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Some estimates of the logarithmic Sobolev constants on manifolds with boundary and an application to the Ising models

By

Yuzuru Inahama

1. Introduction

In this article we will give some estimates of the logarithmic Sobolev constant in terms of geometric quantities under the Neumann boudary condition and will give an application to the logarithmic Sobolev inequality for the Ising models.

Let M be a d-dimensional, smooth, compact and connected Riemannian manifold with smooth boundary ∂M and let m be the Riemannian measure. For a potential function $U \in C^\infty(M)$, set $dm^U = e^{-U}dm$ and L^U by

$$L^U f = \triangle f - (\nabla U|\nabla f)$$

for $f \in C^\infty(M)$ satisfying the Neumann boundary condition. In the first part of this article we will consider the spectral gap and the logarithnic Sobolev inequality for L^U (or equivalently, for $\int_M (\nabla f, \nabla g) dm^U$) and give some estimates of the spectral gap constant and logarithmic Sobolev constant. Note that that the logarithmic Sobolev inequality (hence the spectral gap, too) holds is proved by using the strict positivity of the heat kernel (see Chapter VI of Deuschel and Stroock [3]). In the latter part we will consider the Gibbs measures on $M^{\mathbf{Z}^\nu}$ determined by a finite range and shift-invariant potential and will apply those results for finite dimensional manifolds to the Ising models.

In fact, our article is a generalization of Deuschel and Stroock [2] and based on it. They argued those problems on smooth manifolds (without boundary) by searching for constants which make the Bakry-Emery criterion [1] hold. In our case a similar argument holds with a slight modification by adding a certain term which includes the second fundamental form. When the second fundamental form is non-negative our results in this article are the same as those in [2]. So we will prepare some estimates of the second fundamental form in Lemma 2.4 to have those estimates of the logarithmic Sobolev constant in

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a similar way to Deuschel and Stroock [2]. When the situation is not very bad, we will obtain the estimates in terms of geometric quantities such as Ricci tensor, Hess$U$ and the constants which we have in the estimates of the second fundamental form.

Because logarithmic Sobolev inequalities demonstrate their real ability in infinite dimensional situations, we would like to find such an application. As in Deuschel and Stroock [2], we will consider finite range and shift-invariant potential $\mathfrak{U}$ and the Gibbs states with potential $\mathfrak{U}$. Because various objects which are written in terms of the extreme elements of the Gibbs states with potential $\mathfrak{U}$ are easily obtained by finite dimensional approximation, we will show that logarithmic Sobolev inequalities hold under some additional conditions.

The organization of this paper is as follows. In Section 2 we will prepare some estimates from below of the second fundamental form and by using them we will show spectral gap inequalities on finite dimensional manifolds with boundary. In Section 3 we will show several versions of logarithmic Sobolev inequalities. In Section 4 we will give two concrete examples for which we can apply the results in Sections 2 and 3. In Section 5 we will construct the stochastic Ising model whose spin space is a manifold with boundary and will give a sufficient condition for the logarithmic Sobolev inequality.

2. Spectral gap

In this section we will consider the spectral gap inequality on a finite dimensional manifold with boundary under the Neumann boundary condition. Let N be the inner normal vector on ∂M and $C_N^\infty(M) = \{f \in C^\infty(M) | \nabla_N f = 0\}$. We define a bilinear Markovian form a^U as

$$a^U(f,g) = \int_M (\nabla f | \nabla g) dm^U, \qquad f,g \in C_N^\infty(M),$$

and a second-order differential operator L^U on $C_N^\infty(M)$ as

$$L^U f = \triangle f - (\nabla U | \nabla f), \qquad f \in C_N^\infty(M).$$

Then it is well-known that the closure of a^U is a Dirichlet form whose corresponding generator is the closure of L^U (we denote them again by a^U and L^U, respectively). We will denote the corresponding semigroup by P_t^U.

We say that the spectral gap inequality (or Poincaré inequality) for a^U (or L^U, P_t^U) holds if, for some $C > 0$,

$$\|f - \langle f \rangle_{m^U}\|^2_{L^2(m^U)} \le \frac{1}{C} a^U(f,f), \qquad f \in \mathrm{Dom}(a^U). \tag{2.1}$$

Remark 2.1. The best constant for which (2.1) holds is called the spectral gap constant and is denoted by $C(U)$. We adopt the definition of the spectral gap constant in Deuschel-Stroock [2]. However, it is often defined as the reciprocal number of our definition.

By the general theory (2.1) is equivalent to either (2.2) or (2.3) below:

(2.2)
$$\|P_t^U f - \langle f\rangle_{m^U}\|^2_{L^2(m^U)} \le e^{-Ct}\|f - \langle f\rangle_{m^U}\|^2_{L^2(m^U)}, \quad t>0,\ f\in L^2(m^U);$$

(2.3)
$$\ker(L^U) = \{\text{constant}\}, \quad \mathrm{Spec}(-L^U)\subset\{0\}\cup[C,\infty).$$

However, in order to show the spectral gap inequality, we will use the following form;

(2.4)
$$\int_M |L^U f|^2 dm^U \ge C\int_M \|\nabla f\|^2 dm^U, \qquad f\in C_N^\infty(M).$$

It is easy to see that (2.4) is equivalent to (2.1).

Before we show (2.4), we need some preparation. First we will prepare an integration by parts formula. Though it is an easy formula, it will play a basic role. Note that we will sometimes abuse the notation to write $L^U f$, even if f does not satisfy the Neumann boundary condition.

Lemma 2.2. *Let $d\sigma$ be the surface measure on ∂M and $d\sigma^U = e^{-U}d\sigma$. Then, for any smooth functions f and g, we have*

$$-\int_M fL^U g\,dm^U = \int_M (\nabla f|\nabla g)dm^U + \int_{\partial M} f\nabla_N g\,d\sigma^U.$$

In particular, if $f,g\in C_N^\infty(M)$,

$$-\int_M fL^U g\,dm^U = a^U(f,g).$$

Proof. By a straight-forward calculation we can easily see that the lemma is reduced to the case $U=0$, which is well-known. □

Next we will introduce $\Gamma_2^U(f,f)$. For $f,g\in C_N^\infty(M)$, let us define

$$\Gamma_2^U(f,g) = \frac{1}{2}\{L^U(\nabla f|\nabla g) - (\nabla L^U f|\nabla g) - (\nabla f|\nabla L^U g)\}.$$

Then by the Weitzenböck formula we have the following lemma.

Lemma 2.3. *For any $f\in C_N^\infty(M)$, $\Gamma_2^U(f,f)$ is explicitly known as*

$$\Gamma_2^U(f,f) = \|\mathrm{Hess} f\|^2 + (\mathrm{Ric} + \mathrm{Hess}U)(\nabla f,\nabla f).$$

Proof. This is well-known. So we omit the proof. □

We will introduce a bilinear form on ∂M. For $\theta,\eta\in\Gamma(T^*M)$ satisfying $(\theta|N) = (\eta|N) = 0$, let

$$A(\theta,\eta) = \frac{1}{2}\nabla_N(\theta|\eta).$$

It is known that $A(\cdot,\cdot)$ is tensorial on ∂M in the sense that

$$A(f\theta, g\eta) = fgA(\theta,\eta), \qquad f,g \in C_N^\infty(M).$$

$A(\cdot,\cdot)$ is called the second fundamental form.

Now we will show (2.4). By Lemmas 2.2 and 2.3 we have

$$\begin{aligned}\int_M |L^U f|^2 dm^U &= \int_M \left\{\Gamma_2^U(f,f) - \frac{1}{2}L^U\|\nabla f\|^2\right\} dm^U \\ &= \int_M \{\|\mathrm{Hess} f\|^2 + (\mathrm{Ric} + \mathrm{Hess}U)(\nabla f,\nabla f)\} dm^U \\ &\quad + \int_{\partial M} A(\nabla f,\nabla f) d\sigma^U,\end{aligned} \tag{2.5}$$

for any $f \in C_N^\infty(M)$.

Now we will give a simple estimate of $\int_{\partial M} A(\nabla f,\nabla f)d\sigma^U$.

Lemma 2.4. *For any positive constant $\bar{K_2}$, there exists a constant $\bar{K_1}$ depending on U such that*

$$\int_{\partial M} A(\nabla f,\nabla f)d\sigma^U \geq -\bar{K_2}\int_M \|\nabla^2 f\|^2 dm^U - \bar{K_1}\int_M \|\nabla f\|^2 dm^U, \tag{2.6}$$

for any smooth f satisfying the Neumann boundary condition. Similarly, for any positive constant K_2, there exist a constant K_1 depending on U such that

$$\begin{aligned}\int_{\partial M} A(\nabla f,\nabla f)d\sigma^U \\ \geq -K_2\int_M \left\{\|\nabla^2 f\|^2 - \frac{1}{d}|\triangle f|^2\right\} dm^U - K_1\int_M \|\nabla f\|^2 dm^U,\end{aligned} \tag{2.7}$$

for any smooth f satisfying the Neumann boundary condition.

Proof. It is sufficient to prove (2.7) for $U = 0$. First we write $A(\nabla f,\nabla f)$ in a local coordinate. Fix a point $p \in \partial M$. Then we may choose an open neighbourhood of p coresponding to an open set in $\{x \in \mathrm{R}^d; x^d \geq 0\}$ in which the Riemannian metric tensor $g = (g_{ij})$ satisfies $g_{id} = 0$ if $i \neq d$ and $g_{dd} = 1$.

Since $\nabla_N f = \partial_d|_{x^d=0} f = 0$ and commutativity of $\partial_d|_{x^d=0}$ and $\partial_i (i \neq d)$, we have

$$\begin{aligned}A(\nabla f,\nabla f) &= \frac{1}{2}\nabla_N \|\nabla f\|^2 \\ &= \frac{1}{2}\partial_d|_{x^d=0}\sum_{i,j=1}^{d} g^{ij}\partial_i f\partial_j f \\ &= \frac{1}{2}\sum_{i,j=1}^{d-1}(\partial_d g^{ij})\partial_i f\partial_j f,\end{aligned} \tag{2.8}$$

where $(g^{ij})_{i,j=1}^d$ is the inverse matrix of $(g_{ij})_{i,j=1}^d$.

Since M is compact, we may choose such finitely many connected open sets $\Omega_1,\ldots,\Omega_n$ of ∂M and $\delta_0>0$ satisfying that $\sigma(\partial M)=\sigma(\cup_{i=1}^n\bar\Omega_i)$ and that $\sigma(\bar\Omega_i\cap\bar\Omega_j)=0$ if $i\neq j$ and that each $\bar\Omega_i$ is contained in a coordinate neighborhood as above and so is $\bar\Omega_i\times[0,\delta_0)$.

First we will express Hess$f$ in the local coordinate which contains $\bar\Omega_1$ as follows;

(2.9)

$$\begin{aligned}\mathrm{Hess}f &= \sum_{1\le i,j\le d}\left\{\sum_k g^{jk}\partial_k\partial_i f+\sum_k \partial_i g^{jk}\partial_k f+\sum_{k,s}\Gamma^j_{ki}g^{ks}\partial_s f\right\}\frac{\partial}{\partial x^j}\otimes dx^i\\ &= \sum_{1\le i,j\le d} Z_i^j\frac{\partial}{\partial x^j}\otimes dx^i,\quad \text{say.}\end{aligned}$$

Here Hess$f$ is regarded as a smooth section of $TM\otimes T^*M$ and $\Gamma^j_{ki}=(1/2)\sum_n g^{jn}(\partial_i g_{nk}+\partial_k g_{in}-\partial_n g_{ki})$ is the Christoffel symbol. Set $j=d$ and $i\neq d$. Then we have

$$Z_i^d=\partial_d\partial_i f+\frac{1}{2}\sum_s \partial_d g^{is}\partial_s f. \tag{2.10}$$

Now we will show that

$$c\sum_{i=1}^{d-1}(Z_i^d)^2\le \|\mathrm{Hess}f\|^2-\frac{1}{d}|\triangle f|^2 \tag{2.11}$$

for some positive constant c. Let $\langle e^1(x),\cdots,e^{d-1}(x)\rangle$ be an orthonormal base of T_xM which is obtained from $\{\partial/\partial x^j\}_1^{d-1}$ by the Schmidt orthogonalization. Let $\tilde V(x)$ be a square matrix of size $(d-1)$ which represents the Schmidt orthogonalization above i.e.,

$$\left\langle\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^{d-1}}\right\rangle\tilde V(x)=\langle e^1(x),\ldots,e^{d-1}(x)\rangle.$$

We may assume that $\tilde V(x)$ is continuous in $x\in\bar\Omega_1\times[0,\delta_0]$ and, moreover, there exist a constant $c>0$ such that

$$c\|w\|^2_{\mathbf{R}^{d-1}}\le\|\tilde V^{-1}(x)w\|^2_{\mathbf{R}^{d-1}} \tag{2.12}$$

for every $w\in\mathbf{R}^{d-1}$ and x. If we use the new base $\langle e^1(x),\ldots,e^{d-1}(x),\partial/(\partial x^d)\rangle$ then Hess$f$, which is also regarded as a linear operator of T_xM, is experessed as $V^{-1}(x)Z(x)V(x)$, where $Z=(Z_i^j)_{i,j=1}^d$ and

$$V=\left(\begin{array}{ccc|c} & & & 0\\ & \tilde V & & \vdots\\ & & & 0\\ \hline 0 & \ldots & 0 & 1\end{array}\right).$$

Set $\tilde{Z} = (Z_i^j)_{i,j=1}^{d-1}$, $w = (Z_1^d, \dots, Z_{d-1}^d)^T \in \mathbf{R}^{d-1}$ and $v = (Z_d^1, \dots, Z_d^{d-1})^T \in \mathbf{R}^{d-1}$. Then we have

$$V^{-1}ZV = \left(\begin{array}{c|c} \tilde{V}^{-1}\tilde{Z}\tilde{V} & \tilde{V}^{-1}w \\ \hline v^T\tilde{V} & 1 \end{array}\right). \tag{2.13}$$

Then by (2.12) and (2.13) and the fact that $\triangle f = \mathrm{Trace}(\mathrm{Hess} f)$, we have

$$\begin{aligned} c\sum_{i=1}^{d-1}(Z_i^d)^2 &= c\|w\|^2_{\mathbf{R}^{d-1}} \\ &\le \|\tilde{V}^{-1}(x)w\|^2_{\mathbf{R}^{d-1}} \\ &\le \sum_{i,j=1}^{d}\{(V^{-1}ZV)_i^j\}^2 - \sum_{i=1}^{d}\{(V^{-1}ZV)_i^i\}^2 \\ &\le \|\mathrm{Hess} f\|^2 - \frac{1}{d}|\triangle f|^2, \end{aligned}$$

where $(V^{-1}ZV)_i^j$ is the entities of $V^{-1}ZV$. This shows (2.11).

On the other hand there exists a constant $\delta \in [0, \delta_0]$ such that

$$\begin{aligned} &\sum_{i,j=1}^{d-1}(\partial_d g^{ij})(x^1, \dots, x^{d-1}, \delta)(\partial_i f \partial_j f)(x^1, \dots, x^{d-1}, \delta) \\ &\qquad = \frac{1}{\delta_0}\int_0^{\delta_0}\sum_{i,j=1}^{d-1}\{(\partial_d g^{ij})(\partial_i f \partial_j f)\}dx^d. \end{aligned}$$

Then we have

$$\begin{aligned} &\left|\sum_{i,j=1}^{d-1}(\partial_d g^{ij})(x^1, \dots, x^{d-1}, 0)(\partial_i f \partial_j f)(x^1, \dots, x^{d-1}, 0)\right| \\ &= \left|\frac{1}{\delta_0}\int_0^{\delta_0}\sum_{i,j=1}^{d-1}\{(\partial_d g^{ij})(\partial_i f \partial_j f)\}dx^d - \int_0^{\delta}\sum_{i,j=1}^{d-1}\partial_d\{(\partial_d g^{ij})(\partial_i f \partial_j f)\}dx^d\right| \\ &= \left|\frac{1}{\delta_0}\int_0^{\delta_0}\sum_{i,j=1}^{d-1}\{(\partial_d g^{ij})(\partial_i f \partial_j f)\}dx^d\right. \\ &\qquad \left. - \int_0^{\delta}\sum_{i,j=1}^{d-1}\{(\partial_d^2 g^{ij})(\partial_i f \partial_j f) + 2(\partial_d g^{ij}\partial_j f)(\partial_d \partial_i f)\}dx^d\right| \\ &\le c_1\int_0^{\delta_0}\|\nabla f\|^2 dx^d + c_2\int_0^{\delta_0}\sum_{i=1}^{d-1}\{Z_i^d\}^2 dx^d, \end{aligned} \tag{2.14}$$

where c_1, c_2 are positive constants independent of f. Note that we can take $c_2 > 0$ arbitrarily small (in that case c_1 may be greater) if we use the fact that $2xy \leq \epsilon x^2 + \epsilon^{-1}y^2$ for $x, y \in \mathbf{R}$ and $\epsilon > 0$.

Hence we have from (2.11) and (2.14) that there are positive constants c_3 and c_4 such that

$$\begin{aligned}
&\int_{\bar{\Omega}_1} |A(\nabla f, \nabla f)| d\sigma \\
&= \frac{1}{2} \int_{\bar{\Omega}_1} \Big| \sum_{i,j=1}^{d-1} (\partial_d g^{ij})(x^1, \dots, x^{d-1}, 0)(\partial_i f \partial_j f)(x^1, \dots, x^{d-1}, 0) \Big| \\
&\quad \times \sqrt{\det\{(g_{ij})_{i,j=1}^{d-1}\}(x^1, \dots, x^{d-1}, 0)} dx^1 \cdots dx^{d-1} \\
&\leq c_3 \int_0^{\delta_0} \int_{\bar{\Omega}_1} \|\nabla f\|^2 \sqrt{\det\{(g_{ij})_{i,j=1}^{d}\}(x^1, \dots, x^d)} dx^1 \cdots dx^d \\
&\quad + c_4 \int_0^{\delta_0} \int_{\bar{\Omega}_1} \left\{ |\mathrm{Hess} f|^2 - \frac{1}{d} |\triangle f|^2 \right\} \sqrt{\det\{(g_{ij})_{i,j=1}^{d}\}(x^1, \dots, x^d)} dx^1 \cdots dx^d \\
&\leq c_3 \int_M \|\nabla f\|^2 dm + c_4 \int_M \left\{ |\mathrm{Hess} f|^2 - \frac{1}{d} |\triangle f|^2 \right\} dm.
\end{aligned}$$

Note that we can take $c_4 > 0$ arbitrarily small (in tha case c_3 may be greater). Similarly we obtain the similar estimates for $\bar{\Omega}_2, \dots, \bar{\Omega}_n$. Summing them up, we complete the proof. □

For $\epsilon > 0$ we set

(2.15) $$\bar{K}(U)(\epsilon) = \inf\{\bar{K}_1 \in \mathbf{R} | (2.6) \text{ holds for } \bar{K}_2 = \epsilon \text{ and } \bar{K}_1\}$$

and

(2.16) $$K(U)(\epsilon) = \inf\{K_1 \in \mathbf{R} | (2.7) \text{ holds for } K_2 = \epsilon \text{ and } K_1\}.$$

Lemma 2.5. *$\bar{K}(U)(\epsilon)$ and $K(U)(\epsilon)$ are decresing and continuous function of ϵ.*

Proof. We prove the lemma only for $K(U)$. That $K(U)$ is decreasing is easy. Noting that

$$K(U)(\tau\epsilon + (1-\tau)\epsilon') \leq \tau K(U)(\epsilon) + (1-\tau)K(U)(\epsilon')$$

for $\epsilon, \epsilon' > 0$ and $\tau \in [0, 1]$, we see that $K(U)$ is convex. Hence $K(U)$ is continuous. □

We will show some versions of the spectral gap inequality by estimating the right hand side of (2.5) in terms of $K(U)$, $\bar{K}(U)$ and

$$\rho(U) = \sup\{\rho \in \mathbf{R} | (\mathrm{Ric} + \mathrm{Hess} U)(X, X) \geq \rho \|X\|^2, X \in \Gamma(TM)\}.$$

If the constants $\bar{K_1}$ and $\bar{K_2}$ in (2.6) of Lemma 2.4 can be taken so small as $\bar{K_2} \le 1$ and $\rho(U) > \bar{K_1}$, or equivalently,

$$\rho(U) > \bar{K}(U)(1) \tag{2.17}$$

then we have

$$\begin{aligned}\int_M |L^U f|^2 dm^U &\ge (1-\bar{K_2}) \int_M \|\nabla^2 f\|^2 dm^U + (\rho(U) - \bar{K_1}) \int_M \|\nabla f\|^2 dm^U \\ &\ge (\rho(U) - \bar{K_1}) \int_M \|\nabla f\|^2 dm^U,\end{aligned}$$

which implies

$$C(U) \ge \rho(U) - \bar{K}(U)(1).$$

Next we consider the case $U = 0$. If the constants K_1 and K_2 in (2.7) of Lemma 2.4 can be taken so small as $K_2 \le 1$ and $\rho(0) > K_1$, or equivalently,

$$\rho(0) > K(0)(1) \tag{2.18}$$

then we have

$$\begin{aligned}&\int_M |\triangle f|^2 dm \\ &\ge (1-K_2) \int_M \|\nabla^2 f\|^2 dm + K_2 \int_M \frac{1}{d} |\triangle f|^2 dm + (\rho(0) - K_1) \int_M \|\nabla f\|^2 dm \\ &\ge \frac{1}{d} \int_M |\triangle f|^2 dm + (\rho(0) - K_1) \int_M \|\nabla f\|^2 dm.\end{aligned}$$

which implies

$$C(0) \ge \frac{d}{d-1} (\rho(0) - K(0)(1)). \tag{2.19}$$

At the end of this section we will estimate $C(U)$ in terms of $C(0)$ and $K(U)$ under some assumptions.

Proposition 2.6. *Assume that the constants K_1 and K_2 in* (2.7) *of Lemma* 2.4 *could be taken so small as*

$$1 \ge K_2 \quad \textit{and} \quad \frac{C(0)e^{-\delta(U)}}{d} + \rho(U) - K_1 > 0,$$

or equivalently

$$\frac{C(0)e^{-\delta(U)}}{d} + \rho(U) - K(U)(1) > 0, \tag{2.20}$$

where $\delta(U) = \max_{x \in M} U(x) - \min_{x \in M} U(x)$. Then the spectral gap inequality for $\mathcal{E}^U$ holds and

$$C(U) \ge \frac{C(0)e^{-\delta(U)}}{d} + \rho(U) - K(U)(1).$$

Proof. For $K_2 \le 1$ we have

$$\begin{aligned}
&\int_M |L^U f|^2 dm^U \\
&\ge \int_M \|\nabla^2 f\|^2 dm^U + \rho(U)\int_M \|\nabla f\|^2 dm^U \\
&\quad - K_2 \int_M \left\{ \|\nabla^2 f\|^2 - \frac{1}{d}|\triangle f|^2 \right\} dm^U - K_1 \int_M \|\nabla f\|^2 dm^U \\
&\ge (1-K_2)\int_M \|\nabla^2 f\|^2 dm^U + \frac{e^{-\max U} K_2}{d}\int_M |\triangle f|^2 dm \\
&\quad + (\rho(U) - K_1)\int_M \|\nabla f\|^2 dm^U \\
&\ge \left(\frac{C(0)e^{-\delta(U)}K_2}{d} + \rho(U) - K_1 \right) \int_M \|\nabla f\|^2 dm^U.
\end{aligned}$$

Here we used the fact that $d\|\nabla^2 f\|^2 \ge |\triangle f|^2$ and the spectral gap inequality for $U = 0$ for the last inequality above. Setting $K_2 = 1$ and $K_1 = K(U)(1)$ we complete the proof. □

3. Logarithmic Sobolev inequality

In this section we will show the logarithmic Sobolev inequality with respect to a^U and represent the logarithmic Sobolev constant in terms of $K(U)$, $C(0)$ and $\rho(U)$.

Let L^U, a^U and P_t^U be as in the previous sections. We will say the logarithmic Sobolev inequality for a^U (or L^U, P_t^U) holds if

$$\int_M f^2 \log \frac{f^2}{\|f\|^2_{L^2(m^U)}} dm^U \le \frac{2}{\alpha} a^U(f,f), \qquad f \in \mathrm{Dom}(a^U). \tag{3.1}$$

Remark 3.1. The best constant for which (3.1) holds is called the logarithmic Sobolev constant and is denoted by $\alpha(U)$. We adopt the definition of the logarithmic Sobolev constant in Deuschel-Stroock [2]. However, it is often defined as the reciprocal number of our definition.

It is well-known that the logarithmic Sobolev inequality (3.1) is equivalent to the hypercontractivity;

$$\|P_t^U f\|_{L^q(m^U)} \le \|f\|_{L^p(m^U)}, \qquad f \in L^p(m^U), \tag{3.2}$$

where $q = 1 + (p-1)\exp(2t/\alpha)$ and $p \in (1,\infty)$.

We will show the logarithmic Sobolev inequality of the following form. There is a positive constant α for which

$$\int_M f \log \frac{f}{\langle f \rangle_{m^U}} dm^U \le \frac{2}{\alpha}\int_M (\nabla f^{1/2}, \nabla f^{1/2}) dm^U, \tag{3.3}$$

holds for any strictly positive $f \in C_N^\infty(M)$. It is easy to see that (3.3) is equivalent to (3.1).

Take an arbitrary positive function $f \in C_N^\infty(M)$. Without loss of generality we may assume $\langle f \rangle_{m^U} = 1$. Set $f_t = P_t^U f$ and $H_t = \langle f_t \log f_t \rangle_{m^U}$. Differentiating the both sides by t we have

$$\begin{aligned} H_t' &= \langle L^U f_t \log f_t + L^U f_t \rangle_{m^U} \\ &= -\langle (\nabla f_t | \nabla \log f_t) \rangle_{m^U} \\ &= -\left\langle \frac{\|\nabla f_t\|^2}{f_t} \right\rangle_{m^U} \\ &= -4 \langle \|\nabla f_t^{1/2}\|^2 \rangle_{m^U} \\ &= -\langle f_t \|\nabla \log f_t\|^2 \rangle_{m^U}. \end{aligned}$$

Here we used Lemma 2.2 and that $\nabla_N f_t = 0$.

By this equation we see that the logarithmic Sobolev inequality (3.3) is equivalent to

$$H(0) \le -\frac{1}{2\alpha} H'(0). \tag{3.4}$$

Since $H(t) \to 0$ as $t \to \infty$, in order to show (3.4) it is sufficient to prove

$$-H''(t) \le 2\alpha H'(t). \tag{3.5}$$

Before we caluculate $H''(t)$, we need to compute $L^U(\log f_t)$;

$$L^U(\log f_t) = \frac{L^U f_t}{f_t} - \frac{\|\nabla f_t\|^2}{f_t^2} = \frac{L^U f_t}{f_t} - \|\nabla \log f_t\|^2. \tag{3.6}$$

By (3.6) and integration by parts formula, we compute $H''(t)$ as follows:

$$\begin{aligned} (3.7)\ -H''(t) & \\ &= \frac{d}{dt} \langle (\nabla \log f_t | \nabla f_t) \rangle_{m^U} \\ &= \left\langle \left(\nabla \frac{L^U f_t}{f_t} | \nabla f_t \right) \right\rangle_{m^U} + \langle (\nabla \log f_t | \nabla L^U f_t) \rangle_{m^U} \\ &= \langle (\nabla \{ L^U(\log f_t) + \|\nabla \log f_t\|^2 \} | \nabla f_t) \rangle_{m^U} - \langle L^U(\log f_t) L^U f_t \rangle_{m^U} \\ &= 2 \langle f_t (\nabla L^U(\log f_t) | \nabla \log f_t) \rangle_{m^U} + \langle (\nabla f_t | \nabla \|\nabla \log f_t\|^2) \rangle_{m^U} \\ &= 2 \langle f_t (\nabla L^U(\log f_t) | \nabla \log f_t) \rangle_{m^U} \\ &\quad - \langle f_t L^U \| \log f_t\|^2 \rangle_{m^U} - \int_{\partial M} f_t \nabla_N \|\nabla \log f_t\|^2 d\sigma^U \\ &= -2 \langle f_t \Gamma_2^U(\log f_t, \log f_t) \rangle_{m^U} - 2 \int_{\partial M} f_t A(\nabla \log f_t, \nabla \log f_t) d\sigma^U. \end{aligned}$$

By (3.5) and (3.7), we see that the following inequality (3.8) is sufficient in order to prove the logarithmic Sobolev inequality.

$$\begin{aligned}(3.8)\qquad &\alpha(U)\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U}\\ &\le \langle f\Gamma_2^U(\log f,\log f)\rangle_{m^U}+\int_{\partial M} fA(\nabla\log f,\nabla\log f)d\sigma^U\\ &=\langle f\{\|\mathrm{Hess}(\log f)\|^2+(\mathrm{Ric}+\mathrm{Hess}U)(\nabla\log f,\nabla\log f)\}\rangle_{m^U}\\ &\quad+\frac{1}{2}\int_{\partial M} f\nabla_N\|\nabla\log f\|^2 d\sigma^U.\end{aligned}$$

First we estimate the last term of the right hand side of (3.8). By (2.6) in Lemma 2.4, we obtain

$$\begin{aligned}(3.9)\qquad &\int_{\partial M} fA(\nabla\log f,\nabla\log f)d\sigma^U\\ &=4\int_{\partial M}A(\nabla f^{1/2},\nabla f^{1/2})d\sigma^U\\ &\ge -4\bar{K}_2\int_M\|\nabla^2 f^{1/2}\|^2dm^U-4\bar{K}_1\int_M\|\nabla f^{1/2}\|^2dm^U,\\ &=-4\bar{K}_2\int_M\|\nabla^2 f^{1/2}\|^2dm^U-\bar{K}_1\int_M\frac{\|\nabla f\|^2}{f}dm^U,\end{aligned}$$

or by (2.7) we obtain

$$\begin{aligned}(3.10)\qquad &\int_{\partial M} fA(\nabla\log f,\nabla\log f)d\sigma^U\\ &\ge -4K_2\int_M\left\{\|\nabla^2 f^{1/2}\|^2-\frac{1}{d}|\triangle f^{1/2}|^2\right\}dm^U-K_1\int_M\frac{\|\nabla f\|^2}{f}dm^U.\end{aligned}$$

Lemma 3.2. *Let $f>0$ be a positive smooth function on M satisfying $\nabla_N f=0$. Then we have*

$$\begin{aligned}&\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_m\\ &\quad\ge\frac{4}{d+2}\langle 2\|\mathrm{Hess}f^{1/2}\|^2+(\triangle f^{1/2})^2\rangle_m\\ &\quad=\frac{4}{d+2}\langle 3(\triangle f^{1/2})^2-\mathrm{Ric}(\nabla f^{1/2},\nabla f^{1/2})\rangle_m\\ &\qquad-\frac{8}{d+2}\int_{\partial M}A(\nabla f^{1/2},\nabla f^{1/2})d\sigma.\end{aligned}$$

Proof. The equation above is easily seen by the modified Weitzenböck formula. We will give the same proof for the inequality as that in [2] for readers' convienience. By setting $f^{1/2}=h$ and the fact that

$$\mathrm{Hess}(\log h)=\frac{\mathrm{Hess}h}{h}-\frac{\nabla h\otimes\nabla h}{h^2},$$

we have that

$$f\|\mathrm{Hess}(\log f)\|^2 = 4h^2\|\mathrm{Hess}(\log h)\|^2 = 4\left\{\|\mathrm{Hess}h\|^2 - \frac{2\mathrm{Hess}h(\nabla h,\nabla h)}{h} + \frac{\|\nabla h\|^4}{h^2}\right\}.$$

By integration by parts formula and the fact that $2\mathrm{Hess}h(\nabla h,\nabla h) = (\nabla h, \nabla\|\nabla h\|^2)$, we have that

$$\begin{aligned}(3.11)\qquad &\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_m \\ &= 4\left\{\langle\|\mathrm{Hess}h\|^2\rangle_m - \left\langle\frac{(\nabla h,\nabla\|\nabla h\|^2)}{h}\right\rangle_m + \left\langle\frac{\|\nabla h\|^4}{h^2}\right\rangle_m\right\} \\ &= 4\left\{\langle\|\mathrm{Hess}h\|^2\rangle_m + \langle\triangle\log h\|\nabla h\|^2\rangle_m + \left\langle\frac{\|\nabla h\|^4}{h^2}\right\rangle_m\right\} \\ &= 4\left\{\langle\|\mathrm{Hess}h\|^2\rangle_m + \left\langle\frac{\triangle h\|\nabla h\|^2}{h}\right\rangle_m\right\}.\end{aligned}$$

Here we used (3.6) to the last equality.

From (3.6) it is easy to see that

$$\langle f(\triangle\log f)^2\rangle_m = 4\langle h^2(\triangle\log h)^2\rangle_m = 4\left\{\langle(\triangle h)^2\rangle_m - 2\left\langle\frac{\triangle h\|\nabla h\|^2}{h}\right\rangle_m + \left\langle\frac{\|\nabla h\|^4}{h^2}\right\rangle_m\right\}.$$

Since $(\triangle\log f)^2 \le d\|\mathrm{Hess}(\log f)\|^2$, we have

$$\begin{aligned}&4\left\langle\frac{\triangle h\|\nabla h\|^2}{h}\right\rangle_m \\ &= 2\langle(\triangle h)^2\rangle_m + 2\left\langle\frac{\|\nabla h\|^4}{h^2}\right\rangle_m - \frac{1}{2}\langle f(\triangle\log f)^2\rangle_m \\ &\ge 2\langle(\triangle h)^2\rangle_m - \frac{d}{2}\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_m.\end{aligned}$$

Combining the above inequality with (3.11), we obtain

$$\begin{aligned}&\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_m - 4\langle\|\mathrm{Hess}h\|^2\rangle_m \\ &\qquad \ge 2\langle(\triangle h)^2\rangle_m - \frac{d}{2}\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_m,\end{aligned}$$

which completes the proof. □

Now we will show several versions of the logarithmic Sobolev inequality by Lemma 3.2 above and inequalities (3.9) and (3.10) and have estimates of $\alpha(U)$ in terms of K_i's, $K(U)$, $\bar{K}(U)$, $\rho(U)$ and $C(0)$ by estimating the right hand side of (3.8). We first deal with the terms involving the second order derivatives. Assume there exist $\bar{K_1}$ and $\bar{K_2}$ such that

$$(3.12)\qquad \bar{K_2} \le \frac{2e^{-\delta(U)}}{d+2} \quad\text{and}\quad \frac{C(0)(e^{-\delta(U)} - \bar{K_2})}{d} + \rho(U) - \bar{K_1} > 0.$$

Then we have

(3.13)

$$
\begin{aligned}
&\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_{m^U} + \int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U \\
&\geq e^{-\max U}\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_m - 4\bar{K}_2\langle \|\mathrm{Hess}h\|^2\rangle_{m^U} - \bar{K}_1\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \frac{4e^{-\max U}}{d+2}\langle 2\|\mathrm{Hess}h\|^2 + (\triangle h)^2\rangle_m \\
&\quad - 4e^{-\min U}\bar{K}_2\langle \|\mathrm{Hess}h\|^2\rangle_m - \bar{K}_1\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \left\{\frac{1}{d}\left(\frac{8e^{-\max U}}{d+2} - 4e^{-\min U}\bar{K}_2\right) + \frac{4e^{-\max U}}{d+2}\right\}\langle (\triangle h)^2\rangle_m \\
&\quad - \bar{K}_1\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \frac{4}{d}(e^{-\max U} - \bar{K}_2 e^{-\min U})\langle (\triangle h)^2\rangle_m - \bar{K}_1\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U}.
\end{aligned}
$$

Here we used the fact that $(\triangle h)^2 \leq d\|\mathrm{Hess}h\|^2$ and condition (3.12).

By the spectral gap inequality for $U = 0$, we obtain $\langle (\triangle h)^2\rangle_m \geq C(0)\langle \|\nabla h\|^2\rangle_m$. Hence,

(3.14)

$$
\begin{aligned}
&\langle f\{\|\mathrm{Hess}(\log f)\|^2 + (\mathrm{Ric} + \mathrm{Hess}U)(\nabla \log f, \nabla \log f)\}\rangle_{m^U} \\
&\quad + \int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U \\
&\geq \frac{4C(0)}{d}(e^{-\max U} - \bar{K}_2 e^{-\min U})\langle \|\nabla h\|^2\rangle_m + (\rho(U) - \bar{K}_1)\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&= \frac{C(0)}{d}(e^{-\max U} - \bar{K}_2 e^{-\min U})\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_m + (\rho(U) - \bar{K}_1)\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \left\{\frac{C(0)(e^{-\delta(U)} - \bar{K}_2)}{d} + \rho(U) - \bar{K}_1\right\}\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U}.
\end{aligned}
$$

By (3.8) we see that the logarithmic Sobolev inequality holds under assumption (3.12) with the logarithmic Sobolev constant $\alpha(U)$ satisfying that

$$\alpha(U) \geq \frac{C(0)(e^{-\delta(U)} - \bar{K}_2)}{d} + \rho(U) - \bar{K}_1. \tag{3.15}$$

In a similar way as above, we will obtain another estimate of the right hand side of (3.8) under assumption that there exist K_1 and K_2 such that

$$K_2 \leq \frac{2e^{-\delta(U)}}{d+2} \quad \text{and} \quad \frac{C(0)e^{-\delta(U)}}{d} + \rho(U) - K_1 > 0,$$

or equivalently,

$$\frac{C(0)e^{-\delta(U)}}{d} + \rho(U) - K(U)\left(\frac{2e^{-\delta(U)}}{d+2}\right) > 0. \tag{3.16}$$

Then we have by (3.10) that

$$\begin{aligned}
&\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_{m^U} + \int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U \\
&\geq \frac{4e^{-\max U}}{d+2}\langle 2\|\mathrm{Hess}h\|^2 + (\triangle h)^2\rangle_m \\
&\quad - 4e^{-\min U}K_2\left\langle\left\{\|\mathrm{Hess}h\|^2 - \frac{1}{d}(\triangle h)^2\right\}\right\rangle_m - K_1\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \left\{\left(\frac{8e^{-\max U}}{d+2} - 4e^{-\min U}K_2\right)\right\}\langle\|\mathrm{Hess}h\|^2\rangle_m \\
&\quad + \left\{\frac{4e^{-\max U}}{d+2} + \frac{4e^{-\min U}K_2}{d}\right\}\langle(\triangle h)^2\rangle_m - K_1\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \frac{4e^{-\max U}}{d}\langle(\triangle h)^2\rangle_m - K_1\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&\geq \left\{\frac{C(0)e^{-\delta(U)}}{d} - K_1\right\}\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U}.
\end{aligned} \tag{3.17}$$

Thus we have proved the following: Then the logarithmic Sobolev inequality (3.1) holds with the logarithmic Sobolev constant $\alpha(U)$ satisfying that

$$\alpha(U) \geq \frac{C(0)e^{-\delta(U)}}{d} + \rho(U) - K(U)\left(\frac{2e^{-\delta(U)}}{d+2}\right). \tag{3.18}$$

By the second estimate of $\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_{m^U}$ in Lemma 3.2, we will show another version of the logarithmic Sobolev inequality under some additional assumption on $A(\cdot,\cdot)$. For example, if we assume the non-negativity of A, i.e., $A(\nabla f, \nabla f) \geq 0$ for any $f \in C_N^\infty(M)$, then

$$\alpha(U) \geq \frac{3C(0)e^{-\delta(U)} + N\rho(\frac{N+2}{N}U) + 2(1-e^{-\delta(U)})(\rho(0)\wedge 0)}{d+2}.$$

This is the same result as in [2] and can be proved in a similar way as in [2]. So we omit the proof.

Here we will give a version of the logarithmic Sobolev inequality when the second fundamental form is non-positive, i.e., $A(\nabla f, \nabla f) \leq 0$ for any $f \in C_N^\infty(M)$. Set

$$\theta = \frac{2-(d+2)K_2e^{\delta(U)}}{2(1+K_2)}$$

and

(3.19)

$$\Theta_1 = C(0)e^{-\delta(U)}\left(\frac{1+2\theta}{d+2} + \frac{K_2}{d}\left(1 - \frac{2\theta e^{-\delta(U)}}{d+2}\right)\right) - K_1\left(1 - \frac{2\theta e^{-\delta(U)}}{d+2}\right),$$
$$\Theta_2 = 1 - \frac{2\theta}{d+2},$$
$$\Theta_3 = \frac{2\theta(1-e^{-\delta(U)})}{d+2}.$$

We will assume the following:

$$K_2 \le \frac{2e^{-\delta(U)}}{d+2} \quad \text{and} \quad \Theta_1 + \Theta_2\rho\left(\frac{U}{\Theta_2}\right) + \Theta_3\rho(0) \wedge 0 > 0. \tag{3.20}$$

Then by this assumption we see that $\theta \in [0,1]$.

Applying Lemma 3.2 we have

(3.21)

$$\begin{aligned}
&\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_{m^U} + \int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U \\
&\quad = \theta\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_{m^U} + (1-\theta)\langle f\|\mathrm{Hess}(\log f)\|^2\rangle_{m^U} \\
&\qquad + \int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U \\
&\quad \ge \frac{4\theta e^{-\max U}}{d+2}\langle 3(\triangle h)^2 - 2\mathrm{Ric}(\nabla h, \nabla h)\rangle_m \\
&\qquad - \frac{8\theta e^{-\max U}}{d+2}\int_{\partial M} A(\nabla f^{1/2}, \nabla f^{1/2})d\sigma \\
&\qquad + \frac{4(1-\theta)e^{-\max U}}{d+2}\langle 2\|\mathrm{Hess}h\|^2 + (\triangle h)^2\rangle_m + \int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U \\
&\quad \ge \frac{4(1+2\theta)e^{-\max U}}{d+2}\langle(\triangle h)^2\rangle_m - \frac{8\theta e^{-\max U}}{d+2}\langle \mathrm{Ric}(\nabla h, \nabla h)\rangle_m \\
&\qquad + \frac{8(1-\theta)e^{-\max U}}{d+2}\langle\|\mathrm{Hess}h\|^2\rangle_m \\
&\qquad + \left(1 - \frac{2\theta e^{-\delta(U)}}{d+2}\right)\int_{\partial M} fA(\nabla \log f, \nabla \log f)d\sigma^U.
\end{aligned}$$

Here we have used the non-positivity of the second fundamental form for the last inequality. Applying (3.10) and the definition of θ (θ is determined so that the coefficient of the term $\|\mathrm{Hess}h\|^2$ should vanish in the following), the right hand side of (3.21) is larger than or equal to

$$\begin{aligned}
&\left\{\frac{4(1+2\theta)e^{-\max U}}{d+2} + \frac{4K_2 e^{-\max U}}{d}\left(1 - \frac{2\theta e^{-\delta(U)}}{d+2}\right)\right\}\langle(\triangle h)^2\rangle_m \\
&\quad - \frac{8\theta e^{-\max U}}{d+2}\langle \mathrm{Ric}(\nabla h, \nabla h)\rangle_m - K_1\left(1 - \frac{2\theta e^{-\delta(U)}}{d+2}\right)\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U}.
\end{aligned} \tag{3.22}$$

By the spectral gap inequality for $U=0$ and the fact that

$$
\begin{aligned}
(3.23)\quad & -\langle e^{-\max U}\mathrm{Ric}(\nabla h,\nabla h)\rangle_m \\
&\geq -\langle e^{-U}\mathrm{Ric}(\nabla h,\nabla h)\rangle_m + (1-e^{-\delta(U)})\langle e^{-U}\mathrm{Ric}(\nabla h,\nabla h)\wedge 0\rangle_m,
\end{aligned}
$$

we see that (3.22) is larger than or equal to

$$
\begin{aligned}
(3.24)\quad & \Theta_1\left\langle \frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} \\
&-\frac{8\theta}{d+2}\{\langle\mathrm{Ric}(\nabla h,\nabla h)\rangle_{m^U} - (1-e^{-\delta(U)})\langle\mathrm{Ric}(\nabla h,\nabla h)\wedge 0\rangle_{m^U}\}.
\end{aligned}
$$

Combining (3.21), (3.22) and (3.24), we finally obtain an estimate of the right hand side of (3.8) as follows;

(3.25)

$$
\begin{aligned}
&\langle f\{\|\mathrm{Hess}(\log f)\|^2 + (\mathrm{Ric}+\mathrm{Hess}U)(\nabla\log f,\nabla\log f)\}\rangle_{m^U} \\
&\quad + \int_{\partial M} fA(\nabla\log f,\nabla\log f)d\sigma^U \\
&\geq \Theta_1\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U} + \left\langle\frac{1}{f}\left\{\left(1-\frac{2\theta}{d+2}\right)\mathrm{Ric}+\mathrm{Hess}U\right\}(\nabla f,\nabla f)\right\rangle_{m^U} \\
&\quad + \frac{2\theta(1-e^{-\delta(U)})}{d+2}\left\langle\frac{1}{f}\mathrm{Ric}(\nabla f,\nabla f)\wedge 0\right\rangle_{m^U} \\
&\geq \left\{\Theta_1+\Theta_2\rho\left(\frac{U}{\Theta_2}\right)+\Theta_3\rho(0)\wedge 0\right\}\left\langle\frac{\|\nabla f\|^2}{f}\right\rangle_{m^U},
\end{aligned}
$$

where the constants Θ_1, Θ_2 and Θ_3 are given in (3.19). Here we have another version of the logarithmic Sobolev inequality with the logarithmic Sobolev constant $\alpha(U)$ satisfying that

$$
(3.26)\qquad \alpha(U)\geq\Theta_1+\Theta_2\rho\left(\frac{U}{\Theta_2}\right)+\Theta_3\rho(0)\wedge 0.
$$

Summing up the results in this section, we have proved the following proposition.

Proposition 3.3. *Assume* (3.12)*, then we have*

$$
\alpha(U)\geq\frac{C(0)(e^{-\delta(U)}-\bar{K_2})}{d}+\rho(U)-\bar{K_1}.
$$

Assume (3.16)*, then we have*

$$
\alpha(U)\geq\frac{C(0)e^{-\delta(U)}}{d}+\rho(U)-K(U)\left(\frac{2e^{-\delta(U)}}{d+2}\right).
$$

Assume (3.20) *and the non-positivity of the second fundamental form, then we have*

$$\alpha(U) \geq \Theta_1 + \Theta_2 \rho \left(\frac{U}{\Theta_2} \right) + \Theta_3 \rho(0) \wedge 0.$$

4. Examples

Example 4.1. Let M be the closed interval $[0,1]$ in R and $U = 0$. We calculate the logarithmic Sobolev constant for this case. In this case we see by direct computation that $A(\nabla f, \nabla f) = 0$ for any smooth function which satisfies the Neumann boundary condition. Applying (3.18) with $d = 1$ and $\rho(U) = 0$, we obtain $\alpha(0) \geq C(0)$. Since $\alpha(0) \leq C(0)$ always holds, we obtain $\alpha(0) = C(0)$.

The spectral gap constant $C(0)$ is computed by solving the following eigenvalue problem;

$$\begin{cases} -f''(x) = \lambda f(x) \\ f'(0) = f'(1) = 0. \end{cases}$$

We see that the smallest non-zero eigenvalue is $\lambda = \pi^2$, which is the spectral gap $C(0)$. Thus we have $\alpha(0) = C(0) = \pi^2$.

Example 4.2. Let $S^2 = \{(x,y,z) \in \mathrm{R}^3; x^2 + y^2 + z^2 = 1\}$ be the 2-dimensional sphere in R^3. We introduce the polar coordinate on S^2 as follows. Any point $(x,y,z) \in S^2$ is written as $(x,y,z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, for some $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.

Let $M = \{(x,y,z) \in S^2; z \geq \cos\phi_0\}$ and $U = 0$, where $\phi_0 \in (\pi/2, \pi)$ is a constant sufficiently close to $\pi/2$ (When $\phi_0 \in (0, \pi/2]$ the second fundamental form is non-negative and the problem is easy. When ϕ_0 is close to π our method, as we will see later, cannot be applied.).

Here we will give several facts without proofs. First $\mathrm{Ric} = Id$ at every point in S^2, so $\rho(0) = 1$. Next the Riemannian measure is written as $dm = \sin\phi d\phi d\theta$ and the surface measure as $d\sigma = \sin\phi d\theta$.

We express several terms in the polar coordinate;

$$\|\nabla f\|^2 = (\partial_\phi f)^2 + \left(\frac{1}{\sin\phi} \partial_\theta f \right)^2$$

$$\|\nabla^2 f\|^2 = (\partial_\phi^2 f)^2 + 2 \left| \partial_\phi \left(\frac{\partial_\theta f}{\sin\phi} \right) \right|^2 + \left(\frac{1}{\sin^2\phi} \partial_\theta^2 f + \frac{\cos\phi}{\sin\phi} \partial_\phi f \right)^2.$$

We calculate $A(\nabla f, \nabla f)$ and find it negative as follows;

$$\begin{aligned} A(\nabla f, \nabla f) &= \frac{1}{2} \nabla_N \|\nabla f\|^2 \\ &= -\frac{1}{2} \partial_\phi|_{\phi=\phi_0} \left\{ \left(\frac{\partial_\theta f}{\sin\phi} \right)^2 + (\partial_\phi f)^2 \right\} \\ &= \frac{\cos\phi_0}{\sin\phi_0} \left(\frac{\partial_\theta f}{\sin\phi_0} \right)^2 \leq 0. \end{aligned}$$

By the mean value theorem there is a constant $\xi \in (\phi_1, \phi_0)$ such that

$$\frac{\partial_\theta f(\theta,\xi)}{\sin\xi} = \frac{1}{2|\cos\phi_0|}\int_{\phi_1}^{\phi_0}\left(\frac{\partial_\theta f}{\sin\phi}\right)\sin\phi d\phi,$$

where $\phi_1 = \pi - \phi_0$.

Then we have

$$\begin{aligned}\frac{\partial_\theta f(\theta,\phi_0)}{\sin\phi_0} &= \frac{\partial_\theta f(\theta,\xi)}{\sin\xi} + \int_\xi^{\phi_0}\partial_\phi\left(\frac{\partial_\theta f}{\sin\phi}\right)d\phi \\ &= \frac{1}{2|\cos\phi_0|}\int_{\phi_1}^{\phi_0}\left(\frac{\partial_\theta f}{\sin\phi}\right)\sin\phi d\phi + \int_\xi^{\phi_0}\partial_\phi\left(\frac{\partial_\theta f}{\sin\phi}\right)d\phi.\end{aligned}$$

By the Schwarz inequality we have

$$\begin{aligned}\left|\frac{\partial_\theta f(\theta,\phi_0)}{\sin\phi_0}\right|^2 \le \frac{1+c}{2|\cos\phi_0|}\int_{\phi_1}^{\phi_0}\left(\frac{\partial_\theta f}{\sin\phi}\right)^2\sin\phi d\phi \\ + \left(1+\frac{1}{c}\right)\int_{\phi_1}^{\phi_0}\frac{1}{\sin\phi}d\phi\int_{\phi_1}^{\phi_0}\left|\partial_\phi\left(\frac{\partial_\theta f}{\sin\phi}\right)\right|^2\sin\phi d\phi,\end{aligned}$$

where c is a positive constant which will be determined later.

We have an estimate the integral on the boundary as follows;

$$\begin{aligned}&\left|\int_{\partial M} A(\nabla f,\nabla f)d\sigma\right| \\ &= \int_0^{2\pi}\frac{|\cos\phi_0|}{\sin\phi_0}\left(\frac{\partial_\theta f}{\sin\phi_0}\right)^2\sin\phi_0 d\theta \\ &\le \frac{1+c}{2}\int_0^{2\pi}\int_{\phi_1}^{\phi_0}\left(\frac{\partial_\theta f}{\sin\phi}\right)^2\sin\phi d\phi d\theta \\ &\quad + \left(1+\frac{1}{c}\right)\frac{|\cos\phi_0|}{2}\int_{\phi_1}^{\phi_0}\frac{1}{\sin\phi}d\phi\int_0^{2\pi}\int_{\phi_1}^{\phi_0}2\left|\partial_\phi\left(\frac{\partial_\theta f}{\sin\phi}\right)\right|^2\sin\phi d\phi d\theta \\ &\le \frac{1+c}{2}\langle\|\nabla f\|^2\rangle_m + \left(1+\frac{1}{c}\right)|\cos\phi_0|\int_{\pi/2}^{\phi_0}\frac{1}{\sin\phi}d\phi\left\{\left\langle\|\nabla^2 f\| - \frac{1}{2}|\triangle f|^2\right\rangle_m\right\}.\end{aligned}$$

If there exist $c > 0$, for given ϕ_0, which satisfies,

$$1 - \frac{1+c}{2} > 0, \tag{4.1}$$

$$\left(1+\frac{1}{c}\right)|\cos\phi_0|\int_{\pi/2}^{\phi_0}\frac{1}{\sin\phi}d\phi \le 1,$$

then by (2.19), we see that spectral gap with the spectral gap constant $C(0)$ such that

$$C(0) \ge 2\left\{1 - \frac{1}{2(1-|\cos\phi_0|\int_{\pi/2}^{\phi_0}\frac{1}{\sin\phi}d\phi)}\right\}.$$

Suppose that there is the spectral gap constant $C(0)$. If there exist $\hat{c} > 0$, for given ϕ_0, which satisfies,

$$\frac{C(0)}{2} + 1 - \frac{1+\hat{c}}{2} > 0, \tag{4.2}$$

$$\left(1 + \frac{1}{\hat{c}}\right) |\cos\phi_0| \int_{\pi/2}^{\phi_0} \frac{1}{\sin\phi} d\phi \leq \frac{1}{2},$$

then we see from (3.18) that the logarithmic Sobolev inequality holds with the logarithmic Sobolev constant

$$\alpha(0) \geq \frac{C(0)}{2} + 1 - \frac{1}{2(1 - 2|\cos\phi_0| \int_{\pi/2}^{\phi_0} \frac{1}{\sin\phi} d\phi)}.$$

Though we do not completely know for which ϕ_0 the conditions (4.1) and (4.2) hold, we make sure by direct computation that the both conditions are true at least for $\phi_0 \in [\pi/2, 2\pi/3]$. For example, if $\phi_0 = 2\pi/3$, then we obtain

$$|\cos\phi_0| \int_{\pi/2}^{\phi_0} \frac{1}{\sin\phi} d\phi \leq \frac{1}{2} \cdot \frac{\pi}{12} \left\{ \left(\sin\frac{2\pi}{3}\right)^{-1} + \left(\sin\frac{7\pi}{12}\right)^{-1} \right\} \leq 0.287$$

and conditions (4.1) and (4.2) are satisfied and

$$C(0) \geq 0.597 \qquad \text{and} \qquad \alpha(0) \geq 0.125.$$

Unfortunately, our method cannot be applied when ϕ_0 is close to π, at least when $\phi_0 \in [3\pi/4, \pi)$.

5. Applications to the stochastic Ising model

Let M be as in the preceding with the normalized Riemannian measure m and let ν be some fixed positive natural number. Set $E = M^{\mathbf{Z}^\nu}$ and endow E the product topology. Let $\mathbf{m} = m^{\mathbf{Z}^\nu}$ be the product measure on $(E, \mathcal{B}_E)$. Given a non-empty subset $\Lambda \subset \mathbf{Z}^\nu$, set $E_\Lambda = M^\Lambda$. Let $\mathbf{x} \in E \mapsto \mathbf{x}_\Lambda \in E_\Lambda$ denote the natural projection. We define a mapping $\Phi_\Lambda : E \times E \to E$ so that

$$\Phi_\Lambda(\mathbf{x}|\mathbf{y})_\mathbf{k} = \begin{cases} \mathbf{x}_\mathbf{k} & \text{if} \quad \mathbf{k} \in \Lambda, \\ \mathbf{y}_\mathbf{k} & \text{if} \quad \mathbf{k} \notin \Lambda. \end{cases}$$

We will sometimes think of $\Phi_\Lambda(\cdot|\mathbf{y})$ or $\Phi_\Lambda(\mathbf{x}|\cdot)$ as a function on E_Λ or E_{Λ^C}. As usual when $\Lambda = \{\mathbf{k}\}$ is a one-point set, we will write $E_\mathbf{k}$, $\mathbf{x}_\mathbf{k}$ and $\Phi_\mathbf{k}$ for simplicity. For $\Lambda \subset \mathbf{Z}^\nu$ we set $\mathcal{B}^\Lambda = \sigma\{\mathbf{x}_\mathbf{k} | \mathbf{k} \in \Lambda\}$ and $\tilde{\mathcal{B}}^\Lambda = \sigma\{\mathbf{x}_\mathbf{k} | \mathbf{k} \notin \Lambda\}$.

We introduce a differential structure in the same way as in [2]. Let $C^\infty(E)$ be the space of all continuous functions f on E such that, for each $\phi \neq \Lambda \subset\subset \mathbf{Z}^\nu$ (i.e., a finite non-empty subset),

$$\mathbf{x} \in E_{\Lambda^c} \mapsto f \circ \Phi_\Lambda(\cdot|\mathbf{x}) \in C^\infty(E_\Lambda)$$

is continuous. We define differential operations on $C^\infty(E)$ as the lift of those on $C^\infty(M)$. In other words differential operators on $C^\infty(E)$ are the partial differential operators. For example, for $\mathbf{k} \in \mathbf{Z}^\nu$, we define

$$(X_{\mathbf{k}} f)(\mathbf{x}) = [X(f \circ \Phi_{\mathbf{k}}(\cdot|\mathbf{x}))](\mathbf{x}_{\mathbf{k}}) \qquad \text{for} \qquad X \in \Gamma(TM),$$
$$(\nabla_{\mathbf{k}} f)(\mathbf{x}) = [\nabla(f \circ \Phi_{\mathbf{k}}(\cdot|\mathbf{x}))](\mathbf{x}_{\mathbf{k}}), \qquad \text{etc.}$$

Next we introduce measures on E. Let

$$\mathfrak{U} = \{J_F : \phi \neq F \subset\subset \mathbf{Z}^\nu\}$$

be a shift-invariant, finite range potential on E. That is, for each F, $J_F \in C^\infty(E)$ which depends only on $\mathbf{x}_F$, $J_{F+\mathbf{k}} = J_F \circ S^{-\mathbf{k}}$, $\mathbf{k} \in \mathbf{Z}^\nu$, where S is the natural shift on $\mathbf{Z}^\nu$, and there is a fixed $\Lambda \subset\subset \mathbf{Z}^\nu$ which contains 0 with the property that, if $F \supset\supset \Lambda$, then $J_F \equiv 0$ for any F containing 0.

Given $\Lambda \subset\subset \mathbf{Z}^\nu$, we set

$$U_\Lambda(\mathbf{x}|\mathbf{y}) = \sum_{F : F \cap \Lambda \neq \emptyset} J_F \circ \Phi_\Lambda(\mathbf{x}|\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in E,$$

and the probability measure $\mu_\Lambda(\cdot|\mathbf{y})$ on $(E, \mathcal{B}^\Lambda)$ (or E_Λ) by

$$\mu_\Lambda(d\mathbf{x}|\mathbf{y}) = \frac{\exp[-U_\Lambda(\mathbf{x}|\mathbf{y})]}{Z_\Lambda(\mathbf{y})} \mathbf{m}(d\mathbf{x}),$$

where $Z_\Lambda(\mathbf{y}) = \int_E \exp[-U_\Lambda(\mathbf{x}|\mathbf{y})]\mathbf{m}(d\mathbf{x})$ is the normalizing constant. When $\Lambda = \{\mathbf{k}\}$ is a one-point set, we simply write $U_{\mathbf{k}}$, $\mu_{\mathbf{k}}$ and $Z_{\mathbf{k}}$. It is easily verified that $\{\mu_\Lambda\}_{\Lambda \subset\subset \mathbf{Z}^\nu}$ are consistent family of finite dimensional distributions.

Definition 5.1. We will say that a probability measure μ on $(E, \mathcal{B}_E)$ is a Gibbs state with potential $\mathfrak{U}$ and we will write $\mu \in \mathfrak{G}(\mathfrak{U})$ if

$$\int_E f d\mu = \int_E \Big(\int_E f \circ \Phi_\Lambda(\mathbf{x}|\mathbf{y}) \mu_\Lambda(d\mathbf{x}|\mathbf{y}) \Big) \mu(d\mathbf{y})$$

holds for any $f \in C(E)$ and $\Lambda \subset\subset \mathbf{Z}^\nu$.

We will introduce the operator $\mathcal{L}^{\mathfrak{U}}$, which is the analogue of L^U in the finite dimensional case. We set the space $\mathfrak{G}(E)$ of cylinder functions as follows;

$$\mathfrak{G}(E) = \{f \in C^\infty(E) | f(\mathbf{x}) = \bar{f}(\mathbf{x}_\Lambda) \text{ for some } \Lambda \subset\subset \mathbf{Z}^d \text{ and } \bar{f} \in C_N^\infty(E_\Lambda)\}.$$

We then define the operator $\mathcal{L}^{\mathfrak{U}}$ on $\mathfrak{G}(E)$ by

$$\mathcal{L}^{\mathfrak{U}} f = \sum_{\mathbf{k} \in \mathbf{Z}^\nu} e^{U_{\mathbf{k}}} \Big(\mathrm{div}_{\mathbf{k}} (e^{-U_{\mathbf{k}}} \nabla_{\mathbf{k}} f) \Big) = \sum_{\mathbf{k} \in \mathbf{Z}^\nu} \Big(\triangle_{\mathbf{k}} f - (\nabla_{\mathbf{k}} U_{\mathbf{k}} | \nabla_{\mathbf{k}} f) \Big),$$

where $U_{\mathbf{k}} = U_{\mathbf{k}}(\mathbf{x}|\mathbf{x})$. As in the finite dimensional case it is easy to see from Definition 5.1 that, for $\mu \in \mathfrak{G}(\mathfrak{U})$ and $f, g \in \mathfrak{G}(E)$,

$$(5.1) \qquad -\int_E (\mathcal{L}^{\mathfrak{U}} f) g d\mu = \sum_{\mathbf{k} \in \mathbf{Z}^\nu} \int_E (\nabla_{\mathbf{k}} f | \nabla_{\mathbf{k}} g) d\mu.$$

In particular $\mathcal{L}^{\mathfrak{U}}$ is symmetric in $L^2(\mu)$.

Next we will show that (the closure of) the right hand side of (5.1) is a Dirichlet form coresponding to $\mathcal{L}^{\mathfrak{U}}$ for given $\mu \in \mathfrak{G}(\mathfrak{U})$. Set, for $h = 1, 2, 3, \ldots$,

$$a_h(f, g) = \sum_{|\mathbf{k}| \le h} \int_E (\nabla_{\mathbf{k}} f | \nabla_{\mathbf{k}} g) d\mu.$$

with its domain $\mathrm{Dom}(a_h) = \mathfrak{G}(E)$. It is easy to see that $\{a_h\}_{h=1,2,3,\ldots}$ is a family of Markovian forms on $L^2(\mu)$ and that $\{a_h\}_{h=1,2,3,\cdots}$ is asymptotically regular in $L^2(\mu)$, i.e., there exists a dense subset $\mathcal{C} \subset C(E)$, such that for every $f \in \mathcal{C}$ we have

$$\liminf_{h \to \infty} a_h(f_h, f_h) < \infty, \quad \text{for some} \quad f_h \to f \text{ in } L^2(\mu)$$

(In our case we set $\mathcal{C} = \mathfrak{G}(E)$).

By the results in Section 4 of Mosco [7], we obtain the following (see Mosco [7] for the precise definition of the terms in the following propositions):

Proposition 5.2. *There exists a densely defined Dirichlet form a in $L^2(\mu)$, and a subsequence $\{a_{h'}\}$ of $\{a_h\}$, such that $a_{h'}$ Γ-converges to a in $L^2(\mu)$ as $h' \to \infty$. Moreover $\mathcal{C} \subset \mathrm{Dom}(a)$ and $\bar{a_{\mathcal{C}}}$ is a regular Dirichlet form with core $\mathcal{C}$ in $L^2(\mu)$ and a is an extension of $\bar{a_{\mathcal{C}}}$.*

Proposition 5.3. *Let $\{a_h\}$ be a sequence of strongly local, closable Markovian forms defined on a common domain $\mathcal{C} = \mathrm{Dom}(a_h)$, $\mathcal{C}$ being the dense separating subalgebra of $C(E)$. Let a_h Γ-converge to a in $L^2(\mu)$ as $h \to \infty$. If the sequence of energy measures are bounded and absolutely continuous with respect to μ in E, then $\mathcal{C} \subset \mathrm{Dom}(a)$ and $\bar{a_{\mathcal{C}}}$ is also strongly local.*

Since the energy measure is $\sum_{|\mathbf{k}| \le h} (\nabla_{\mathbf{k}} f | \nabla_{\mathbf{k}} g) d\mu$ in our case, we may apply the propositions above. It is obvious that $\bar{a_{\mathcal{C}}}$ is the desired Dirichlet form and

$$\bar{a_{\mathcal{C}}}(f, f) = \sum_{\mathbf{k} \in \mathbf{Z}^\nu} \int_E (\nabla_{\mathbf{k}} f | \nabla_{\mathbf{k}} f) d\mu, \qquad f \in \mathfrak{G}(E). \tag{5.2}$$

We denote by $P_t^{\mathfrak{U},\mu}$ the semigroup on $L^2(\mu)$ corresponding to $\bar{a_{\mathcal{C}}}$. In the same way as in Holley-Stroock [6] and Dobrushin [4], [5] we obtain the following proposition:

Proposition 5.4. *$\mathfrak{G}(\mathfrak{U})$ is a convex and compact subset of the totality of probability measures on E which is equipped with the weak topology. Moreover, $\mu \in \mathfrak{G}(\mathfrak{U})$ is an extreme element if and only if the tail field $\mathcal{B}_T = \cap_{\Lambda \subset\subset \mathbf{Z}^\nu} \tilde{\mathcal{B}}^\Lambda$ of μ is trivial. It is also known that*

$$\lim_{t \to \infty} \| P_t^{\mathfrak{U},\mu} f - \mu[f | \mathcal{B}_T] \|_{L^2(\mu)} = 0 \qquad \textit{for every} \quad f \in L^2(\mu).$$

Therefore

$$\lim_{t \to \infty} \| P_t^{\mathfrak{U},\mu} f - \langle f \rangle_\mu \|_{L^2(\mu)} = 0 \qquad \textit{for every} \quad f \in L^2(\mu) \tag{5.3}$$

if and only if μ is an extreme element.

Since the logarithmic Sobolev inequality or the spectral gap holds only if (5.3) holds, we will think only of the extreme elements of $\mathfrak{G}(\mathfrak{U})$ in the following.

Let $\Lambda \subset\subset \mathbf{Z}^\nu$. We will consider the problems on E_Λ. Let $\mu_\Lambda(\cdot|\mathbf{y})$ be the regular conditional distribution given the projection $\mathbf{x} \in E \mapsto \mathbf{x}_\Lambda \in E_\Lambda$ as in Definition 5.1. However, we regard $\mu_\Lambda(\cdot|\mathbf{y})$ as a probability measure on E_Λ. We define an operator $L_\Lambda^{\mathbf{y},\mathfrak{U}} = L_\Lambda$ as follows:

$$L_\Lambda f(x) = \sum_{\mathbf{k}\in\Lambda}\Big(\triangle_{\mathbf{k}} f - (\nabla_{\mathbf{k}} U_{\mathbf{k}}(x|\Phi_\Lambda(x|\mathbf{y}))|\nabla_{\mathbf{k}} f)\Big), \tag{5.4}$$

for $f \in C_N^\infty(E_\Lambda)$ (we will sometimes abuse the notation to denote $L_\Lambda f$ for a smooth function f, even if f does not satisfy the boundary condition and will write $U_{\mathbf{k}} = U_{\mathbf{k}}(x|\Phi_\Lambda(x|\mathbf{y}))$ for simplicity). Then by the consistency of the conditional distributions we obtain an integration by parts formula as follows:

$$\begin{aligned} &-\int_{E_\Lambda}(f\cdot L_\Lambda g)(x)\mu_\Lambda(dx|\mathbf{y}) \\ &\quad= \sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda}(\nabla_{\mathbf{k}} f|\nabla_{\mathbf{k}} g)(x)\mu_\Lambda(dx|\mathbf{y}) \\ &\quad\quad+ \sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda}\mu_\Lambda(dx|\mathbf{y})\int_{\partial E_{\mathbf{k}}}(f\cdot\nabla_{\mathbf{k},N} g)(x)\sigma_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_\Lambda(x|\mathbf{y})), \end{aligned} \tag{5.5}$$

for $f, g \in C^\infty(E_\Lambda)$ and $\mathbf{y} \in E$. Here $\sigma_{\mathbf{k}}(dz|\mathbf{w}) = Z_{\mathbf{k}}(\mathbf{w})^{-1}e^{-U_{\mathbf{k}}(z|\mathbf{w})}\sigma(dz)$ is the surface measure corresponding to $\mu_{\mathbf{k}}(dz|\mathbf{w})$.

In order to make sure the validity of the finite dimensional approximation we need to prove the following Lemma 5.5.

Lemma 5.5. *Let $\mu \in \mathfrak{G}(\mathfrak{U})$ be an extreme element and f a bounded function on E.*

If we set $\Lambda_h = \{\mathbf{k} \in \mathbf{Z}^\nu | |\mathbf{k}| \le h\}$ $(h = 1, 2, 3, \dots)$ and

$$G_h f(\mathbf{y}) = \int_E f\circ\Phi_{\Lambda_h}(\mathbf{x}|\mathbf{y})\mu_{\Lambda_h}(d\mathbf{x}|\mathbf{y}),$$

then $G_h f \to \langle f\rangle_\mu$ as $h \to \infty$ μ-almost surely and in $L^1(\mu)$ (as a result in $L^p(\mu)$, $p \in [1,\infty)$).

Proof. Since $G_h f = \mu[f|\tilde{\mathcal{B}}^{\Lambda_h}]$, $\{G_h f\}_{h=1,2,3,\dots}$ is a equi-continuous "reversed time" martingale. Hence $G_h f$ converges μ-almost surely and in $L^1(\mu)$. Since $\lim G_h f$ is measurable with respect to the tail field $\mathcal{B}_T$, it is easily verified that $\lim G_h f = \langle f\rangle_\mu$ by Proposition 5.4. □

By Lemma 5.5 we may reduce the problems (the logarithmic Sobolev inequality and the spectral gap) for $\bar{a_{\mathcal{C}}}(\cdot,\cdot)$ in (5.2) to the finite dimensional cases. Indeed, suppose that we prove the logarithmic Sobolev inequality for

$(E_\Lambda, \mu_\Lambda(dx|\mathbf{y}))$ and $L_\Lambda^{y,\mathfrak{U}}$ as follows;

$$\int_E f \log f \mu_\Lambda(dx|\mathbf{y}) - \int_E f \mu_\Lambda(dx|\mathbf{y}) \log \int_E f \mu_\Lambda(dx|\mathbf{y}) \leq \frac{2}{\alpha} \int_E \sum_{\mathbf{k}\in\Lambda} \|\nabla_{\mathbf{k}} f^{1/2}\|^2 \mu_\Lambda(dx|\mathbf{y}), \tag{5.6}$$

for every strictly positive $f \in C_N^\infty(E_\Lambda)$, with α independent of Λ and y. Then by letting $\Lambda = \Lambda_h$, taking the expectation with respect to μ and letting $h \to \infty$ we obtain

$$\int_E f \log f d\mu - \int_E f d\mu \log \int_E f d\mu \leq \frac{2}{\alpha} \int_E \sum_{\mathbf{k}} \|\nabla_{\mathbf{k}} f^{1/2}\|^2 d\mu. \tag{5.7}$$

By the denseness of the cylinder functions, this shows that the logarithmic Sobolev inequality for $\bar{a_{\mathcal{C}}}(\cdot,\cdot)$ holds. The same method works for the spectral gap inequality (however, we omit the proof).

Let $L_\Lambda^{\mathbf{y},\mathfrak{U}} = L_\Lambda$ be as in (5.4). Then the square field operator in this case is defined as follows;

$$\Gamma_2^\Lambda(f,f) = \frac{1}{2} \sum_{\mathbf{k}\in\Lambda} L_\Lambda(\nabla_{\mathbf{k}} f|\nabla_{\mathbf{k}} f) - \sum_{\mathbf{k}\in\Lambda} (\nabla_{\mathbf{k}} L_\Lambda f|\nabla_{\mathbf{k}} f).$$

As in Deuchel-Stroock [2] we define

$$\mathrm{Hess}_{\mathbf{k},\mathbf{l}}(f)(X_{\mathbf{k}}, Y_{\mathbf{l}})(\mathbf{x}) = \begin{cases} \mathrm{Hess}(f \circ \Phi(\cdot|\mathbf{x}))(X,Y)(\mathbf{x_k}) & (\text{if } \mathbf{k} = \mathbf{l}) \\ X_{\mathbf{k}} \circ Y_{\mathbf{l}} f(\mathbf{x}) & (\text{if } \mathbf{k} \neq \mathbf{l}), \end{cases}$$

for $f \in C^\infty(E)$ and X, Y are smooth vector fields on E, and

$$\mathrm{Hess}_{\mathbf{k},\mathbf{l}}(\mathfrak{U}) = \sum_{F: F \ni \mathbf{k},\mathbf{l}} \mathrm{Hess}_{\mathbf{k},\mathbf{l}}(J_F),$$

for $\mathbf{k}, \mathbf{l} \in \mathbf{Z}^\nu$. We also define

$$\Gamma_{2,\mathbf{k}}(f,f) = \|\mathrm{Hess}_{\mathbf{k},\mathbf{k}}(f)\|^2 + (\mathrm{Ric}_{\mathbf{k}} + \mathrm{Hess}_{\mathbf{k},\mathbf{k}}(\mathfrak{U}))(\nabla_{\mathbf{k}} f, \nabla_{\mathbf{k}} f),$$

where $\mathrm{Ric}_{\mathbf{k}}$ be the Ricci tensor on $E_{\mathbf{k}}$ and

$$R^\Lambda(f,f) = \sum_{\mathbf{k},\mathbf{l}\in\Lambda, \mathbf{k}\neq\mathbf{l}} \mathrm{Hess}_{\mathbf{k},\mathbf{l}}(\mathfrak{U})(\nabla_{\mathbf{k}} f, \nabla_{\mathbf{l}} f).$$

Lemma 5.6. *For every $f \in \mathfrak{G}(E)$ we obtain*

$$\Gamma_2^\Lambda(f,f) \geq \sum_{\mathbf{k}\in\Lambda} \Gamma_{2,\mathbf{k}}(f,f) + R^\Lambda(f,f).$$

Proof. Let $L_{\mathbf{i}}f = \triangle_{\mathbf{i}}f - (\nabla_{\mathbf{i}}U_{\mathbf{i}}|\nabla_{\mathbf{i}}f)$. If $\mathbf{i} \neq \mathbf{j}$, then $[\nabla_{\mathbf{j}}, \nabla_{\mathbf{i}}] = [\nabla_{\mathbf{j}}, \triangle_{\mathbf{i}}] = 0$. Hence

$$\begin{aligned}
&\frac{1}{2}L_{\mathbf{i}}(\nabla_{\mathbf{j}}f|\nabla_{\mathbf{j}}f) - (\nabla_{\mathbf{j}}L_{\mathbf{i}}f|\nabla_{\mathbf{j}}f) \\
&\quad = \frac{1}{2}\triangle_{\mathbf{i}}(\nabla_{\mathbf{j}}f|\nabla_{\mathbf{j}}f) - \frac{1}{2}\Big(\nabla_{\mathbf{i}}U_{\mathbf{i}} \mid \nabla_{\mathbf{i}}(\nabla_{\mathbf{j}}f|\nabla_{\mathbf{j}}f)\Big) \\
&\qquad -(\triangle_{\mathbf{i}}\nabla_{\mathbf{j}}f|\nabla_{\mathbf{j}}f) + \Big(\nabla_{\mathbf{j}}(\nabla_{\mathbf{i}}U_{\mathbf{i}}|\nabla_{\mathbf{i}}f) \mid \nabla_{\mathbf{j}}f\Big) \\
&\quad = \|\nabla_{\mathbf{i}}\nabla_{\mathbf{j}}f\|^2 + (\nabla_{\mathbf{j}}\nabla_{\mathbf{i}}U_{\mathbf{i}}|\nabla_{\mathbf{i}}f \otimes \nabla_{\mathbf{j}}f).
\end{aligned}$$

If $\mathbf{i} = \mathbf{j}$, then by the same way as in the finite dimensional case we have

$$\frac{1}{2}L_{\mathbf{j}}(\nabla_{\mathbf{j}}f|\nabla_{\mathbf{j}}f) - (\nabla_{\mathbf{j}}L_{\mathbf{j}}f|\nabla_{\mathbf{j}}f) = \|\mathrm{Hess}_{\mathbf{j}}(f)\|^2 + (\mathrm{Ric}_{\mathbf{j}} + \mathrm{Hess}_{\mathbf{j}}U_{\mathbf{j}})(\nabla_{\mathbf{j}}f, \nabla_{\mathbf{j}}f).$$

Summing up these results we see that Lemma 5.6 holds. □

Let $\mathbf{y} \in E$ and $\Lambda \subset\subset \mathbf{Z}^\nu$ be arbitrary. As we showed in the previous sections, the spectral gap

$$(5.8) \qquad \int_{E_\Lambda} (f - \langle f\rangle_{\mu_\Lambda(\cdot|\mathbf{y})})^2 \mu_\Lambda(dx|\mathbf{y}) \leq \frac{1}{C}\sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda} \|\nabla_{\mathbf{k}}f\|^2 \mu_\Lambda(dx|\mathbf{y}),$$

for every $f \in C_N^\infty(E_\Lambda)$ is equivalent to

$$\begin{aligned}
(5.9) \qquad C\sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda} \|\nabla_{\mathbf{k}}f\|^2(x)\mu_\Lambda(dx|\mathbf{y}) &\leq \int_{E_\Lambda} \Gamma_2^\Lambda(f,f)(x)\mu_\Lambda(dx|\mathbf{y}) \\
&+ \sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda}\mu_\Lambda(dx|\mathbf{y})\int_{\partial E_{\mathbf{k}}} A_{\mathbf{k}}(\nabla_{\mathbf{k}}f, \nabla_{\mathbf{k}}f)(x)\sigma_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi(x|\mathbf{y}))
\end{aligned}$$

for every $f \in C_N^\infty(E_\Lambda)$.

Similarly, to prove the logarithmic Sobolev inequality (5.6), it is sufficient to show

$$\begin{aligned}
(5.10) \qquad \alpha\sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda} \frac{\|\nabla_{\mathbf{k}}f\|^2}{f}(x)\mu_\Lambda(dx|\mathbf{y}) &\leq \int_{E_\Lambda} f\Gamma_2^\Lambda(\log f, \log f)(x)\mu_\Lambda(dx|\mathbf{y}) \\
&+ \sum_{\mathbf{k}\in\Lambda}\int_{E_\Lambda}\mu_\Lambda(dx|\mathbf{y})\int_{\partial E_{\mathbf{k}}} fA_{\mathbf{k}}(\nabla_{\mathbf{k}}\log f, \nabla_{\mathbf{k}}\log f)(x)\sigma_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi(x|\mathbf{y}))
\end{aligned}$$

for every strictly positive $f \in C_N^\infty(E_\Lambda)$.

Remark 5.7. First we prepare notations. set $S_f = \cap\{\Omega \subset \mathbf{Z}^\nu | f$ is $\mathcal{B}^\Omega$-measurable$\}$ and $|||f||| = \sum_{\mathbf{i}\in\Lambda}\|\nabla_{\mathbf{i}}f\|_\infty$. Moreover, let us denote by d_M the the Riemannian distance of M and by $d_{\mathbf{Z}^\nu}$ the distance of $\mathbf{Z}^\nu$ which is defined by $d_{\mathbf{Z}^\nu}(\mathbf{k}, \mathbf{k}') = \sum_{i=1}^\nu |\mathbf{k}_i - \mathbf{k}'_i|$ for $\mathbf{k}, \mathbf{k}' \in \mathbf{Z}^\nu$. As usual we define $d_{\mathbf{Z}^\nu}(\Lambda, \mathbf{k}') = \inf\{d_{\mathbf{Z}^\nu}(\mathbf{k}, \mathbf{k}')|\mathbf{k} \in \Lambda\}$.

By Stroock and Zegarlinski [8], [9], [10], Yoshida [11] etc., we see that the following three conditions are equivalent;

1. there exists a constant $C > 0$ such that (5.8) holds for any $\mathbf{y} \in E$ and $\Lambda \subset\subset \mathbf{Z}^\nu$.

2. there exists a constant $\alpha > 0$ such that (5.6) holds for any $\mathbf{y} \in E$ and $\Lambda \subset\subset \mathbf{Z}^\nu$.

3. there exists a constant $c > 0$ such that if $\Lambda \subset\subset \mathbf{Z}^\nu$, $f \in C_N^\infty(E_\Lambda)$, $\mathbf{k} \notin \Lambda$ and $\mathbf{y} = \mathbf{y}'$ off $\mathbf{k}$, then

$$|\langle f\rangle_{\mu_\Lambda(dx|\mathbf{y})} - \langle f\rangle_{\mu_\Lambda(dx|\mathbf{y}')}| \leq B(f) d_M(\mathbf{y_k}, \mathbf{y'_k}) \exp(-d_{\mathbf{Z}^\nu}(S_f, \mathbf{k})/c),$$

where $B(f)$ is a positive constant depending only on $|S_f|$ and $|||f|||$.

Moreover, if one of the three conditions holds, then $|\mathfrak{G}(\mathfrak{U})| = 1$. (The proofs for the case of compact manifolds without boundary in [8], [9], [10], [11] depend on finite dimensional analysis and applicable to our case with slight modifications).

Now we define

$$\begin{aligned}
\delta(\mathfrak{U}) &= \sup\{\delta(U_{\mathbf{0}}(\cdot|y))|\mathbf{y} \in E\}, \\
\rho(\mathfrak{U}) &= \inf\{\rho(U_{\mathbf{0}}(\cdot|y))|\mathbf{y} \in E\}, \\
\beta(\mathfrak{U}) &= \sup\left\{\beta \in \mathbf{R} | R^{\mathbf{Z}^\nu}(f,f) \geq \beta \sum_{\mathbf{k}\in\mathbf{Z}^\nu} \|\nabla_{\mathbf{k}} f\|^2, \quad f \in \mathfrak{G}(E)\right\}, \\
K(\mathfrak{U})(\epsilon) &= \sup\{K(U_0(\cdot|\mathbf{y}))(\epsilon)|\mathbf{y} \in E\}.
\end{aligned}$$

Let us consider the spectral gap (5.9). Let

$$C(\mathfrak{U}) = \inf\{C(U_{\mathbf{0}}(\cdot|y))|\mathbf{y} \in E\}.$$

We can estimate $C(\mathfrak{U})$ from below by using the finite dimensional case. For example we have by Proposition 2.6,

$$C(\mathfrak{U}) \geq \frac{C(0)e^{-\delta(\mathfrak{U})}}{d} + \rho(\mathfrak{U}) - K(\mathfrak{U})(1).$$

Then for every $\mathbf{k} \in \Lambda_h$ and $f \in C_N^\infty(E_{\Lambda_h})$, we have by the shift-invariance that

$$\begin{aligned}
C(\mathfrak{U}) &\int_{E_{\mathbf{k}}} \|\nabla_{\mathbf{k}} f\|^2(x) \mu_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_{\mathbf{k}}(x|\mathbf{y})) \\
&\leq \int_{E_{\mathbf{k}}} \Gamma_{2,\mathbf{k}}(f,f)(x)\mu_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_{\mathbf{k}}(x|\mathbf{y})) \\
&\quad + \int_{\partial E_{\mathbf{k}}} A_{\mathbf{k}}(\nabla_{\mathbf{k}} f, \nabla_{\mathbf{k}} f)(x)\sigma_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_{\mathbf{k}}(x|\mathbf{y})).
\end{aligned}$$

Integrating the both sides by $\mu_{\Lambda_h}(dx|\mathbf{y})$ and summing them up with respect to $\mathbf{k}$, we see by Lemma 5.6 that (5.9) holds with $C \geq C(\mathfrak{U}) + \beta(\mathfrak{U})$.

We have an estimate of $\beta(\mathfrak{U})$ as follows;

Remark 5.8. Let

$$\gamma(\mathbf{k}) = \sup\{\|\mathrm{Hess}_{\mathbf{0},\mathbf{k}}(\mathfrak{U})(X_{\mathbf{0}}, Y_{\mathbf{k}})\|_{C(E)} \\ |X, Y \text{ are smooth vector field on } E \text{ and } \||X\||_{C(E)} \vee \||Y\||_{C(E)} \le 1\},$$

and

$$\gamma(\mathfrak{U}) = \sum_{\mathbf{k}\neq\mathbf{0}} \gamma(\mathbf{k}).$$

In the same way as in Deuschel-Stroock [2], $\beta(\mathfrak{U}) \ge -\gamma(\mathfrak{U})$ (see Deuschel-Stroock [2] for the proof).

Proposition 5.9. *Suppose*

$$\frac{C(0)e^{-\delta(\mathfrak{U})}}{d} + \rho(\mathfrak{U}) - K(\mathfrak{U})(1) - \gamma(\mathfrak{U}) > 0.$$

Then spectral gap inequality (5.8) *holds for any* $\mathbf{y} \in E$ *and* $\Lambda \subset\subset \mathbf{Z}^\nu$ *with*

$$C \ge \frac{C(0)e^{-\delta(\mathfrak{U})}}{d} + \rho(\mathfrak{U}) - K(\mathfrak{U})(1) - \gamma(\mathfrak{U}).$$

In that case, $|\mathfrak{G}(\mathfrak{U})| = 1$ *by Remarks* 5.7 *and* 5.8 *and the following spectral gap inequality holds*:

$$\int_E (f - \langle f\rangle_\mu)^2 d\mu \le \frac{1}{C} \sum_{\mathbf{k}\in\mathbf{Z}^\nu} \int_E \|\nabla_{\mathbf{k}} f\|^2 d\mu, \qquad f \in \mathfrak{G}(E).$$

Next we will consider the logarithmic Sobolev inequality. Though there are several versions as we showed in the finite dimensional case, here we will show the simplest case only.

Let

$$\alpha(\mathfrak{U}) = \frac{C(0)e^{-\delta(\mathfrak{U})}}{d} + \rho(\mathfrak{U}) - K(\mathfrak{U}) \left(\frac{2e^{-\delta(\mathfrak{U})}}{d+2}\right).$$

Then by the shift-invariance

$$\begin{aligned} \alpha(\mathfrak{U}) &\int_{E_{\mathbf{k}}} \frac{\|\nabla_{\mathbf{k}} f\|^2}{f}(x)\mu_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_{\mathbf{k}}(x|\mathbf{y})) \\ &\le \int_{E_{\mathbf{k}}} f\Gamma_{2,\mathbf{k}}(\log f, \log f)(x)\mu_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_{\mathbf{k}}(x|\mathbf{y})) \\ &\quad + \int_{\partial E_{\mathbf{k}}} fA_{\mathbf{k}}(\nabla_{\mathbf{k}} \log f, \nabla_{\mathbf{k}} \log f)(x)\sigma_{\mathbf{k}}(dx_{\mathbf{k}}|\Phi_{\mathbf{k}}(x|\mathbf{y})) \end{aligned}$$

holds for every strictly positive $f \in C_N^\infty(E_{\Lambda_h})$. Then integrating the both sides by $\mu_{\Lambda_h}(dx|\mathbf{y})$ and summing them up with respect to $\mathbf{k}$, we see by Lemma 5.6 that (5.10) holds with $\alpha \ge \alpha(\mathfrak{U}) + \beta(\mathfrak{U})$. Thus we have proved the following result.

Proposition 5.10. *Suppose $\alpha(\mathfrak{U}) - \gamma(\mathfrak{U}) > 0$, Then the logarithmic Sobolev inequality* (5.6) *holds for any $\mathbf{y} \in E$ and $\Lambda \subset\subset \mathbf{Z}^\nu$ with $\alpha \geq \alpha(\mathfrak{U}) - \gamma(\mathfrak{U})$. In particular, $|\mathfrak{G}(\mathfrak{U})| = 1$ by Remark* 5.7 *and the following logarithmic Sobolev inequality holds with $\alpha \geq \alpha(\mathfrak{U}) - \gamma(\mathfrak{U})$;*

$$\langle f \log f \rangle_\mu - \langle f \rangle_\mu \log \langle f \rangle_\mu \leq \frac{2}{\alpha} \sum_{\mathbf{k} \in \mathbf{Z}^\nu} \langle \|\nabla_{\mathbf{k}} f^{1/2}\|^2 \rangle_\mu, \qquad 0 < f \in \mathfrak{G}(E). \tag{5.11}$$

In order to show that Proposition 5.10 is not useless, here we will give a very simple example.

Example 5.1. Let M and $U \in C^\infty(M)$ satisfy that

$$\frac{C(0)e^{\delta(U)}}{d} + \rho(U) - K(U)\left(\frac{2e^{-\delta(U)}}{d+2}\right) > 0.$$

We assume further for simplicity that $A(\nabla f, \nabla f) \leq 0$ for any $f \in C_N^\infty(M)$ (such M and U exist as we saw in Example 4.2). Now we construct measures on $M^{\mathbf{Z}}$. For $v, w \in C^\infty(M)$ and $\epsilon > 0$, set $J_F(\mathbf{x}_i, \mathbf{x}_{i+1}) = U(\mathbf{x}_i) + \epsilon v(\mathbf{x}_i) w(\mathbf{x}_{i+1})$ for $F = \{i, i+1\}$ (for $i \in \mathbf{Z}$) and otherwise $J_F = 0$. Clearly this defines the Gibbs state with potential $\mathfrak{U}_\epsilon$. Note that when $\epsilon = 0$, the Gibbs state is the product measure. By a straight forward computation we have

$$|\gamma(\mathfrak{U}_\epsilon)| \leq 2\epsilon \|\nabla v\|_\infty \cdot \|\nabla w\|_\infty \tag{5.12}$$

$$|\delta(U) - \delta(\mathfrak{U}_\epsilon)| \leq \epsilon(\delta(v)|w|_\infty + \delta(w)|v|_\infty) \tag{5.13}$$

$$|\rho(U) - \rho(\mathfrak{U}_\epsilon)| \leq \epsilon(\|\mathrm{Hess} v\|_\infty |w|_\infty + \|\mathrm{Hess} w\|_\infty |v|_\infty). \tag{5.14}$$

We see by (5.13) and Lemma 2.5 that $|K(U)(2e^{-\delta(U)}/d+2) - K(U)(2e^{-\delta(\mathfrak{U}_\epsilon)}/d+2)|$ goes to 0 as ϵ goes to 0. Because we assumed the non-positivity of A, we can easily see that

$$K(U)(t)e^{-\delta(W)} \leq K(U+W)(t) \leq K(U)(t)e^{\delta(W)}$$

for any $U, W \in C^\infty(M)$ and $t > 0$. Combining this with (5.13) we have

$$\left| K(\mathfrak{U}_\epsilon)\left(\frac{2e^{-\delta(\mathfrak{U}_\epsilon)}}{d+2}\right) - K(U)\left(\frac{2e^{-\delta(\mathfrak{U}_\epsilon)}}{d+2}\right) \right|$$

goes to 0 as ϵ goes to 0. Combining them with (5.12), (5.13) and (5.14), we see that

$$\frac{C(0)e^{\delta(\mathfrak{U}_\epsilon)}}{d} + \rho(\mathfrak{U}_\epsilon) - K(\mathfrak{U}_\epsilon)\left(\frac{2e^{-\delta(\mathfrak{U}_\epsilon)}}{d+2}\right) > 0$$

for sufficiently small $\epsilon > 0$ and we can apply Proposition 5.10.

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Department of Mathematics
Graduate School of Science
Kyoto University
Kyoto 606-8502 Japan

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