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The hyperbolic metric and spherically convex regions

By

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Abstract

There are a number of characterizations of convex subregions Ω of the complex plane $\mathbb{C}$ in terms of the density $\lambda_\Omega(w)$ of the hyperbolic metric $\lambda_\Omega(w)|dw|$ for Ω. We derive analogous characterizations for spherically convex regions Ω on the Riemann sphere $\mathbb{P}$ in terms of the spherical density $\mu_\Omega(w) = (1 + |w|^2)\lambda_\Omega(w)$ of the hyperbolic metric. A proper subregion Ω of $\mathbb{P}$ is spherically convex if for all pairs A, B of points in Ω the spherical geodesic (the shorter arc of the great circle) joining A and B lies in Ω. As a limiting case of our results we obtain known characterizations of convex regions in $\mathbb{C}$.

1. Introduction

Characterizations of convex regions Ω in the complex plane $\mathbb{C}$ in terms of analytic or geometric properties of the density λ_Ω of the hyperbolic metric $\lambda_\Omega(w)|dw|$ on Ω are known ([H], [KM], [MW], [Y_1], [Y_2], [Y_3]). For example, Ω is convex if and only if $1/\lambda_\Omega$ is concave. Some of these characterizations were originally obtained separately; a unified, geometric approach to some of them occurs in [KM].

We obtain analogous characterizations of regions Ω on the Riemann sphere $\mathbb{P}$ that are convex relative to spherical geometry. A region Ω on $\mathbb{P}$ is called spherically convex if for all pairs A, B of distinct points in Ω the spherical geodesic (the shorter arc of the great circle) joining A and B lies in Ω. The Riemann sphere $\mathbb{P}$ is spherically convex. In the following we only consider proper subregions of $\mathbb{P}$ which are spherically convex. A standard example of a spherically convex region is a hemisphere; for instance, the unit disk $\mathbb{D}$ is spherically convex. When considering regions Ω on $\mathbb{P}$ it is natural to consider the so-called spherical density $\mu_\Omega(w) = (1+|w|^2)\lambda_\Omega(w)$ of the hyperbolic metric [M_1]. By using a systematic approach we derive a number of characterizations of spherically convex regions in terms of the spherical density. For example, Ω

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is spherically convex if and only if $1/\mu_\Omega$ is spherically concave. This means that if $\gamma : w = w(s)$ is any arc of a great circle in Ω that is parametrized by spherical arclength and if $v(s) = 1/\mu_\Omega(w(s))$, then v satisfies the differential inequality $v''(s) + 4v(s) \le 0$. The significance of the factor 4 in this differential inequality is that the spherical metric has curvature 4. Also, if Ω is a hemisphere and γ is any great circular arc passing through the spherical center of the hemisphere, then $v''(s) + 4v(s) = 0$. A consequence is that in any spherically convex region Ω the spherical density μ_Ω attains its minimum value at a unique point.

An important tool in our approach is the use of certain natural differential operators. These operators are natural for hyperbolic or spherical geometry in the sense that they are essentially invariant under isometries of the geometry. The use of these differential operators greatly simplifies calculations and makes possible the concise expression of our results. These operators should be of independent interest.

Finally, we indicate how corresponding characterizations of euclidean convex regions in $\mathbb{C}$ can be viewed as limiting cases of our results. The use of differential operators makes this connection with euclidean results easy to see.

2. Hyperbolic and spherical geometry

We recall basic facts about some standard conformal metrics.

The hyperbolic metric on the unit disk is $\lambda_{\mathbb{D}}(w) = |dw|/(1-|w|^2)$. It has curvature -4:

$$\kappa(w, \lambda_{\mathbb{D}}) = -\frac{\Delta \log \lambda_{\mathbb{D}}(w)}{\lambda_{\mathbb{D}}(w)^2} = -4,$$

where Δ is the Laplacian. A region Ω on the Riemann sphere is called hyperbolic if $\mathbb{P}\backslash\Omega$ contains at least three points. The hyperbolic metric $\lambda_\Omega(w)|dw|$ on Ω is uniquely determined from $f^*(\lambda_\Omega(w)|dw|) = \lambda_{\mathbb{D}}(z)|dz|$, where $w = f(z)$ is any meromorphic universal covering projection of $\mathbb{D}$ onto Ω and

$$f^*(\lambda_\Omega(w)|dw|) = \lambda_\Omega(f(z))|f'(z)||dz|.$$

The hyperbolic metric on $\mathbb{D}$ is invariant under $\mathrm{Aut}(\mathbb{D})$, the group of conformal automorphisms of $\mathbb{D}$. This means that if $T(z) = e^{i\theta}(z-a)/(1-\overline{a}z)$, where $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$, then $T^*(\lambda_{\mathbb{D}}(w)|dw|) = \lambda_{\mathbb{D}}(z)|dz|$. In other words, $\mathrm{Aut}(\mathbb{D})$ is the isometry group for $\lambda_{\mathbb{D}}(w)|dw|$. For $A, B \in \Omega$ the hyperbolic distance between A and B is

$$d_\Omega(A, B) = \inf \int_\gamma \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all paths γ in Ω joining A and B. For the unit disk

$$d_{\mathbb{D}}(A, B) = \operatorname{arctanh}\left|\frac{A-B}{1-\overline{A}B}\right|.$$

For a hyperbolic region $\Omega \subset \mathbb{C}$ the (euclidean) density of the hyperbolic metric is

$$\lambda_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_{\mathbb{C}}(w)|dw|},$$

where $\lambda_{\mathbb{C}}(w)|dw| = 1|dw|$ denotes the euclidean metric on $\mathbb{C}$. The density λ_Ω is a smooth function on Ω which is invariant under euclidean motions; that is, if $T(z) = e^{i\theta}(z-a)$, where $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$, then $\lambda_{T(\Omega)}(T(z)) = \lambda_\Omega(z)$.

The spherical metric on $\mathbb{P}$ is $\lambda_{\mathbb{P}}(w)|dw| = |dw|/(1+|w|^2)$. It has curvature $+4$. The spherical metric is invariant under the group of rotations of $\mathbb{P}$; that is for any rotation T, $T^*(\lambda_{\mathbb{P}}(w)|dw|) = \lambda_{\mathbb{P}}(z)|dz|$. A rotation of $\mathbb{P}$ has the form $T(z) = e^{i\theta}(z-a)/(1+\overline{a}z)$, where $a \in \mathbb{P}$ and $\theta \in \mathbb{R}$. The spherical distance between $A, B \in \mathbb{P}$ is

$$d_{\mathbb{P}}(A,B) = \inf \int_\gamma \lambda_{\mathbb{P}}(w)|dw|,$$

where the infimum is taken over all paths γ connecting A and B. The explicit formula for spherical distance is

$$d_{\mathbb{P}}(A,B) = \arctan\left|\frac{A-B}{1+\overline{A}B}\right|.$$

Spherical distance is invariant under rotations; that is, if T is any rotation of $\mathbb{P}$, then $d_{\mathbb{P}}(T(A),T(B)) = d_{\mathbb{P}}(A,B)$. Geometrically, $d_{\mathbb{P}}(A,B)$ is one-half of the angle subtended at the center of the sphere by any geodesic arc joining A and B. A path δ joining A and B is called a spherical geodesic if

$$d_{\mathbb{P}}(A,B) = \int_\delta \lambda_{\mathbb{P}}(w)|dw|.$$

For distinct $A, B \in \mathbb{P}$ spherical geodesics always exist. The spherical geodesic is the shorter arc between A and B of the unique great circle determined by A and B. Note that $d_{\mathbb{P}}(A,B) \le \pi/2$ with equality if and only if A, B are antipodal.

For a hyperbolic region Ω on $\mathbb{P}$ it is natural to consider the spherical density

$$\mu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_{\mathbb{P}}(w)|dw|} = (1+|w|^2)\lambda_\Omega(w)$$

of the hyperbolic metric. This defines a function μ_Ω on Ω which is invariant under rotations of $\mathbb{P}$. The spherical density of the hyperbolic metric was used in [M_1] and [MO]. In [MO] it was noted that $\mu_\Omega(w) \to \infty$ whenever $w \to \partial\Omega$. This implies that μ_Ω always attains a minimum value on Ω. If $f : \mathbb{D} \to \Omega$ is a meromorphic universal covering, then

$$\mu_\Omega(f(z)) = \frac{1+|f(z)|^2}{(1-|z|^2)|f'(z)|} = \frac{1}{(1-|z|^2)f^\sharp(z)},$$

where $f^\sharp(z) = |f'(z)|/(1+|f(z)|^2)$ denotes the spherical derivative of f.

We require some additional information about the spherical density on spherically convex regions. If Ω is spherically convex, then Ω is contained in a hemisphere. For the unit disk, $\mu_{\mathbb{D}}(w) = (1+|w|^2)/(1-|w|^2) \geq 1$ with strict inequality unless $w = 0$, so $\mu_\Omega \geq 1$ for any hemisphere since the spherical density is rotationally invariant. Because the hyperbolic metric depends monotonically on the region, it follows that the spherical density also depends monotonically. Thus, $\Omega \subset \Delta$ implies $\mu_\Omega \geq \mu_\Delta$ with strict inequality unless $\Omega = \Delta$. Since each spherically convex region Ω is contained in a hemisphere, we conclude that $\mu_\Omega(w) \geq 1$, $w \in \Omega$, with strict inequality unless Ω is a hemisphere and w is the spherical center of the hemisphere.

3. Invariant differential operators

We make use of several differential operators that are (essentially) invariant with respect to isometries of either hyperbolic or spherical geometry. A number of analytic results can be expressed concisely in terms of these differential operators. Computations that are lengthy without these operators become quite manageable by making use of them. Also, their use makes evident the similarities between the known euclidean results and our spherical results.

We start by recalling basic (euclidean) differential operators and their invariance properties.

3.1. The euclidean differential operators

Suppose Ω is a region in $\mathbb{C}$ and $r : \Omega \to \mathbb{R}$ is of class C^2. Then for $w = u+iv$

$$\partial r = \frac{\partial r}{\partial w} = \frac{1}{2}\left(\frac{\partial r}{\partial u} - i\frac{\partial r}{\partial v}\right),$$
$$\partial^2 r = \frac{\partial^2 r}{\partial w^2} = \partial(\partial r),$$

and

$$\Delta r = \frac{\partial^2 r}{\partial u^2} + \frac{\partial^2 r}{\partial v^2} = 4\frac{\partial^2 r}{\partial w \partial \overline{w}} .$$

If $T(z) = e^{i\theta}(z-a)$ is any euclidean motion of $\mathbb{C}$, then

$$\begin{aligned}
\partial(r \circ T) &= e^{i\theta}(\partial r) \circ T = T'[(\partial r) \circ T],\\
\partial^2(r \circ T) &= (e^{i\theta})^2(\partial^2 r) \circ T = (T')^2[(\partial^2 r) \circ T],\\
\Delta(r \circ T) &= (\Delta r) \circ T.
\end{aligned}$$

Thus, the Laplacian Δ is invariant under euclidean motions. Since $|\partial(r \circ T)| = |\partial r| \circ T$ and $|\partial^2(r \circ T)| = |\partial^2 r| \circ T$, the absolute values $|\partial r|$ and $|\partial^2 r|$ are invariant under euclidean motions.

3.2. The hyperbolic differential operator

We define analogous differential operators relative to hyperbolic geometry on $\mathbb{D}$. For a C^2 function $r : \mathbb{D} \to \mathbb{R}$ set

$$\begin{aligned}\partial_h r &= \frac{\partial r}{\lambda_{\mathbb{D}}},\\ \partial_h^2 r &= \frac{1}{\lambda_{\mathbb{D}}^2}[\partial^2 r - 2(\partial \log \lambda_{\mathbb{D}})(\partial r)],\\ \Delta_h r &= \frac{\Delta r}{\lambda_{\mathbb{D}}^2}.\end{aligned}$$

The operator Δ_h is the invariant Laplacian relative to the hyperbolic metric. At the origin each of these hyperbolic differential operators coincides with the corresponding euclidean differential operator; for example, $\partial_h r(0) = \partial r(0)$. If $T(z) = e^{i\theta}(z-a)/(1-\overline{a}z)$ is an isometry of the hyperbolic metric on $\mathbb{D}$, then it is straightforward to verify that

$$\begin{aligned}\partial_h(r \circ T) &= \frac{T'}{|T'|}[(\partial_h r) \circ T],\\ \partial_h^2(r \circ T) &= \left(\frac{T'}{|T'|}\right)^2 [(\partial_h^2 r) \circ T],\\ \Delta_h(r \circ T) &= (\Delta_h r) \circ T.\end{aligned}$$

The quantities $|\partial_h r|$ and $|\partial_h^2 r|$ are invariant under hyperbolic isometries since $|\partial_h(r \circ T)| = |\partial_h r| \circ T$ and $|\partial_h^2(r \circ T)| = |\partial_h^2| \circ T$. Unlike the euclidean setting, ∂_h^2 is not equal to $\partial_h \circ \partial_h$, but there is a simple relationship between these two operators:

$$\partial_h^2 r(w) = \partial_h(\partial_h r(w)) - \overline{w}\partial_h r(w).$$

It is elementary to verify that the product rule is valid for ∂_h.

3.3. The spherical differential operator

There are analogous differential operators relative to spherical geometry. Suppose Ω is a region on $\mathbb{P}$ and $r : \Omega \to \mathbb{R}$ is of class C^2. Set

$$\begin{aligned}\partial_s r &= \frac{\partial r}{\lambda_{\mathbb{P}}},\\ \partial_s^2 r &= \frac{1}{\lambda_{\mathbb{P}}^2}[\partial^2 r - 2(\partial \log \lambda_{\mathbb{P}})(\partial r)],\\ \Delta_s r &= \frac{\Delta r}{\lambda_{\mathbb{P}}^2}.\end{aligned}$$

$\Delta_s r$ is the spherical Laplacian. At the origin all of these coincide with the corresponding euclidean differential operators. If $T(z) = e^{i\theta}(z-a)/(1+\overline{a}z)$, is an isometry of the spherical metric or equivalently, a rotation of $\mathbb{P}$, then

$$\begin{aligned}
\partial_s(r\circ T) &= \frac{T'}{|T'|}[(\partial_s r)\circ T],\\
\partial_s^2(r\circ T) &= \left(\frac{T'}{|T'|}\right)^2[(\partial_s^2 r)\circ T],\\
\Delta_s(r\circ T) &= (\Delta_s r)\circ T.
\end{aligned}$$

Hence, the spherical Laplacian is invariant under rotations of the sphere, and it follows from the preceding identities that $|\partial_s r|$ and $|\partial_s^2 r|$ are invariant under spherical isometries. Analogous to the hyperbolic situation,

$$\partial_s^2 r(w) = \partial_s(\partial_s r(w)) + \overline{w}\partial_s r(w)$$

and the product rule is valid for ∂_s.

3.4. The differential operators D_j $(j = 1, 2, 3)$

Now we introduce the differential operators D_j $(j = 1, 2, 3)$ which give

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1-|z|^2)^2 S_f(z),$$

where

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f. S_f are defined and continuous at simple poles. For a meromorphic function $f : \mathbb{D} \to \Omega$ set

$$\begin{aligned}
D_1 f(z) &= \frac{(1-|z|^2)f'(z)}{1+|f(z)|^2},\\
D_2 f(z) &= \frac{(1-|z|^2)^2 f''(z)}{1+|f(z)|^2} - \frac{2\overline{z}(1-|z|^2)f'(z)}{1+|f(z)|^2}\\
&\quad - \frac{2(1-|z|^2)^2\overline{f(z)}f'(z)^2}{(1+|f(z)|^2)^2}
\end{aligned}$$

and

$$\begin{aligned}
D_3 f(z) = {}& \frac{(1-|z|^2)^3 f'''(z)}{1+|f(z)|^2} - \frac{6(1-|z|^2)^3\overline{f(z)}f'(z)f''(z)}{1+|f(z)|^2}\\
& - \frac{6\overline{z}(1-|z|^2)^2 f''(z)}{1+|f(z)|^2} + \frac{6\overline{z}^2(1-|z|^2)f'(z)}{1+|f(z)|^2}\\
& + \frac{12\overline{z}(1-|z|^2)^2\overline{f(z)}f'(z)^2}{(1+|f(z)|^2)^2} + \frac{6(1-|z|^2)^3\overline{f(z)}^2 f'(z)^3}{(1+|f(z)|^3)^3}.
\end{aligned}$$

The reader should note that $D_j f$ has a different meaning in [KM]. Note that if $f(0) = 0$, then $D_j f(0) = f^{(j)}(0)$ $(j = 1, 2, 3)$. These differential operators were employed in [MM] and are invariant in the sense that $|D_j(R\circ f\circ T)| = |D_j f|\circ T$ $(j = 1, 2, 3)$, whenever T is a conformal automorphism of $\mathbb{D}$ and R is a rotation of $\mathbb{P}$. Note that if $f : \mathbb{D} \to \Omega$ is a covering, then $\mu_\Omega \circ f = 1/|D_1 f|$. In case f has a pole at a point z, we set $D_j f(z) = (D_j(1/f))(z)$ $(j = 1, 2, 3)$. Then $D_j f$ $(j = 1, 2, 3)$ is not continuous at a simple pole while $|D_j f|$ is. We also define

$$Q_f(z) = \frac{D_2 f(z)}{D_1 f(z)} = (1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z} - \frac{2(1 - |z|^2)\overline{f(z)}f'(z)}{1 + |f(z)|^2}$$

3.5. Chain rules

There are simple chain rules relating to hyperbolic and spherical differential operators when a function $r : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{P}$, is composed with a meromorphic function $f : \mathbb{D} \to \Omega$. The chain rules are:

$$\partial_h(r \circ f) = [(\partial_s r) \circ f]D_1 f, \tag{1}$$

$$\partial_h^2(r \circ f) = [(\partial_s^2 r) \circ f](D_1 f)^2 + [(\partial_s r) \circ f]D_2 f. \tag{2}$$

These are exact analogs of the usual chain rules,

$$\partial(r \circ f) = [(\partial r) \circ f]f',$$
$$\partial^2(r \circ f) = [(\partial^2 r) \circ f](f')^2 + [(\partial r) \circ f]f''.$$

From (1) and (2) we obtain

$$[(\partial_s^2 r) \circ f](D_1 f)^2 = \partial_h^2(r \circ f) - \partial_h(r \circ f) \cdot Q_f. \tag{3}$$

4. Characterizations of spherically convex regions

Now we are in a position to obtain a number of characterizations of spherically convex regions. An important ingredient in our proofs is the application of some of the results in the preceding section to the specific function $r = 1/\mu_\Omega$. If Ω is a hyperbolic region on $\mathbb{P}$ and $f : \mathbb{D} \to \Omega$ is a meromorphic universal covering projection, then $(1/\mu_\Omega) \circ f = |D_1 f|$. We note that

$$\partial_h|D_1 f| = \frac{1}{2}|D_1 f|Q_f, \tag{4}$$

and

$$\partial_h^2|D_1 f| = \frac{1}{2}|D_1 f|\left[\frac{D_3 f}{D_1 f} - \frac{1}{2}\left(\frac{D_2 f}{D_1 f}\right)^2\right]. \tag{5}$$

The identity (4) is straightforward to verify while (5) can be obtained as follows.

$$\begin{aligned}\partial_h^2|D_1 f(z)| &= \partial_h(\partial_h|D_1 f(z)|) - \overline{z}\partial_h|D_1 f(z)|\\ &= \partial_h\left(\frac{1}{2}|D_1 f(z)|Q_f(z)\right) - \overline{z}\frac{1}{2}|D_1 f(z)|Q_f(z)\\ &= \frac{1}{2}Q_f(z)\partial_h|D_1 f(z)| + \frac{1}{2}|D_1 f(z)|\partial_h Q_f(z) - \frac{1}{2}\overline{z}|D_1 f(z)|Q_f(z).\end{aligned}$$

Direct calculation gives

$$\partial_h Q_f(z) = \frac{D_3 f(z)}{D_1 f(z)} - \left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 + \overline{z} Q_f(z),$$

so (5) now follows.

We also need the following identities to shorten the lengthy calculations in the proof for the characterizations of spherical convexity.

Lemma 1. *Let Ω be a hyperbolic region on $\mathbb{P}$ and $f : \mathbb{D} \to \Omega$ a meromorphic universal covering projection. Then we obtain the following identities.*

$$\left|\partial_s \frac{1}{\mu_\Omega}\right| \circ f = \frac{1}{2}|Q_f|, \tag{6}$$

$$\left|\frac{1}{\mu_\Omega}\left(\partial_s^2 \frac{1}{\mu_\Omega}\right)\right| \circ f = \frac{1}{2}\left|\frac{D_3 f}{D_1 f} - \frac{3}{2}\left(\frac{D_2 f}{D_1 f}\right)^2\right|, \tag{7}$$

$$\Delta_s \left(\frac{1}{\mu_\Omega}\right) = 4\mu_\Omega \left[\left|\partial_s \frac{1}{\mu_\Omega}\right|^2 - \frac{1}{\mu_\Omega^2} - 1\right]. \tag{8}$$

Proof. The chain rule (1) and (4) give

$$\left[\left(\partial_s \frac{1}{\mu_\Omega}\right) \circ f\right] D_1 f = \partial_h |D_1 f| = \frac{1}{2}|D_1 f| Q_f$$

so that we obtain (6).

Similarly, (3), (4) and (5) give

$$\begin{aligned}\left[\left(\partial_s^2 \frac{1}{\mu_\Omega}\right) \circ f\right] (D_1 f)^2 &= \partial_h^2 |D_1 f| - \partial_h |D_1 f| \cdot Q_f \\ &= \frac{1}{2}|D_1 f| \left[\frac{D_3 f}{D_1 f} - \frac{3}{2}\left(\frac{D_2 f}{D_1 f}\right)^2\right],\end{aligned}$$

so we have (7).

The identity (8) can be demonstrated as follows. Since $\lambda_{\mathbb{P}}(w)|dw|$ has curvature 4 and $\lambda_\Omega(w)|dw|$ has curvature -4,

$$\begin{aligned}\Delta_s \log \mu_\Omega &= \frac{1}{\lambda_{\mathbb{P}}^2} \Delta \log \mu_\Omega \\ &= \frac{1}{\lambda_{\mathbb{P}}^2}[\Delta \log \lambda_\Omega - \Delta \log \lambda_{\mathbb{P}}] \\ &= \frac{1}{\lambda_{\mathbb{P}}^2}[4\lambda_\Omega^2 + 4\lambda_{\mathbb{P}}^2] \\ &= 4[\mu_\Omega^2 + 1].\end{aligned}$$

Also,

$$\begin{aligned}
\Delta_s \log \mu_\Omega &= \frac{1}{\lambda_{\mathbb{P}}^2}\Delta \log \mu_\Omega = \frac{4}{\lambda_{\mathbb{P}}^2}\frac{\partial^2}{\partial \overline{w}\partial w}\log \mu_\Omega \\
&= \frac{4}{\lambda_{\mathbb{P}}^2}\frac{\partial}{\partial \overline{w}}\left[\frac{\frac{\partial \mu_\Omega}{\partial w}}{\mu_\Omega}\right] = \frac{4}{\lambda_{\mathbb{P}}^2}\frac{\partial}{\partial \overline{w}}\left[-\mu_\Omega \frac{\partial}{\partial w}\left(\frac{1}{\mu_\Omega}\right)\right] \\
&= -\frac{4\mu_\Omega}{\lambda_{\mathbb{P}}^2}\frac{\partial^2}{\partial \overline{w}\partial w}\left(\frac{1}{\mu_\Omega}\right) - \frac{4}{\lambda_{\mathbb{P}}^2}\frac{\partial \mu_\Omega}{\partial \overline{w}}\frac{\partial}{\partial w}\left(\frac{1}{\mu_\Omega}\right) \\
&= -\mu_\Omega \Delta_s\left(\frac{1}{\mu_\Omega}\right) + \frac{4\mu_\Omega^2}{\lambda_{\mathbb{P}}^2}\frac{\partial}{\partial \overline{w}}\left(\frac{1}{\mu_\Omega}\right)\cdot \frac{\partial}{\partial w}\left(\frac{1}{\mu_\Omega}\right) \\
&= -\mu_\Omega \Delta_s\left(\frac{1}{\mu_\Omega}\right) + 4\mu_\Omega^2\left|\partial_s\left(\frac{1}{\mu_\Omega}\right)\right|^2.
\end{aligned}$$

The desired identity (8) follows from our two expressions for $\Delta_s \log \mu_\Omega$. □

Earlier, we noted that $\mu_\Omega(w) \geq 1$, $w \in \Omega$, if Ω is spherically convex. If $f : \mathbb{D} \to \Omega$ is a covering, this is equivalent to $|D_1 f(z)| \leq 1$. Now, we give characterizations for spherically convex regions on $\mathbb{P}$.

Theorem 1. *Suppose Ω is a hyperbolic region on $\mathbb{P}$ and μ_Ω denotes the spherical density of the hyperbolic metric. The following are equivalent*:

(i) *Ω is spherically convex,*

(ii) $\left|\partial_s \dfrac{1}{\mu_\Omega}\right|^2 \leq 1 - \dfrac{1}{\mu_\Omega^2}$,

(iii) $\dfrac{1}{\mu_\Omega}\left|\partial_s^2 \dfrac{1}{\mu_\Omega}\right| + \left|\partial_s \dfrac{1}{\mu_\Omega}\right|^2 \leq 1 - \dfrac{1}{\mu_\Omega^2}$,

(iv) $\Delta_s \dfrac{1}{\mu_\Omega} \leq -\dfrac{8}{\mu_\Omega}$.

Proof. Let $f : \mathbb{D} \to \Omega$ be a meromorphic universal covering projection. Then

$$\begin{aligned}
\frac{1}{\mu_\Omega(f(z))} &= |D_1 f(z)|, \\
\left|\left(\partial_s \frac{1}{\mu_\Omega}\right)(f(z))\right| &= \frac{1}{2}\left|Q_f(z)\right|, \\
\frac{1}{\mu_\Omega(f(z))}\left|\left(\partial_s^2 \frac{1}{\mu_\Omega}\right)(f(z))\right| &= \frac{1}{2}(1-|z|^2)^2|S_f(z)|.
\end{aligned}$$

(i) $\Longleftrightarrow$ (ii) Note that (ii) is equivalent to

$$|Q_f(z)|^2 \leq 4(1 - |D_1 f(z)|^2),$$

or

$$|Q_f(z)|^2 + 4|D_1 f(z)|^2 \leq 4.$$

This inequality is equivalent to $\Omega = f(\mathbb{D})$ being spherically convex [MM, Theorem 4].

(i) $\Longleftrightarrow$ (iii) Similarly, (iii) is equivalent to

$$(1-|z|^2)^2|S_f(z)| + \frac{1}{2}|Q_f(z)|^2 \leq 2(1-|D_1 f(z)|^2),$$

or

$$(1-|z|^2)^2|S_f(z)| \leq 2\left(1-|D_1 f(z)|^2 - \frac{1}{4}|Q_f(z)|^2\right),$$

which is equivalent to $\Omega = f(\mathbb{D})$ being spherically convex ([W], [MM, Theorem 6]).

(ii) $\Longleftrightarrow$ (iv) From (6) we see that condition (iv) is equivalent to

$$\left|\partial_s \frac{1}{\mu_\Omega}\right|^2 - \frac{1}{\mu_\Omega^2} - 1 \leq -\frac{2}{\mu_\Omega^2},$$

or

$$\left|\partial_s \frac{1}{\mu_\Omega}\right|^2 \leq 1 - \frac{1}{\mu_\Omega^2},$$

which is (ii). □

Remark. We recall the euclidean analogs for the parts of Theorem 1. A hyperbolic region Ω in $\mathbb{C}$ is convex if and only if $|\partial(1/\lambda_\Omega)| \leq 1$ ([H], [Y_1]). Necessary and sufficient for the convexity of Ω is $(1/\lambda_\Omega)|\partial^2(1/\lambda_\Omega)| + |\partial(1/\lambda_\Omega)|^2 \leq 1$ ([KM], [Y_1]). Finally, Ω is convex if and only if $1/\lambda_\Omega$ is superharmonic; that is, $\Delta(1/\lambda_\Omega) \leq 0$ [Y_3].

Next, we compute derivatives of a function along paths parameterized by spherical arclength by the use of spherical differential operators and then we introduce "spherical concavity" for a real-valued function.

Lemma 2. *Assume Ω is a region on $\mathbb{P}$ and $r : \Omega \to \mathbb{R}$. Suppose $\gamma : w = w(s)$ is a spherical geodesic in Ω that is parametrized by spherical arclength. Let $v(s) = r(w(s))$. Then*

$$v'(s) = 2\,\mathrm{Re}\{e^{i\theta(s)}(\partial_s r)(w(s))\}, \tag{9}$$

and

$$v''(s) = 2\,\mathrm{Re}\{e^{2i\theta(s)}\partial_s^2 r(w(s))\} + \frac{1}{2}\Delta_s r(w(s)) \tag{10}$$

where $e^{i\theta(s)}$ is a unit tangent vector for γ at $w(s)$.

Proof. Since γ is parametrized by spherical arclength, $w'(s) = (1 + |w(s)|^2)e^{i\theta(s)}$. r is real-valued and so

$$\begin{aligned} v'(s) &= \frac{\partial r}{\partial w}(w(s))w'(s) + \frac{\partial r}{\partial \overline{w}}(w(s))\overline{w'(s)} \\ &= 2\operatorname{Re}\left\{\frac{\partial r}{\partial w}(w(s))w'(s)\right\} \\ &= 2\operatorname{Re}\left\{(1+|w(s)|^2)\frac{\partial r}{\partial w}(w(s))e^{i\theta(s)}\right\}, \end{aligned}$$

or

$$v'(s) = 2\operatorname{Re}\{e^{i\theta(s)}(\partial_s r)(w(s))\}.$$

Calculation of $v''(s)$ involves the spherical curvature of γ. Recall that the spherical curvature of γ at the point $w(s)$ is

$$\begin{aligned} \kappa_s(w(s),\gamma) &= \frac{\kappa_e(w(s),\gamma) + 2\operatorname{Im}\left\{\frac{\partial \log \lambda_{\mathbb{P}}}{\partial w}(w(s))\frac{w'(s)}{|w'(s)|}\right\}}{\lambda_{\mathbb{P}}(w(s))} \\ &= (1+|w(s)|^2)\kappa_e(w(s),\gamma) - 2\operatorname{Im}\{\overline{w}(s)e^{i\theta(s)}\}, \end{aligned}$$

where

$$\kappa_e(w(s),\gamma) = \frac{\operatorname{Im}\left\{\frac{w''(s)}{w'(s)}\right\}}{|w'(s)|} = \frac{\operatorname{Im}\left\{\frac{w''(s)}{w'(s)}\right\}}{1+|w(s)|^2}$$

is the euclidean curvature of γ at $w(s)$ (see [MM] or [M$_2$]). Thus,

$$\operatorname{Im}\left\{\frac{w''(s)}{w'(s)}\right\} = \kappa_s(w(s),\gamma) + 2\operatorname{Im}\{\overline{w}(s)e^{i\theta(s)}\}.$$

From $w'(s) = (1+|w(s)|^2)e^{i\theta(s)}$ we get

$$\operatorname{Re}\left\{\frac{w''(s)}{w'(s)}\right\} = 2\operatorname{Re}\{\overline{w}(s)e^{i\theta(s)}\}.$$

Hence,

$$\frac{w''(s)}{w'(s)} = 2\overline{w}(s)e^{i\theta(s)} + i\kappa_s(w(s),\gamma)$$

and so

$$w''(s) = 2\overline{w}(s)(1+|w(s)|^2)e^{2i\theta(s)} + i(1+|w(s)|^2)\kappa_s(w(s),\gamma)e^{i\theta(s)}.$$

Now,

$$\begin{aligned}v''(s) &= 2\operatorname{Re}\left\{\frac{\partial^2 r}{\partial w^2}(w(s))w'(s)^2\right\} + 2\operatorname{Re}\left\{\frac{\partial r}{\partial w}(w(s))w''(s)\right\}\\ &\quad +2\frac{\partial^2 r}{\partial w\partial\overline{w}}(w(s))|w'(s)|^2\\ &= 2\operatorname{Re}\left\{\left[(1+|w(s)|^2)^2\frac{\partial^2 r}{\partial w^2}(w(s))+2\overline{w}(s)(1+|w(s)|^2)\frac{\partial r}{\partial w}(w(s))\right]e^{2i\theta(s)}\right\}\\ &\quad +2(1+|w(s)|^2)^2\frac{\partial^2 r}{\partial w\partial\overline{w}}(w(s))\\ &\quad -2\kappa_s(w(s),\gamma)\operatorname{Im}\left\{(1+|w(s)|^2)\frac{\partial r}{\partial w}(w(s))e^{i\theta(s)}\right\}\end{aligned}$$

so that

$$\begin{aligned}v''(s) = 2\operatorname{Re}\left\{e^{2i\theta(s)}\partial_s^2 r(w(s))\right\} + \frac{1}{2}\Delta_s r(w(s))\\ -2\kappa_s(w(s),\gamma)\operatorname{Im}\{e^{i\theta(s)}\partial_s r(w(s))\}.\end{aligned}$$

γ is a spherical geodesic and hence $\kappa_s(w(s),\gamma)=0$. Therefore we obtain

$$v''(s) = 2\operatorname{Re}\{e^{2i\theta(s)}\partial_s^2 r(w(s))\} + \frac{1}{2}\Delta_s r(w(s)). \qquad \square$$

Now we introduce the notion of "spherical concavity" for a real-valued function before stating our next result. For a positive function this is stronger than the usual notion of concavity. Suppose Ω is a region on $\mathbb{P}$ and $r:\Omega\to\mathbb{R}$ is of class C^2. The function r is called *spherically concave* if $v''+4v\le 0$ whenever $\gamma: w=w(s)$ is a spherical geodesic arc in Ω that is parametrized by spherical arclength and $v(s)=r(w(s))$. The factor "4" in the definition of spherical concavity is due to the fact that the spherical metric has curvature 4. Relative to euclidean geometry a function is concave if $v''\le 0$; that is, $v''+0v\le 0$ and the coefficient 0 of v is the curvature of the euclidean metric.

It is easy to find an equivalent description of spherical concavity. For $v(s)=r(w(s))$, formula (10) yields

$$v''(s)+4v(s) = 2\operatorname{Re}\left\{e^{2i\theta(s)}\partial_s^2 r(w(s))\right\} + \frac{1}{2}\Delta_s r(w(s)) + 4r(w(s)).$$

Through every point $w\in\Omega$ there is a spherical geodesic in each direction; that is, given any unit vector at w there is a spherical geodesic through w with the given vector as tangent vector at w. Therefore, $v''+4v\le 0$ along all spherical geodesic arcs in Ω if and only if

$$2|\partial_s^2 r(w)| + \frac{1}{2}\Delta_s r(w) + 4r(w) \le 0, \quad w\in\Omega.$$

(Relative to the euclidean geometry a function r is concave if and only if $|\partial^2 r(w)| \le -(1/4)\Delta r(w)$, $w\in\Omega$ [KM].)

It is known that a hyperbolic region Ω in $\mathbb{C}$ is convex if and only if $1/\lambda_\Omega$ is concave ([KM], [MW]). We now give a spherical analog.

Theorem 2. *Suppose Ω is a hyperbolic region on $\mathbb{P}$. Ω is spherically convex if and only if $1/\mu_\Omega$ is spherically concave.*

Proof. The function $1/\mu_\Omega$ is spherically concave precisely when

$$2\left|\partial_s^2 \frac{1}{\mu_\Omega}\right| + \frac{1}{2}\Delta_s \frac{1}{\mu_\Omega} + \frac{4}{\mu_\Omega} \le 0$$

on Ω. By making use of the identity (8) we see that the preceding inequality is equivalent to part (iii) of Theorem 1. Thus, Theorem 2 gives a geometric interpretation of Theorem 1 (iii). $\square$

Corollary 1. *Suppose Ω is a spherically convex region. Then μ_Ω attains its minimum value at a unique point of Ω.*

Proof. It is known that μ_Ω always attains its minimum value on any hyperbolic region [MO]. Suppose Ω is spherically convex and μ_Ω attained its mininum value at two distinct points $A, B \in \Omega$. Then $1/\mu_\Omega$ has a maximum value at both A and B. Let $\gamma : w = w(s)$ be the spherical geodesic arc from A to B parametrized by spherical arclength. Then γ lies in Ω because Ω is spherically convex and $v(s) = 1/\mu_\Omega(w(s))$ satisfies $v''(s) + 4v(s) \le 0$, or $v''(s) \le -4v(s) < 0$, along γ. Thus, v is strictly concave along γ which is inconsistent with the assumption that v attains its maximum value at the endpoints A and B. In other words, μ_Ω must attain its minimum value at a unique point of Ω. $\square$

The euclidean analog of the Corollary is not so simply stated. The density λ_Ω of the hyperbolic metric on a hyperbolic region $\Omega \subset \mathbb{C}$ need not attain a minimum value even if Ω is convex. For example, if $\mathbb{H} = \{w : \operatorname{Im}(w) > 0\}$, then $\lambda_\mathbb{H}(w) = 1/[2\operatorname{Im}(w)]$ does not attain a minimum value on $\mathbb{H}$. But on any bounded convex region Ω, the density λ_Ω does attain its minimum value at a unique point of Ω [MW]. Minimum points and critical points of the density of the hyperbolic metric have been investigated in [CO], [COP], [MO] and [Y_4].

Yamashita [Y_3] proved that a region Ω in $\mathbb{C}$ is convex if and only if

$$\left|\frac{1}{\lambda_\Omega(A)} - \frac{1}{\lambda_\Omega(B)}\right| \le 2|A - B|$$

for all $A, B \in \Omega$. The next result is a spherical analog.

Theorem 3. *Let Ω be a hyperbolic region on $\mathbb{P}$. Then Ω is spherically convex if and only if*

$$\left|\arcsin\left(\frac{1}{\mu_\Omega(A)}\right) - \arcsin\left(\frac{1}{\mu_\Omega(B)}\right)\right| \le 2d_\mathbb{P}(A, B)$$

for all $A, B \in \Omega$.

Proof. First, suppose Ω is spherically convex. Fix $A, B \in \Omega$. Let $\gamma :$ $w = w(s)$, $0 \leq s \leq L$, be the spherical geodesic from A to B parametrized by spherical arclength. Then $\gamma \subset \Omega$ and $L = d_{\mathbb{P}}(A, B)$. If $v(s) = 1/\mu_\Omega(w(s))$, then formula (9) gives

$$|v'(s)| \leq 2 \left| \partial_s \frac{1}{\mu_\Omega}(w(s)) \right|.$$

By making use of Theorem 1 (ii) we obtain

$$|v'(s)| \leq 2\sqrt{1 - \frac{1}{\mu_\Omega^2(w(s))}} = 2\sqrt{1 - v^2(s)},$$

or

$$-2 \leq \frac{v'(s)}{\sqrt{1 - v^2(s)}} \leq 2.$$

By integrating these inequalities over $[0, L]$ we obtain

$$|\arcsin v(L) - \arcsin v(0)| \leq 2L,$$

or

$$\left| \arcsin\left(\frac{1}{\mu_\Omega(B)}\right) - \arcsin\left(\frac{1}{\mu_\Omega(A)}\right) \right| \leq 2 d_{\mathbb{P}}(A, B).$$

Next, we demonstrate that if the preceding inequality holds for all $A, B \in \Omega$, then Ω is spherically convex. Note that in order for this inequality to make sense we must have $\mu_\Omega \geq 1$. We show that this inequality implies

$$\left| \partial_s \frac{1}{\mu_\Omega} \right|^2 \leq 1 - \frac{1}{\mu_\Omega^2}$$

and so Ω is spherically convex by Theorem 1. Fix $w_0 \in \Omega$. Let $\gamma : w = w(s)$ be a spherical geodesic arc parametrized by spherical arclength on some interval containing 0 with $w_0 = w(0)$ and

$$\mathrm{Re}\left\{ e^{i\theta(0)} \partial_s \frac{1}{\mu_\Omega}(w_0) \right\} = \left| \partial_s \frac{1}{\mu_\Omega}(w_0) \right|;$$

of course, γ is assumed to be parametrized by spherical arclength. If $v(s) = 1/\mu_\Omega(w(s))$, then we have

$$|\arcsin v(s) - \arcsin v(0)| \leq 2 d_{\mathbb{P}}(w(s), w(0)) = 2s$$

for all s sufficiently small. If we divide by s and let s tend to 0, we obtain

$$\frac{|v'(0)|}{\sqrt{1-v^2(0)}} \leq 2.$$

From formula (9)

$$v'(0) = 2\,\mathrm{Re}\left\{e^{i\theta(0)}\partial_s \frac{1}{\mu_\Omega}(w_0)\right\} = 2\left|\partial_s \frac{1}{\mu_\Omega}(w_0)\right|,$$

so

$$\left|\partial_s \frac{1}{\mu_\Omega(w_0)}\right| \leq \sqrt{1-\frac{1}{\mu_\Omega^2(w_0)}},$$

which implies that Ω is spherically convex. □

5. Connection with euclidean convexity

We now indicate how our results for spherical convexity contain the corresponding results for euclidean convexity as limiting cases. In order to do this we consider a one-parameter family

$$\sigma_R(w)|dw| = \frac{R^2|dw|}{R^2+|w|^2}$$

of conformal metrics defined for $R>0$. The spherical metric corresponds to $R=1$ while the euclidean metric arises as the limiting case $R\to\infty$. Our results for spherical convexity are readily translated into corresponding results for convexity relative to $\sigma_R(w)|dw|$ for $R>0$. Since there is no real difference between the special case $R=1$ and the general case $R>0$, we felt it was better to consider only the case $R=1$ in the earlier parts of the paper in order to simplify the exposition. But to show that our results contain the euclidean results as a limiting case it is now necessary to consider arbitrary $R>0$. By considering arbitrary $R>0$ we can see how the euclidean results which sometimes seem quite different from the spherical results are really limiting cases. Loosely speaking, the differences are due to the curvature of the metrics involved.

The metric $\sigma_R(w)|dw|$ arises from stereographically projecting a sphere of radius $R/2$ that is tangent to the complex plane $\mathbb{C}$ at the origin. The metric $\sigma_R(w)|dw|$ has curvature $4/R^2$. The isometries of the metric are $T(z) = e^{i\theta}R^2(z-a)/(R^2+\overline{a}z)$, where $a\in\mathbb{P}$ and $\theta\in\mathbb{R}$. The associated distance function is

$$d_R(A,B) = R\arctan\frac{R|A-B|}{|R^2+\overline{A}B|}.$$

The "density" of the hyperbolic metric relative to $\sigma_R(w)|dw|$ is

$$\mu_R(w) = \frac{\lambda_\Omega(w)|dw|}{\sigma_R(w)|dw|}.$$

Note that $\mu_R \to \lambda_\Omega$ when $R \to \infty$. Differential operators relative to $\sigma_R(w)|dw|$ are defined by

$$\begin{aligned}
\partial_R r &= \frac{\partial r}{\sigma_R}, \\
\partial_R^2 r &= \frac{\partial^2 r - 2(\partial \log \sigma_R)(\partial r)}{\sigma_R^2}, \\
\Delta_R r &= \frac{\Delta r}{\sigma_R^2}.
\end{aligned}$$

Although we do not explicitly state them, we remark that these differential operators have the same type of invariance properties relative to the isometries of $\sigma_R(w)|dw|$ that the spherical differential operators have relative to the isometries of the spherical metric. When $R \to \infty$ these differential operators tend to the corresponding euclidean differential operators.

Next, we make precise the relationship between $\sigma_R(w)|dw|$ and $\lambda_{\mathbb{P}}(w)|dw|$. If $h_R(z) = z/R$, then

$$h_R^*(R\lambda_{\mathbb{P}}(w)|dw|) = \sigma_R(z)|dz|.$$

In words, h_R is an isometry from $\mathbb{P}$ with the conformal metric $\sigma_R(z)|dz|$ to $\mathbb{P}$ with the conformal metric $R\lambda_{\mathbb{P}}(w)|dw|$. This makes it easy to determine the geodesics relative to $\sigma_R(w)|dw|$. Note that $R\lambda_{\mathbb{P}}(w)|dw|$ and $\lambda_{\mathbb{P}}(w)|dw|$ have the same geodesics. Therefore, a region is convex relative to $R\lambda_{\mathbb{P}}(w)|dw|$ if and only if it is convex relative to $\lambda_{\mathbb{P}}(w)|dw|$. A path γ is a geodesic relative to $\sigma_R(z)|dz|$ if and only if $h_R \circ \gamma$ is a spherical geodesic. Hence, a region Ω on $\mathbb{P}$ is convex relative to $\sigma_R(z)|dz|$ exactly when $h_R(\Omega) = \Omega_R$ is spherically convex.

In order to translate our theorems on spherical convexity to results on convexity relative to $\sigma_R(w)|dw|$ we need to indicate the connection between spherical differential operators and our new differential operators. It is not difficult to check that $\mu_{\Omega_R} \circ h_R = R\mu_R$, or

$$\frac{1}{\mu_R} = \frac{R}{\mu_{\Omega_R}} \circ h_R.$$

Also,

$$\begin{aligned}
\partial_R \frac{1}{\mu_R} &= \left(\partial_s \frac{1}{\mu_{\Omega_R}}\right) \circ h_R, \\
\partial_R^2 \frac{1}{\mu_R} &= \frac{1}{R}\left(\partial_s^2 \frac{1}{\mu_{\Omega_R}}\right) \circ h_R, \\
\frac{1}{\mu_R}\partial_R^2 \frac{1}{\mu_R} &= \left(\frac{1}{\mu_{\Omega_R}}\partial_s^2 \frac{1}{\mu_{\Omega_R}}\right) \circ h_R, \\
\Delta_R \frac{1}{\mu_R} &= \frac{1}{R}\Delta_s \frac{1}{\mu_{\Omega_R}}.
\end{aligned}$$

Now, we can reformulate Theorem 1.

Theorem 4. *Suppose Ω is a hyperbolic region on $\mathbb{P}$ and μ_R is the density of the hyperbolic metric relative to $\sigma_R(w)|dw|$. The following are equivalent:*

(i) *Ω is convex relative to $\sigma_R(w)|dw|$,*

(ii) $\left|\partial_R \dfrac{1}{\mu_R}\right|^2 \le 1 - \dfrac{1}{R^2\mu_R^2}$,

(iii) $\dfrac{1}{\mu_R}\left|\partial_R^2 \dfrac{1}{\mu_R}\right| + \left|\partial_R \dfrac{1}{\mu_R}\right|^2 \le 1 - \dfrac{1}{R^2\mu_R^2}$,

(iv) $\Delta_R \dfrac{1}{\mu_R} \le -\dfrac{8}{R^2\mu_R}$.

Before translating Theorem 2, we need to define the appropriate notion of concavity. A function $r : \Omega \to \mathbb{R}$ is called concave relative to $\sigma_R(w)|dw|$ if $v'' + (4/R^2)v \le 0$ whenever $\gamma : w = w(s)$ is a geodesic arc relative to $\sigma_R(w)|dw|$ in Ω that is parametrized by arclength relative to $\sigma_R(w)|dw|$ (that is, $\sigma_R(w(s))|w'(s)| = 1$) and $v(s) = r(w(s))$. We find that $1/\mu_R$ is concave relative to $\sigma_R(w)|dw|$ if and only if

$$2R\left|\partial_R^2 \frac{1}{\mu_R}\right| + \frac{R}{2}\Delta_R \frac{1}{\mu_R} + \frac{4}{R\,\mu_R} \le 0,$$

or equivalently,

$$2\left|\partial_R^2 \frac{1}{\mu_R}\right| + \frac{1}{2}\Delta_R \frac{1}{\mu_R} + \frac{4}{R^2\mu_R} \le 0.$$

For $R \to \infty$ we obtain the usual euclidean notion of concavity ($v'' \le 0$).

Theorem 5. *Suppose Ω is a hyperbolic region on $\mathbb{P}$. Ω is convex relative to $\sigma_R(w)|dw|$ if and only if $1/\mu_R$ is concave relative to $\sigma_R(w)|dw|$.*

Theorem 6. *Suppose Ω is a hyperbolic region on $\mathbb{P}$. Ω is convex relative to $\sigma_R(w)|dw|$ if and only if*

$$\left|R\arcsin\left(\frac{1}{R\,\mu_R(A)}\right) - R\arcsin\left(\frac{1}{R\,\mu_R(B)}\right)\right| \le 2d_R(A,B)$$

for all $A, B \in \Omega$.

Of course, for $R = 1$ Theorems 4, 5 and 6 are just Theorems 1, 2 and 3. If $\Omega \subset \mathbb{C}$, then the limiting cases ($R \to \infty$) of Theorems 4, 5 and 6 are known characterizations of euclidean convex regions in terms of the usual density λ_Ω of the hyperbolic metric.

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