J. Math. Kyoto Univ. (JMKYAZ)
41-4 (2002), 627–667

Values of the Epstein zeta functions at the critical points

By

Akio Fujii

1. Introduction

We continue our study on the distribution of the zeros of the Epstein zeta functions attached to the positive definite quadratic forms. Here we are concerned only with the vanishing or the non-vanishing of the following type of Epstein zeta functions at the critical points. For each integer $K \geq 1$ and for any positive number d, we define $F_d(s,K)$ by

$$F_d(s,K) = \sum{}' \frac{1}{(m_1^2 + m_2^2 + \cdots + m_K^2 + d(m_{K+1}^2 + m_{K+2}^2 + \cdots + m_{2K}^2))^s},$$

for $\Re(s) > K$, the dash indicating that m_j's run over the integers excluding the case $(m_1, m_2, m_3, \ldots, m_{2K}) = (0,0,0,\ldots,0)$. We have seen in Fujii [7] that $F_d(s,K)$ can be continued analytically to the whole complex plane with a simple pole at $s = K$ and has the following functional equation:

$$\left(\frac{\pi}{\sqrt{d}}\right)^{-s} \Gamma(s) F_d(s,K) = \left(\frac{\pi}{\sqrt{d}}\right)^{-(K-s)} \Gamma(K-s) F_d(K-s,K),$$

where $\Gamma(s)$ is the Γ-function. Thus the critical point of $F_d(s,K)$ is $s = K/2$. The purpose of the present article is to prove the following.

Main Theorem. *For all pairs (K,d) of integers $K \geq 1$ and positive real numbers d, we have*

$$F_d\left(\frac{K}{2}, K\right) \neq 0,$$

except exactly for eight pairs.

This is much sharper than what we have claimed and expected in our previous work Fujii [7].

Main Theorem consists of the following two theorems. Theorem 1 describes the exceptional eight cases. Theorem 2 describes that they are really exceptional.

Communicated by Prof. Y. Yoshida, June 14, 1999

Theorem 1. *For each integer $K = 1, 2, 3$ and 4, there exist two real numbers $D_1(K)$ and $D_2(K)$ such that $1/(16\pi^2) < D_1(K) < D_2(K)$ and they satisfy the following three properties.*

$$(1) \qquad F_d\left(\frac{K}{2}, K\right) = 0 \quad \text{when} \quad d = D_1(K) \quad \text{and} \quad d = D_2(K).$$

Moreover $s = K/2$ is a double zero of $F_d(s, K)$ for both cases.

$$(2) \qquad F_d\left(\frac{K}{2}, K\right) > 0 \quad \text{when} \quad 0 < d < D_1(K) \quad \text{or} \quad d > D_2(K).$$

Hence in this case, $F_d(s, K)$ has one real zero in $0 < \Re(s) < K/2$, one real zero in $K/2 < \Re(s) < K$ and $F_d(s, K) \neq 0$ for $s = 0, K/2$ and K.

$$(3) \qquad F_d\left(\frac{K}{2}, K\right) < 0 \quad \text{when} \quad D_1(K) < d < D_2(K).$$

This theorem has been proved in Fujii [5] for $K = 1$ and 2 and in Fujii [7] for $K = 4$. In this article, we shall prove the case for $K = 3$.

Concerning the numerical values of $D_1(K)$ and $D_2(K)$, our rough computations, by Mathematica, show that

$$\begin{aligned} D_1(1) &= 0.1417332\cdots, & D_2(1) &= 7.0555079\cdots, \\ D_1(2) &= 0.165\cdots, & D_2(2) &= 6.039\cdots, \\ D_1(3) &= 0.3\cdots, & D_2(3) &= 2.7\cdots, \\ D_1(4) &= 0.61\cdots, & D_2(4) &= 1.620\cdots \quad . \end{aligned}$$

Remark 1. Since $F_d(s, K) = d^{-s} F_{d^{-1}}(s, K)$, one has

$$D_1(K) \cdot D_2(K) = 1$$

in the above theorem. This is noticed by Yoshida [16].

When $K \geq 5$, then the situation becomes different. We have the following theorem.

Theorem 2. *For any integer $K \geq 5$ and for any positive real number d, there exists a positive absolute constant A such that*

$$\frac{F_d(\frac{K}{2}, K)}{A(K)} \geq A,$$

where we put

$$A(K) = \frac{\pi^{\frac{K}{2}}}{\Gamma(\frac{K}{2})},$$

which will be introduced also in the next section.

Thus for any integer $K \geq 5$ and for any positive d, $F_d(s,K)$ has one real zero in $0 < \Re(s) < K/2$, one real zero in $K/2 < \Re(s) < K$ and

$$F_d(s,K) \neq 0$$

for $s = 0, K/2$ and K.

Remark 2. We can take

$$\begin{aligned} A &= 1.8 \quad \text{for} \quad K \geq 8, \\ A &= 0.9 \quad \text{for} \quad K = 7, \\ A &= 0.5 \quad \text{for} \quad K = 6 \end{aligned}$$

and

$$A = 0.06 \quad \text{for} \quad K = 5.$$

It is clear that we can refine these numerical values further as our method in Section 3 shows.

Remark 3. Theorems 1 and 2 should be compared with the following result (cf. Theorem VI in p. 169 of Fujii [7]): For each $K \geq 2$, when d is sufficiently large, there exist two real zeros $\rho_d(K)$ and $K - \rho_d(K)$ of $F_d(s,K)$ such that

$$\rho_d(K) = \frac{\pi^K}{d^{\frac{K}{2}}\Gamma(K)Z_K(K)}\left(1 + O\left(\frac{\log d}{d^{\frac{K}{2}}}\right)\right)$$

as $d \to \infty$, where we put, for any integer $K \geq 1$ and for $\Re(s) > K/2$,

$$Z_K(s) = \sum\nolimits'_{-\infty < m_1, \dots, m_K < +\infty} \frac{1}{(m_1^2 + m_2^2 + \cdots + m_K^2)^s},$$

the dash indicating that we omit $(m_1, \dots, m_K) = (0, 0, \dots, 0)$.

Combining Theorems 1 and 2, we get our Main Theorem.

We shall prove our theorems as an application of our extensions of Chowla-Selberg's formula (cf. Lemmas 2 through 4 in the next section).

After submission of the original version of the present article, the author has obtained [15]. There we have shown that

(i) $Z_K(K/4) > 0$ for all integer $K \geq 10$

and

(ii) $Z_K(K/4) < 0$ for all integer $1 \leq K \leq 9$,

where $K/4$ is the critical point of $Z_K(s)$.

Later, after reading the original version of the present article and [15], Yoshida [16] has informed the author a simple and different proof of the positivity of both $Z_K(K/4)$ for any integer $K \geq 10$ and that of $F_d(K/2, K)$ for $K \geq 6$. In stead of Chowla-Selberg's formula, he has started from the following expression of $Z_K(s)$, for example, which holds for any integer $K \geq 1$ and for any complex s:

$$\pi^{-s}\Gamma(s)Z_K(s) = -\left(\frac{1}{s} + \frac{1}{\frac{K}{2} - s}\right) + \int_1^\infty (\vartheta(t)^K - 1)(t^{s-1} + t^{\frac{K}{2}-1-s})\, dt$$

with

$$\vartheta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi t}.$$

It reminds us a simple proof of the negativity of the Riemann zeta function $\zeta(s) = (1/2)Z_1(s/2)$ for $0 < s < 1$. For example, the corresponding result is a direct consequence of any of the following simpler expressions which holds for $0 < s < 1$.

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{x - [x]}{x^{s+1}}\, dx \ \left(< \frac{-1}{1-s} + 1 < 0\right)$$

with Gauss symbol $[x]$ or

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

Although the application of our extensions of Chowla-Selberg's formula is not so simple, it gives explicit lower bounds like Remark 2 stated above. It gives also more explicit description of real zeros as stated in Remark 3 above. Furthermore, it explains some background of the arithmetic nature of the values $Z_K(K/4)$, which has been shown in Fujii [15].

We shall give preliminary lemmas in Section 2. We shall prove Theorem 2 in Section 3 and Theorem 1 for $K = 3$ in Section 4. In Section 5, we shall give some corrections to Fujii [5], [7], some notice on the complex zeros of $F_d(s, 8)$ and some other supplemental remarks.

Finally, the author wishes to express his thanks to Prof. Hiroyuki Yoshida who has sent [16] to the author and given valuable comments to both the original manuscript of the present article and [15].

2. Preliminaries for the proof of Theorems 1 and 2

We start with describing an explicit expression of $F_d(K/2, K)$, which we have derived in our previous article [7]. For this purpose we shall use $Z_K(s)$ introduced above. We put for any integer $K \geq 1$ and for any integer $n \geq 0$,

$$r_K(n) = \sum_{n = m_1^2 + m_2^2 + \cdots + m_K^2} 1.$$

We put $r(n) = r_2(n)$. It is critical in our investigations that we have the following simple expressions:

$$Z_1(s) = 2\zeta(2s),$$
$$Z_2(s) = 4\zeta(s)L(s,\chi),$$
$$Z_4(s) = 8(1-2^{2-2s})\zeta(s)\zeta(s-1)$$

and

$$Z_8(s) = 16(1-2^{1-s}+2^{4-2s})\zeta(s)\zeta(s-3),$$

where χ is the non-principal Dirichlet character mod 4 and $L(s,\chi)$ is the corresponding Dirichlet L-function. For convenience (cf. definition of $A_0(K)$ in (ii)-2 of Lemma 1) we put

$$Z_0(s) = 0$$

and

$$r_0(n) = 0 \quad \text{for any integer } n \geq 1.$$

As is well-known, $Z_K(s)$ satisfies the following functional equation

$$\pi^{-s}\Gamma(s)Z_K(s) = \pi^{-(\frac{K}{2}-s)}\Gamma\left(\frac{K}{2}-s\right)Z_K\left(\frac{K}{2}-s\right).$$

This gives the analytic continuation of $Z_K(s)$ to the whole complex plane with a simple pole at $s = K/2$. We describe its Laurent expansion as follows.

$$Z_K(s) = \frac{A(K)}{s-\frac{K}{2}} + A_0(K) + A_1(K)\left(s-\frac{K}{2}\right) + A_2(K)\left(s-\frac{K}{2}\right)^2 + \cdots \quad .$$

$A(K)$ and $A_0(K)$ can be evaluated explicitly as follows, where we have corrected some mistakes in the description in pp. 158–159 of Fujii [7].

Lemma 1. *Suppose that K is an integer ≥ 2. Then we have the following.*

(i)
$$A(K) = \frac{\pi^{\frac{K}{2}}}{\Gamma(\frac{K}{2})}.$$

(ii-1)
$$A_0(2) = \pi(2C_0 - \log 4 - 2\log|\eta(i)|^2),$$

where C_0 is the Euler constant and we put

$$\eta(z) = e^{\frac{\pi iz}{12}}\prod_{m=1}^{\infty}(1-e^{2\pi imz})$$

for complex $z = x+iy$ with $y > 0$.

(ii-2) *For any integer $K \geq 3$, we put $L_0 = 2^{N_0}$ with N_0 defined by*

$$N_0 = \begin{cases} [\frac{\log K}{\log 2}] & \textit{if} \quad K \neq 2^N \quad \textit{for any integer} \quad N \geq 2, \\ N & \textit{if} \quad K = 2^N \quad \textit{for some integer} \quad N \geq 2. \end{cases}$$

Then we have

$$\begin{aligned}A_0(K) &= Z_{K-L_0}\left(\frac{K}{2}\right) + A(K)\sum_{k=1}^{N_0}\frac{1}{A\left(2\frac{L_0}{2^k}\right)}Z_{\frac{L_0}{2^k}}\left(\frac{L_0}{2^k}\right)\\ &\quad + A(K)\left\{C_0 - 2\log 2 - \frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right\}\\ &\quad + 2A(K)\sum_{k=1}^{N_0}\left(\sum_{n=1}^{\infty} n^{\frac{1}{4}\frac{L_0}{2^k}}\left(\sum_{m|n}\frac{r_{\frac{L_0}{2^k}}(m)r_{\frac{L_0}{2^k}}\left(\frac{n}{m}\right)}{m^{\frac{1}{2}\cdot\frac{L_0}{2^k}}}\right)K_{\frac{1}{2}\frac{L_0}{2^k}}(2\pi\sqrt{n})\right)\\ &\quad + 2A(K)\sum_{n=1}^{\infty} n^{\frac{L_0}{4}}\left(\sum_{m|n}\frac{r_{L_0}(m)r_{K-L_0}\left(\frac{n}{m}\right)}{m^{\frac{L_0}{2}}}\right)K_{\frac{L_0}{2}}(2\pi\sqrt{n}),\end{aligned}$$

where we put

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-z\frac{u+u^{-1}}{2}}u^{\nu-1}\,du$$

for arbitrary ν *and* $|\arg z| < \pi/2$.

Here we mention the following formula (cf. Theorem I in p. 164 of Fujii [7]) which is an extension of Chowla-Selberg's formula (cf. Selberg [12]).

Lemma 2. *For each integer* $K \geq 2$, *we have*

$$F_d(s,K) = Z_K(s) + \frac{\pi^{\frac{K}{2}}}{d^{s-\frac{K}{2}}}\frac{\Gamma(s-\frac{K}{2})}{\Gamma(s)}Z_K\left(s-\frac{K}{2}\right) + \left(\frac{\pi}{\sqrt{d}}\right)^s\frac{2d^{\frac{K}{4}}}{\Gamma(s)}E(s,d,K),$$

where we put

$$E(s,d,K) = \sum_{n=1}^{\infty} n^{\frac{s-\frac{K}{2}}{2}}\left(\sum_{m|n}\frac{r_K(m)r_K(\frac{n}{m})}{m^{s-\frac{K}{2}}}\right)K_{s-\frac{K}{2}}(2\pi\sqrt{dn}).$$

For $Z_K(s)$, we have the following formula (cf. p. 179 of Fujii [7] for $L = L_0$).

Lemma 3. *For any integer* $K \geq 2$ *and for any decomposition of* K *into* $K = (K-L) + L$ *with an integer* $0 \leq L \leq K$, *we have*

$$\begin{aligned}Z_K(s) &= Z_{K-L}(s) + \pi^{\frac{K-L}{2}}\frac{\Gamma(s-\frac{K-L}{2})}{\Gamma(s)}Z_L\left(s-\frac{K-L}{2}\right)\\ &\quad + \frac{2\pi^s}{\Gamma(s)}\sum_{n=1}^{\infty} n^{\frac{s-\frac{K-L}{2}}{2}}\left(\sum_{m|n}\frac{r_L(m)r_{K-L}(\frac{n}{m})}{m^{s-\frac{K-L}{2}}}\right)K_{s-\frac{K-L}{2}}(2\pi\sqrt{n}).\end{aligned}$$

Using Lemma 2, we have derived the following expression of $F_d(K/2,K)$ (cf. Theorem IV-(i) in p. 165 of Fujii [7]).

Lemma 4. *For any integer $K \geq 2$, we have*

$$
\begin{aligned}
F_d\left(\frac{K}{2}, K\right) &= 2Z_{K-L_0}\left(\frac{K}{2}\right) + 2A(K)\sum_{k=1}^{N_0} \frac{1}{A\left(2\frac{L_0}{2^k}\right)} Z_{\frac{L_0}{2^k}}\left(\frac{L_0}{2^k}\right) \\
&\quad + 2A(K)\left\{C_0 - 2\log 2 - \frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right\} \\
&\quad + 4A(K)\sum_{k=1}^{N_0}\left(\sum_{n=1}^{\infty} n^{\frac{1}{4}\frac{L_0}{2^k}}\left(\sum_{m|n}\frac{r_{\frac{L_0}{2^k}}(m)r_{\frac{L_0}{2^k}}\left(\frac{n}{m}\right)}{m^{\frac{1}{2}\cdot\frac{L_0}{2^k}}}\right)K_{\frac{1}{2}\frac{L_0}{2^k}}(2\pi\sqrt{n})\right) \\
&\quad + 4A(K)\sum_{n=1}^{\infty} n^{\frac{L_0}{4}}\left(\sum_{m|n}\frac{r_{L_0}(m)r_{K-L_0}\left(\frac{n}{m}\right)}{m^{\frac{L_0}{2}}}\right)K_{\frac{L_0}{2}}(2\pi\sqrt{n}) \\
&\quad - A(K)\left(\log\frac{\pi^2}{d} - 2\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right) + 2A(K)E(d,K) \\
&= 2A_0(K) - A(K)\left(\log\frac{\pi^2}{d} - 2\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right) + 2A(K)E(d,K),
\end{aligned}
$$

where L_0 and N_0 are chosen as in Lemma 1 *above and we put*

$$
E(d,K) = E\left(\frac{K}{2}, d, K\right) = \sum_{n=1}^{\infty}\left(\sum_{m|n} r_K(m)r_K\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn}).
$$

Using again Lemma 2, we have derived the following formula which is Theorem IV′-(i) of Fujii [7].

Lemma 5. *Suppose that K is an integer ≥ 2. Then we have the following formulas.*

$$
\text{(i)}\quad F_d'\left(\frac{K}{2}, K\right) = A_0(K)\left(\log\frac{\pi^2}{d} - 2\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right) - A(K)\left(\frac{1}{2}\left(\log\frac{\pi^2}{d}\right)^2 - 2\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\log\frac{\pi^2}{d} + 2\left(\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right)^2\right) + 2A(K)\left(\log\frac{\pi}{\sqrt{d}} - \frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right)E(d,K),
$$

where $E(d,K)$ is defined in the statement of Lemma 4.

$$\text{(ii)}\quad F_d''\left(\frac{K}{2},K\right)=2\left\{-A(K)\cdot\left(\frac{1}{6}\left(\log\frac{\pi^2}{d}\right)^3-\left(\log\frac{\pi^2}{d}\right)^2\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right.\right.$$
$$+2\cdot\log\frac{\pi^2}{d}\left(\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right)^2-\frac{1}{3}\frac{\Gamma'''}{\Gamma}\left(\frac{K}{2}\right)$$
$$\left.+\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\frac{\Gamma''}{\Gamma}\left(\frac{K}{2}\right)-2\left(\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right)^3\right)$$
$$+A_0(K)\cdot\left(-2\log\frac{\pi^2}{d}\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)+\frac{1}{2}\left(\log\frac{\pi^2}{d}\right)^2+2\left(\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right)^2\right)$$
$$\left.-A_1(K)\cdot\left(-2\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)+\log\frac{\pi^2}{d}\right)+2A_2(K)\right\}$$
$$+2A(K)E(d,K)\left(\frac{1}{4}\left(\log\frac{\pi^2}{d}\right)^2-\left(\log\frac{\pi^2}{d}\right)\cdot\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right.$$
$$\left.-\frac{\Gamma''}{\Gamma}\left(\frac{K}{2}\right)+2\left(\frac{\Gamma'}{\Gamma}\left(\frac{K}{2}\right)\right)^2\right)$$
$$+2A(K)E^{(2)}(d,K),$$

where we put

$$E^{(2)}(d,K)=\sum_{n=1}^{\infty}\left(\sum_{m|n}\log m\cdot\left(\log m-\frac{1}{2}\log n\right)r_K(m)r_K\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn})$$
$$+\sum_{n=1}^{\infty}\left(\sum_{m|n}r_K(m)r_K\left(\frac{n}{m}\right)\right)\cdot\frac{1}{2}\int_0^{\infty}e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log^2 y}{y}\,dy.$$

To prove Theorem 2, we shall use alternative expressions of $F_d(K/2,K)$ for $K=4$ and $K=8$. Namely, we use the following.

Lemma 6. *For any positive d, we have*

$$F_d(2,4)=\pi^2\left(\log d+2+\frac{4}{3}\log 2-2\log\pi+\frac{12\zeta'(2)}{\pi^2}+2E(d,4)\right).$$

and

$$F_d(4,8)=\frac{\pi^4}{6}\left(\log d+\frac{11}{3}-2\log\pi+\frac{180}{\pi^4}\zeta'(4)+2E(d,8)\right).$$

The former is Theorem IV-(ii) in p. 166 of Fujii [7]. We shall give a proof to the latter, for completeness. First, Lemma 4 implies, directly, that

$$F_d(4,8)=2A_0(8)-A(8)\left(\log\frac{\pi^2}{d}-2\frac{\Gamma'}{\Gamma}(4)\right)+2A(8)E(d,8).$$

Hence, we have only to evaluate $A_0(8)$ in an alternative way. Here we use the expression

$$Z_8(s) = 16(1 - 2^{1-s} + 2^{4-2s})\zeta(s)\zeta(s-3)$$

which has been noticed above. Using the Laurent expansion of $\zeta(s)$ at $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + C_0 + C_1(s-1) + C_2(s-1)^2 + \cdots \quad ,$$

we get at $s = 4$,

$$\begin{aligned} Z_8(s) = {} & \frac{15 \cdot \zeta(4)}{s-4} + \{C_0\zeta(4)15 + \zeta'(4)15\} \\ & + (s-4)\left\{C_1\zeta(4)15 + C_0\zeta'(4)15 + \frac{\zeta''(4)}{2}15 + \log^2 2 \cdot \zeta(4)\right\} \\ & + (s-4)^2 \left\{\frac{\zeta'''(4)}{6}15 - \log^3 2 \cdot \zeta(4) + C_0 \log^2 2 \cdot \zeta(4) + C_0\frac{\zeta''(4)}{2}15 \right. \\ & \left. + \log^2 2 \cdot \zeta'(4) + C_1\zeta'(4)15 + C_2\zeta(4)15 \right\} + \cdots \quad . \end{aligned}$$

Hence, we get

$$A_0(8) = C_0\zeta(4)15 + \zeta'(4)15.$$

Since $A(8) = \pi^4/6$, $(\Gamma'/\Gamma)(4) = -C_0 + 11/6$ and $\zeta(4) = \pi^4/90$, we get the above expression.

To prove Theorem 1 for $K = 3$, we need to evaluate $A_1(3)$ and $A_2(3)$. We have two expressions corresponding to the two decompositions

$$\begin{aligned} 3 &= (3-2) + 2 \\ &= (3-1) + 1. \end{aligned}$$

Namely, by Lemma 3, we have two decompositions of $Z_3(s)$ as follows. To the former, it corresponds the following.

$$\begin{aligned} Z_3(s) &= Z_1(s) + \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} Z_2\left(s - \frac{1}{2}\right) \\ &\quad + \frac{2\pi^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{\frac{s-\frac{1}{2}}{2}} \left(\sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m^{s-\frac{1}{2}}}\right) K_{s-\frac{1}{2}}(2\pi\sqrt{n}) \\ &= 2\zeta(2s) + \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} 4\zeta\left(s-\frac{1}{2}\right) L\left(s-\frac{1}{2}, \chi\right) \\ &\quad + \frac{2\pi^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{\frac{s-\frac{1}{2}}{2}} \left(\sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m^{s-\frac{1}{2}}}\right) K_{s-\frac{1}{2}}(2\pi\sqrt{n}). \end{aligned}$$

To the latter, it corresponds the following.

$$
\begin{aligned}
Z_3(s) &= Z_2(s) + \pi\frac{\Gamma(s-1)}{\Gamma(s)}Z_1(s-1) \\
&\quad + \frac{2\pi^s}{\Gamma(s)}\sum_{n=1}^{\infty} n^{\frac{s-1}{2}}\left(\sum_{m|n}\frac{r_1(m)r_2(\frac{n}{m})}{m^{s-1}}\right)K_{s-1}(2\pi\sqrt{n}) \\
&= 4\zeta(s)L(s,\chi) + \frac{2\pi}{s-1}\zeta(2s-2) \\
&\quad + 2\frac{\pi^s}{\Gamma(s)}\sum_{n=1}^{\infty} n^{\frac{s-1}{2}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u^{2s-2}}\int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})}y^{s-2}\,dy \\
&= 4\zeta(s)L(s,\chi) + \frac{2\pi}{s-1}\zeta(2s-2) + \Psi(s), \quad \text{say.}
\end{aligned}
$$

The last expression is nothing but the Chowla-Selberg's formula (45) with $d = 1$ in p. 533 of Selberg [13]. A comparison of these two expressions will be touched slightly in the last section. Here we start with the latter, we get the following three expressions.

Lemma 7-(i).

$$
A_0(3) = 4\pi C_0 - 4\pi + 4\zeta\left(\frac{3}{2}\right)L\left(\frac{3}{2},\chi\right) + 4\pi\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}.
$$

Lemma 7-(ii).

$$
\begin{aligned}
A_1(3) = 8\pi C_1 - 8\pi C_0 + 8\pi &+ 4\zeta'\left(\frac{3}{2}\right)L\left(\frac{3}{2},\chi\right) + 4\zeta\left(\frac{3}{2}\right)L'\left(\frac{3}{2},\chi\right) \\
&+ 4\pi(\log\pi + C_0 + 2\log 2 - 2)\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u} \\
&+ 4\pi\sum_{n=1}^{\infty}\frac{1}{2}\log n\cdot e^{-2\pi\sqrt{n}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u} \\
&+ 4\pi\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}}\sum_{u^2|n}(-2\log u)\frac{r(\frac{n}{u^2})}{u} \\
&+ 4\pi\sum_{n=1}^{\infty} n^{\frac{1}{4}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}\int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})}\frac{\log y}{\sqrt{y}}\,dy.
\end{aligned}
$$

Lemma 7-(iii).

$$\begin{aligned}
A_2(3) &= 2\pi(8C_0 - 8C_1 - 8 + 8C_2) \\
&\quad + 2\left(\zeta''\left(\frac{3}{2}\right) L\left(\frac{3}{2},\chi\right) + 2\zeta'\left(\frac{3}{2}\right) L'\left(\frac{3}{2},\chi\right) + \zeta\left(\frac{3}{2}\right) L''\left(\frac{3}{2},\chi\right)\right) \\
&\quad + 2\pi\left(\log^2\pi - 2\log\pi\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) - \frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right) + 2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2\right) \\
&\quad\quad \times \sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\
&\quad + 2\pi\left(\log\pi - \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right) \cdot \sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} (\log n - 4\log u) \cdot \frac{r(\frac{n}{u^2})}{u} \\
&\quad + \frac{\pi}{2}\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} (\log n - 4\log u)^2 \cdot \frac{r(\frac{n}{u^2})}{u} \\
&\quad + 2\pi\sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} (\log n - 4\log u)\frac{r(\frac{n}{u^2})}{u} \int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})}\frac{\log y}{\sqrt{y}}\,dy \\
&\quad + 4\pi\left(\log\pi - \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right) \cdot \sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})}\frac{\log y}{\sqrt{y}}\,dy \\
&\quad + 2\pi\sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})}\frac{\log^2 y}{\sqrt{y}}\,dy.
\end{aligned}$$

Finally, we shall mention the following estimates (cf. Lemma 3 of Bateman and Grosswald [1]), which will be used frequently below.

For $0 \le \nu \le 1/2$ and for $z > 0$, we have

$$0 < \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} e^z K_\nu(z) \le 1$$

and

$$1 - \frac{1-4\nu^2}{8z} \le \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} e^z K_\nu(z) \le 1 - \frac{1-4\nu^2}{8z} + \frac{(1-4\nu^2)(9-4\nu^2)}{2!(8z)^2}.$$

In particular, we have for any $z > 0$,

$$e^{-z}\sqrt{\frac{\pi}{2z}}\left(1 - \frac{1}{8z}\right) \le K_0(z) \le e^{-z}\sqrt{\frac{\pi}{2z}}\left(1 - \frac{1}{8z} + \frac{9}{128z^2}\right).$$

In a similar manner, we have a more precise approximation as follows.

$$\begin{aligned}
e^{-z}\sqrt{\frac{\pi}{2z}}\left(1 - \frac{1}{8z} + \frac{9}{128z^2} - \frac{75}{1024z^3}\right) &\le K_0(z) \\
&\le e^{-z}\sqrt{\frac{\pi}{2z}}\left(1 - \frac{1}{8z} + \frac{9}{128z^2} - \frac{75}{1024z^3} + \frac{3675}{32768z^4}\right).
\end{aligned}$$

3. Proof of Theorem 2 and Remark 2

We have only to prove that

$$\frac{F_d(\frac{K}{2},K)}{A(K)} \geq A \quad \text{for any} \quad d \geq 1.$$

Because we have for any $d \geq 1$,

$$\frac{F_{d^{-1}}(\frac{K}{2},K)}{A(K)} = \frac{d^{\frac{K}{2}}F_d(\frac{K}{2},K)}{A(K)} \geq \frac{F_d(\frac{K}{2},K)}{A(K)} \geq A.$$

Let K be an integer ≥ 2. Let N_0 and L_0 be the same as in Lemmas 1 and 4 above. Using Lemma 6, we see that if $4 \leq K \leq 7$, then

$$\begin{aligned} S(N_0) &\equiv 2\sum_{j=0}^{N_0-1} \frac{1}{A(2^{j+1})} Z_{2^j}(2^j) + 2C_0 - 4\log 2 - 2\log \pi + \log d \\ &\quad + 4\sum_{j=0}^{N_0-1}\left(\sum_{n=1}^{\infty} n^{2^{j-2}}\left(\sum_{m|n}\frac{r_{2^j}(m)r_{2^j}(\frac{n}{m})}{m^{2^{j-1}}}\right)K_{2^{j-1}}(2\pi\sqrt{n})\right) \\ &\geq S(2) = \frac{F_d(2,4)}{A(4)} - 2E(d,4) = \log d + 2 + \frac{4}{3}\log 2 - 2\log\pi + \frac{12\zeta'(2)}{\pi^2} \\ &= \Theta_K(d), \quad \text{say.} \end{aligned}$$

Similarly, if $K \geq 8$, then we have

$$\begin{aligned} S(N_0) \geq S(3) &= \frac{F_d(4,8)}{A(8)} - 2E(d,8) = -2\log\pi + \log d + \frac{11}{3} + \frac{180}{\pi^4}\zeta'(4) \\ &= \Theta_8(d) = \Theta_K(d), \quad \text{say.} \end{aligned}$$

Thus we get by Lemma 4 for any $K \geq 4$,

$$\frac{F_d\left(\frac{K}{2},K\right)}{A(K)} \geq \Phi_1(K) + \Theta_K(d) + B(K) + 2E(d,K), \quad \text{say,}$$

where we put

$$\Phi_1(K) = \frac{2Z_{K-L_0}(\frac{K}{2})}{A(K)}$$

and

$$B(K) = 4\sum_{n=1}^{\infty} n^{\frac{L_0}{4}}\left(\sum_{m|n}\frac{r_{L_0}(m)r_{K-L_0}(\frac{n}{m})}{m^{\frac{L_0}{2}}}\right)K_{\frac{L_0}{2}}(2\pi\sqrt{n}).$$

We shall use the following simple lower bounds.

$$
\begin{aligned}
2E(d,K) &\geq \sum_{n=1}^{\infty}\left(\sum_{m|n} r_K(m) r_K\left(\frac{n}{m}\right)\right) e^{-2\pi\sqrt{dn}}\frac{1}{(dn)^{\frac{1}{4}}}\left(1-\frac{1}{16\pi\sqrt{dn}}\right)\\
&\geq 4K^2 e^{-2\pi\sqrt{d}}\frac{1}{(d)^{\frac{1}{4}}}\left(1-\frac{1}{16\pi\sqrt{d}}\right)\\
&\quad + 8K^2(K-1)e^{-2\pi\sqrt{2d}}\frac{1}{(2d)^{\frac{1}{4}}}\left(1-\frac{1}{16\pi\sqrt{2d}}\right)\\
&\quad + \frac{16}{3}K^2(K-1)(K-2)e^{-2\pi\sqrt{3d}}\frac{1}{(3d)^{\frac{1}{4}}}\left(1-\frac{1}{16\pi\sqrt{3d}}\right)\\
&\quad + \left(4K^2(K-1)^2+\frac{8}{3}K^2(K-1)(K-2)(K-3)\right)\\
&\quad \times e^{-2\pi\sqrt{4d}}\frac{1}{(4d)^{\frac{1}{4}}}\left(1-\frac{1}{16\pi\sqrt{4d}}\right)\\
&= \Xi_K(d), \quad \text{say, for } 5\leq K\leq 7
\end{aligned}
$$

and

$$
\begin{aligned}
2E(d,8) &\geq \sum_{n=1}^{50}\left(\sum_{m|n} r_8(m) r_8\left(\frac{n}{m}\right)\right) e^{-2\pi\sqrt{dn}}\frac{1}{(dn)^{\frac{1}{4}}}\left(1-\frac{1}{16\pi\sqrt{dn}}\right)\\
&= \Xi_8(d), \quad \text{say.}
\end{aligned}
$$

The values of $\sum_{m|n} r_8(m)r_8(n/m)$ for $n = 1, 2, 3, \cdots, 50$ are

$$
\begin{aligned}
&256, 3584, 14336, 48896, 64512, 200704, 176128, 552960, 588288, 903168,\\
&681984, 2738176, 1125376, 2465792, 3612672, 5775616, 2515968, 8236032,\\
&3512320, 12321792, 9863168, 9547776, 6230016, 30965760, 12128768,\\
&15755264, 21317632, 3360448, 12487680, 50577408, 15253504, 57135616,\\
&38191104, 35223552, 44384256, 112363008, 25934848, 49172480, 63021056,\\
&139345920, 35288064, 138084352, 40708096, 130258944, 148248576,\\
&87220224, 53157888, 323434496, 90706432, 169802752,
\end{aligned}
$$

respectively.

Moreover, when $5 \leq K \leq 7$,

$$
\begin{aligned}
B(K) &= 4\sum_{n=1}^{\infty} n\left(\sum_{m|n}\frac{r_4(m)r_{K-4}(\frac{n}{m})}{m^2}\right)K_2(2\pi\sqrt{n})\\
&\geq 4r_4(1)r_{K-4}(1)K_2(2\pi) \geq 64(K-4)e^{-2\pi}\left(\frac{1}{\pi}-\frac{1}{\pi^2}\right) = \Phi_2(K), \text{ say,}
\end{aligned}
$$

since we have

$$K_2(2\pi) \geq \frac{1}{2}e^{-\pi}\int_1^\infty e^{-\pi y}\left(\frac{1}{y^3}+y\right)dy \geq e^{-2\pi}\left(\frac{1}{\pi}-\frac{1}{\pi^2}\right).$$

We put

$$\tilde{\Theta}_K(d) = \begin{cases} \Phi_1(K)+\Theta_K(d)+\Phi_2(K) & \text{for} \quad K=5,6 \text{ and } 7, \\ \Theta_8(d) & \text{for} \quad K=8. \end{cases}$$

Now we put

$$\Delta(5)=0.06, \quad \Delta(6)=0.5, \quad \Delta(7)=0.9 \quad \text{and} \quad \Delta(8)=1.8.$$

Here we notice that since

$$\frac{d\,E(z,K)}{dz} = -\frac{\pi}{\sqrt{z}}\sum_{n=1}^\infty \sqrt{n}\left(\sum_{m|n} r_K(m) r_K\left(\frac{n}{m}\right)\right)K_1(2\pi\sqrt{zn}) < 0$$

for any $z>0$, $E(d,K)$ is monotone decreasing for any $d>0$ and for any integer $K(\geq 1)$. For each $K=5,6,7$ and 8, we choose $d_j(K)$ for $j=1,2,\ldots,J(=J(K))$ as follows. We can find $d_1(K)(\geq 1)$ such that

$$\tilde{\Theta}_K(d) \geq \tilde{\Theta}_K(d_1(K)) \geq \Delta(K) \quad \text{for any} \quad d \geq d_1(K).$$

Then we can find $d_2(K)(\leq d_1(K))$ such that for any d in $d_2(K)\leq d \leq d_1(K)$,

$$\begin{aligned}\tilde{\Theta}_K(d)+2E(d,K) &\geq \tilde{\Theta}_K(d)+2E(d_1(K),K) \\ &\geq \tilde{\Theta}_K(d_2(K))+\Xi_K(d_1(K)) \geq \Delta(K).\end{aligned}$$

We repeat this and find $d_j(K)$ for $j\geq 2$ such that for any $d_j(K)\leq d\leq d_{j-1}(K)$,

$$\tilde{\Theta}_K(d)+2E(d,K) \geq \tilde{\Theta}_K(d_j(K))+\Xi_K(d_{j-1}(K)) \geq \Delta(K).$$

We stop at $d_J(K)$ if $d_J(K)\leq 1$ with the above property. In this way we get, for $K=5,6,7$ and 8,

$$\frac{F_d(\frac{K}{2},K)}{A(K)} \geq \Delta(K) \quad \text{for any} \quad d>0.$$

This proves Theorem 2 and Remark 2. For $K\geq 8$, we have, further,

$$\begin{aligned}\frac{F_d(\frac{K}{2},K)}{A(K)} &\geq \Phi_1(K)+\frac{F_d(4,8)}{A(8)}+B(K)+2(E(d,K)-E(d,8)) \\ &\geq 1.8+\Phi_1(K)+B(K)+2(E(d,K)-E(d,8)).\end{aligned}$$

The actual process is as follows, for example. We use also the following lower bound.

$$\Phi_1(5)=\frac{4\zeta(5)\Gamma(\frac{5}{2})}{\pi^{\frac{5}{2}}}=\frac{3}{\pi^2}\zeta(5)\geq 0.315188,$$

$$\Phi_1(6) = \frac{16}{\pi^3}\zeta(3)L(3,\chi) = \frac{1}{2}\zeta(3) \geq 0.60102$$

and

$$\Phi_1(7) = \frac{15}{4\pi^3}Z_3\left(\frac{7}{2}\right) \geq \frac{15}{4\pi^3}\cdot\sum_{n=1}^{55}\frac{r_3(n)}{n^{\frac{7}{2}}} \geq \frac{15}{4\pi^3}\cdot 7.466737 \geq 0.90305.$$

j	$d_j(5)$	$d_j(6)$	$d_j(7)$	$d_j(8)$
1	1.2512	1.423	1.53	1.8
2	1.11	1.2785	1.366	1.62
3	1.0252	1.195	1.2599	1.55
4	0.95	1.1241	1.1535	1.489
5		1.047	1.003	1.442
6		0.938	0.698	1.399
7				1.35
8				1.285
9				1.182
10				0.98

Finally, we notice that since we have

$$\tilde{\Theta}_K(1) \geq 0.14 \quad \text{for} \quad K = 6 \text{ and } 7$$

and

$$\tilde{\Theta}_8(1) \geq 1.24,$$

we do not need the above process only to show that

$$\frac{F_d(\frac{K}{2},K)}{A(K)} \geq 0.14 \quad \text{(or, 1.24) for } 6 \leq K \leq 7 \quad \text{(or, for } K \geq 8\text{, respectively).}$$

4. Proof of Theorem 1

We shall first write down $F_d(3/2,3)$ explicitly using Lemmas 1 and 4 with $K = 3$ and $L_0 = 2$. We have first

$$A_0(3) = A(3)\left\{\frac{2\zeta(3)}{A(3)} + \frac{2\zeta(2)}{\pi} + 2C_0 - 2 + 4\cdot\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n} + \frac{1}{2}B(3)\right\},$$

where $B(3)$ is introduced in Section 3 and we have used

$$\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) = -C_0 - 2\log 2 + 2,$$

and

$$\sum_{n=1}^{\infty} n^{\frac{1}{4}}\sum_{m|n}\frac{r_1(m)r_1(\frac{n}{m})}{\sqrt{m}}K_{\frac{1}{2}}(2\pi\sqrt{n}) = 2\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n}.$$

Since

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n}=\frac{1}{4\pi}\left(-\frac{\pi^2}{3}-2\pi\log\mid\eta(i)\mid^2\right),$$

we get as a result of Lemma 4,

$$\begin{aligned}F_d\left(\frac{3}{2}\right)&\equiv F_d\left(\frac{3}{2},3\right)=2A_0(3)-A(3)\left(\log\frac{\pi^2}{d}-2\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)+2A(3)E(d,3)\\&=A(3)\left\{\frac{2\zeta(3)}{\pi}+2C_0-4\log 2-4\log\mid\eta(i)\mid^2\right.\\&\qquad\left.-\log\frac{\pi^2}{d}+B(3)+2E(d)\right\}\\&=A(3)\{-\beta(d)+2E(d)\},\quad\text{say,}\end{aligned}$$

where we put $E(d)=E(d,3)=\sum_{n=1}^{\infty}(\sum_{m|n}r_3(m)r_3(n/m))K_0(2\pi\sqrt{dn})$.

We start our analysis from this expression. We may suppoose that $d\geq 1$. We shall prove first that

$$\frac{F_1(\frac{3}{2})}{A(3)}=-\beta(1)+2E(1)<0.$$

Using an upper bound for $K_\nu(z)$ described in Section 2, we get

$$\begin{aligned}E(1)&=\sum_{n=1}^{\infty}\left(\sum_{n=kl}r_3(k)r_3(l)\right)K_0(2\pi\sqrt{n})\leq\frac{1}{2}\sum_{n=1}^{\infty}\left(\sum_{n=kl}r_3(k)r_3(l)\right)\frac{e^{-2\pi\sqrt{n}}}{n^{\frac14}}\\&\leq\frac{1}{2}\sum_{n=1}^{16}\left(\sum_{n=kl}r_3(k)r_3(l)\right)\frac{e^{-2\pi\sqrt{n}}}{n^{\frac14}}+\frac{66\cdot e^{-8\pi}}{2}\sum_{n=17}^{\infty}\left(\sum_{n=kl}r_3(k)r_3(l)\right)\frac{1}{n^2}\\&\leq\frac{1}{2}\sum_{n=1}^{16}\left(\sum_{n=kl}r_3(k)r_3(l)\right)\frac{e^{-2\pi\sqrt{n}}}{n^{\frac14}}\\&\quad+\frac{66}{2}e^{-8\pi}\left(Z_3^2(2)-\sum_{n=1}^{16}\frac{(\sum_{n=kl}r_3(k)r_3(l))}{n^2}\right),\end{aligned}$$

where we have used the estimate

$$\frac{e^{-2\pi\sqrt{n}}}{n^{\frac14}}\leq 66\cdot\frac{e^{-8\pi}}{n^2}\qquad\text{for}\quad n\geq 17.$$

Here we need an upper bound for $Z_3(2)$. Using the same notation as introduced in Section 2,

$$Z_3(2)=4\zeta(2)L(2,\chi)+2\pi\zeta(2)+\Psi(2).$$

We use the following crude estimate for sufficiently large z.

$$K_1(z)=\frac{1}{2}\int_1^{\infty}e^{-z\frac{y+y^{-1}}{2}}\left(1+\frac{1}{y^2}\right)dy\leq\int_1^{\infty}e^{-\frac{zy}{2}}\,dy\leq\frac{2e^{-\frac{z}{2}}}{z}.$$

Since
$$e^{-\pi\sqrt{n}} \le 197 \cdot \frac{e^{-4\pi}}{n^2} \qquad \text{for} \quad n \ge 17,$$
we get
$$\begin{aligned}\Psi(2) &\le 4\pi^2 \sum_{n=1}^{16} \sqrt{n} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u^2} K_1(2\pi\sqrt{n}) + 4\pi e^{-4\pi} \cdot 197 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u^2} \\ &\le 4\pi^2 \sum_{n=1}^{16} \sqrt{n} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u^2} K_1(2\pi\sqrt{n}) + 4\pi e^{-4\pi} \cdot 197 \cdot \zeta(6) \cdot 4\zeta(2) L(2,\chi) \\ &\le 0.22302,\end{aligned}$$
where we have used the following lemma for the numerical computation of $K_1(z)$ for a smaller $z > 0$.

Lemma 8. *For any positive z, ε and T satisfying $0 < \varepsilon < T$, we have*
$$\begin{aligned}K_1(z) \le \frac{1}{2} e^{-z} \int_{\varepsilon}^{T} e^{-\frac{z}{2}\mu} \frac{\sqrt{\mu}}{\sqrt{\mu+4}}\, d\mu + e^{-z} \int_{\varepsilon}^{T} e^{-\frac{z}{2}\mu} \frac{1}{\sqrt{\mu}\sqrt{\mu+4}}\, d\mu \\ + e^{-z} \left\{ \sqrt{\varepsilon} \left(1 + \frac{\varepsilon}{6}(1+2z)\right) + \frac{1}{z} e^{-\frac{z}{2}T} \left(1 + \frac{2}{\sqrt{T}\sqrt{T+4}}\right) \right\}\end{aligned}$$
and
$$K_1(z) \ge \frac{1}{2} e^{-z} \int_{\varepsilon}^{T} e^{-\frac{z}{2}\mu} \frac{\sqrt{\mu}}{\sqrt{\mu+4}}\, d\mu + e^{-z} \int_{\varepsilon}^{T} e^{-\frac{z}{2}\mu} \frac{1}{\sqrt{\mu}\sqrt{\mu+4}}\, d\mu.$$

We shall give a proof to this for completeness. By the change of variable as in p. 528 of Selberg [12], we have
$$\begin{aligned}K_1(z) &= \frac{1}{4} e^{-z} \int_0^{\infty} e^{-\frac{z}{2}\mu} \frac{\mu+2}{\sqrt{\mu}\sqrt{1+\frac{\mu}{4}}}\, d\mu \\ &= \frac{1}{2} e^{-z} \int_0^{\infty} e^{-\frac{z}{2}\mu} \frac{\sqrt{\mu}}{\sqrt{\mu+4}}\, d\mu + e^{-z} \int_0^{\infty} e^{-\frac{z}{2}\mu} \frac{1}{\sqrt{\mu}\sqrt{\mu+4}}\, d\mu.\end{aligned}$$
For any ε, we have
$$\int_0^{\varepsilon} e^{-\frac{z}{2}\mu} \frac{\sqrt{\mu}}{\sqrt{\mu+4}}\, d\mu \le \frac{1}{2} \int_0^{\varepsilon} \sqrt{\mu}\, d\mu \le \frac{1}{3} \varepsilon^{\frac{3}{2}}$$
and
$$\int_0^{\varepsilon} e^{-\frac{z}{2}\mu} \frac{1}{\sqrt{\mu}\sqrt{\mu+4}}\, d\mu \le \frac{1}{2} \int_0^{\varepsilon} e^{-\frac{z}{2}\mu} \frac{1}{\sqrt{\mu}}\, d\mu = \sqrt{\varepsilon} + \frac{z}{3} \varepsilon^{\frac{3}{2}}.$$
For any $T > 0$, we have
$$\int_T^{\infty} e^{-\frac{z}{2}\mu} \frac{\sqrt{\mu}}{\sqrt{\mu+4}}\, d\mu \le \int_T^{\infty} e^{-\frac{z}{2}\mu}\, d\mu = \frac{2}{z} e^{-\frac{z}{2}T}$$

and

$$\int_T^\infty e^{-\frac{z}{2}\mu}\frac{1}{\sqrt{\mu}\sqrt{\mu+4}}\,d\mu \le \frac{2}{z\sqrt{T}\sqrt{T+4}}e^{-\frac{z}{2}T}.$$

Combining these, we get our assertion as described in Lemma 8.

By Lemma 8, we have, for example,

$$0.00098693 < K_1(2\pi) < 0.000986997.$$

Thus we get

$$Z_3(2) \le 16.585278.$$

Since

$$\frac{1}{2}\sum_{n=1}^{16}\left(\sum_{n=kl} r_3(k)r_3(l)\right)\frac{e^{-2\pi\sqrt{n}}}{n^{\frac{1}{4}}} - \frac{66}{2}e^{-8\pi}\sum_{n=1}^{16}\frac{(\sum_{n=kl} r_3(k)r_3(l))}{n^2}$$
$$\le 0.04306497,$$

we get finally,

$$E(1) \le 0.0430653,$$

where the values of $\sum_{m|n} r_3(m)r_3(n/m)$ for $n = 1, 2, 3, \ldots, 16$ are

$$36, 144, 96, 216, 288, 480, 0, 288, 424, 864, 288, 768, 288, 576, 384, 396,$$

respectively.

On the other hand, we have

$$\beta(1) = -\left\{\frac{2\zeta(3)}{\pi} + 2C_0 - 4\log 2 - 4\log|\,\eta(i)\,|^2 - \log\frac{\pi^2}{1} + B(3)\right\}$$
$$\ge 0.154 > 2E(1).$$

This proves our inequality.

Now, let $E'(d)$ and $E''(d)$ be the first and the second derivative of $E(d)$ with respect to d, respectively. Then we see easily that

$$E''(d) = \frac{\pi}{2d^{\frac{3}{2}}}\sum_{n=1}^{\infty}\sqrt{n}\left(\sum_{m|n} r_3(m)r_3\left(\frac{n}{m}\right)\right)K_1(2\pi\sqrt{dn})$$
$$+\frac{\pi^2}{2d}\sum_{n=1}^{\infty} n\left(\sum_{m|n} r_3(m)r_3\left(\frac{n}{m}\right)\right)(K_2(2\pi\sqrt{dn}) + K_0(2\pi\sqrt{dn}))$$
$$> 0 \quad \text{for any} \quad d > 0.$$

We have seen already that $E'(d) < 0$ for $d > 0$. Hence both $\beta(d)$ and $E(d)$ are monotone decreasing convex and continuous function of positive real d.

Since

$$\beta(1) > 2E(1),$$

$$E(d) > 0 \quad \text{for any} \quad d > 0,$$

and

$$\beta(d) \to -\infty \quad \text{as} \quad d \to \infty,$$

the equation

$$F_d\left(\frac{3}{2}\right) = 0$$

has at least one real solution $d = D_2$ in $d > 1$. We notice next that

$$\frac{dF_z(\frac{3}{2})}{dz} = \frac{\pi^2}{\sqrt{z}} \left\{ \frac{1}{\sqrt{z}} - 2\pi \sum_{n=1}^{\infty} \sqrt{n} \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) K_1(2\pi\sqrt{zn}) \right\}.$$

Since

$$\begin{aligned}
&\frac{d}{dz}\left(\sqrt{z} \sum_{n=1}^{\infty} \sqrt{n} \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) K_1(2\pi\sqrt{zn}) \right) \\
&= \frac{1}{2\sqrt{z}} \sum_{n=1}^{\infty} \sqrt{n} \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) \frac{1}{2} \int_0^{\infty} e^{-\pi\sqrt{zn}(y+y^{-1})}\, dy \\
&\quad - \frac{\pi}{2} \sum_{n=1}^{\infty} n \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) \frac{1}{2} \int_0^{\infty} e^{-\pi\sqrt{zn}(y+y^{-1})} (y + y^{-1})\, dy.
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\pi}{2} \sum_{n=1}^{\infty} n \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) \frac{1}{2} \int_0^{\infty} e^{-\pi\sqrt{zn}(y+y^{-1})} (y + y^{-1})\, dy \\
&\quad \geq \frac{2\pi}{2} \sum_{n=1}^{\infty} n \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) \frac{1}{2} \int_0^{\infty} e^{-\pi\sqrt{zn}(y+y^{-1})}\, dy \\
&\quad > \frac{1}{2\sqrt{z}} \sum_{n=1}^{\infty} \sqrt{n} \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) \frac{1}{2} \int_0^{\infty} e^{-\pi\sqrt{zn}(y+y^{-1})}\, dy,
\end{aligned}$$

provided that $1/\sqrt{z} < 2\pi$, namely that $1/4\pi^2 < z$,

$$\sqrt{z} \sum_{n=1}^{\infty} \sqrt{n} \left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right) \right) K_1(2\pi\sqrt{zn})$$

is monotone decreasing for $z > 1/4\pi^2$. Hence, the equation

$$\frac{dF_z(\frac{3}{2})}{dz} = 0$$

has at most one real solution in $z > 1/4\pi^2$.

Consequently, we see that the equation

$$F_d\left(\frac{3}{2}\right) = 0$$

has exactly one real solution $d = D_2$ in $d > 1$. Namely, there exist exactly two real solutions $d = D_1 = 1/D_2$ and $d = D_2$ of the equation $F_d(3/2) = 0$ in $d > 0$.

If $d > D_2$, then $F_d(3/2) > 0$. Hence since

$$\lim_{\sigma \to 3-0} F_d(\sigma) = -\infty,$$

$F_d(\sigma)$ must have a zero in the interval $\sigma \in (3/2, 3)$.

If $0 < d < D_1$, we have also

$$F_d\left(\frac{3}{2}\right) > 0$$

and $F_d(\sigma)$ must have a zero in the interval $\sigma \in (3/2, 3)$.

This proves (2) of Theorem 1.

We get at the same time (3) of Theorem 1.

We shall next prove that $s = 3/2$ is a double zero for $F_{D_2}(s)$, hence also for $F_{D_1}(s)$.

We shall notice first that $s = 3/2$ is not a simple zero for $F_{D_2}(s)$. Suppose that d satisfies

$$F_d\left(\frac{3}{2}\right) = 0,$$

namely, that d satisfies

$$\frac{2\zeta(3)}{\pi} + 2C_0 - 4\log 2 - 4\log \mid \eta(i) \mid^2 - \log\frac{\pi^2}{d} + B(3) + 2E(d) = 0.$$

Then one can see easily, using Lemma 4, that

$$\begin{aligned} F_d'\left(\frac{3}{2}, 3\right) &= A_0(3)\left(\log\frac{\pi^2}{d} - 2\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right) - A(3)\left(\frac{1}{2}\left(\log\frac{\pi^2}{d}\right)^2\right. \\ &\quad \left. - 2\,\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\log\frac{\pi^2}{d} + 2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2\right) - A(3)\left(\log\frac{\pi}{\sqrt{d}} - \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right) \\ &\quad \times \left\{\frac{2\zeta(3)}{\pi} + 2C_0 - 4\log 2 - 4\log \mid \eta(i) \mid^2 - \log\frac{\pi^2}{d} + B(3)\right\} = 0. \end{aligned}$$

To proceed further we need to locate D_2 more precisely.

We shall use a more precise upper bound of $E(d)$ as follows.

$$E(d) \le \frac{1}{2}\sum_{n=1}^{16}\left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right)\right) e^{-2\pi\sqrt{dn}} \frac{1}{d^{\frac{1}{4}} n^{\frac{1}{4}}}\left(1 - \frac{1}{16\pi\sqrt{dn}} + \frac{9}{512\pi^2 dn} - \frac{75}{8192\pi^3 (dn)^{\frac{3}{2}}} + \frac{3675}{524288\pi^4 (dn)^2}\right)$$
$$+ \frac{1}{2}\sum_{n=17}^{\infty}\left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right)\right) e^{-2\pi\sqrt{dn}} \frac{1}{d^{\frac{1}{4}} n^{\frac{1}{4}}}\left(1 - \frac{1}{16\pi\sqrt{dn}} + \frac{9}{512\pi^2 dn} - \frac{75}{8192\pi^3 (dn)^{\frac{3}{2}}} + \frac{3675}{524288\pi^4 (dn)^2}\right)$$
$$= E_1 + E_2, \quad \text{say.}$$

Since for any $n \ge 17$,

$$e^{-2\pi\sqrt{dn}} \le 0.001390088 \cdot \frac{e^{-\pi\sqrt{17d}}}{n^{\frac{9}{4}}},$$

we have

$$E_2 \le 0.001390088 \cdot \frac{e^{-\pi\sqrt{17d}}}{2d^{\frac{1}{4}}} \sum_{n=17}^{\infty} \frac{\sum_{m|n} r_3(m) r_3(\frac{n}{m})}{n^{\frac{5}{2}}}\left(1 - \frac{1}{16\pi\sqrt{dn}} + \frac{9}{512\pi^2 dn} - \frac{75}{8192\pi^3 (dn)^{\frac{3}{2}}} + \frac{3675}{524288\pi^4 (dn)^2}\right)$$
$$= 0.001390088 \cdot \frac{e^{-\pi\sqrt{17d}}}{2d^{\frac{1}{4}}}\left\{Z_3(2)^2 - \frac{1}{16\pi\sqrt{d}} Z_3\left(\frac{5}{2}\right)^2 + \frac{9}{512\pi^2 d} Z_3(3)^2 - \frac{75}{8192\pi^3 d^{\frac{3}{2}}} Z_3\left(\frac{7}{2}\right)^2 + \frac{3675}{524288\pi^4 d^2} Z_3(4)^2\right\}$$
$$- 0.001390088 \cdot \frac{e^{-\pi\sqrt{17d}}}{2d^{\frac{1}{4}}}\left\{H(2) - \frac{1}{16\pi\sqrt{d}} H\left(\frac{5}{2}\right) + \frac{9}{512\pi^2 d} H(3) - \frac{75}{8192\pi^3 d^{\frac{3}{2}}} H\left(\frac{7}{2}\right) + \frac{3675}{524288\pi^4 d^2} H(4)\right\} = q_2^*(d), \quad \text{say,}$$

where we put

$$H(x) = \sum_{n=1}^{16} \frac{\sum_{m|n} r_3(m) r_3(\frac{n}{m})}{n^x}.$$

Using the values of $\sum_{m|n} r_3(m) r_3(n/m)$ mentioned above, we get

$$H(2) = 155.0045\cdots, \quad H\left(\frac{5}{2}\right) = 95.3702\cdots, \quad H(3) = 68.6771\cdots,$$
$$H\left(\frac{7}{2}\right) = 55.4113\cdots, \quad H(4) = 48.1768\cdots \quad .$$

We have already given an upper bound of $Z_3(2)$ and a lower bound of $Z_3(7/2)$. We shall give a lower bound to $Z_3(5/2)$ and also upper bounds to $Z_3(3)$ and $Z_3(4)$ in a simpler way, in stead of using the representation of $Z_3(s)$ written in Section 2.

First, we have

$$Z_3\left(\frac{5}{2}\right) \geq \sum_{n=1}^{55} \frac{r_3(n)}{n^{5/2}} \geq 10.2793.$$

Now to get upper bounds of $Z_3(3)$ and $Z_3(4)$, it is better to use $\sum_{n=1}^{55}(r_3(n)/n^x)$ in stead of the zeta functions which come from an extension of Chowla-Selberg's formula (cf. Section 5.2. below). Moreover we shall use the following crude estimate.

$$Z_3(3) \leq \sum_{n=1}^{55} \frac{r_3(n)}{n^3} + \frac{1}{56}\sum_{n=56}^{\infty} \frac{r_3(n)}{n^2} \leq \sum_{n=1}^{55} \frac{r_3(n)}{n^3} - \frac{1}{56}\sum_{n=1}^{55} \frac{r_3(n)}{n^2} + \frac{1}{56}Z_3(2).$$

Combining this with our upper bound of $Z_3(2)$ given above, we get

$$Z_3(3) \leq 8.424793.$$

Finally, we have in the same manner, using the upper bound of $Z_3(3)$ just obtained above,

$$Z_3(4) \leq \sum_{n=1}^{55} \frac{r_3(n)}{n^4} - \frac{1}{56}\sum_{n=1}^{55} \frac{r_3(n)}{n^3} + \frac{1}{56}Z_3(3) \leq 6.9463842.$$

Consequently, we get

$$Z_3(2)^2 \leq 275.0715, \qquad Z_3\left(\frac{5}{2}\right)^2 \geq 105.6, \qquad Z_3(3)^2 \leq 70.97713,$$
$$Z_3\left(\frac{7}{2}\right)^2 \geq 55.75216, \qquad Z_3(4)^2 \leq 48.25226.$$

Hence, we have

$$\begin{aligned} q_2^*(d) \leq{}& 0.001390088 \cdot \frac{e^{-\pi\sqrt{17d}}}{2d^{\frac{1}{4}}}\left\{275.0715 - 105.6 \cdot \frac{1}{16\pi\sqrt{d}} + 70.97713\right. \\ &\left. \times \frac{9}{512\pi^2 d} - 55.75216 \cdot \frac{75}{8192\pi^3 d^{\frac{3}{2}}} + 48.25226 \cdot \frac{3675}{524288\pi^4 d^2}\right\} \\ &- 0.001390088 \cdot \frac{e^{-\pi\sqrt{17d}}}{2d^{\frac{1}{4}}}\left\{155.004 - 95.37 \cdot \frac{1}{16\pi\sqrt{d}} + 68.677\right. \\ &\left. \times \frac{9}{512\pi^2 d} - 55.411 \cdot \frac{75}{8192\pi^3 d^{\frac{3}{2}}} + 48.177 \cdot \frac{3675}{524288\pi^4 d^2}\right\} = q_2^{**}(d), \text{ say.} \end{aligned}$$

We denote E_1 by $q_1^*(d)$.

We shall use the following lower bound for $E(d)$.

$$E(d) \geq \frac{1}{2}\sum_{n=1}^{16}\left(\sum_{m|n} r_3(m) r_3\left(\frac{n}{m}\right)\right) e^{-2\pi\sqrt{dn}} \frac{1}{d^{\frac{1}{4}} n^{\frac{1}{4}}}\left(1 - \frac{1}{16\pi\sqrt{dn}} + \frac{9}{512\pi^2 dn} - \frac{75}{8192\pi^3 (dn)^{\frac{3}{2}}}\right) = q_3^{**}(d), \text{ say.}$$

Now we have the following inequality

$$q_*(d) \leq F_d(2) \leq q^*(d),$$

where we put

$$\begin{aligned} q_*(d) = &- 2\pi \log \frac{\pi^2}{d} + 2\pi \frac{2\zeta(3)}{\pi} + 4\pi C_0 - 8\pi \log 2 \\ &- 2\pi \cdot 4 \log \mid \eta(i) \mid^2 \; + 2\pi B(3) + 4\pi q_3^{**}(d) \end{aligned}$$

and

$$\begin{aligned} q^*(d) = &- 2\pi \log \frac{\pi^2}{d} + 2\pi \frac{2\zeta(3)}{\pi} + 4\pi C_0 - 8\pi \log 2 \\ &- 2\pi \cdot 4 \log \mid \eta(i) \mid^2 \; + 4\pi \left(q_1^*(d) + q_2^{**}(d) + \frac{1}{2} B(3) \right). \end{aligned}$$

We are left to estimate $B(3)$.

$$\begin{aligned} B(3) &= 4\left(\sum_{n=1}^{19} + \sum_{n=20}^{\infty}\right) \sqrt{n} \left(\sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m}\right) K_1(2\pi\sqrt{n}) \\ &= B_1(3) + B_2(3), \quad \text{say.} \end{aligned}$$

The values of $\sum_{m|n}(r_2(m) r_1(n/m)/m)$ for $n = 1, 2, 3, \dots, 19$ are

$$8, 4, 0, 10, \frac{16}{5}, 0, 0, 5, \frac{80}{9}, \frac{8}{5}, 0, 0, \frac{16}{13}, 0, 0, \frac{21}{2}, \frac{16}{17}, \frac{40}{9}, 0,$$

respectively. These give the lower and upper bounds of $B_1(3)$. For $B_1(3)$, we use Lemma 8. For the upper bound of $B_2(3)$, we have

$$\begin{aligned} B_2(3) &\leq 4 \sum_{n=20}^{\infty} \sqrt{n} \sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m} \frac{e^{-\pi\sqrt{n}}}{\pi\sqrt{n}} \\ &\leq \frac{4}{\pi} \cdot 0.00001583 \cdot \sum_{n=20}^{\infty} \frac{1}{n} \sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m} \\ &\leq \frac{4}{\pi} \cdot 0.00001583 \cdot \left(Z_2(2) Z_1(1) - \sum_{n=1}^{19} \frac{\left(\sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m}\right)}{n} \right) \\ &= \frac{4}{\pi} \cdot 0.00001583 \cdot \left(8\zeta^2(2) L(2,\chi) - \sum_{n=1}^{19} \frac{\left(\sum_{m|n} \frac{r_2(m) r_1(\frac{n}{m})}{m}\right)}{n} \right), \end{aligned}$$

where

$$e^{-\pi\sqrt{n}} \le \frac{0.00001583}{n} \quad \text{for} \quad n \ge 20.$$

Thus we get

$$0.03306113959 \le B(3) \le 0.0331416562.$$

By the numerical computations, we get

$$q^*(2.7154) = 0.00001911395\cdots$$
$$q^*(2.7153) = -0.00001752549\cdots$$

and

$$q_*(2.71558) = 4.536\cdots \times 10^{-6}$$
$$q_*(2.71556) = -2.791\cdots \times 10^{-6}.$$

These imply that $F_d(3/2) = 0$ at $d = D_1$ and $d = D_2$, where

$$2.7153 < D_2 = 2.7\cdots < 2.71558$$

and

$$D_1 = \frac{1}{D_2} = 0.3\cdots .$$

Hence to complete the proof of (1) of Theorem 1, we have to prove that

$$F_d''\left(\frac{3}{2}\right) \ne 0 \quad \text{for } d = D_2 \text{ (and hence for } d = D_1).$$

For this purpose, we notice first that by Lemma 5 in Section 2, we have

$$\begin{aligned}
F_d''\left(\frac{3}{2},3\right) = 2\Bigg\{ & -A(3)\cdot\left(\frac{1}{6}\left(\log\frac{\pi^2}{d}\right)^3 - \left(\log\frac{\pi^2}{d}\right)^2\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right. \\
& + 2\cdot\log\frac{\pi^2}{d}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2 - \frac{1}{3}\frac{\Gamma'''}{\Gamma}\left(\frac{3}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right) \\
& \left. - 2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^3\right) + A_0(3)\cdot\left(-2\log\frac{\pi^2}{d}\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\frac{1}{2}\left(\log\frac{\pi^2}{d}\right)^2\right. \\
& \left. + 2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2\right) - A_1(3)\cdot\left(-2\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) + \log\frac{\pi^2}{d}\right) + 2A_2(3)\Bigg\} \\
& + 2A(3)E(d,3)\left(\frac{1}{4}\left(\log\frac{\pi^2}{d}\right)^2 - \left(\log\frac{\pi^2}{d}\right)\cdot\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) - \frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right)\right. \\
& \left. + 2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2\right) + 2A(3)E^{(2)}(d,3),
\end{aligned}$$

where we put

$$E^{(2)}(d,3)=\sum_{n=1}^{\infty}\left(\sum_{m|n}\log m\cdot\left(\log m-\frac{1}{2}\log n\right)r_3(m)r_3\left(\frac{n}{m}\right)\right)K_0(2\pi\sqrt{dn})$$
$$+\sum_{n=1}^{\infty}\left(\sum_{m|n}r_3(m)r_3\left(\frac{n}{m}\right)\right)\cdot\frac{1}{2}\int_0^{\infty}e^{-2\pi\sqrt{dn}\frac{y+y^{-1}}{2}}\frac{\log^2 y}{y}\,dy.$$

Since we suppose that $F_d(3/2,3)=0$, we have now

$$2E(d)=-\frac{2\zeta(3)}{\pi}-2C_0+4\log 2+4\log\mid\eta(i)\mid^2+\log\frac{\pi^2}{d}-B(3).$$

Substituting this into the above expression of $F_d''(3/2,3)$, simplifying and writing

$$X=\log\frac{\pi^2}{d},$$

we get

$$F_d''\left(\frac{3}{2},3\right)=-\frac{1}{6}\pi X^3+X^2\left\{A_0(3)+\frac{\pi}{2}(-2C_0+4\log 2+\tilde{B})+2\pi\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right\}$$
$$+X\left\{-4\pi\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2-4A_0(3)\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)-2A_1(3)\right.$$
$$\left.-2\pi\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)(-2C_0+4\log 2+\tilde{B})-2\pi\frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right)\right\}$$
$$+\left\{-4\pi\left(-\frac{1}{3}\frac{\Gamma'''}{\Gamma}\left(\frac{3}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right)-2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2\right)\right.$$
$$+4A_0(3)\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2+4A_1(3)\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)+4A_2(3)$$
$$\left.+2\pi(-2C_0+4\log 2+\tilde{B})\left(-\frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right)+2\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2\right)\right\}+4\pi E^{(2)}(d,3)$$
$$=f(X)+4\pi E^{(2)}(d,3),\quad\text{say},$$

where we put

$$\tilde{B}=-\frac{2\zeta(3)}{\pi}+4\log\mid\eta(i)\mid^2-B(3).$$

Here we notice that

$$\frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right)=\left(\frac{\Gamma'}{\Gamma}\right)'\left(\frac{3}{2}\right)+\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\right)^2$$
$$=\frac{\pi^2}{2}+C_0^2+4\log^2 2+4C_0\log 2-4C_0-8\log 2$$

and

$$\frac{\Gamma'''}{\Gamma}\left(\frac{3}{2}\right) = \left(\frac{\Gamma'}{\Gamma}\right)''\left(\frac{3}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\frac{\Gamma''}{\Gamma}\left(\frac{3}{2}\right) + 2\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right)\left(\frac{\Gamma'}{\Gamma}\right)'\left(\frac{3}{2}\right)$$
$$= -2\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{3}{2}\right)^3} + (-C_0 - 2\log 2 + 2)\cdot\left(\frac{\pi^2}{2} + C_0^2 + 4\log^2 2\right.$$
$$\left. + 4C_0\log 2 - 4C_0 - 8\log 2\right) + 2(-C_0 - 2\log 2 + 2)\cdot\left(\frac{\pi^2}{2} - 4\right)$$

with

$$-2\sum_{k=0}^{\infty}\frac{1}{(k+\frac{3}{2})^3} = -2\zeta\left(3, \frac{1}{2}\right) + 16,$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta function defined for $\Re(s) > 1$ by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s} \quad \text{for any} \quad 0 < \alpha < 1.$$

Then $f(X)$ can be rewritten as follows.

$$f(X) = -\frac{\pi}{6}X^3 + aX^2 + bX + c$$

with

$$a = A_0(3) + \frac{\pi}{2}(-2C_0 + 4\log 2 + \tilde{B}) + 2\pi(-C_0 - 2\log 2 + 2),$$
$$b = -4\pi(-C_0 - 2\log 2 + 2)^2 - 4A_0(3)(-C_0 - 2\log 2 + 2) - 2A_1(3)$$
$$- 2\pi(-C_0 - 2\log 2 + 2)(-2C_0 + 4\log 2 + \tilde{B})$$
$$- 2\pi\left(\frac{\pi^2}{2} + C_0^2 + 4\log^2 2 + 4C_0\log 2 - 4C_0 - 8\log 2\right)$$

and

$$c = -4\pi\left\{-\frac{1}{3}\left\{-2\zeta\left(3, \frac{1}{2}\right) + 16 + (-C_0 - 2\log 2 + 2)\cdot\left(\frac{\pi^2}{2} + C_0^2 + 4\log^2 2\right.\right.\right.$$
$$\left.\left. + 4C_0\log 2 - 4C_0 - 8\log 2\right) + 2(-C_0 - 2\log 2 + 2)\cdot\left(\frac{\pi^2}{2} - 4\right)\right\}$$
$$+ (-C_0 - 2\log 2 + 2)\left(\frac{\pi^2}{2} + C_0^2 + 4\log^2 2 + 4C_0\log 2 - 4C_0 - 8\log 2\right)$$
$$\left. - 2(-C_0 - 2\log 2 + 2)^2\right\} + 4A_0(3)(-C_0 - 2\log 2 + 2)^2$$
$$+ 4A_1(3)(-C_0 - 2\log 2 + 2) + 4A_2(3) + 2\pi(-2C_0 + 4\log 2 + \tilde{B})$$
$$\times\left(-\frac{\pi^2}{2} - C_0^2 - 4\log^2 2 - 4C_0\log 2 + 4C_0 + 8\log 2\right.$$
$$\left. + 2(-C_0 - 2\log 2 + 2)^2\right).$$

Suppose that there exist the upper bounds a_1 and b_1 such that

$$a < a_1$$

and

$$b < b_1.$$

Then we have

$$f(X) - f_1(X) = X^2(a - a_1) + X(b - b_1) < 0$$

for any $X > 0$, where we put

$$f_1(X) = -\frac{\pi}{6}X^3 + a_1X^2 + b_1X + c.$$

To get a_1 and b_1, we need to estimate $A_0(3)$ and $A_1(3)$. We shall estimate first $A_0(3)$.

By Lemma 7-(i), we know that

$$A_0(3) = 4\pi C_0 - 4\pi + 4\zeta\left(\frac{3}{2}\right) L\left(\frac{3}{2}, \chi\right) + 4\pi\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u}.$$

Since

$$e^{-2\pi\sqrt{n}} \leq \frac{134 \cdot e^{-8\pi}}{n^2} \quad \text{for} \quad n \geq 17,$$

we have

$$\begin{aligned}
&4\pi\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\
&\leq 4\pi\sum_{n=1}^{16}\left(e^{-2\pi\sqrt{n}} - \frac{134 \cdot e^{-8\pi}}{n^2}\right)\sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} + 4\pi \cdot 134 \cdot e^{-8\pi} Z_2(2)\zeta(5).
\end{aligned}$$

On the other hand, we have

$$4\pi\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \geq 4\pi\sum_{n=1}^{16} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \geq 0.10116646479.$$

For $n = 1, 2, 3, \ldots, 16$, the values of

$$\sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u}$$

are

$$4, 4, 0, 6, 8, 0, 0, 6, \frac{16}{3}, 8, 0, 0, 8, 0, 0, 7.$$

Using these values, we have

$$4\pi \sum_{n=1}^{16} \left(e^{-2\pi\sqrt{n}} - \frac{134 \cdot e^{-8\pi}}{n^2} \right) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} = 0.101166341728\cdots$$

and

$$\begin{aligned} &4\pi \sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\ &\leq 4\pi \sum_{n=1}^{16} \left(e^{-2\pi\sqrt{n}} - \frac{134 \cdot e^{-8\pi}}{n^2} \right) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} + 4\pi \cdot 134 \cdot e^{-8\pi} \zeta(5) \cdot 4\zeta(2) L(2,\chi) \\ &= 0.101166469708\cdots. \end{aligned}$$

Since

$$4\pi C_0 - 4\pi + 4\zeta\left(\frac{3}{2}\right) L\left(\frac{3}{2},\chi\right) = 3.72075703819135133247 98\cdots,$$

we have

$$3.82192338 < A_0(3) < 3.82192351.$$

Hence, we have

$$\begin{aligned} 2.025475 < a = A_0(3) &+ \frac{\pi}{2}(-2C_0 + 4\log 2 + \tilde{B}) \\ &+ 2\pi(-C_0 - 2\log 2 + 2) \\ &< 2.02561 = a_1. \end{aligned}$$

To estimate b, we need to estimate $A_1(3)$. We have first, by Lemma 7-(ii),

$$\begin{aligned} -2A_1(3) &= -16\pi C_1 + 16\pi C_0 - 16\pi - 8\zeta'\left(\frac{3}{2}\right) L\left(\frac{3}{2},\chi\right) - 8\zeta\left(\frac{3}{2}\right) L'\left(\frac{3}{2},\chi\right) \\ &\quad - 8\pi(\log\pi + C_0 + 2\log 2 - 2) \sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\ &\quad + 4\pi \sum_{n=1}^{\infty} \log n \cdot e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} - 8\pi \sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \log\frac{n}{u^2} \frac{r(\frac{n}{u^2})}{u} \\ &\quad + 8\pi \sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_1^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})} \frac{\log y}{y^{\frac{3}{2}}}\, dy \\ &\quad - 8\pi \sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_1^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})} \frac{\log y}{\sqrt{y}}\, dy. \end{aligned}$$

We shall first get an upper bound.

$$\begin{aligned}-2A_1(3) \leq & -16\pi C_1 + 16\pi C_0 - 16\pi - 8\zeta'\left(\frac{3}{2}\right) L\left(\frac{3}{2}, \chi\right) - 8\zeta\left(\frac{3}{2}\right) L'\left(\frac{3}{2}, \chi\right) \\ & - 8\pi(\log \pi + C_0 + 2\log 2 - 2) \sum_{n=1}^{16} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\ & + 4\pi \sum_{n=1}^{\infty} \log n \cdot e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} - 8\pi \sum_{n=1}^{16} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \log \frac{n}{u^2} \frac{r(\frac{n}{u^2})}{u} \\ & + 8\pi \sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} e^{-\pi\sqrt{n}} \left\{ \frac{1}{\pi^2 n} - \frac{5}{2}\frac{1}{\pi^3 n^{\frac{3}{2}}} + \frac{35}{4}\frac{1}{\pi^4 n^2} \right\} \\ & - 8\pi \sum_{n=1}^{16} n^{\frac{1}{4}} e^{-2\pi\sqrt{n}} \left(\frac{1}{2\pi\sqrt{n}} + \frac{1}{8\pi^2 n} \right) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u}.\end{aligned}$$

Since

$$e^{-2\pi\sqrt{n}} \log n \leq 378 \cdot \frac{e^{-8\pi}}{n^2} \quad \text{for} \quad n \geq 17,$$

we get

$$\begin{aligned} & 4\pi \sum_{n=1}^{\infty} \log n \cdot e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\ & \leq 4\pi \cdot 378 \cdot e^{-8\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} + 4\pi \sum_{n=1}^{16} \Bigg(\log n \cdot e^{-2\pi\sqrt{n}} \\ & \quad - \frac{378 \cdot e^{-8\pi}}{n^2} \Bigg) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\ & = 4\pi \cdot 378 \cdot e^{-8\pi} \zeta(5) Z_2(2) + 4\pi \sum_{n=1}^{16} \left(\log n \cdot e^{-2\pi\sqrt{n}} - \frac{378 \cdot e^{-8\pi}}{n^2} \right) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\ & \leq 0.0053172. \end{aligned}$$

Since

$$e^{-\pi\sqrt{n}} \leq 24 \cdot \frac{e^{-4\pi}}{n^{\frac{5}{4}}} \quad \text{for} \quad n \geq 17,$$

we get

$$
\begin{aligned}
&8\pi\sum_{n=1}^{\infty}n^{\frac{1}{4}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}e^{-\pi\sqrt{n}}\left\{\frac{1}{\pi^2 n}-\frac{5}{2}\frac{1}{\pi^3 n^{\frac{3}{2}}}+\frac{35}{4}\frac{1}{\pi^4 n^2}\right\}\\
&\le 8\pi\cdot 24\cdot e^{-4\pi}\sum_{n=1}^{\infty}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}\left\{\frac{1}{\pi^2 n^2}-\frac{5}{2}\frac{1}{\pi^3 n^{\frac{5}{2}}}+\frac{35}{4}\frac{1}{\pi^4 n^3}\right\}\\
&\quad+8\pi\sum_{n=1}^{16}n^{\frac{1}{4}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}e^{-\pi\sqrt{n}}\left\{\frac{1}{\pi^2 n}-\frac{5}{2}\frac{1}{\pi^3 n^{\frac{3}{2}}}+\frac{35}{4}\frac{1}{\pi^4 n^2}\right\}\\
&\quad-8\pi\cdot 24\cdot e^{-4\pi}\sum_{n=1}^{16}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}\left\{\frac{1}{\pi^2 n^2}-\frac{5}{2}\frac{1}{\pi^3 n^{\frac{5}{2}}}+\frac{35}{4}\frac{1}{\pi^4 n^3}\right\}\\
&=8\pi\cdot 24\cdot e^{-4\pi}\left\{\frac{1}{\pi^2}\zeta(5)Z_2(2)-\frac{5}{2}\frac{1}{\pi^3}\zeta(6)Z_2\left(\frac{5}{2}\right)+\frac{35}{4}\frac{1}{\pi^4}\zeta(7)Z_2(3)\right\}\\
&\quad+8\pi\sum_{n=1}^{16}n^{\frac{1}{4}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}e^{-\pi\sqrt{n}}\left\{\frac{1}{\pi^2 n}-\frac{5}{2}\frac{1}{\pi^3 n^{\frac{3}{2}}}+\frac{35}{4}\frac{1}{\pi^4 n^2}\right\}\\
&\quad-8\pi\cdot 24\cdot e^{-4\pi}\sum_{n=1}^{16}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}\left\{\frac{1}{\pi^2 n^2}-\frac{5}{2}\frac{1}{\pi^3 n^{\frac{5}{2}}}+\frac{35}{4}\frac{1}{\pi^4 n^3}\right\}\\
&\le 0.5564001.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
-2A_1(3)\le&-16\pi C_1+16\pi C_0-16\pi-8\zeta'\left(\frac{3}{2}\right)L\left(\frac{3}{2},\chi\right)-8\zeta\left(\frac{3}{2}\right)L'\left(\frac{3}{2},\chi\right)\\
&-8\pi(\log\pi+C_0+2\log 2-2)\sum_{n=1}^{16}e^{-2\pi\sqrt{n}}\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}\\
&+0.005372-8\pi\sum_{n=1}^{16}e^{-2\pi\sqrt{n}}\sum_{u^2|n}\log\frac{n}{u^2}\frac{r(\frac{n}{u^2})}{u}\\
&+0.5564001-8\pi\sum_{n=1}^{16}n^{\frac{1}{4}}e^{-2\pi\sqrt{n}}\left(\frac{1}{2\pi\sqrt{n}}+\frac{1}{8\pi^2 n}\right)\sum_{u^2|n}\frac{r(\frac{n}{u^2})}{u}.
\end{aligned}
$$

The values of $\sum_{u^2|n}\log(n/u^2)\cdot(r(n/u^2)/u)$ at $n=1,2,3,\ldots,16$ are

$$
\begin{gathered}
0.4\log 2, 0, 4\log 4, 8\log 5, 0, 0, 14\log 2,\\
4\log 9, 8\log 10, 0, 0, 8\log 13, 0, 0, 5\log 16.
\end{gathered}
$$

Here we have

$$
8\pi\sum_{n=1}^{16}e^{-2\pi\sqrt{n}}\sum_{u^2|n}\log\frac{n}{u^2}\cdot\frac{r(\frac{n}{u^2})}{u}\ge 0.010389543.
$$

We have also

$$8\pi \sum_{n=1}^{16} n^{\frac{1}{4}} e^{-2\pi\sqrt{n}} \left(\frac{1}{2\pi\sqrt{n}} + \frac{1}{8\pi^2 n} \right) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \geq 0.03430233.$$

Using also the estimates given before, we get

$$-2A_1(3) < -0.0821041.$$

We shall next get an lower bound for $-2A_1(3)$ in a similar manner.
First, we have

$$\begin{aligned}
-2A_1(3) \geq & -16\pi C_1 + 16\pi C_0 - 16\pi - 8\zeta'\left(\frac{3}{2}\right) L\left(\frac{3}{2},\chi\right) - 8\zeta\left(\frac{3}{2}\right) L'\left(\frac{3}{2},\chi\right) \\
& - 8\pi(\log\pi + C_0 + 2\log 2 - 2) \sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\
& + 4\pi \sum_{n=1}^{16} \log n \cdot e^{-2\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \\
& - 8\pi \left\{ \sum_{n=1}^{16} \left(e^{-2\pi\sqrt{n}} - \frac{134 \cdot e^{-8\pi}}{n^2} \right) \sum_{u^2|n} \log \frac{n}{u^2} \frac{r(\frac{n}{u^2})}{u} \right. \\
& \left. + 134 \cdot e^{-8\pi} \zeta(5) \cdot (-4) \cdot (\zeta'(2) L(2,\chi) + \zeta(2) L'(2,\chi)) \right\} \\
& - \frac{8}{\pi} \left\{ \sum_{n=1}^{16} \left(\frac{e^{-\pi\sqrt{n}}}{n^{\frac{3}{4}}} - \frac{24 \cdot e^{-4\pi}}{n^2} \right) \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \right. \\
& \left. + 24 \cdot e^{-4\pi} \zeta(5) \cdot 4 \cdot \zeta(2) L(2,\chi) \right\}.
\end{aligned}$$

This is

$$\begin{aligned}
\geq & -16\pi C_1 + 16\pi C_0 - 16\pi - 8\zeta'\left(\frac{3}{2}\right) L\left(\frac{3}{2},\chi\right) - 8\zeta\left(\frac{3}{2}\right) L'\left(\frac{3}{2},\chi\right) \\
& - 2(\log\pi + C_0 + 2\log 2 - 2) \cdot 0.101166469708 \\
& + 0.0053172 - 0.010389454 - \frac{8}{\pi} \cdot 0.207283736 \\
\geq & -1.13204.
\end{aligned}$$

Hence, we get

$$-1.13204 \leq -2A_1(3) \leq -0.0821041.$$

By numerical computations, we have

$$\begin{aligned}
&-4\pi(-C_0-2\log 2+2)^2-4A_0(3)(-C_0-2\log 2+2)\\
&\quad -2\pi(-C_0-2\log 2+2)(-2C_0+4\log 2+\tilde{B})\\
&\quad -2\pi\left(\frac{\pi^2}{2}+C_0^2+4\log^2 2+4C_0\log 2-4C_0-8\log 2\right)\\
&> -6.1608259
\end{aligned}$$

and

$$\begin{aligned}
&-4\pi(-C_0-2\log 2+2)^2-4A_0(3)(-C_0-2\log 2+2)\\
&\quad -2\pi(-C_0-2\log 2+2)(-2C_0+4\log 2+\tilde{B})\\
&\quad -2\pi\left(\frac{\pi^2}{2}+C_0^2+4\log^2 2+4C_0\log 2-4C_0-8\log 2\right)\\
&< -6.16080749.
\end{aligned}$$

Combining with our estimates on $A_1(3)$, we get

$$-7.29285<b<-6.24293=b_1.$$

We now study the function

$$f_1(X)=-\frac{\pi}{6}X^3+2.02561\cdot X^2-6.24293\cdot X+c.$$

Since

$$f_1'(X)=-\frac{1}{2}\pi X^2+2.02561\cdot 2\cdot X-6.24293<0$$

for real X, $f_1(X)$ is monotone decreasing.

We shall give an upper bound of the value $f_1(1.29)$, where we have

$$\log\frac{\pi^2}{2.71558}=1.29045\cdots, \qquad \log\frac{\pi^2}{2.7153}=1.29055\cdots.$$

For this purpose we need to get an upper bound of $A_2(3)$. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}e^{-2\pi\sqrt{n}}\sum_{u^2|n}(\log n-4\log u)\cdot\frac{r(\frac{n}{u^2})}{u}\\
&\le\sum_{n=1}^{7}e^{-2\pi\sqrt{n}}\sum_{u^2|n}(\log n-4\log u)\cdot\frac{r(\frac{n}{u^2})}{u}+2.5472\times 10^{-6}\cdot\zeta(5)Z_2(2)\\
&\le 0.00041934686,
\end{aligned}$$

$$\sum_{n=1}^{\infty} e^{-2\pi\sqrt{n}} \sum_{u^2|n} (\log n - 4\log u)^2 \cdot \frac{r(\frac{n}{u^2})}{u}$$
$$\leq \sum_{n=1}^{7} e^{-2\pi\sqrt{n}} \sum_{u^2|n} (\log n - 4\log u)^2 \cdot \frac{r(\frac{n}{u^2})}{u} + 5.2966 \times 10^{-6} \cdot \zeta(5) Z_2(2)$$
$$\leq 0.000355581,$$

$$\sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} (\log n - 4\log u) \frac{r(\frac{n}{u^2})}{u} \int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})} \frac{\log y}{\sqrt{y}}\, dy$$
$$\leq \frac{1}{\pi^2} \sum_{n=1}^{7} \frac{\log n}{n^{\frac{3}{4}}} e^{-\pi\sqrt{n}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} + \frac{1}{\pi^2} \times 0.003871 \cdot \zeta(5) Z_2(2)$$
$$\leq 0.0053193,$$

$$\sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})} \frac{\log y}{\sqrt{y}}\, dy$$
$$\leq \sum_{n=1}^{16} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_1^{1000} e^{-\pi\sqrt{n}(y+y^{-1})} \log y \cdot \left(\frac{1}{\sqrt{y}} - \frac{1}{y^{\frac{3}{2}}}\right) dy$$
$$+ \frac{1}{\pi^2} \times 0.0000817699 \cdot \zeta(5) Z_2(2) + 10^{-8}$$
$$\leq 0.00013832 \times 4 + 0.0005464685 + \frac{1}{\pi^2} \times 0.0000817699 \cdot \zeta(5) Z_2(2)$$
$$\leq 0.00115152,$$

and

$$\sum_{n=1}^{\infty} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_0^{\infty} e^{-\pi\sqrt{n}(y+y^{-1})} \frac{\log^2 y}{\sqrt{y}}\, dy$$
$$\leq \sum_{n=1}^{16} n^{\frac{1}{4}} \sum_{u^2|n} \frac{r(\frac{n}{u^2})}{u} \int_1^{1000} e^{-\pi\sqrt{n}(y+y^{-1})} \log^2 y \cdot \left(\frac{1}{\sqrt{y}} + \frac{1}{y^{\frac{3}{2}}}\right) dy$$
$$+ \frac{4}{\pi^3} \times 0.000019832101 \cdot \zeta(5) Z_2(2) + 10^{-8}$$
$$\leq 0.000286461 \times 4 + 0.00112123 + \frac{4}{\pi^3} \times 0.000019832101 \cdot \zeta(5) Z_2(2)$$
$$\leq 0.0022831.$$

Using these estimates, we get

$$A_2(3) \leq 2\pi(8C_0 - 8C_1 - 8 + 8C_2)$$

$$
\begin{aligned}
&+2\left(\zeta''\left(\frac{3}{2}\right)L\left(\frac{3}{2},\chi\right)+2\zeta'\left(\frac{3}{2}\right)L'\left(\frac{3}{2},\chi\right)+\zeta\left(\frac{3}{2}\right)L''\left(\frac{3}{2},\chi\right)\right)\\
&+2\pi(\log^2\pi-2\log\pi(-C_0-2\cdot\log 2+2)+2(-C_0-2\cdot\log 2+2)^2\\
&-\frac{\pi^2}{2}-C_0^2-4\log^2 2-4C_0\log 2+4C_0+8\log 2)\cdot\frac{0.101166469708}{4\pi}\\
&+2\pi(\log\pi-(-C_0-2\cdot\log 2+2))\cdot 0.00041934686\\
&+\frac{\pi}{2}\cdot 0.00035581+2\pi\cdot 0.0053193\\
&+4\pi(\log\pi-(-C_0-2\cdot\log 2+2))\cdot 0.00115152\\
&+2\pi\cdot 0.0022831\\
\le&-0.0008795.
\end{aligned}
$$

Consequently, we get

$$c \le 4.2242$$

and

$$f_1(1.29) \le -1.58237.$$

Consequently, we get also

$$f(y) \le -1.58237 \quad \text{for} \quad \log\frac{\pi^2}{2.71558} \le y \le \log\frac{\pi^2}{2.7153}.$$

In the same manner, we get

$$4\pi\cdot E^{(2)}(2.71,3) \le 0.01.$$

Hence, we get finally

$$F_d''\left(\frac{3}{2},3\right) \le -1.5 \quad \text{for} \quad \log\frac{\pi^2}{2.71558} \le d \le \log\frac{\pi^2}{2.7153}.$$

This proves our Theorem 1 for $K=3$.

5. Supplemental Remarks

5.1. Complex zeros of $F_d(s,8)$

We shall briefly discuss the distribution of the complex zeros of $F_d(s,8)$ in the same manner as in our previous works [4], [5] and [7], where we have treated that of $F_d(s,K)$ for $K=1,2$ and 4.

First of all we shall show that

$$\text{if } \kappa=\sqrt{d}>1.084, \text{ then } F_d(s,8) \text{ has no zeros in } \Re(s)=\sigma\ge\frac{17}{2}.$$

We start from the following decomposition for $\Re(s)=\sigma>8$

$$F_d(s,8)=Z_8(s)+E_0(s,d,8),$$

where we put

$$E_0(s,d,8)=\sum_{m=1}^{\infty} r_8(m) \sum_{-\infty<m_1,m_2,\cdots,m_8<\infty} (m_1^2+m_2^2+\cdots+m_8^2+dm)^{-s}.$$

Suppose that $\Re(s)=\sigma>8$. We notice first that

$$\begin{aligned}|Z_8(s)| &\geq 16 \mid 1-2^{1-s}+2^{4-2s} \mid \cdot \mid \zeta(s) \mid \cdot \mid \zeta(s-3) \mid \\ &\geq 16(1-16\cdot 2^{-2\sigma}-2\cdot 2^{-\sigma})\cdot \prod_p \left(1+\frac{1}{p^\sigma}\right)^{-1} \prod_p \left(1+\frac{1}{p^{\sigma-3}}\right)^{-1} \\ &\geq 16(1-16\cdot 2^{-2\sigma}-2\cdot 2^{-\sigma})\cdot \frac{\zeta(2\sigma)}{\zeta(\sigma)}\frac{\zeta(2\sigma-6)}{\zeta(\sigma-3)},\end{aligned}$$

where p runs over the prime numbers.
On the other hand,

$$\begin{aligned}|E_0(s,d,8)| \leq & \sum_{m=1}^{\infty} r_8(m) \sum_{v=0}^{\infty} r_8(v)(v+dm)^{-\sigma} \\ &\leq \frac{1}{d^\sigma}\sum_{m=1}^{\infty}\frac{r_8(m)}{m^\sigma} + \sum_{m=1}^{\infty} r_8(m) \sum_{v=1}^{\infty} r_8(v)(v+dm)^{-\sigma} \\ &\leq \frac{1}{d^\sigma}\sum_{m=1}^{\infty}\frac{r_8(m)}{m^\sigma} + \sum_{m=1}^{\infty} r_8(m) \sum_{v=1}^{\infty} \frac{r_8(v)}{(4vdm)^{\frac{\sigma}{2}}} \\ &\leq \frac{1}{d^\sigma}\sum_{m=1}^{\infty}\frac{r_8(m)}{m^\sigma} + \frac{1}{2^\sigma d^{\frac{\sigma}{2}}}\left(\sum_{m=1}^{\infty}\frac{r_8(m)}{m^{\frac{\sigma}{2}}}\right)^2 \\ &\leq \frac{1}{d^\sigma}16\cdot(1+2^{1-\sigma}+2^{4-2\sigma})\cdot\zeta(\sigma)\zeta(\sigma-3) \\ &\quad + \frac{1}{2^\sigma d^{\frac{\sigma}{2}}}\left(16\cdot(1+2^{1-\frac{\sigma}{2}}+2^{4-\sigma})\cdot\zeta\left(\frac{\sigma}{2}\right)\zeta\left(\frac{\sigma}{2}-3\right)\right)^2.\end{aligned}$$

Thus

$$\mid Z_8(s) \mid \ > \ \mid E_o(s,d,8) \mid,$$

provided that

$$\begin{aligned}&16(1-16\cdot 2^{-2\sigma}-2\cdot 2^{-\sigma})\cdot \frac{\zeta(2\sigma)}{\zeta(\sigma)}\frac{\zeta(2\sigma-6)}{\zeta(\sigma-3)} \\ &\quad \geq \frac{1}{d^\sigma}16\cdot(1+2^{1-\sigma}+2^{4-2\sigma})\cdot\zeta(\sigma)\zeta(\sigma-3) \\ &\qquad + \frac{1}{2^\sigma d^{\frac{\sigma}{2}}}\left(16\cdot(1+2^{1-\frac{\sigma}{2}}+2^{4-\sigma})\cdot\zeta\left(\frac{\sigma}{2}\right)\zeta\left(\frac{\sigma}{2}-3\right)\right)^2.\end{aligned}$$

The last condition is satisfied if

$$d > \left(\frac{A_1(\sigma)}{A_2(\sigma)}\right)^{\frac{2}{\sigma}} = d_o(\sigma), \quad \text{say},$$

where we put

$$\begin{aligned} A_1(\sigma) \ &= \frac{1}{2^\sigma}\left(16\cdot(1+2^{1-\frac{\sigma}{2}}+2^{4-\sigma})\cdot\zeta\left(\frac{\sigma}{2}\right)\zeta\left(\frac{\sigma}{2}-3\right)\right)^2 \\ &\quad+\left\{\frac{1}{2^{2\sigma}}\left(16\cdot\left(1+2^{1-\frac{\sigma}{2}}+2^{4-\sigma}\right)\cdot\zeta\left(\frac{\sigma}{2}\right)\zeta\left(\frac{\sigma}{2}-3\right)\right)^4\right. \\ &\quad+4\cdot 16(1-16\cdot 2^{-2\sigma}-2\cdot 2^{-\sigma})\cdot\frac{\zeta(2\sigma)}{\zeta(\sigma)}\frac{\zeta(2\sigma-6)}{\zeta(\sigma-3)}\cdot 16 \\ &\quad\left.\times(1+2^{1-\sigma}+2^{4-2\sigma})\cdot\zeta(\sigma)\zeta(\sigma-3)\ \right\}^{\frac{1}{2}} \end{aligned}$$

and

$$A_2(\sigma)=2\cdot 16(1-16\cdot 2^{-2\sigma}-2\cdot 2^{-\sigma})\cdot\frac{\zeta(2\sigma)}{\zeta(\sigma)}\frac{\zeta(2\sigma-6)}{\zeta(\sigma-3)}.$$

Thus we see that for any $\sigma>8$, if $d>d_o(\sigma)$, then

$$F_d(\sigma+it)\neq 0 \quad \text{for any} \quad t.$$

Now to prove the above statement on the zero free region, suppose that

$$d_1=d_o(\sigma_1)$$

for any $\sigma_1>8$. Then for $d>d_1$ and for $\sigma\geq\sigma_1$, we have

$$\begin{aligned} \mid Z_8(\sigma+it)\mid \ &\geq 16(1-16\cdot 2^{-2\sigma}-2\cdot 2^{-\sigma})\cdot\frac{\zeta(2\sigma)}{\zeta(\sigma)}\frac{\zeta(2\sigma-6)}{\zeta(\sigma-3)} \\ &\geq 16(1-16\cdot 2^{-2\sigma_1}-2\cdot 2^{-\sigma_1})\cdot\frac{\zeta(2\sigma_1)}{\zeta(\sigma_1)}\frac{\zeta(2\sigma_1-6)}{\zeta(\sigma_1-3)} \\ &\geq \frac{1}{d_1^{\sigma_1}}16\cdot(1+2^{1-\sigma_1}+2^{4-2\sigma_1})\cdot\zeta(\sigma_1)\zeta(\sigma_1-3) \\ &\quad+\frac{1}{2^{\sigma_1}d_1^{\frac{\sigma_1}{2}}}\left(16\cdot(1+2^{1-\frac{\sigma_1}{2}}+2^{4-\sigma_1})\cdot\zeta\left(\frac{\sigma_1}{2}\right)\zeta\left(\frac{\sigma_1}{2}-3\right)\right)^2 \\ &\geq \frac{1}{d^{\sigma_1}}16\cdot(1+2^{1-\sigma_1}+2^{4-2\sigma_1})\cdot\zeta(\sigma_1)\zeta(\sigma_1-3) \\ &\quad+\frac{1}{2^{\sigma_1}d^{\frac{\sigma_1}{2}}}\left(16\cdot(1+2^{1-\frac{\sigma_1}{2}}+2^{4-\sigma_1})\cdot\zeta\left(\frac{\sigma_1}{2}\right)\zeta\left(\frac{\sigma_1}{2}-3\right)\right)^2 \\ &\geq E_0(\sigma_1,d,8)\geq E_0(\sigma,d,8)\geq\ \mid E_0(\sigma+it,d,8)\mid. \end{aligned}$$

Thus we see that for any for $d>d_1=d_o(\sigma_1)$ and for $\sigma\geq\sigma_1$,

$$F_d(\sigma+it)\neq 0.$$

In particular, since

$$d_o\left(\frac{17}{2}\right)=1.1754\cdots \quad ,$$

$$F_d(s)\neq 0 \quad \text{for any } d>1.1754 \text{ and for any } \Re s=\sigma\geq\frac{17}{2}.$$

In the above argument, we have used the fact that

$$(1 - 16 \cdot 2^{-2\sigma} - 2 \cdot 2^{-\sigma}) \cdot \frac{\zeta(2\sigma)}{\zeta(\sigma)} \frac{\zeta(2\sigma - 6)}{\zeta(\sigma - 3)} = a(\sigma), \quad \text{say,}$$

is monotone increasing for $\sigma > 8$. This can be seen easily by showing that $a'(\sigma)/a(\sigma) > 0$ for $\sigma > 8$.

We recall next Stark's result on the low lying zeros of the Epstein zeta function $\zeta(s, Q)$ defined by

$$\zeta(s, Q) = \frac{1}{2}\sum\nolimits'_{x,y} Q(x, y)^{-s} \quad \text{for} \quad \Re(s) > 1,$$

where x, y runs over all integers excluding $(x, y) = (0, 0)$, $Q(x, y) = ax^2 + bxy + cy^2$ is a positive definite quadratic form with discriminant $\Delta = b^2 - 4ac$, a, b and c are real numbers, $a > 0$ and we put

$$k = \frac{\sqrt{|\,\Delta\,|}}{2a}.$$

Stark has shown "k-analogue" of the Riemann hypothesis. Namely, he has shown that for sufficiently large k, all the zeros of $\zeta(s, Q)$ in the region $-1 < \sigma < 2$, $-2k \le t \le 2k$ are simple zeros; with the exception of two real zeros between 0 and 1, all are on the line $\sigma = 1/2$.

Stark's result has been extended to $F_d(s, K)$ for $K = 2$ in Fujii [5] and $K = 4$ in Fujii [7]. In the same manner, we can extend it to the case for $K = 8$. Namely, the "κ-analogue" of the "Riemann Hypothesis" holds in the following form: *there exists a number T_0 such that if $\kappa = \sqrt{d} > T_0$, then all the zeros of $F_d(s, 8)$ in the region $-1/2 < \sigma < 17/2$, $-\kappa \le t \le \kappa$ are simple zeros*; *with the exception of two real zeros between* 0 *and* 8, *all are on the line $\sigma = 4$.*

An extension of the Riemann-von Mangoldt formula may be stated as follows.

Let $N(T, F)$ denote the number of the zeros of $F_d(s, 8)$ in the region $-1/2 < \sigma < 17/2$, $0 \le t \le T$. If $\kappa > T_0$ and $0 < T \le \kappa$, then

$$\begin{aligned} N(T, F) &= \frac{1}{\pi}\arg\left(\left(\frac{\kappa}{\pi}\right)^{4+iT}\Gamma(4 + iT)\right) + \frac{1}{\pi}\arg\zeta(1 + iT) + O(1) \\ &= \frac{T}{\pi}\log\frac{\kappa T}{\pi e} + O(\log\log T), \end{aligned}$$

uniformly for κ.

Finally we shall state the failure of the "κ-analogue" of the "GUE law" for $F_d(s, 8)$ in the following form (cf. Fujii [4], [5], [7]).

For $\kappa > T_0$ and $0 < T \le \kappa$, we have, for some positive absolute constant

C,

$$\frac{1}{\frac{T}{2}}\int_{\frac{T}{2}}^{T-1}\left(S_F\left(t+\frac{\alpha\pi}{\log\frac{\kappa T}{\pi}}\right)-S_F(t)\right)^2 dt \le C$$

uniformly for $0<\alpha\ll T\log(\kappa T/\pi)$, *where we put*

$$S_F(t)=N(t,F)-\frac{1}{\pi}\arg\left(\left(\frac{\kappa}{\pi}\right)^{4+it}\Gamma(4+it)\right)=\frac{1}{\pi}\arg\zeta(1+it)+\Delta_F(t),$$

with

$$|\ \Delta_F(t)\ |\le C.$$

Hence we see that as $\kappa\to\infty$,

$$\frac{1}{\frac{\kappa}{2}}\int_{\frac{\kappa}{2}}^{\kappa-1}\left(S_F\left(t+\frac{\alpha\pi}{\log\frac{\kappa^2}{\pi}}\right)-S_F(t)\right)^2 dt \le C$$

uniformly for $0<\alpha\ll\kappa\log(\kappa^2/\pi)$.

5.2. A comparison of two expressions of $Z_3(s)$

We see easily, for example, by drawing the graph, that when real s is bigger,

$$4\zeta(s)L(s,\chi)+\frac{2\pi}{s-1}\zeta(2s-2)$$

gives a better approximation than

$$2\zeta(2s)+\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}4\zeta\left(s-\frac{1}{2}\right)L\left(s-\frac{1}{2},\chi\right).$$

On the other hand, concerning a better approximation to $Z_3(s)$ for $\Re(s)>3/2$, there is another candidate, which is the partial sum of $Z_3(s)$. For our purpose we have used the partial sum

$$\sum_{n=1}^{55}\frac{r_3(n)}{n^s}.$$

We can see this, for example, by drawing the graph of

$$\left\{4\zeta(s)L(s,\chi)+\frac{2\pi}{s-1}\zeta(2s-2)\right\}-\sum_{n=1}^{55}\frac{r_3(n)}{n^s}\quad\text{for}\quad s\ge\frac{3}{2}.$$

5.3. Evaluations and estimations of the values of the derivatives of $\zeta(s)$ and $L(s,\chi)$ at certain points

We have used at several places in this article the numerical values of the derivatives of $\zeta(s)$ and $L(s,\chi)$ at certain points, for example, $\zeta'(2)$ and $L'(2,\chi)$.

It is noticed by Siegel (cf. p. 75 of Siegel [13]) that the following formula

$$L'(1,\chi) = \frac{\pi}{4}(C_0 - 2\log 2 - 4\log |\, \eta(i)\, |),$$

which is a consequence of Kronecker's limit formula for $Z_2(s)$, gives a very effective method to get a numerical value of $L'(1,\chi)$.

We might expect that Kronecker's limit formulas for $Z_K(s)$ for $K \geq 3$ play also the same role. We mention here only the following results, although we have not used these in this article.

Example 1.

(i)

$$\frac{1}{2\pi}\zeta(3) = \frac{1}{\pi}\zeta\left(\frac{3}{2}\right) L\left(\frac{3}{2},\chi\right) + 1 + 2\log |\, \eta(i)\, | + \sum_{n=1}^{\infty}\sum_{u^2|n} \frac{1}{u} r_2\left(\frac{n}{u^2}\right) e^{-2\pi\sqrt{n}} - \sum_{n=1}^{\infty}\sqrt{n}\sum_{m|n}\frac{r_2(m)r_1(\frac{n}{m})}{m} K_1(2\pi\sqrt{n}).$$

(ii)

$$-\frac{3}{\pi^2}\zeta'(2) + \frac{1}{3}L(2,\chi) = -\frac{C_0}{2} + \frac{4}{3}\log 2 + \frac{1}{2} + 2\log |\, \eta(i)\, | - \sum_{n=1}^{\infty}\sqrt{n}\sum_{m|n}\frac{r_2(m)r_2(\frac{n}{m})}{m} K_1(2\pi\sqrt{n}).$$

(i) is a consequence of Kronecker's limit formulas for $Z_3(s)$.
(ii) is a consequence of Kronecker's limit formulas for $Z_4(s)$.

It might be natural to think that not only the Kronecker's limit formulas but also the evaluations of the residues of $Z_K(s)$ play certain role in this context (cf. also p. 209 of Hua [10]). It might be not worthless to write down that a simple counting formula for $X \geq 1$, $\sum_{1\leq n\leq X} 1 = X + O(1)$, and the explicit formulas of $r_K(n)$ for $K = 2, 4, 6, 8, 10$ and 12 give us the following well known evaluations.

Example 2.

$$L(1,\chi) = \frac{\pi}{4}, \quad \zeta(2) = \frac{\pi^2}{6}, \quad L(3,\chi) = \frac{\pi^3}{32},$$
$$\zeta(4) = \frac{\pi^4}{90}, \quad L(5,\chi) = \frac{5\pi^5}{1536} \quad \text{and} \quad \zeta(6) = \frac{\pi^6}{945}.$$

In principle, we can expand the list in Example 2 indefinitely. Namely, the list include the values of $L(2n+1,\chi)$ for all integers $n \geq 0$ and $\zeta(2n)$ for all integers $n \geq 1$.

Concerning the values of $\zeta(2n+1)$, $n \geq 1$, we have noticed in [15] that they are connected with the values $Z_K(K/4)$, $K \geq 6$. In addition to this, we see easily that

$$Z_8(1) = -\frac{16}{\pi^2}\zeta(3) \qquad \text{and} \qquad Z_8(3) = -8\zeta(3).$$

For the numerical computations which we have used in the present article, the analogues of Example 1 do not help very much. In stead of these, we have used the Euler-Maclaurin summation formula in a standard way and used the following numerical values.

$$\zeta\left(\frac{3}{2}\right) = 2.61237534\cdots, \qquad \zeta\left(\frac{5}{2}\right) = 1.34148725\cdots,$$
$$\zeta(3) = 1.202056903\cdots, \qquad \zeta(5) = 1.036927755\cdots,$$
$$\zeta(7) = 1.00834927\cdots, \qquad \zeta'\left(\frac{3}{2}\right) = -3.93223973\cdots,$$
$$\zeta'(2) = -0.9375482\cdots, \qquad \zeta'(4) = -0.068911265\cdots,$$
$$\zeta''\left(\frac{3}{2}\right) = 15.98955637\cdots, \qquad \zeta''(2) = 1.989280\cdots,$$
$$L\left(\frac{3}{2},\chi\right) = 0.86450265\cdots, \qquad L(2,\chi) = 0.91596559\cdots,$$
$$L\left(\frac{5}{2},\chi\right) = 0.94862217\cdots, \qquad L'\left(\frac{3}{2},\chi\right) = 0.12721993\cdots,$$
$$L'(2,\chi) = 0.081580736\cdots, \qquad L''\left(\frac{3}{2},\chi\right) = -0.1097970321\cdots,$$
$$L''(2,\chi) = -0.0744152124\cdots.$$

5.4. Correction to Fujii [5]

"$D_1 = 0.156\cdots$" in p. 710 should be "$D_1 = 0.165\cdots$" as in p. 734.

5.5. Corrections to Fujii [7]

(1) Lemma B in p. 158 should be corrected like Lemma 4 of the present article, although we have changed the notations slightly.

(2) e^{C_0} in l. 21 of p. 167 and l. 10 and l. 16 of p. 218 should be e^{-C_0}.

(3) $2.4\cdots\times 10^{30}$ in l. 21 of p. 167 should be replaced by $2.99\cdots\times 10^{9}$.

(4) $r_K(n)$ in l. 4 of p. 175 should be $r_K(m)$.

(5) Γ in l. 3 of p. 176 should be erased.

(6) $\Gamma(K/2)-\Gamma'(K_0/2)$ in l. 7 of p. 179 should be $\Gamma(K/2)\Gamma'(K_0/2)$.

(7) $K_{K_0/2^k}(2\pi\sqrt{n})$ in l. 10 and l. 12 of p. 180 and l. 1 and l. 5 of p. 181 should be $K_{(1/2)(K_0/2^k)}(2\pi\sqrt{n})$.

(8) $A(K_0/2^k)$ in l. 11 and l. 13 of p. 180, l. 3 of p. 181 and l. 22 of p. 216 should be $A(2(K_0/2^k))$.

(9) In l. 10 and 12 of p. 180 and l. 1 of p. 181, "$+2C_0$" should be added.

(10) K_0 in l. 4 of p. 181 should be K.

(11) In l. 6 of p. 181, "$+A(K)2C_0$" should be added.

(12) "$+2C_0$" in l. 21 of p. 216, l. 10 of p. 217 and l. 9 of p. 219 should be "$-2C_0$".

(13) $m^{K_0/2^k}$ in l. 1 of p. 217 should be $m^{(1/2)(K_0/2^k)}$.

(14) e^{2C_0} in l. 12 of p. 217 and l. 8 of p. 218 should be e^{-2C_0}.

Department of mathematics
Rikkyo University
Tokyo 171-8501, Japan

References

[1] P. T. Bateman and E. Grosswald, On Epstein's zeta functions, Acta Arith., **9** (1964), 365–373.

[2] E. Bombieri and D. Hejhal, Sur les zeros des fonctions zeta d'Epstein, C. R. Acad. Sci. Paris, **304** (1987), 213–217.

[3] A. Fujii, Some observations concerning the distribution of the zeros of the zeta functions I, Adv. Stud. Pure Math., **21** (1992), 237–280.

[4] A. Fujii, Number variance of the zeros of the Epstein zeta functions, Proc. Japan Acad., **70** (1994), 140–145.

[5] A. Fujii, On the zeros of the Epstein zeta functions, J. Math. Kyoto Univ., **36**-4 (1996), 697–770.

[6] A. Fujii, On the Berry Conjecture, J. Math. Kyoto Univ., **37**-1 (1996), 55–98.

[7] A. Fujii, On the zeros of the Epstein zeta functions (II), Comment. Math. Univ. St. Paul., **47**-2 (1998), 155–236.

[8] E. Grosswald, Representations of Integers as Sums of Squares, Springer, 1985.

[9] D. Hejhal, Zeros of Epstein zeta functions and supercomputers, Proc. Int. Congress of Math. Berkeley, 1988, 1362–1384.

[10] L. K. Hua, Introduction to Number Theory, Springer-Verlag, 1982.

[11] M. E. Low, Real zeros of the Dedekind zeta function of an imaginary quadratic fields, Acta Arith., **XIV** (1968), 117–140.

[12] A. Selberg, Collected Papers, Springer-Verlag, vol. 1, 1989; vol. 2, 1991.

[13] C. L. Siegel, Lectures on Advanced Analytic Number Theory, Tata, Bombay, 1963.

[14] H. M. Stark, On the zeros of Epstein's zeta functions, Mathematika, **14** (1967), 47–55.

[15] A. Fujii, A note on the values of the Epstein zeta functions, Comment. Math. Univ. St. Paul., **49**-2 (2000), 195–225.

[16] H. Yoshida, A letter to the author and a note, Nov. 13, 2000.