tilde{R} = (\tilde{R}_{\rho\phi})_{\phi \in C_e^*, \rho \in \Pi^* - B}$$

which is the inverse of $R|\check{B}$. Remark that each component of $(\tilde{R}_{\rho\phi})$ is a meromorphic function on $\mathbf{D}$. We have for any $\phi \in C_e^*$

(3.13)
$$\begin{aligned} \sum_{\rho \in \Pi^* - B} (\theta^\rho(y^-, \gamma') - \theta^\rho(x^+, \gamma')) \tilde{R}_{\rho\phi} &= \sum_{\rho \in \Pi^* - B} \sum_{\phi' \in C_e^*} \lambda^{\phi'}(\gamma, \gamma') R_{\phi'\rho} \tilde{R}_{\rho\phi} \\ &= \lambda^\phi(\gamma, \gamma'). \end{aligned}$$

Thus $\lambda^{\phi}(\gamma,\gamma')$ is a meromorphic function on $\mathbf{D}$. Moreover,

$$\begin{aligned}\Delta\lambda^{\phi}(\gamma,\gamma') &= \Delta\sum_{\rho\in\Pi^*-B}(\theta^{\rho}(y^-,\gamma')-\theta^{\rho}(x^+,\gamma'))\tilde{R}_{\rho\phi}\\ &= \sum_{\rho\in\Pi^*-B}(\theta^{\rho}(y^-,\gamma')-\theta^{\rho}(x^+,\gamma'))\tilde{R}_{\rho\phi}\det R|B/\det H|B.\end{aligned}$$

Since $\tilde{R}_{\rho\phi}\det R|B$ is holomorphic on $\mathbf{D}$, we conclude that $\Delta\lambda^{\phi}(\gamma,\gamma')$ is holomorphic on $\mathbf{D}$. Hence a pole of $\lambda^{\phi}(\gamma,\gamma')$ is a zero of Δ.

Suppose $\Delta(\alpha_1,\alpha_2,\ldots,\alpha_N)=0$. Then there is a column vector $\mathbf{a}=[a_\rho]_{\rho\in\Pi^*}\in\mathbf{C}^{\Pi^*}$, at least one of a_ρ is nonzero, such that $\mathbf{g}=[G_\phi]_{\phi\in C_e^*}=R\mathbf{a}$ is a vector each component of which is holomorphic function with zero at $(\alpha_1,\alpha_2,\ldots,\alpha_N)$. We can assume that $a_\rho=0$ for $\rho\in B$. It is clear that $\tilde{R}\mathbf{g}=[a_\rho]_{\rho\in\Pi^*-B}$. By (3.13),

$$\sum_{\phi\in C_e^*}G_\phi\lambda^{\phi}(\gamma,\gamma')=\sum_{\rho\in\Pi^*-B}(\theta^{\rho}(y^-,\gamma')-\theta^{\rho}(x^+,\gamma'))a_\rho$$

for any $\gamma'\in Q$ and any $\gamma=(x,y)<\gamma'$. Suppose $\lambda^{\phi}(\gamma,\gamma')$ does not have a pole at $(\alpha_1,\alpha_2,\ldots,\alpha_N)$ for any $\phi\in C_e^*$ and any $\gamma'\in Q^*$. Then

$$\sum_{\rho\in\Pi^*-B}(\theta^{\rho}(y^-,\gamma')-\theta^{\rho}(x^+,\gamma'))a_\rho=0$$

at $(\alpha_1,\alpha_2,\ldots,\alpha_N)$ for any $\gamma'\in Q^*$ and any $x\neq y\in\gamma'$. From this, it follows that

$$S(\gamma')=\sum_{\rho\in\Pi^*-B}\theta^{\rho}(x^{\pm},\gamma';\alpha_1,\alpha_2,\ldots,\alpha_N)a_\rho$$

is independent of $x\in\gamma'$.

Let $\rho\in\Pi^*$, and $\gamma\in Q^*, x\in\gamma$. From Corollary 3.36 we have

$$\theta^{\rho'}(x^{\pm},\gamma)=\begin{cases}1\ +\ e(\rho)X_{Y(\rho)}\theta^{\rho'}(\tilde{\mu}^1(x^{\pm},\gamma)) & \text{if } \rho'=\rho\\ e(\rho)X_{Y(\rho)}\theta^{\rho'}(\tilde{\mu}^1(x^{\pm},\gamma)) & \text{if } \rho'\neq\rho\end{cases}.$$

Thus

$$S(\gamma)=a_\rho+e(\rho)\alpha_{Y(\rho)}S(\eta_*^1(x^{\pm},\gamma)).$$

Consequently, if we take $\gamma=\gamma(\rho)$, then we have

$$\sum_{\gamma'\in Q^*}\big(h_{\gamma'}(\gamma(\rho))\,S(\gamma(\rho))-e(\rho)\alpha_{Y(\rho)}h_{\gamma'}(\eta_*(\rho))\,S(\gamma(\eta_*(\rho)))\big)=a_\rho,$$

or equivalently

$$HS=\mathbf{a}, \tag{3.14}$$

where $S=[S(\gamma(\rho))]_{\rho\in\Pi^*}\in\mathbf{C}^{\Pi^*}$. In particular,

$$(H|B)S=[a_\rho]_{\rho\in B}=0.$$

Since $\mathbf{a} \neq 0$, we have $S \neq 0$ from (3.14). But $\det H|B \neq 0$. This is a contradiction. $\square$

Proof of Lemma 3.19. From Lemma 3.44,

$$2v(\gamma) = \overline{\Omega(x^+, \gamma')} + \overline{\Omega(y^-, \gamma')} + \sum_{\phi \in C_e^*} \lambda^\phi(\gamma, \gamma')\overline{\Psi(\phi)}.$$

Thus $v(\gamma)$ is extended to a meromorphic function on $\mathbf{D}$. Similarly, we can prove that $v(T)$ is also extended to be meromorphic on $\mathbf{D}$. When we consider

$$u(T)(t) = v(T)(\alpha_1 t, \alpha_2 t, \ldots, \alpha_N t) \text{ and } u(\gamma)(t) = v(\gamma)(\alpha_1 t, \alpha_2 t, \ldots, \alpha_N t)$$

as functions of one variable t, they are meromorphic on $\{|t| < 1/(\max_i \alpha_i)\}$. Therefore

$$\frac{u(\gamma)}{u(T)}$$

is meromorphic on $\{|t| < 1/(\max_i \alpha_i)\}$. But this function does not have a pole at $t = 1$, because $u(\gamma)/u(T)$ is bounded for $0 < t < 1$. Hence it is holomorphic near $t = 1$, and so the limit $\lim_{t \to 1-} u(\gamma)/u(T)$ exists.

Suppose that $u(T)(t)$ converges at $t = 1$. Then it also converges on the circle $|t| = 1$, since the coefficients of $u(T)$ are non-negative. Therefore $u(T)$ is holomorphic near $|t| = 1$. This contradicts the fact that the radius of convergence of $u(T)$ is one. This completes the proof.

Theorem 3.25 is a consequence of the following.

Lemma 3.54. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ be a polyratio. If $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_N) = 0$ and if $\Delta(\alpha'_1, \alpha'_2, \ldots, \alpha'_N) \neq 0$ for any $0 < \alpha'_i < \alpha_i$, then there exists $\gamma \in Q$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a polyradius of convergence of the power series $v(\gamma)$.*

Proof. By Lemma 3.53, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a pole of $\lambda^\phi(\gamma, \gamma')$ for some ϕ, γ, γ', moreover $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_N)$ is not a pole of $\lambda^\phi(\gamma, \gamma')$ for any ϕ, γ' if $0 < \alpha'_i < \alpha_i$. Thus the series $\lambda^\phi(\gamma, \gamma'; \alpha')$ is convergent for any ϕ, γ' if $0 < \alpha'_i < \alpha_i$. Note that $\lambda^\phi(\gamma, \gamma'; \alpha') > 0$ for any ϕ, γ' if $0 < \alpha'_i < \alpha_i$. Therefore the series $v(\gamma)(\alpha')$ is convergent if $0 < \alpha'_i < \alpha_i$, and $v(\gamma)(\alpha)$ is divergent. $\square$

Example 3.55. This is continued from Example 3.45.

(1) From

$$R = \left(-1 + X_2 \sum_{k=1}^{\infty} X_1^k \quad 1 - X_1 \sum_{k=1}^{\infty} X_2^k \right) \text{ and } H = \begin{pmatrix} 1 - X_1 \\ 1 - X_2 \end{pmatrix},$$

we have

$$\Delta = \frac{1 - X_1 - X_2}{(1 - X_1)(1 - X_2)}.$$

Thus the set of critical ratios is $\mathrm{CR} = \{(\alpha_1, \alpha_2) \in \mathbf{Ra}_2 \mid \alpha_1 + \alpha_2 = 1\}$.

(2) From

$$R = \begin{pmatrix} -1+(X_1+X_2)X_1^2 & 1 & X_1G & -(X_1+X_2)X_1 & X_1+X_2-G \\ X_1^2X_2 & -X_1 & -1+X_1^3+X_1G & 1-X_1X_2 & X_2-X_1^2-G \end{pmatrix}$$

and

$$H = \begin{pmatrix} 1 & 0 & X_2 \\ 1 & 0 & -X_1 \\ 0 & 1 & X_2 \\ X_1 & 1 & 0 \\ 0 & X_1 & 1 \end{pmatrix},$$

where $G = (X_1+X_2)X_1 \sum_{k=1}^{\infty}(X_1X_2)^k$, we have

$$\Delta = \frac{1 - X_1X_2 - X_1^3 - X_1^2X_2}{1 - X_1X_2}.$$

Thus the set of critical ratios is $\mathrm{CR} = \{(\alpha_1, \alpha_2) \in \mathbf{Ra}_2 \mid \alpha_1\alpha_2 + \alpha_1^3 + \alpha_1^2\alpha_2 = 1\}$.

Appendix

In Appendix, we prove results on the arcwise connectedness of topological self-similar sets and the uniqueness of paths in self-similar sets. These result are used in Section 3 to construct a self-similar metric.

First we show the arcwise connectedness of connected components of self-similar sets. Recall that a connected and locally connected metric space is arcwise connected (for example, see [19]).

Proposition A.1. *Let $(K, \{F_i\}_{i=1}^N)$ be a topological self-similar system. Suppose that $K(U) \cap K(V)$ has at most finite number of components for any n and any distinct words $U, V \in \mathcal{W}_n$. Then each component of K is locally connected. In particular, each component of K is arcwise connected.*

Proof. Let $x \in K$. Note that for $n \geq 0$,

$$Q_n = \bigcup_{\substack{U \in \mathcal{W}_n \\ x \notin K(U)}} K(U) \cap L_n(x)$$

has at most finite components by assumption. We denote, by X, the component of K containing x, and by X_n, the component of $L_n(x)$ containing x. Clearly, X_n is a subset of X. We show that X_n is a neighborhood of x in X. If X consists of one point, then X is locally connected. We assume that X contains more than one point.

Assume that X_n is not a neighborhood of x in X. Then for any integer $k \geq 0$ there exists a point y_k in $(X \cap L_k(x)) - X_n$. Let $Y(k)$ denote the component of $L_n(x) \cap X$ containing y_k. Since $y_k \notin X_n$, we see that $Y(k) \cup X_n$ is not connected. If A is an open and closed subset in $L_n(x)\cap X$, then $A\cap Q_n \neq \emptyset$. Indeed, let B be an open set such that $B\cap L_n(x)\cap X = A$. If $A\cap Q_n = \emptyset$, then

$$\left(B - \bigcup_{\substack{U\in\mathcal{W}_n \\ x\notin K(U)}} K(U) \right) \cap X = \left(B - \bigcup_{\substack{U\in\mathcal{W}_n \\ x\notin K(U)}} K(U) \right) \cap L_n(x) \cap X = A.$$

Thus A is open in X. Since A is closed, it is closed in X. This contradicts the connectedness of X, and hence $A\cap Q_n \neq \emptyset$. For any open set B including $Y(k)$, there exists an open and closed set A in $L_n(x)\cap X$ such that $Y(k)\subset A\subset B$. Therefore $Y(k)\cap Q_n \neq \emptyset$.

Since Q_n has at most finite components, there exists a component P of Q_n such that $Y(k_i)\cap Q_n \subset P$ for a sequence $k_1 < k_2 < \cdots$. Then $Y(k_1) = \bigcup_{i=1}^{\infty} Y(k_i)$. The sequence $\{y_{k_i}\}_i$ converges to x, but $x \notin Y(k_1)$. This is a contradiction. Consequently, X_n is a neighborhood of x in X. That means the local connectedness of X at x. $\square$

Immediately, by the above proposition we obtain the following.

Corollary A.2. *If the critical set is finite, then each component of K is arcwise connected.*

For a finitely ramified topological self-similar system $(K, \{F_i\}_{i=1}^N)$ with Condition A, we show the uniqueness of a simple path between two points in K.

Lemma A.3. *Let $X = X_1 \cup X_2 \cup \cdots \cup X_n$ be an arcwise connected metric space. Suppose that each X_i is compact and that $C = \bigcup_{i\neq j} X_i \cap X_j$ is a finite set. Let $\gamma : [0,1] \to X$ be a continuous path between $x \in X_1$ and $y \in X_n$. Then $[0,1]$ is divided into finite intervals $I_\gamma(1), I_\gamma(2), \ldots, I_\gamma(l_\gamma)$ such that $I_\gamma(k)$ is a maximal interval satisfying $\gamma(I_\gamma(k)) \subset X_i$ for some $i = i_\gamma(k)$. We write*

$$C(\gamma) = (i_\gamma(1), i_\gamma(2), \ldots, i_\gamma(l_\gamma); \gamma(a_1), \gamma(a_2), \ldots, \gamma(a_{l_\gamma - 1})),$$

where $I_\gamma(k) = [a_{k-1}, a_k]$. If γ' is a simple path between x and y which is homotopic to γ with the endpoints x, y fixed, then $C(\gamma) = C(\gamma')$.

Proof. Let us consider, for $i = 1, 2, \ldots n$, the set

$$Q(i) = \left\{ (i_1, i_2, \ldots, i_{l+1}; x_1, x_2, \ldots, x_l) \,\middle|\, \begin{array}{l} i_1 = 1, i_{l+1} = i, i_k \neq i_{k+1}, x_k \neq x_{k+1}, \\ x_k \in X_{i_k} \cap X_{i_{k+1}} \ (k = 1, 2, \ldots, l) \end{array} \right\}$$

in

$$\bigcup_{l=0}^{\infty} (\{1, 2, \ldots, n\}^{l+1} \times X^l),$$

where we set $X^0 = \{\emptyset\}$. Note that $Q(1)$ contains the member $(1;\emptyset)$. The set $Q(i)$ has the discrete topology. We define

$$\tilde{X} = \left(\bigcup_{i=1}^{n} X_i \times Q(i) \right) / \sim,$$

where the equivalence relation $\sim$ is defined by

$$(x_l, (i_1, i_2, \ldots, i_{l+1}; x_1, x_2, \ldots, x_l)) \sim (x_l, (i_1, i_2, \ldots, i'_{l+1}; x_1, x_2, \ldots, x_l))$$

and

$$(x_{l+1}, (i_1, i_2, \ldots, i_{l+1}; x_1, x_2, \ldots, x_l)) \\ \sim (x_{l+1}, (i_1, i_2, \ldots, i_{l+1}, i_{l+2}; x_1, x_2, \ldots, x_l, x_{l+1})).$$

Then the projection $\rho : \tilde{X} \ni (x, *) \mapsto x \in X$ is a covering, that is, for $x \in X$ there exists a neighborhood U such that $\rho^{-1}(U)$ is a union of disjoint open sets on each of which ρ is homeomorphic. Indeed, it suffices to take U to be the ϵ-neighborhood of x, where ϵ is the minimum of the distances between x and a point in $\bigcup_{k \neq m}(X_k \cap X_m) - \{x\}$.

For $x \in X_1$ we take $\tilde{x} = (x, (1;\emptyset)) \in \tilde{X}$. If γ is a path between x and y, then there uniquely exists a path $\tilde{\gamma} : [0,1] \to \tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{x}$ and $\rho \circ \tilde{\gamma} = \gamma$. Moreover if $h : [0,1] \times [0,1] \to X$ is a homotopy between γ and γ', then there exists a homotopy $\tilde{h} : [0,1] \times [0,1] \to \tilde{X}$ between $\tilde{\gamma}$ and $\tilde{\gamma}'$ such that $\rho \circ \tilde{h} = h$.

Let γ and γ' be simple paths between x and y. Then $\tilde{y} = \tilde{\gamma}(1) = (y, C(\gamma)), \tilde{y}' = \tilde{\gamma}'(1) = (y, C(\gamma'))$. If γ and γ' are homotopic, then $\tilde{y}=\tilde{y}'$. Thus $C(\gamma) = C(\gamma')$. □

Let $\gamma : [0,1] \to K$ be a simple path between x and y. For $n = 1, 2, \ldots,$ the interval $[0,1]$ is uniquely divided into finite intervals $\{I_\gamma(n,i)\}_{i=1}^{l_\gamma(n)}$, where $I_\gamma(n,i)$ is a maximal interval such that $\gamma(I_\gamma(n,i)) \subset K(U)$ for some $U = U_\gamma(n,i) \in \mathcal{W}_n$.

Proposition A.4. *Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system. If two simple paths γ, γ' are homotopic with the endpoints fixed, then $\gamma([0,1]) = \gamma'([0,1])$.*

Proof. For any n the partition $K = \bigcup_{U \in \mathcal{W}_n} K(U)$ satisfies the condition of Lemma A.3. Therefore $l_\gamma(n) = l_{\gamma'}(n)$ and $U_\gamma(n,i) = U_{\gamma'}(n,i)$ $(i = 1, 2, \ldots, l_\gamma(n))$. Consequently,

$$\gamma([0,1]) = \bigcap_{n>0} \bigcup_{i=1}^{l_\gamma(n)} K(U_\gamma(n,i)) = \gamma'([0,1]).$$

□

Corollary A.5. *Let $(K, \{F_i\}_{i=1}^N)$ be a finitely ramified topological self-similar system. Suppose each component of K is simply connected. Then for two points x, y in a component of K there uniquely exists a simple path joining x and y.*

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References

[1] V. Baladi and D. Ruelle, An extension of the theorem of Milnor and Thurston on the zeta functions of interval maps, Ergod. Th. Dyn. Sys., **14** (1994), 621–632.

[2] P. Collet and J.-P. Eckmann, Iterated Maps of the Interval as Dynamical Systems, Birkhaüser, Boston, 1980.

[3] L. Carleson and T. W. Gamelin, Complex Dynamics, Springer-Verlag, 1993.

[4] M. Hata, On the structure of self-similar sets, Japan J. Appl. Math., **3** (1985), 381–414.

[5] W. V. D. Hodge and D. Pedoe, Methods of Algebraic Geometry Volume I, Cambridge University Press, Cambridge, 1947.

[6] J. E. Hutchinson, Fractals and self-similarity, Indiana Math., J. **30** (1981), 713–743

[7] A. Kameyama, Self-similar sets from the topological point of view, Japan J. Ind. Appl. Math., **10** (1993), 85–95.

[8] A. Kameyama, Julia sets of postcritically finite rational maps and topological self-similar sets, Nonlinearity, **13** (2000), 165–188.

[9] A. Kameyama, in preparation.

[10] J. L. Kelley, General Topology, Van Nostrand, Toront, 1955.

[11] J. Kigami, Harmonic calculus on p.c.f. self-similar sets, Trans. Amer. Math. Soc., **335** (1993), 721–755.

[12] J. Kigami, Hausdorff dimension of self-similar sets and shortest path metrics, J. Math. Soc. Japan, **47** (1995), 381–404.

[13] T. Lindstrøm, Brownian motion on nested fractals, Mem. Amer. Math. Soc., **420**-83 (1990).

[14] W. de Melo and S. van Strien, One-Dimensional Dynamics, Springer-Verlag, 1993.

[15] P. A. P. Moran, Additive functions of intervals and Hausdorff measure, Proc. Camb. Phil. Soc., **42** (1946), 15–23.

[16] J. Milnor and W. Thurston, On iterated maps of the interval, Lecture Note in Math. vol. 1342, Springer, 1988.

[17] M. Urbański, Geometry and ergodic theory of conformal non-recurrent dynamics, Ergod. Th. Dyn. Sys., **17** (1997), 1449–1476.

[18] S. van Strien, Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition, Commun. Math. Phys., **128** (1990), 437–496.

[19] G. T. Whyburn, Topological Analysis, Princeton University Press, Princeton, 1958.

[20] R. F. Williams, Composition of contractions, Bol. Soc. Brasil. Mat., **2** (1971), 55–59.