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A fixed point formula for compact almost complex manifolds

By

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Abstract

In this paper, using the group structures of the spheres S^1, S^3 and the results of Atiyah-Patodi-Singer, Donnelly and Morita, we introduce a fixed point formula for periodic automorphisms of compact almost complex manifolds. Our main result is Theorem 1.3. The theorem is refined for a certain case if the almost complex manifold admits an Einstein-Kähler metric.

1. Introduction and Main Theorem

Let M be a compact $2m$-dimensional almost complex manifold with the almost complex structure J and $P \to M$ the associated principal $GL(m;\mathbb{C})$-bundle of M. We call a diffeomorphism $\psi : M \longrightarrow M$ an automorphism of M if ψ commutes with J and denote the topological group consisting of all automorphisms of M by $A(M)$. The group $A(M)$ naturally acts on P on the left.

Definition 1.1. Let $S(n)$ be the set of symmetric homogeneous polynomials in $x_1, x_2, \ldots, x_m$ of order n with integral coefficients. Let

$$\phi = \phi(\tau_1, \tau_2, \ldots, \tau_m)$$

be any element of $S(n)$ where $\tau_j = \sigma_j(x_1, x_2, \ldots, x_m)$ is the j-th elmentary symmetric polynomial in $\{x_i\}_{i=1}^m$, whose degree is equal to j. Let V_ϕ be the element of the representation ring $R(GL(m;\mathbb{C}))$ of $GL(m;\mathbb{C})$ defined by

$$V_\phi = \phi(\tau_1, \tau_2, \ldots, \tau_m) \in R(GL(m;\mathbb{C})) \subset R(T^m) = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_m, t_m^{-1}],$$

where T^m is the maximal torus of $GL(m;\mathbb{C})$, $t_i : T^m \to S^1$ is the i-th factor projection and $\tau_j = \sigma_j(t_1-1, t_2-1, \ldots, t_m-1)$ is the j-th elementary symmetric polynomial in $t_1 - 1, t_2 - 1, \ldots, t_m - 1$. Note that $\sigma_j = \sigma_j(t_1, t_2, \ldots, t_m)$ is isomorphic to the $GL(m;\mathbb{C})$-representation $\wedge^j \mathbb{C}^m$. Hence, setting

$$\hat{\phi}(\sigma_1, \sigma_2, \ldots, \sigma_m) = \phi(\tau_1, \tau_2, \ldots, \tau_m)\,,$$

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we have

$$V_\phi = \hat{\phi}(\wedge^1\mathbb{C}^m, \wedge^2\mathbb{C}^m, \dots, \wedge^m\mathbb{C}^m) \in R(GL(m;\mathbb{C}))\,.$$

Using this virtual $GL(m;\mathbb{C})$-representation V_ϕ, we can define a virtual complex vector bundle E_ϕ on M by

$$E_\phi = P \times_{GL(m;\mathbb{C})} V_\phi = \hat{\phi}(\wedge^1 TM, \wedge^2 TM, \dots, \wedge^m TM) \in K(M) \tag{1.1}$$

where TM is the tangent bundle of M and $K(M)$ is the K-group of M. Then the action of $A(M)$ on P naturally defines the action of $A(M)$ on E_ϕ and E_ϕ is a virtual complex $A(M)$-vector bundle.

Definition 1.2. Let a be any periodic element of $A(M)$, G the cyclic subgroup of $A(M)$ generated by a and Ω the fixed point set of a consisting of compact connected submanifolds N of M. Then the restriction of J defines an almost complex structure of N and the Todd class $\mathrm{Td}(TN)$ of TN is defined by

$$\mathrm{Td}(TN) = \prod_{k=1}^{d} \frac{x_k}{1-e^{-x_k}} \in H^*(N;\mathbb{C}),$$

where $2d$ is the dimension of N and $\prod_{k=1}^{d}(1+x_k)$ equals to the total Chern class of TN. Note that $\mathrm{Td}(TN) = 1$ if N is a point. On the other hand, a complex G-vector bundle E over N is decomposed into the direct sum of subbundles

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_s,$$

where a acts on the subbundle E_j via multiplication by $e^{\sqrt{-1}\theta_j}$. Then we can define the characteristic class $\mathrm{Ch}(E,a)$ by

$$\mathrm{Ch}(E,a) = \sum_{j=1}^{s} e^{\sqrt{-1}\theta_j}\mathrm{Ch}(E_j) \in H^*(N;\mathbb{C}),$$

where $\mathrm{Ch}(E_j)$ is the Chern character of E_j. This definition is extended to the case of virtual vector bundles by

$$\mathrm{Ch}(E-F,a) = \mathrm{Ch}(E,a) - \mathrm{Ch}(F,a) \in H^*(N;\mathbb{C})$$

and $\mathrm{Ch}(*\,,a)$ defines a ring homomorphism

$$\mathrm{Ch}(*\,,a) \,:\, K(N) \longrightarrow H^*(N;\mathbb{C})\,,$$

namely, satisfies the following equalities:

(1.2)

$$\mathrm{Ch}(E\pm F,a) = \mathrm{Ch}(E,a) \pm \mathrm{Ch}(F,a)\,, \quad \mathrm{Ch}(E\otimes F,a) = \mathrm{Ch}(E,a)\,\mathrm{Ch}(F,a)\,.$$

We can also define the characteristic class $\mathfrak{U}(E,a)$ by

$$\mathfrak{U}(E,a) = \prod_{j=1}^{s}\prod_{k=1}^{r_j} \frac{1}{1-e^{-x_k-\sqrt{-1}\theta_j}} \in H^*(N;\mathbb{C}),$$

where $r_j = \mathrm{rank}(E_j)$ and $\prod_{k=1}^{r_j}(1+x_k)$ equals to the total Chern class of E_j.

Our main result is the following theorem.

Theorem 1.3. *Let ℓ be 0, 1 or 2 and ϕ any element of $S(n)$. Let ψ be any periodic element of $A(M)$ and assume that the order of ψ is p. Let γ be any natural number which is prime to p. Let $\Omega(k)$ be the fixed point set of ψ^k $(1 \leqq k \leqq p-1)$ consisting of compact connected almost complex manifolds N, $\nu(N, M)$ the normal bundle of N in M and $[N]$ the fundamental cycle of N. Then the equality*

$$\sum_{k=1}^{p-1} C_\ell(k,\gamma) \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\, \mathrm{Td}(TN)\, \mathfrak{U}(\nu(N,M), \psi^k)[N] \equiv 0 \qquad (\mathrm{mod}\ p)$$

holds for any $n > m + \ell$, where

$$C_0(k,\gamma) = 1\ , \quad C_1(k,\gamma) = \frac{1}{1 - e^{-2\pi\sqrt{-1}\gamma k/p}}\ , \quad C_2(k,\gamma) = \frac{1}{|1 - e^{-2\pi\sqrt{-1}\gamma k/p}|^2}\ .$$

Let N be a connected component of the fixed point set of the action of a periodic automorphism a of M. Assume that the restriction of the tangent bundle TM to N splits into the direct sum of complex line bundles

$$TM|_N = L_1 \oplus \cdots \oplus L_m$$

where a acts on L_j via multiplication by $e^{\sqrt{-1}\theta_j}$. Let σ_j be the j-th elementary symmetric polynomial in $\{e^{\sqrt{-1}\theta_j} e^{c_1(L_j)}\}_{j=1}^m$ and τ_j the j-th elementary symmetric polynomial in $\{e^{\sqrt{-1}\theta_j} e^{c_1(L_j)} - 1\}_{j=1}^m$. Then since

$$\mathrm{Ch}(L_j, a) = e^{\sqrt{-1}\theta_j} e^{c_1(L_j)}\ ,$$

it follows from (1.1) and (1.2) that

$$\mathrm{Ch}(E_\phi|_N, a) = \hat{\phi}(\sigma_1, \sigma_2, \ldots, \sigma_m) = \phi(\tau_1, \tau_2, \ldots, \tau_m)\ . \tag{1.3}$$

The next corollary is deduced from Theorem 1.3 and (1.3).

Corollary 1.4. *Assume that $\Omega(k)$ in Theorem 1.3 consists of points $\{q_s\}_{s=1}^{N(k)}$ for any k. Then the automorphism ψ^k acts on the tangent space $T_{q_s}M$ via multiplication by some periodic diagonal unitary matrix, which we assume is the diagonal matrix with diagonal entries $\{e^{2\pi\sqrt{-1}h_{js}^k/p}\}_{j=1}^m$ $(h_{js}^k \in \mathbb{Z})$. Let τ_j be the j-th elementary symmetric polynomial in $\{e^{2\pi\sqrt{-1}h_{js}^k/p} - 1\}_{j=1}^m$. Then under the notation in Theorem 1.3, the equality*

$$\sum_{k=1}^{p-1} C_\ell(k,\gamma) \sum_{s=1}^{N(k)} \phi(\tau_1, \tau_2, \ldots, \tau_m) \prod_{j=1}^m \frac{1}{1 - e^{-2\pi\sqrt{-1}h_{js}^k/p}} \equiv 0 \qquad (\mathrm{mod}\ p)$$

holds for any $n > m + \ell$.

Proof. For any $q_s \in \Omega(k)$, the tangent space $T_{q_s}M$ splits into the direct sum of m-copies of $\mathbb{C}^1$

$$T_{q_s}M = \mathbb{C}^1_1 \oplus \mathbb{C}^1_2 \oplus \cdots \oplus \mathbb{C}^1_m,$$

where ψ^k acts on $\mathbb{C}^1_j$ via multiplication by $e^{2\pi\sqrt{-1}h^k_{js}/p}$. Hence it follows from (1.3) that

$$\mathrm{Ch}(E_\phi|_{q_i}, \psi^k) = \phi(\tau_1, \tau_2, \dots, \tau_m),$$

where τ_j is the j-th elementary symmetric polynomial in $\{e^{2\pi\sqrt{-1}h^k_{js}/p} - 1\}^m_{j=1}$. Moreover, since $\mathrm{Td}(Tq_s) = 1$ for any s, the equality in Corollary 1.4 immediately follows from the equality in Theorem 1.3. □

Remark 1.5. As we will see in Remarks 3.3 and 4.2, the equality in Theorem 1.3 does not hold in general if $n = m + \ell$.

Remark 1.6. The author does not know whether the equality in Theorem 1.3 holds for $\ell \geqq 3$ by introducing some appropriate $C_\ell(k, \gamma)$.

2. Proof of the Theorem

In this section we give the proof of Theorem 1.3. Let G be the cyclic subgroup of $A(M)$ generated by ψ. We give a G-invariant Hermitian metric on M and let $Q \longrightarrow M$ be the subbundle of P consisting of unitary frames with respect to the metric. Let ∇ be a G-invariant connection in Q. Then since V_ϕ is considered as a virtual representation of $U(m)$ and E_ϕ equals to $Q \times_{U(m)} V_\phi$, the natural $U(m)$-invariant inner product in V_ϕ defines a G-invariant inner product in E_ϕ and ∇ defines a unitary connection of E_ϕ. The connection ∇ also defines a G-invariant connection of the half spinor bundles $S^\pm = Q \times_{U(m)} \triangle^\pm$ over M where $\triangle^\pm$ are the half spin representations of $\mathrm{spin}^c(2m)$. (For details of spinor bundles and spin^c-Dirac operators, see [6].) Using the connections defined above, we can define the G-equivariant spin^c-Dirac (Dolbeault) operator

$$D \;:\; \Gamma(S^+ \otimes E_\phi) \longrightarrow \Gamma(S^- \otimes E_\phi)$$

and it follows from the Riemann-Roch theorem (see (4.3) in [2]) that

$$\text{(2.1)}\quad \mathrm{Index}(D) := \dim\ker(D) - \dim\mathrm{coker}(D) = \int_M \mathrm{Ch}(E_\phi, \nabla)\,\mathrm{Td}(TM, \nabla),$$

where $\mathrm{Ch}(E_\phi, \nabla)$ is the Chern character form of E_ϕ with respect to ∇, $\mathrm{Td}(TM, \nabla)$ is the Todd form of TM with respect to ∇. Here for any $1 \leqq j \leqq m$, we can see that

$$\mathrm{Ch}(\wedge^j TM, \nabla) = \sigma_j(e^{x_1}, e^{x_2}, \dots, e^{x_m}),$$

where by definition the j-th Chern form $c_j(TM, \nabla)$ is the j-th elementary symmetric polynomial in $x_1, x_2, \dots, x_m$. Hence it follows from (1.1) and (1.2) that

$$\mathrm{Ch}(E_\phi, \nabla) = \hat{\phi}(\sigma_1, \sigma_2, \dots, \sigma_m) = \phi(\tau_1, \tau_2, \dots, \tau_m)$$

where τ_j is the j-th elementary symmetric polynomial in $e^{x_1}-1, e^{x_2}-1, \dots,$ $e^{x_m}-1$ for $1 \leqq j \leqq m$. Since

$$\begin{aligned}\tau_j &= \sigma_j(e^{x_1}-1, e^{x_2}-1, \dots, e^{x_m}-1) \\ &= \sigma_j(x_1, x_2, \dots, x_m) + \text{higher order terms} \\ &= c_j(TM, \nabla) + \text{higher order terms}\,,\end{aligned} \tag{2.2}$$

we have

$$\mathrm{Ch}(E_\phi, \nabla) = \phi(c_1(TM,\nabla), c_2(TM,\nabla), \dots, c_m(TM,\nabla)) + \text{higher order terms}$$

and therefore it follows that

$$\int_M \mathrm{Ch}(E_\phi, \nabla)\,\mathrm{Td}(TM, \nabla) = 0 \tag{2.3}$$

because the order of ϕ is greater than m and the dimension of M is $2m$. On the other hand, it follows from (4.6) in [2] that

$$\begin{aligned}\mathrm{Index}(D, \psi^k) &:= \mathrm{Tr}(\psi^k|_{\ker(D)}) - \mathrm{Tr}(\psi^k|_{\mathrm{coker}(D)}) \\ &= \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M), \psi^k)[N]\end{aligned} \tag{2.4}$$

for $1 \leqq k \leqq p-1$. Now let V be any finite dimensional complex G-module and β an eigenvalue of $\psi|_V$. Then since $\beta^p = 1$, it follows that

$$\sum_{k=1}^{p} \beta^k \equiv 0 \qquad (\mathrm{mod}\ p)$$

and hence it follows that

$$\sum_{k=1}^{p} \mathrm{Tr}(\psi^k|_V) \equiv 0 \qquad (\mathrm{mod}\ p)\,. \tag{2.5}$$

Therefore we have

$$\sum_{k=1}^{p} \mathrm{Index}(D, \psi^k) \equiv 0 \qquad (\mathrm{mod}\ p)$$

and hence it follows from (2.1), (2.3) and (2.4) that

$$\begin{aligned}&\sum_{k=1}^{p-1} \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M), \psi^k)[N] \\ &= \sum_{k=1}^{p-1} \mathrm{Index}(D, \psi^k) = \sum_{k=1}^{p} \mathrm{Index}(D, \psi^k) \equiv 0 \qquad (\mathrm{mod}\ p)\end{aligned}$$

because $\mathrm{Index}(D,\psi^p)=\mathrm{Index}(D)=0$. This completes the proof of the equality in Theorem 1.3 for $\ell=0$.

Now assume that $\ell=1$ or 2 and let $D^{2\ell}$ and $\partial D^{2\ell}=S^{2\ell-1}$ be the unit disk and the unit sphere in $\mathbb{C}^\ell$ respectively. Let $\mathbb{H}$ be the set of quaternions, which is identified with $\mathbb{C}^2$ as follows:

$$\mathbb{H}\ni a+bi+cj+dk=(a+bi)+(c+di)j \longleftrightarrow (a+bi,c+di)\in\mathbb{C}^2\,.$$

Then $\mathbb{C}^1$ is contained in $\mathbb{H}$ by $a+bi=a+bi+0j+0k$. Let $\alpha:=e^{2\pi\sqrt{-1}/p}$ be the primitive p-th root of 1. Then G acts on $\mathbb{H}$ by $\psi\cdot h=h\alpha^\gamma$ $(h\in\mathbb{H})$, which corresponds to the $SU(2)$-transformation

$$\mathbb{C}^2\ni(z_1,z_2)\longrightarrow(\alpha^\gamma z_1,\overline{\alpha}^\gamma z_2)\in\mathbb{C}^2$$

under the identification above because $j\alpha^\gamma=\overline{\alpha}^\gamma j$. This G-action defines G-actions on $D^{2\ell}$, $S^{2\ell-1}$ for $\ell=1$, 2. We give the standard metric on $S^{2\ell-1}$, which is G-invariant, and give a G-invariant Hermitian metric on $D^{2\ell}$ such that it is a product metric of $S^{2\ell-1}\times[0,\delta)$ near $\partial D^{2\ell}=S^{2\ell-1}$. Here since ℓ equals to 1 or 2, the sphere $S^{2\ell-1}$ has a group structure. Actually the group structure of S^3 is induced from the multiplication in the quaternions $\mathbb{H}$ and S^1 is the subgroup of S^3 consisting of complex numbers. Using this group structure, we can construct a global orthonormal frame field $\{F_A^1,F_A^2,F_A^3\}_{A\in S^3}$ on S^3 as follows:

$$F_A^1=i\cdot A\,,\ F_A^2=j\cdot A\,,\ F_A^3=k\cdot A\in\mathbb{H}\,.$$

It is clear that $\{F_A^1\}_{A\in S^1}$ defines a global orthonormal frame field on S^1. Now considering the associativity of the multiplication in $\mathbb{H}$, we can see that the frame field above is invariant under the action of G. Hence the trivialization of the tangent bundle TS^3:

$$TS^3\ni(A,w=aF_A^1+bF_A^2+cF_A^3)\longrightarrow(A,(a,b,c))\in S^3\times\mathbb{R}^3$$

$(A\in S^3,\ w\in T_AS^3)$ is G-invariant and therefore the unique trivial spin^c-structure of $S^{2\ell-1}$ is G-invariant. Moreover $F_A^0:=A$ defines the outward unit normal vector field on $S^{2\ell-1}$ and the trivialization of $TD^4|_{S^3}$:

$$\begin{aligned}TD^4|_{S^3}\ni&(A,v=aF_A^0+bF_A^1+cF_A^2+dF_A^3)\\&\longrightarrow(A,((a+bi),(c+di)))\in S^3\times\mathbb{C}^2\end{aligned}$$

$(A\in S^3,\ v\in T_AD^4)$ is G-invariant. Therefore the quotient $(TS^{2\ell-1})/G$ is the trivial real vector bundle and the quotient $(TD^{2\ell}|_{S^{2\ell-1}})/G$ is the trivial complex vector bundle.

Set $X=M\times D^{2\ell}$ and $Y=\partial X=M\times S^{2\ell-1}$. Then the metric on M and the metrics on $D^{2\ell}$, $S^{2\ell-1}$ define the G-invariant product metrics on X, Y respectively and the G-actions on $D^{2\ell}$, $S^{2\ell-1}$ define the diagonal G-actions on X, Y as follows:

$$\psi\cdot(q\,,\,h)=(\psi\cdot q\,,\,h\alpha^\gamma)\quad(q\in M,\,h\in\mathbb{H})\,. \tag{2.6}$$

Moreover the tangent bundle TX, TY splits as

$$TX = q_X^* TM \oplus r_X^* TD^{2\ell} = q_X^* TM \oplus \varepsilon_{\mathbb{C}}^{\ell}\,,$$
$$TY = q_Y^* TM \oplus r_Y^* TS^{2\ell-1} = q_Y^* TM \oplus \varepsilon^{2\ell-1},$$

where $q_X : X \longrightarrow M$, $q_Y : Y \longrightarrow M$ denote the first factor projections, $r_X : X \longrightarrow D^{2\ell}$, $r_Y : Y \longrightarrow S^{2\ell-1}$ denote the second factor projections and $\varepsilon_{\mathbb{C}}^k$ (ε^k) denotes the trivial complex (real) vector bundle of rank k with a G-invariant trivialization. Therefore spin^c-structures on X, Y are defined by the $U(m)$-structures q_X^*Q, q_Y^*Q respectively and connections ∇^X, ∇^Y in q_X^*Q, q_Y^*Q are induced from the connection ∇ in Q. These connections ∇^X, ∇^Y define G-invariant metric connections of TX, TY, which are the direct sum of the connection ∇ of TM and the globally flat connections of the trivial bundles. These connections ∇^X, ∇^Y also define G-invariant connections of the half spinor bundles $S_X^{\pm} = q_X^*Q \times_{U(m)} \triangle^{\pm}$ over X and a G-invariant connection of the spinor bundle $S_Y = S_X^+|_Y = S_X^-|_Y = q_Y^*Q \times_{U(m)} \triangle$ over Y where $\triangle^{\pm}$ are the half spin representations of spin$^c(2m+2\ell)$ and $\triangle$ is the spin representation of spin$^c(2m+2\ell-1)$.

Set $E_{\phi,X} = q_X^* E_\phi = q_X^*Q \times_{U(m)} V_\phi$ and $E_{\phi,Y} = q_Y^* E_\phi = q_Y^*Q \times_{U(m)} V_\phi$. Then $E_{\phi,X}$ and $E_{\phi,Y}$ are virtual G-vector bundles with G-invariant unitary connections ∇^X, ∇^Y and the restriction of $E_{\phi,X}$ to Y coincides with $E_{\phi,Y}$. Using the spin^c-structures and the connections defined above, we can define the G-equivariant spin^c-Dirac operators

$$D_X \;:\; \Gamma(S_X^+ \otimes E_{\phi,X}) \longrightarrow \Gamma(S_X^- \otimes E_{\phi,X})\,,$$
$$D_Y \;:\; \Gamma(S_Y \otimes E_{\phi,Y}) \longrightarrow \Gamma(S_Y \otimes E_{\phi,Y})\,.$$

Since the metric and the connection ∇^X is product near $\partial X = Y$, D_X can be expressed as

$$D_X = \sigma\left(\frac{\partial}{\partial u} + D_Y\right)$$

on the collar $Y \times [0,\delta) \subset X$ where u is the coordinate of $[0,\delta)$ and σ is a bundle isomorphism defined by the Clifford multiplication (see [1]). Hence the following equality is deduced from (4.3) in [1] (see also (4.6) in [2] and Lemma 3.5.4 in [6]):

$$\text{(2.7)}\quad \operatorname{Index}(D_X) = \int_X \operatorname{Ch}(E_{\phi,X},\nabla^X)\operatorname{Td}(TX,\nabla^X) - \frac{1}{2}(\eta_Y + \dim\ker D_Y),$$

where $\operatorname{Index}(D_X)$ is the index of D_X with a certain global boundary condition, which is an integer, $\operatorname{Ch}(E_{\phi,X},\nabla^X)$ is the Chern character form of $E_{\phi,X}$ with respect to ∇^X, $\operatorname{Td}(TX,\nabla^X)$ is the Todd form of TX with respect to ∇^X and η_Y is the eta invariant of D_Y. (For details of eta invariants, see [1], [3].) Here the same argument as was used to prove (2.3) shows that

$$\text{(2.8)}\qquad \int_X \operatorname{Ch}(E_{\phi,X},\nabla^X)\operatorname{Td}(TX,\nabla^X) = 0$$

because the order of ϕ is greater than $m+\ell$ and the dimension of X is $2m+2\ell$. Therefore it follows from (2.7) that

$$\frac{1}{2}\eta_Y = -\mathrm{Index}(D_X) - \frac{1}{2}\dim\ker D_Y \,. \tag{2.9}$$

Let O be the origin of $\mathbb{C}^\ell$. Then M is regarded as an almost complex submanifold of X by the identification of M with $M\times\{O\}$ and hence N is also regarded as an almost complex submanifold of X. Note that the fixed point set of the G-action on X is contained in M and coincides with the fixed point set of the G-action on M. Let $\nu(N,X)$ be the normal bundle of N in X. Then $\nu(N,X)$ is decomposed into the direct sum of complex subbundles

$$\nu(N,X) = \nu(N,M)\oplus\varepsilon^\ell_{\mathbb{C}} = \oplus_j\nu_j(N,M)\oplus\varepsilon^\ell_{\mathbb{C}}\,,$$

where ψ^k acts on $\nu_j(N,M)$ via multiplication by $e^{\sqrt{-1}\theta_j}$ and acts on the trivial complex line bundle $\varepsilon^\ell_{\mathbb{C}} = N\times\mathbb{C}^\ell$ by

$$\psi^k\cdot(q,(z_1,\dots,z_\ell)) = \begin{cases}(q,(\alpha^{\gamma k}z_1)) & (\ell=1)\,,\\ (q,(\alpha^{\gamma k}z_1,\overline{\alpha}^{\gamma k}z_2)) & (\ell=2)\end{cases}$$

($q\in N$, $(z_1,\dots,z_\ell)\in\mathbb{C}^\ell$). Hence the following equality is deduced from Theorem 1.2 in [3] (see also (4.6) in [2] and Lemma 3.5.4 in [6]):

$$\begin{aligned}&\mathrm{Index}(D_X,\psi^k)\\ &= \sum_{N\subset\Omega(k)}\mathrm{Ch}(E_\phi|_N,\psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M),\psi^k)C_\ell(k,\gamma)[N]\\ &\quad-\frac{1}{2}\{\eta_Y(\psi^k)+\mathrm{Tr}(\psi^k|_{\ker D_Y})\}\end{aligned} \tag{2.10}$$

for $1\leqq k\leqq p-1$, where $\mathrm{Index}(D_X,\psi^k)$ is the index of D_X with a certain global boundary condition evaluated at ψ^k, namely,

$$\mathrm{Index}(D_X,\psi^k) := \mathrm{Tr}(\psi^k|_{\ker D_X}) - \mathrm{Tr}(\psi^k|_{\mathrm{coker}\,D_X})\,,$$

$\eta_Y(\psi^k)$ is the eta invariant of D_Y evaluated at ψ^k and

$$C_1(k,\gamma) = \frac{1}{1-\alpha^{-\gamma k}}\,,\quad C_2(k,\gamma) = \frac{1}{1-\alpha^{-\gamma k}}\frac{1}{1-\overline{\alpha}^{-\gamma k}} = \frac{1}{|1-\alpha^{-\gamma k}|^2}\,.$$

Note that $\mathrm{Index}(D_X,\psi^p)$, $\eta_Y(\psi^p)$ coincide with $\mathrm{Index}(D_X)$, η_Y in (2.7) respectively.

Since the restriction of the G-action to Y is free and preserves the metric and the spin^c-structure of Y, the quotient space $M_S = Y/G$ is a smooth manifold with the metric and the spin^c-structure inherited from those of Y. The quotient space X/G also has the metric and the spin^c-structure inherited from those of X near $\partial(X/G) = M_S$, whose restriction to M_S coincides with those of M_S. Moreover the G-invariant metric connections ∇^Y, ∇^X of TY, TX define a metric connection ∇^S of TM_S, a unitary connection $\nabla^{X/G}$ of $T(X/G)$ near M_S respectively. We can show that M_S is the boundary of an almost complex manifold W as follows. Let ε^1 be the normal bundle of $S^{2\ell-1}$ in $\mathbb{C}^\ell$, which has a G-invariant trivialization, and ε^1_S the quotient bundle $(r_Y^*\varepsilon^1)/G$. Note that both of ε^1 and ε^1_S are trivial real line bundles. Since $TS^{2\ell-1}\oplus\varepsilon^1 = TD^{2\ell}|_{S^{2\ell-1}}$ has the standard complex structure, which is invariant under the action of G,

$$\begin{aligned} TM_S\oplus\varepsilon^1_S &\cong (q_Y^*TM\oplus r_Y^*TS^{2\ell-1}\oplus r_Y^*\varepsilon^1)/G \\ &\cong (q_Y^*TM\oplus r_Y^*(TD^{2\ell}|_{S^{2\ell-1}}))/G \end{aligned}$$

has a complex structure. Hence the $(2m+2\ell-1)$-dimensional compact manifold M_S is stably almost complex manifold and therefore it follows from the result of Morita [8] that there exists a compact $(2m+2\ell)$-dimensional almost complex manifold W such that $\partial W = M_S$ and $W = X/G$ near M_S as an almost complex manifold with Hermitian metric. The Hermitian metric of X/G near M_S is extended to a Hermitian metric on W. Let Q^W be the principal $U(m+\ell)$-bundle of unitary frames on W. Then the connection $\nabla^{X/G}$ extends to a unitary connection ∇^W in Q^W. On the other hand, we can see that $TW|_{M_S} = (TX/G)|_{M_S}$ is orthogonally decomposed into

$$TW|_{M_S}\cong(q_Y^*TM\oplus r_Y^*(TD^{2\ell}|_{S^{2\ell-1}}))/G\cong(TM)_S\oplus\varepsilon^\ell_{\mathbb{C}}\,, \tag{2.11}$$

where $(TM)_S$ is the vector bundle over M_S defined by $(TM)_S = (q_Y^*TM)/G$ and $\varepsilon^\ell_{\mathbb{C}}$ is the trivial complex line bundle of rank ℓ. Then the connection ∇^W splits according to (2.11) as

$$\nabla^W|_{TM_S} = \nabla^{X/G}|_{TM_S} = \nabla^{(TM)_S}\oplus\nabla^0, \tag{2.12}$$

where $\nabla^{(TM)_S}$ denotes the connection of $(TM)_S$ naturally defined by ∇ and ∇^0 denotes the globally flat connection of $\varepsilon^\ell_{\mathbb{C}}$. Now let V_ϕ^W be the element of the representation ring $R(U(m+\ell))$ defined by

$$\begin{aligned} V_\phi^W = \phi(\tau_1,\tau_2,\dots,\tau_m) &\in R(U(m+\ell)) \\ &\subset \mathbb{Z}[t_1,t_1^{-1},\dots,t_m,t_m^{-1},\dots,t_{m+\ell},t_{m+\ell}^{-1}]\,, \end{aligned}$$

where $\tau_j = \sigma_j(t_1-1,\dots,t_m-1,t_{m+1}-1,\dots,t_{m+\ell}-1)$ and set

$$E_\phi^W = Q^W\times_{U(m+\ell)}V_\phi^W\,.$$

Then the connection ∇^W naturally defines a unitary connection of E_ϕ^W and the E_ϕ^W-valued spin^c-Dirac operator D_W is defined.

On the other hand, the quotient bundle $E_{\phi,S} = E_{\phi,Y}/G$ is a virtual complex vector bundle over M_S with a unitary connection and the G-equivariant Dirac operator D_Y naturally defines a differential operator D_S, which is the $E_{\phi,S}$-valued spin^c-Dirac operator on M_S. Since $Q_S = (q_Y^*Q)/G$ is the unitary frame bundle associated to $(TM)_S$, it follows from (2.11) and (2.12) that

$Q^W|_{M_S}$ is reducible to Q_S with the connection. Since V_ϕ^W is isomorphic to V_ϕ as a virtual $U(m)$-representation, it follows that

$$\begin{aligned} E_\phi^W|_{M_S} &\cong (Q^W|_{M_S}) \times_{U(m+\ell)} V_\phi^W \cong Q_S \times_{U(m)} V_\phi^W \cong Q_S \times_{U(m)} V_\phi \\ &\cong q_Y^*(Q \times_{U(m)} V_\phi)/G = (q_Y^* E_\phi)/G = E_{\phi,Y}/G = E_{\phi,S}\,, \end{aligned}$$

where $\cong$ denotes the isomorphism as a virtual vector bundle with an inner product and a unitary connection. Hence, on the collar $M_S \times [0,\delta) \subset W$, D_W can be expressed as

$$D_W = \sigma\left(\frac{\partial}{\partial u} + D_S\right),$$

where u is the coordinate of $[0,\delta)$ and σ is a bundle isomorphism defined by the Clifford multiplication. Hence the following equality is deduced from (4.3) in [1] as well as in (2.7):

$$\text{(2.13)} \quad \operatorname{Index}(D_W) = \int_W \operatorname{Ch}(E_\phi^W, \nabla^W)\operatorname{Td}(TW, \nabla^W) - \frac{1}{2}(\eta_S + \dim\ker D_S),$$

where $\operatorname{Index}(D_W)$ is the index of D_W with a certain global boundary condition, $\operatorname{Ch}(E_\phi^W, \nabla^W)$ is the Chern character form of E_ϕ^W, $\operatorname{Td}(TW, \nabla^W)$ is the Todd form of TW and η_S is the eta invariant of D_S. Here since the spin^c-structure of M_S comes from the $U(m)$-structure of Y which is naturally defined by that of M, the spinor bundle $S_{M_S} = S_Y/G$ on M_S splits into $S_{M_S} = S_{M_S}^+ \oplus S_{M_S}^-$ and D_S splits into $D_S = D_S^+ \oplus D_S^-$, where

$$\text{(2.14)} \quad \begin{aligned} D_S^+ &: \Gamma(S_{M_S}^+ \otimes E_{\phi,S}) \longrightarrow \Gamma(S_{M_S}^- \otimes E_{\phi,S})\,, \\ D_S^- = (D_S^+)^* &: \Gamma(S_{M_S}^- \otimes E_{\phi,S}) \longrightarrow \Gamma(S_{M_S}^+ \otimes E_{\phi,S})\,. \end{aligned}$$

Hence we have

$$\dim\ker D_S = \dim\ker D_S^+ + \dim\ker D_S^-\,.$$

On the other hand, since the dimension of Y is odd, it follows that

$$\text{(2.15)} \quad \operatorname{Index}(D_S^+) = \dim\ker D_S^+ - \dim\ker(D_S^+)^* = 0$$

(see Proposition 9.2 in [2]). Therefore we have

$$\dim\ker D_S^- = \dim\ker(D_S^+)^* = \dim\ker D_S^+$$

and hence it follows that

$$\frac{1}{2}\dim\ker D_S = \dim\ker D_S^+ \equiv 0 \pmod{\mathbb{Z}}\,.$$

Moreover it follows from (3.6) in [3] that

$$\frac{1}{2}\eta_S = \frac{1}{p}\sum_{k=1}^{p}\frac{1}{2}\eta_Y(\psi^k)\,.$$

Hence it follows from (2.13) that

(2.16)
$$\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{2}\,\eta_Y(\psi^k)+\frac{1}{p}\frac{1}{2}\,\eta_Y\equiv\int_W \mathrm{Ch}(E_\phi^W,\nabla^W)\,\mathrm{Td}(TW,\nabla^W)\qquad(\mathrm{mod}\ \mathbb{Z})\,.$$

Here it follows from (2.9) and (2.10) that

$$\begin{aligned}(2.17)\quad &\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{2}(\eta_Y(\psi^k))+\frac{1}{p}\frac{1}{2}\,\eta_Y\\ &=\frac{1}{p}\sum_{k=1}^{p-1}C_\ell(k,\gamma)\sum_{N\subset\Omega(k)}\mathrm{Ch}(E_\phi|_N,\psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M),\psi^k)[N]\\ &\quad-\frac{1}{p}\sum_{k=1}^{p}\mathrm{Index}(D_X,\psi^k)-\frac{1}{p}\sum_{k=1}^{p}\frac{1}{2}\mathrm{Tr}(\psi^k|_{\ker D_Y})\,.\end{aligned}$$

Here since the spin^c-structure of Y comes from the $U(m)$-structure of M, the spinor bundle S_Y splits into $S_Y=S_Y^+\oplus S_Y^-$ and D_Y splits into $D_Y=D_Y^+\oplus D_Y^-$ where

$$\begin{aligned}D_Y^+&:\Gamma(S_Y^+\otimes E_{\phi,Y})\longrightarrow\Gamma(S_Y^-\otimes E_{\phi,Y})\,,\\ D_Y^-=(D_Y^+)^*&:\Gamma(S_Y^-\otimes E_{\phi,Y})\longrightarrow\Gamma(S_Y^+\otimes E_{\phi,Y})\end{aligned}$$

as in (2.14). Here since ψ^k $(1\le k\le p-1)$ acts freely on Y, it follows from the fixed point formula in [2] that

$$\mathrm{Index}(D_Y^+,\psi^k):=\mathrm{Tr}(\psi^k|_{\ker D_Y^+})-\mathrm{Tr}(\psi^k|_{\ker(D_Y^+)^*})=0$$

for any $1\leqq k\leqq p-1$. Moreover, since the dimension of Y is odd, it follows as in (2.15) that

$$\mathrm{Index}(D_Y^+)=\mathrm{Tr}(\psi^p|_{\ker D_Y^+})-\mathrm{Tr}(\psi^p|_{\ker(D_Y^+)^*})=0$$

and hence that

$$\begin{aligned}\sum_{k=1}^{p}\frac{1}{2}\mathrm{Tr}(\psi^k|_{\ker D_Y})&=\sum_{k=1}^{p}\frac{1}{2}\{\mathrm{Tr}(\psi^k|_{\ker D_Y^+})+\mathrm{Tr}(\psi^k|_{\ker D_Y^-})\}\\ &=\sum_{k=1}^{p}\frac{1}{2}\{\mathrm{Tr}(\psi^k|_{\ker D_Y^+})+\mathrm{Tr}(\psi^k|_{\ker(D_Y^+)^*})\}=\sum_{k=1}^{p}\mathrm{Tr}(\psi^k|_{\ker D_Y^+})\,.\end{aligned}$$

Therefore it follows from (2.5) that

$$\begin{aligned}(2.18)\quad &\sum_{k=1}^{p}\mathrm{Index}(D_X,\psi^k)+\sum_{k=1}^{p}\frac{1}{2}\mathrm{Tr}(\psi^k|_{\ker D_Y})\\ &=\sum_{k=1}^{p}\mathrm{Index}(D_X,\psi^k)+\sum_{k=1}^{p}\mathrm{Tr}(\psi^k|_{\ker D_Y^+})\equiv 0\qquad(\mathrm{mod}\ p)\,.\end{aligned}$$

Hence it follows from (2.16), (2.17) and (2.18) that

$$\begin{aligned}(2.19)\quad & \frac{1}{p}\sum_{k=1}^{p-1} C_\ell(k,\gamma) \sum_{N\subset\Omega(k)} \mathrm{Ch}(E_\phi|_N,\psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M),\psi^k)[N] \\ & \equiv \int_W \mathrm{Ch}(E_\phi^W,\nabla^W)\,\mathrm{Td}(TW,\nabla^W) \qquad (\mathrm{mod}\ \mathbb{Z})\,.\end{aligned}$$

Here the same argument as was used to prove (2.3) shows that

$$(2.20)\qquad \int_W \mathrm{Ch}(E_\phi^W,\nabla^W)\,\mathrm{Td}(TW,\nabla^W) = 0$$

because the order of ϕ is greater than $m+\ell$ and the dimension of W is $2m+2\ell$. Now the equality in Theorem 1.3 is deduced from (2.19) and (2.20). This completes the proof of Theorem 1.3.

3. Examples

In this section, applying Theorem 1.3, we give certain fixed point formulae for the standard torus T^2, the sphere S^6 and the complex projective space $\mathbf{C}\mathbb{P}^m$, which can be verified by direct computation.

Example 3.1. Let T^2 be the standard torus defined by $T^2 = \mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z})$. Let ψ be the automorphism of T^2 defined by the $\pi/2$-rotation with center at $(1+i)/2$. Then the order of ψ is 4 and the fixed point set $\Omega(k)$ of ψ^k is as follows:

$$\begin{aligned}\Omega(1)=\Omega(3) &= \left\{A=\frac{1+i}{2},\ B=1+i\right\}\ , \\ \Omega(2) &= \left\{A=\frac{1+i}{2},\ B=1+i,\ C=\frac{1}{2}+i,\ D=1+\frac{i}{2}\right\}\,.\end{aligned}$$

Set $\ell=2$, $\gamma=3$ and $\phi = x_1^n = \tau_1^n \in S(n)$. Since ψ^k acts on T_AT^2, T_BT^2 via multiplication by i^k for $1 \leqq k \leqq 3$ and ψ^2 acts on T_CT^2, T_DT^2 via multiplication by -1, it follows from Corollary 1.4 that the equality

$$\begin{aligned}(3.1)\quad & \frac{1}{|1-i^{-3}|^2}\left(2(i-1)^n\frac{1}{1-i^{-1}}\right) \\ & +\frac{1}{|1-i^{-6}|^2}\left(2(i^2-1)^n\frac{1}{1-i^{-2}} + 2(-1-1)^n\frac{1}{1-(-1)^{-1}}\right) \\ & +\frac{1}{|1-i^{-9}|^2}\left(2(i^3-1)^n\frac{1}{1-i^{-3}}\right) \equiv 0 \qquad (\mathrm{mod}\ 4)\end{aligned}$$

holds for any $n > m+\ell = 3$. The equality above can be easily verified as follows:

$$\begin{aligned}\text{the left-hand side of (3.1)} &= i(i-1)^{n-1} + (-2)^n + \overline{(i(i-1)^{n-1})} \\ &= 2\mathrm{Re}\left(i(i-1)^{n-1}\right) + (-2)^n \equiv 0 \qquad (\mathrm{mod}\ 4),\end{aligned}$$

where Re denotes the real part because we can show that both of the real part and the imaginary part of $i(i-1)^{n-1}$ are even for $n \geqq 3$ by induction.

Example 3.2. Let $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ be the set of octonions with multiplication defined by the rule

$$x \cdot x' = (q_1, q_2) \cdot (q_1', q_2') \equiv (q_1 q_1' - \overline{q_2'} q_2 \,,\, q_2' q_1 + q_2 \overline{q_1'})$$

for any x, $x' \in \mathbb{O}$ (see [7]). The conjugation $\overline{x}$ and the real part $\mathrm{Re}(x)$ of $x = (q_1, q_2) \in \mathbb{O}$ are defined by $\overline{x} = (\overline{q_1}, -q_2)$ and $\mathrm{Re}(x) = \mathrm{Re}(q_1)$ respectively. Moreover the standard Euclidean inner product $\langle x, x' \rangle$ and its norm $|x|$ are defined for x, $x' \in \mathbb{O}$ by

$$\langle x, x' \rangle = \mathrm{Re}(x \cdot \overline{x'}) = \mathrm{Re}(\overline{x} \cdot x') \,, \quad |x| = \sqrt{\langle x, x \rangle} = \sqrt{x \cdot \overline{x}} = \sqrt{\overline{x} \cdot x}$$

respectively. The map

$$\mathbb{O} \ni ((z_1, z_2), (z_3, z_4)) \longrightarrow (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$$

gives an isomorphism as a complex vector space. We denote $((z_1, z_2), (z_3, z_4))$ by (z_1, z_2, z_3, z_4) hereafter. Let $\mathrm{Im}(\mathbb{O})$ be the set of pure imaginary octonions, namely,

$$\mathrm{Im}(\mathbb{O}) = \{x \in \mathbb{O} \,|\, \overline{x} = -x\} \,,$$

which is isomorphic to $\mathbb{R}^7$ as a real vector space and S^6 the standard 6-dimensional sphere defined by

$$S^6 = \{A = (z_1, z_2, z_3, z_4) \in \mathrm{Im}(\mathbb{O}) \,|\, |A| = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1\} \,.$$

Then, for any point $A \in S^6$, the tangent space $T_A S^6$ is given by

$$T_A S^6 = \{B \in \mathrm{Im}(\mathbb{O}) \,|\, \langle A, B \rangle = 0\} \,.$$

For any $A \in S^6$, $B \in T_A S^6$, set $J_A(B) = A \cdot B$. Then since the equality $\overline{x} \cdot (x \cdot y) = (\overline{x} \cdot x) \cdot y = |x|^2 y$ holds for any x, $y \in \mathbb{O}$, we have

$$J_A(J_A(B)) = A \cdot (A \cdot B) = -\overline{A} \cdot (A \cdot B) = -(\overline{A} \cdot A) \cdot B = -|A|^2 B = -B \,,$$

which implies that $J_A(B) \in T_A S^6$ and $J_A^2 = -1$ because $\overline{A} \cdot (A \cdot B) = B$ implies that $\langle A, A \cdot B \rangle = \mathrm{Re}(\overline{A} \cdot (A \cdot B)) = \mathrm{Re}(B) = 0$. Hence this J defines an almost complex structure of S^6. Let p be any natural number, $\alpha = e^{2\pi\sqrt{-1}/p}$ the primitive p-th root of 1 and ψ the periodic $\mathbb{C}$-linear map of $\mathbb{O}$ of order p defined by

$$\mathbb{O} \ni (q_1, q_2) = (z_1, z_2, z_3, z_4) \longrightarrow (q_1, \alpha q_2) = (z_1, z_2, \alpha z_3, \alpha z_4) \,.$$

Then ψ maps S^6 to S^6 and we have

$$\begin{aligned} \psi(x) \cdot \psi(y) &= (q_1, \alpha q_2) \cdot (q_1', \alpha q_2') = (q_1 q_1' - \overline{\alpha q_2'} \alpha q_2 \,,\, \alpha q_2' q_1 + \alpha q_2 \overline{q_1'}) \\ &= (q_1 q_1' - \overline{q_2'} \overline{\alpha} \alpha q_2 \,,\, \alpha q_2' q_1 + \alpha q_2 \overline{q_1'}) = (q_1 q_1' - \overline{q_2'} q_2 \,,\, \alpha(q_2' q_1 + q_2 \overline{q_1'})) \\ &= \psi(x \cdot y) \end{aligned}$$

for any $x = (q_1, q_2)$, $y = (q_1', q_2') \in \mathbb{O}$. Hence it follows that

$$J_{\psi(A)}(\psi_*(B)) = J_{\psi(A)}(\psi(B)) = \psi(A) \cdot \psi(B) = \psi(A \cdot B) = \psi_*(J_A(B))$$

for any $A \in S^6$, $B \in T_A S^6$, which implies that ψ commutes with J. Hence ψ defines an automorphism of the almost complex manifold S^6. The fixed point set of ψ^k is independent of k and coincides with the standard 2-dimensional sphere

$$S^2 = \{(z_1, z_2, z_3, z_4) \in S^6 \mid z_3 = z_4 = 0\}$$

for any $1 \leqq k \leqq p-1$. The normal bundle $\nu(S^2, S^6)$ is the trivial complex vector bundle of rank 2 and ψ^k acts on $\nu(S^2, S^6)$ via multiplications by α^k.

Set $\ell = 0$ and $\phi = (x_1 x_2 x_3)^n = \tau_3^n \in S(3n)$. Since $TS^6|_{S^2}$ splits into the direct sum $TS^2 \oplus \nu(S^2, S^6)$ and ψ^k acts on TS^2 via multiplication by 1, it follows from (1.3) that

$$\mathrm{Ch}(E_\phi|_{S^2}, \psi^k) = \{(e^{c_1(TS^2)} - 1)(\alpha^k - 1)^2\}^n = \{(e^{2x} - 1)(\alpha^k - 1)^2\}^n,$$

where x denotes the positive generator of $H^2(S^2) \cong \mathbb{Z}$. Hence it follows from Theorem 1.3 that the equality

$$\begin{aligned} (3.2)\quad & \sum_{k=1}^{p-1} \{(e^{2x} - 1)(\alpha^k - 1)^2\}^n \frac{x}{1 - e^{-x}} \left(\frac{1}{1 - \alpha^{-k}}\right)^2 [S^2] \\ &= 2^n \sum_{k=1}^{p-1} (\alpha^k - 1)^{2n} \left(\frac{1}{1 - \alpha^{-k}}\right)^2 (x^n + \text{higher order terms})[S^2] \\ &\equiv 0 \qquad (\mathrm{mod}\ p) \end{aligned}$$

holds for any n such that $3n > m + \ell = 3$. The equality above can also be easily verified because $3n > 3$ implies that $n \geqq 2$ and hence that

$$(x^n + \text{higher order terms})[S^n] = 0\,.$$

Remark 3.3. If $3n = m + \ell = 3 \Longleftrightarrow n = 1$, it follows from (2.5) and (3.2) that

$$\begin{aligned} & \sum_{k=1}^{p-1} (e^{2x} - 1)(\alpha^k - 1)^2 \frac{x}{1 - e^{-x}} \left(\frac{1}{1 - \alpha^{-k}}\right)^2 [S^2] \\ &= 2 \sum_{k=1}^{p-1} \alpha^{2k} = 2\left(\sum_{k=1}^{p} \alpha^{2k} - 1\right) \equiv -2 \neq 0 \qquad (\mathrm{mod}\ p) \end{aligned}$$

if $p \neq 2$.

Example 3.4. Let M be the m-dimensional complex projective space $\mathbf{C}\mathbb{P}^m$, p any natural number, $\alpha = e^{2\pi\sqrt{-1}/p}$ the primitive p-th root of 1 and ψ the periodic automorphism of $\mathbf{C}\mathbb{P}^m$ of order p defined by

$$\mathbf{C}\mathbb{P}^m \ni [z_0 : z_1 : \cdots : z_m] \longrightarrow [\alpha z_0 : z_1 : \cdots : z_m]\,.$$

Then the fixed point set of ψ^k is independent of k and coincides with the disjoint union of the point $q = [1 : 0 : \cdots : 0]$ and the hyperplane $\mathbf{C}\mathbb{P}^{m-1}$ defined by $z_0 = 0$. Set $\ell = 1$, $\gamma = 1$ and $\phi = (x_1 + x_2 + \cdots + x_m)^n = \tau_1^n \in S(n)$. Then it follows that $\phi = (t_1 + t_2 + \cdots + t_m - m)^n$ and hence that

$$E_\phi = \otimes^n (T\mathbf{C}\mathbb{P}^m - \varepsilon_{\mathbb{C}}^m),$$

where ψ^k acts on the trivial bundle $\varepsilon_{\mathbb{C}}^m$ via multiplication by 1. Here ψ^k acts on $\nu(q, \mathbf{C}\mathbb{P}^m) \cong \mathbb{C}^m$ via multiplication by α^{-k} and hence we have

$$\mathrm{Ch}(T\mathbf{C}\mathbb{P}^m|_q, \psi^k) = m\alpha^{-k}\ , \quad \mathrm{Td}(Tq) = 1\ , \quad \mathfrak{U}(\nu(q, M), \psi^k) = \left(\frac{1}{1-\alpha^k}\right)^m .$$

On the other hand, the normal bundle $\nu(\mathbf{C}\mathbb{P}^{m-1}, \mathbf{C}\mathbb{P}^m)$ is isomorphic to the restriction of the hyperplane bundle L to $\mathbf{C}\mathbb{P}^{m-1}$ and ψ^k acts on $\nu(\mathbf{C}\mathbb{P}^{m-1}, \mathbf{C}\mathbb{P}^m)$ $\cong L|_{\mathbf{C}\mathbb{P}^{m-1}}$ via multiplication by α^k. Let x be the positive generator of $H^2(\mathbf{C}\mathbb{P}^{m-1}) \cong \mathbb{Z}$ which equals to the first Chern class $c_1(L|_{\mathbf{C}\mathbb{P}^{m-1}})$. Then since

$$T\mathbf{C}\mathbb{P}^m|_{\mathbf{C}\mathbb{P}^{m-1}} = T\mathbf{C}\mathbb{P}^{m-1} \oplus (L|_{\mathbf{C}\mathbb{P}^{m-1}}),$$

where ψ^k acts on $T\mathbf{C}\mathbb{P}^{m-1}$ via multiplication by 1, it follows that

$$\mathrm{Ch}(T\mathbf{C}\mathbb{P}^m|_{\mathbf{C}\mathbb{P}^{m-1}}, \psi^k) = me^x - 1 + \alpha^k e^x\ , \quad \mathrm{Td}(T\mathbf{C}\mathbb{P}^{m-1}) = \left(\frac{x}{1-e^{-x}}\right)^m ,$$
$$\mathfrak{U}(\nu(\mathbf{C}\mathbb{P}^{m-1}, \mathbf{C}\mathbb{P}^m), \psi^k) = \frac{1}{1-\alpha^{-k}e^{-x}} .$$

Hence it follows from Theorem 1.3 that the equality

(3.3)
$$\sum_{k=1}^{p-1} \frac{1}{1-\alpha^{-k}} \left(m\alpha^{-k} - m\right)^n \left(\frac{1}{1-\alpha^k}\right)^m + \varphi(x)[\mathbf{C}\mathbb{P}^{m-1}] \equiv 0 \qquad (\mathrm{mod}\ p)$$

holds for any $n > m + \ell = m + 1$, where

$$\varphi(x) = \sum_{k=1}^{p-1} \frac{1}{1-\alpha^{-k}} (me^x - 1 + \alpha^k e^x - m)^n \left(\frac{x}{1-e^{-x}}\right)^m \frac{1}{1-\alpha^{-k}e^{-x}} .$$

We can verify (3.3) as follows.

(3.4)

$\varphi(x)[\mathbf{C}\mathbb{P}^{m-1}] = x^{m-1}$-coefficient of $\varphi(x)$

$= x^{-1}$-coefficient of

$$\frac{\varphi(x)}{x^m} = \sum_{k=1}^{p-1} \frac{(me^x - 1 + \alpha^k e^x - m)^n}{(1-\alpha^{-k})(1-e^{-x})^m (1-\alpha^{-k}e^{-x})}$$

$$= \sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1}((m+\alpha^k)e^x - 1 - m)^n \frac{(e^x)^m}{(e^x-1)^m} \frac{1}{\alpha^k e^x - 1} e^x$$

$$= \frac{1}{2\pi i} \oint_{C(x)} \sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1}((m+\alpha^k)e^x - 1 - m)^n \frac{(e^x)^m}{(e^x-1)^m} \frac{1}{\alpha^k e^x - 1} e^x \, dx$$

($C(x)$ is a sufficiently small counterclockwise simple loop around $0 \in \mathbb{C}$)

$$= \frac{1}{2\pi i} \oint_{C(y)} \sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1}((m+\alpha^k)(y+1) - 1 - m)^n \frac{(y+1)^m}{y^m} \frac{1}{\alpha^k (y+1) - 1} \, dy$$

($y = e^x - 1\,,\ C(y)$ is a counterclockwise simple loop around $0 \in \mathbb{C}$)

$= y^{-1}$-coefficient of

$$\sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1}((m+\alpha^k)(y+1) - 1 - m)^n \frac{(y+1)^m}{y^m} \frac{1}{\alpha^k (y+1) - 1}$$

$= y^{m-1}$-coefficient of

$$\sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1}((m+\alpha^k)y + \alpha^k - 1)^n (y+1)^m \frac{1}{\alpha^k - 1 + \alpha^k y}$$

$= y^{m-1}$-coefficient of

$$\sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{(\alpha^k - 1)^2} \sum_{i=0}^{n} \binom{n}{i} (m+\alpha^k)^i y^i (\alpha^k - 1)^{n-i} \sum_{j=0}^{m} \binom{m}{j} y^j \sum_{s=0}^{\infty} \left(\frac{-\alpha^k y}{\alpha^k - 1} \right)^s$$

$= y^{m-1}$-coefficient of

$$\sum_{k=1}^{p-1} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{s=0}^{\infty} \binom{n}{i} \binom{m}{j} (-1)^s (\alpha^k)^{s+2} (m+\alpha^k)^i (\alpha^k - 1)^{n-i-s-2} y^{i+j+s}$$

$$= \sum_{k=1}^{p-1} \sum_{i=0}^{m-1} \sum_{s=0}^{m-1-i} \binom{n}{i} \binom{m}{m-1-i-s} (-1)^s (\alpha^k)^{s+2} (m+\alpha^k)^i (\alpha^k - 1)^{n-i-s-2}$$

$$= \sum_{k=1}^{p-1} (\alpha^k - 1) R(\alpha^k),$$

where $R(z)$ is an integral polynomial defined by

$$R(z) = \sum_{i=0}^{m-1} \sum_{s=0}^{m-1-i} \binom{n}{i} \binom{m}{m-1-i-s} (-1)^s z^{s+2} (m+z)^i (z-1)^{n-i-s-3}.$$

(Note that $n > m+1$ and $i+s \leqq m-1$ imply that $n-i-s-3 \geqq 0$.)

Now since $(\alpha^{\pm\nu})^p = 1$ for any nonnegative integer ν, it follows that

$$\sum_{k=1}^{p-1}(\alpha^{\pm k})^\nu = \sum_{k=1}^{p-1}(\alpha^{\pm\nu})^k = \sum_{k=1}^{p}(\alpha^{\pm\nu})^k - 1 \equiv -1 \qquad (\mathrm{mod}\ p)$$

(see (2.5)). Hence, for any integral polynomial $Q(z)$, we can see that

$$\sum_{k=1}^{p-1} Q(\alpha^k) \equiv \sum_{k=1}^{p-1} Q(\alpha^{-k}) \equiv -Q(1) \qquad (\mathrm{mod}\ p) \tag{3.5}$$

and therefore it follows from (3.4) that

$$\varphi(x)[\mathbf{C}\mathbb{P}^{m-1}] = \sum_{k=1}^{p-1}(\alpha^k - 1)R(\alpha^k) \equiv -(1-1)R(1) = 0 \qquad (\mathrm{mod}\ p)\,.$$

On the other hand, it follows from (3.5) that

$$\begin{aligned}&\sum_{k=1}^{p-1} \frac{1}{1-\alpha^{-k}} \left(m\alpha^{-k} - m\right)^n \left(\frac{1}{1-\alpha^k}\right)^m \\ &= \sum_{k=1}^{p-1}(-m^n)(\alpha^{-k})^m(\alpha^{-k}-1)^{n-m-1} \equiv m^n \cdot 1^m \cdot (1-1)^{n-m-1} = 0 \qquad (\mathrm{mod}\ p)\end{aligned}$$

because $n-m-1 > 0$. Hence the equality (3.7) is verified.

4. Relation to the Einstein-Kähler metrics

If the Ricci form $\rho(\omega)$ of the Kähler form ω on a Kähler manifold M is a constant multiple of ω, M is called an Einstein-Kähler manifold and the metric corresponding to ω is called an Einstein-Kähler metric. In this section, we refine the result of Theorem 1.3 for $\ell = 1$, $\gamma = 1$ and $\phi = \tau_1^{m+1}$ in the case that M is an Einstein-Kähler manifold.

Let M be an m-dimensional complex manifold and $A(M)$ the complex Lie group consisting of all biholomorphic automorphisms of M. Assume that the periodic element $\psi \in A(M)$ of order p is contained in the identity component of $A(M)$ and hence is expressed as $\psi = \exp v$ by a holomorphic vector field v on M. Then using the result of Futaki in [4] and the result in [9], we can prove the next theorem.

Theorem 4.1. *If M admits an Einstein-Kähler metric, then under the notation in Theorem* 1.3 *the equality*

$$\sum_{k=1}^{p-1} C_1(k,1) \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\,\mathrm{Td}(TN)\,\mathfrak{U}(\nu(N,M), \psi^k)[N] \equiv 0 \qquad (\mathrm{mod}\ p) \tag{4.1}$$

holds for $\phi = (x_1 + x_2 + \cdots + x_m)^{m+1} = \tau_1^{m+1} \in S(m+1)$.

Proof. Let G be the cyclic subgroup of $A(M)$ generated by ψ. Set $\ell = 1$, $\gamma = 1$ and $\phi = (x_1 + x_2 + \cdots + x_m)^{m+1} = \tau_1^{m+1} \in S(m+1)$. Then G acts freely on $Y = M \times S^1$ by

$$\psi \cdot (q\,,\, z) = (\psi \cdot q\,,\, z\alpha) \quad (q \in M,\, z \in \mathbb{C})\,.$$

Let M_S be the quotient space Y/G and W the $(2m+2)$-dimensional almost complex manifold whose boundary is M_S as in Section 2. Then it follows from Theorem 1.6 and Lemma 2.1 in [9] that the equality

$$f(v) \equiv \int_W c_1(TW, \nabla^W)^{m+1} \pmod{\mathbb{Z}} \tag{4.2}$$

holds, where $f(v)$ is the Futaki invariant of v (see [4]). Since

$$\mathrm{Ch}(E_\phi^W, \nabla^W) = c_1(TW, \nabla^W)^{m+1} + \text{higher order terms}$$

(see (2.2)) it follows from (4.2) that

$$f(v) \equiv \int_W \mathrm{Ch}(E_\phi^W, \nabla^W)\,\mathrm{Td}(E_\phi^W, \nabla^W) \pmod{\mathbb{Z}}\,.$$

Therefore it follows from (2.19) and the equality above that

$$f(v) \equiv \frac{1}{p}\sum_{k=1}^{p-1} C_1(k,1) \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M), \psi^k)[N] \pmod{\mathbb{Z}}\,. \tag{4.3}$$

On the other hand, Futaki proved in [4] (see also [5]) that $f(v) = 0$ for any holomorphic vector field v if M admits an Einstein-Kähler metric. Hence it follows that

$$\frac{1}{p}\sum_{k=1}^{p-1} C_1(k,1) \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M), \psi^k)[N] \equiv 0 \pmod{\mathbb{Z}}$$

if M admits an Einstein-Kähler metric. This completes the proof of Theorem 4.1. □

Remark 4.2. Note that it follows from Theorem 1.3 that the equality (4.1) holds for any almost complex manifold M if $\phi \in S(n)$ and $n > m + \ell = m+1$. On the other hand, the equality in Theorem 1.3 does not hold in general if $n = m+\ell$. For example, let M be the blowing-up of $\mathbf{C}\mathbb{P}^2$ at one point. Then as was seen in [9] (see Theorem 1.6 and p. 215 in [9]) there exists a periodic biholomorphic automorphism $\psi = \exp v \in A(M)$ such that $f(v)$ is not an integer. Hence it follows from (4.3) that

$$\sum_{k=1}^{p-1} C_1(k,1) \sum_{N \subset \Omega(k)} \mathrm{Ch}(E_\phi|_N, \psi^k)\,\mathrm{Td}(TN)\mathfrak{U}(\nu(N,M), \psi^k)[N] \not\equiv 0 \pmod{p},$$

where $\phi = (x_1 + x_2 + \cdots + x_m)^{m+1} = \tau_1^{m+1} \in S(m+1) = S(m+\ell)$.

Example 4.3. Let $M = \mathbf{C}\mathbb{P}^m$ and ψ the periodic automorphism defined in Example 3.4. Then the equality (4.1) holds for any periodic $\psi \in A(\mathbf{C}\mathbb{P}^m)$ because $A(\mathbf{C}\mathbb{P}^m)$ is connected and $\mathbf{C}\mathbb{P}^m$ admits an Einstein-Kähler metric. In fact it follows as in (3.3) that the equality

$$\begin{aligned}&\text{the left-hand side of (4.1)}\\&=\sum_{k=1}^{p-1}\frac{1}{1-\alpha^{-k}}\left(m\alpha^{-k}-m\right)^{m+1}\left(\frac{1}{1-\alpha^{k}}\right)^{m}+\varphi(x)[\mathbf{C}\mathbb{P}^{m-1}]\equiv 0 \qquad (\mathrm{mod}\ p)\end{aligned}$$

holds, where

$$\varphi(x)=\sum_{k=1}^{p-1}\frac{1}{1-\alpha^{-k}}(me^{x}-1+\alpha^{k}e^{x}-m)^{m+1}\left(\frac{x}{1-e^{-x}}\right)^{m}\frac{1}{1-\alpha^{-k}e^{-x}}\,.$$

Hence it follows from the same argument as in Example 3.4 and (3.5) that

$$\begin{aligned}&\text{the left-hand side of (4.1)}\\&\equiv m^{m+1}\cdot 1^{m}\cdot(1-1)^{m+1-m-1}\\&\quad-\sum_{i=0}^{m-1}\sum_{s=0}^{m-1-i}\binom{m+1}{i}\binom{m}{m-1-i-s}\\&\qquad\times(-1)^{s}1^{s+2}(m+1)^{i}(1-1)^{m+1-i-s-2}\quad(\mathrm{mod}\ p)\\&=m^{m+1}-\sum_{i=0}^{m-1}\binom{m+1}{i}\binom{m}{0}(-1)^{m-1-i}(m+1)^{i}(1-1)^{0}\\&=m^{m+1}\\&\quad-\sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^{m+1-i}(m+1)^{i}+(m+1)^{m+1}-\binom{m+1}{m}(m+1)^{m}\\&=m^{m+1}-\sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^{m+1-i}(m+1)^{i}=m^{m+1}-\{(m+1)-1\}^{m+1}\\&=m^{m+1}-m^{m+1}=0\,.\end{aligned}$$

Thus the equality (4.1) holds for $M = \mathbf{C}\mathbb{P}^m$ and

$$\psi:\mathbf{C}\mathbb{P}^{m}\ni[z_0:z_1:\cdots:z_m]\longrightarrow[\alpha z_0:z_1:\cdots:z_m]\,.$$

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