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Weak approximation, Brauer and R-equivalence in algebraic groups over arithmetical fields, II

By

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Abstract

In this paper we prove that certain natural birational and arithmetic invariants of connected subgroups of linear algebraic groups all defined over a local or global field of characteristic 0 are bounded in terms of the ambient group and the base field.

Introduction

This paper is a continuation of [T1], [T2], where we set out to study some aspects of weak approximation in relation with Brauer and R-equivalence relation on the subgroups of rational points of linear algebraic groups over arithmetical fields. Here we investigate some quantitative problems naturally arised in the course of study. Let k be a field of characteristic 0 and G a connected linear algebraic group defined over k. There are attached to G some important birational and arithmetic invariants of G: The obstruction to weak approximation $\mathrm{A}(G)$, the group of Brauer equivalence (resp. R-equivalence) classes $G(k)/Br$ (resp. $G(k)/R$) of $G(k)$. One may also define the Tate-Shafarevich group $\mathrm{III}(G)$ of G with respect to the set of all valuations of k, the first Galois cohomology $\mathrm{H}^1(k, \mathrm{Pic}\bar{V}(G))$, where $V(G)$ denotes a k-smooth compactification of G, $\bar{V}(G) = V(G) \times \bar{k}$, and $\mathrm{Pic}(\cdot)$ denotes the Picard group. If k is a number field all these sets are known to be finite abelian groups, and it is natural to inquire about their cardinality. The bigger it is, the worse arithmetic (geometric) property the group G possesses. Some of the above objects may also defined over a local field, and the same question may arise.

In this note we are interested in the following question:

Given a connected linear algebraic group G, and let $P(G)$ be one of the above objects. Is there any bound $C(k, G)$ depending only on k, G, such that $Card(P(H)) < C(k, G)$ for all connected k-subgroups $H \subset G$?

In this note we aim to answer this question for the above mentioned objects.

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Notation and convention. k always denotes either local or global field of characteristic 0, unless otherwise stated. $\mathrm{H}^i(k,\cdot)$ denotes Galois cohomology of $(\cdot)$. If k is a global field, $\text{Ш}(G)$ denotes the Tate-Shafarevich group of G, $\mathrm{A}(G)$ denotes the obstruction to weak approximation $\mathrm{A}(G) = \prod_v G(k_v)/Cl(G(k))$, where $G(k)$ is embedded diagonally into the direct product $\prod_v G(k_v)$ and Cl means taking the closure there.

1. Some invariants over local fields

In this section we are interested in the boundness of Galois cohomology of connected k_v-subgroups of a connected group G defined over a local field k_v. We have

Lemma 1.1. *Let T be a k_v-subtorus of G. Then $Card(\mathrm{H}^1(k_v,T))$ is bounded by a constant depending only on G and k_v.*

Proof. Let n be the smallest natural number such that there exists a k_v-embedding $G \hookrightarrow \mathrm{GL}_n$. It is clear that n depends on G and k_v. Let T' be a maximal k_v-torus of GL_n containing T. Then it is well-known that T' is an induced k_v-torus, so it has a minimal Galois splitting field L/k_v such that $m := Card(Gal(L/k_v))$ is bounded by some function $f(n)$ of n. Therefore T is also split by L. Next we use a Serre's argument in [Se]. Let ϕ be the homomorphism $T \to T$ defined by the rule $\phi : x \mapsto x^m$, $T_m := \mathrm{Ker}(\phi)$. Then we have the following exact sequence of k_v-groups

$$1 \to T_m \to T \to T \to 1,$$

and the exact sequence of Galois cohomology deduced from this:

$$\mathrm{H}^1(k_v,T_m) \xrightarrow{\alpha} \mathrm{H}^1(k_v,T) \xrightarrow{\beta} \mathrm{H}^1(k_v,T).$$

Since T is split over L, it follows that $\mathrm{H}^1(L,T) = 0$. Also the map β is just the multiplication with m, and we have the following exact sequence

$$\mathrm{H}^1(L/k_v,T_m) \xrightarrow{\alpha} \mathrm{H}^1(L/k_v,T) \xrightarrow{\beta} \mathrm{H}^1(L/k_v,T).$$

Now $m\mathrm{H}^1(L/k_v,T) = 0$ so α is surjective, so

$$Card(\mathrm{H}^1(k_v,T)) \le Card(\mathrm{H}^1(k_v,T_m)).$$

Since T_m is a finite multiplicative group of order $\le mn \le nf(n)$, it is clear that $Card(\mathrm{H}^1(k_v,T_m))$ is bounded by some constant depending only on n and k_v, the same is true for $Card(\mathrm{H}^1(k_v,T))$. □

Lemma 1.2. *For any connected k_v-subgroup H of G, $Card(\mathrm{H}^1(k_v,H))$ is bounded by a constant depending only on G and k_v.*

Proof. Since Galois cohomology of unipotent groups is trivial, we may assume that H is a connected reductive k_v-subgroup of G. We fix an embedding $G \hookrightarrow \mathrm{GL}_n$. Here n depends on G. It suffices therefore to prove that $Card(\mathrm{H}^1(k_v, H))$ is bounded by a constant depending only on G, n, k_v. Let $H = TH'$, where T is the connected center of H and $H' = [H, H]$ is the derived subgroup of H. Let $F = T \cap H'$ a finite central subgroup of H'. We embed F diagonally into $T \times H'$ and denote the resulting subgroup again by F. Thus we have the following exact sequence of k_v-groups

$$1 \to F \to T \times H' \to H \to 1.$$

There is a bound $c_1 = c_1(n, k_v)$ for $Card(\mathrm{H}^1(k_v, H'_1))$, where H'_1 runs over all k_v-forms of H', since H' is a semisimple subgroup contained in SL_n and there are only finitely many forms of H' over k_v (see [Se, Chapter III]). Also there is a bound $c_2 = c_2(n, k_v)$ for $Card(\mathrm{H}^1(k_v, T))$ by Lemma 1.1, and there is a bound $c_3 = c_3(n, k_v)$ for $Card(\mathrm{H}^2(k_v, F))$ as it is easy to see (use [Se, Chapter III]). Now we have the following exact sequence

$$\mathrm{H}^1(k_v, F) \to \mathrm{H}^1(k_v, T \times H') \xrightarrow{p} \mathrm{H}^1(k_v, H) \xrightarrow{\Delta} \mathrm{H}^2(k_v, F).$$

We have $\mathrm{Ker}(\Delta) = \mathrm{Im}\, p$, and

$$Card(\mathrm{H}^1(k_v, H)) \leq c_1 c_2 c_3.$$

Indeed, let $Card(\mathrm{Im}(\Delta)) = r$, $\mathrm{Im}(\Delta) = \{f_1, \dots, f_r\}$, and $\Delta(\gamma_i) = f_i$, $\gamma_i \in \mathrm{H}^1(k_v, H)$. Then we have $Card((\mathrm{H}^1(k_v, H)) \leq r \max(Card(\Delta^{-1}(f_i)))$. We know by Serre [Se], Chapter I, that

$$Card(\Delta^{-1}(f_i)) = Card(\mathrm{H}^1(k_v, \ _{c_i}(T \times H'))/\mathrm{H}^1(k_v, F)),$$

where c_i is a cocycle corresponding to γ_i, the subscript denotes the twisting by the cocycle c_i and the latter denotes the factor set with repect to the action of $\mathrm{H}^1(k_v, F)$. In particular,

$$Card(\Delta^{-1}(f_i)) \leq Card(\mathrm{H}^1(k_v, T) \times \mathrm{H}^1(k_v, \ _{c_i}H')),$$

and we are done. □

2. Birational and arithmetic invariants

Let G be an algebraic group defined over a field k, S a finite subset of the set V of all valuations of k. We denote the obstruction to weak approximation in S and over k by

$$\mathrm{A}(S, G) := \prod_{v \in S} G(k_v)/Cl_S(G(k)), \quad \mathrm{A}(G) := \prod_{v \in V} G(k_v)/Cl(G(k)),$$

where $G(k)$ is embedded diagonally into the corresponding direct product and $Cl_S(\cdot)$ (resp. $Cl(\cdot)$) denotes the closure in the corresponding product topology.

It is well-known (see e.g. [Sa]) that for k a number field all these factor sets are in fact finite abelian groups, and that there exists a finite subset S_0 of V depending on G such that $\mathrm{A}(S,G) = \mathrm{A}(G)$ for all subsets S of V containing S_0. Denote by $\mathrm{III}(G)$ the Tate-Shafarevich group of G, i.e.,

$$\mathrm{III}(G) = \mathrm{Ker}(\mathrm{H}^1(k,G) \to \prod_v \mathrm{H}^1(k_v,G)).$$

Let V be a smooth algebraic variety defined over a field k of characteristic 0. We refer to [CTS] for basic notions and results related to Brauer and R-equivalence classes. Recall that the set $V(k)/Br$ (resp. $V(k)/R$) of Brauer (resp. R-) equivalence classes is a birational invariant of V as k-variety. We are interested in these invariants when $V = G$ is a linear algebraic group defined over an arithmetical field k, such as local or global field. It is known by [V2] (resp. [Gi]) that, if k is a local field (resp. global field) then $G(k)/R$ and $G(k)/Br$ are finite groups.

We show that over local and global fields of characteristic 0, the finite abelian groups $G(k)/Br$, and in the case of global field, with respect to any finite set S of valuations of k, the group $\mathrm{A}(S,G)$ are birational invariants *as groups* for connected linear algebraic groups. We say that two k-varieties V, W are birationally stably equivalent if there are integers $n, m \geq 0$ such that $V \times \mathbf{A}^n$ is birationally equivalent to $W \times \mathbf{A}^m$.

Theorem 2.1. *Let G be a connected linear algebraic group defined over a field k of characteristic $0, V(G)$ a smooth compactification of G over k.*

1) *If k is a local field, then there is a canonical isomorphism*

$$G(k)/R = G(k)/Br \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}_G))^*,$$

where $*$ means taking the Pontryagin dual $(\cdot)^* = \mathrm{Hom}(\cdot, \mathbf{Q}/\mathbf{Z})$.

2) *If k is number field, let S_G be the Neron-Severi torus of G, i.e., its dual $\hat{S}_G = \mathrm{Pic}(\bar{V}(G))$. Denote by $\Delta_G : \mathrm{H}^1(k,\hat{S}_G) \to \prod_v \mathrm{H}^1(k,\hat{S}_G)$, the diagonal map, $\lambda_G : \prod_v \mathrm{H}^1(k,\hat{S}_G) \to \prod_v \mathrm{H}^1(k_v,\hat{S}_G)$ the product of restriction maps, and $\mu_G = \lambda_G \circ \Delta_G$. Then we have canonical isomorphism*

$$G(k)/Br \simeq (\mathrm{Im}\,\lambda_G/\,\mathrm{Im}\,\mu_G)^*.$$

3) *Over a local or global field k of characteristic 0, the finite abelian group $G(k)/Br$ is a birational invariant in the class of birationally stably equivalent linear algebraic groups and so is the finite abelian group $\mathrm{A}(S,G)$ for any finite set S of valuations of a number field k.*

Before proving the theorem we need a result due to Borovoi and Kunyavskiĭ. For a k-variety X we denote by $\bar{X} = X \times \bar{k}$. Let G be a connected linear algebraic group defined over a field k of characteristic 0. Let $G = LR_u(G)$, where $R_u(G)$ denotes the unipotent radical of G and L denotes a Levi subgroup of G. Take any z-extension $1 \to Z \to H \to L \to 1$ of L. Let $\tilde{G} = [H,H]$, the semisimple part of H, which is simply connected, and $T = H/[H,H]$, the largest torus quotient. Now we fix a smooth compactification $V(X)$ for $X = G, H, L, T$, and denote by $S_X = \mathrm{Pic}(\bar{V}(X))\hat{}$, the corresponding Neron-Severi torus. We have the following

Theorem 2.2 ([BK]). *With above notation, for any field k of characteristic 0, we have functorial isomorphisms*

$$\mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(G))) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(L))) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(H))) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(T))).$$

Proof. The proof follows directly from the proof of Theorems 2.1 and 2.4 of [BK]. We refer the reader to Sections 2.3 and 2.5, pp. 813–817 of [BK] for more details. □

Proof of Theorem 2.1. 1) Assume that k is a local field. Then we know that (see [T2, Theorem 4.9])

$$G(k)/R = G(k)/Br$$

and by [T2, Theorem 3.4], there are canonical isomorphisms of finite abelian groups

$$G(k)/Br \simeq L(k)/Br \simeq H(k)/Br \simeq T(k)/Br.$$

By [CTS, Proposition 17, Corolary 1], we know that $T(k)/Br \simeq \mathrm{H}^1(k, \hat{S}_T))^*$, and by Theorem 2.2 we know that

$$\mathrm{H}^1(k, \hat{S}_G) \simeq \mathrm{H}^1(k, \hat{S}_L) \simeq \mathrm{H}^1(k, \hat{S}_H) \simeq \mathrm{H}^1(k, \hat{S}_T),$$

hence we have

$$G(k)/R = G(k)/Br \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(G)))^*$$

as required.

2) Let k be a number field. With above notation, let L/k be a common Galois splititng field of G, H, (hence also of T) with (finite) Galois group $\mathcal{G}$. For each valuation v of k denote by $\mathcal{G}_v$ the decomposition group of v. Then the maps Δ_T, λ_T are just the maps

$$\Delta_T : \mathrm{H}^1(\mathcal{G}, \hat{S}_T) \to \prod_v \mathrm{H}^1(\mathcal{G}, \hat{S}_T),$$
$$\lambda_T : \prod_v \mathrm{H}^1(\mathcal{G}, \hat{S}_T) \to \prod_v \mathrm{H}^1(\mathcal{G}_v, \hat{S}_T).$$

Then by [CTS, Proposition 17, Corolary 1], we have $T(k)/Br \simeq (\mathrm{Im}\,\lambda_T/\mathrm{Im}\,\mu_T)^*$, i.e., it can be expressed in terms of $\mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(T))) = \mathrm{H}^1(\mathcal{G}, \mathrm{Pic}(\bar{V}(T)))$. Hence by Theorem 2.2 and the functoriality of the isomorphisms

$$\mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(G))) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(L))) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(H))) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(T))),$$

it follows from above and from [T2, Theorem 3.7], that

$$G(k)/Br \simeq (\operatorname{Im}\lambda_G/\operatorname{Im}\mu_G)^*$$

as desired.

3) The assertion regarding the birational invariance of the *finite abelian groups* $G(k)/Br$ follows directly from 1) and 2), due to the birational invariance of the group $\mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(G)))$. Regarding the birational invariance of $\mathrm{A}(S, G)$, we use the following exact sequence (see [T2, Theorem 3.4]),

$$1 \to G(k)/Br \to \prod_{v\in S} G(k_v)/Br \to \mathrm{A}(S,G) \to 1,$$

thus $\mathrm{A}(S,G)$ can be expressed via $\mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(G)))$ and $\mathrm{H}^1(k_v, \mathrm{Pic}(\bar{V}(G)))$ as it follows from 1) and 2) and we are done. □

Remark. This theorem extends similar formulas for $G(k)/Br$ from [CTS, Proposition 17, Corolary 1] (in the case of tori) to arbitrary connected linear algebraic groups over local or global fields, and that of [Gi, Theorem III. 4.3, a)] (in the case of semisimple groups over local fields) to arbitrary connected linear algebraic groups over local fields. Notice that our formula differs from that of Gille by the Pontryagin dual sign.

3. Bounds for groups of Brauer and R-equivalence classes, obstruction to weak approximation and Hasse principle

We now investigate the boundedness of birational invariants related with Brauer and R-equivalence and of the obstruction to weak approximation and Hasse principle. We have

Proposition 3.1. *Let k be a local field of characteristic 0, G a connected linear algebraic group over k and let H be a connected k-subgroup of G. Then $Card(H(k)/R)(= Card(H(k)/Br))$ is bounded by a constant depending only on G, k.*

Proof. The equality regarding cardinalities follows from [T2, Theorem 4.9]. By [CTS] we may assume that H is reductive. Let $G \hookrightarrow \mathrm{GL}_n$, where n is minimal. Thus n depends only on G and k and we may assume that $G = \mathrm{GL}_n$. We recall that a z-extension of H is a k-group H_1, which is an extension of H by an induced k-torus Z such that H_1' is simply connected. The construction follows from a *cross-diagram* of Ono [O].

Let $1 \to Z \to H_1 \to H \to 1$ be a z-extension of H, and let $T = H_1/H_1'$ be the torus quotient of H_1. Then by [T2, Theorems 3.7 and 4.9] we know that

$$H(k)/R = H(k)/Br = H_1(k)/R = H_1(k)/Br = T(k)/Br.$$

Since $\dim(H)$ is bounded by a constant depending only on n and k, it follows from the construction of H_1 that the smallest value of $\dim(T)$ is also bounded in the same way. Let $1 \to S \to N \to T \to 1$ be a flasque resolution of T, i.e., S is a flasque k-torus and N is an induced k-torus. From the construction of S, N (or of flasque and coflasque resolutions) (due to [EM], see also [CTS, Lemme 3]), it follows that the smallest value of $\dim(S)$ is also bounded by a constant depending only on k, n. From [CTS, Théorème 2] and from results of Section 1, k being a local field, we see that $T(k)/R = T(k)/Br = \mathrm{H}^1(k, S)$ has also bounded cardinality as required. □

The following result is a direct consequence of some main results and arguments from the proofs of results related to Roquette's theorem [L] and of Théorème 1 of [CTS], but for convenience of the reader, we give a detailed proof here.

Lemma 3.2. *Let k be a field finitely generated over the prime field, and let T be a torus over k. Then $Card(T(k)/R)$ is bounded by a constant depending only on k and $n = \dim(T)$.*

Proof. By [CTS, Théorème 1], we know that $T(k)/R = \mathrm{H}^1(k, S)$ is finite, where S is a flasque k-torus coming from a flasque resolution

$$1 \to S \to N \to T \to 1$$

of T over k. From the proof of the finiteness of $\mathrm{H}^1(k, S)$ in (loc. cit), we need only show that $Card(\mathrm{H}^1(k, S))$ is bounded by a constant depending only on k and $\dim(S) = \mathrm{rank}_{\mathbf{Z}}(\hat{S})$, since minimal number among $\dim(S)$ is bounded in term of k, n as we mentioned above. We fix once for all such a flasque resolution of T with minimal number $\dim(S)$. In fact, it is known (essentially due to Roquette, see [L, Chapter II, Section 7]), that for any reduced, normal, commutative $\mathbf{Z}$-algebra A of finite type, its group of units A^* and divisor class group $\mathrm{Div}(A)$ are finitely generated. From the proof of Theorems 7.2, 7.4 and Corollary 7.5 (loc. cit.) we see that

the number of generators of A^* depends only on the finite type of A.

(A more constructive proof of this result, which uses the Dirichlet's Theorem on finitely generation of the group of units of a global field can be found in [Sam, Theorem 1].) Also, by inverting a suitable element $a \in A$ we have $\mathrm{Pic}(A[1/a]) = 0$ (see Corollary 7.7 of [L]). If K/k is a finite Galois extension of k with Galois group $\mathcal{G}$, which splits the tori S, N, T, then one can construct in a canonical (i.e. effective) way a commutative, $\mathcal{G}$-stable $\mathbf{Z}$-algebra A of finite type in K with fraction field K. So the minimal number of generators of A^* depends only on k, $\mathcal{G}$, i.e., only on k, n. Then the singular locus (i.e. non-regular part) of $\mathrm{Spec}(A)$ can be effectively determined, which is a proper subscheme of $\mathrm{Spec}(A)$, by a well-known result of Nagata. By passing to a localization of A (which can be also effectively determined) we will get a regular algebra A', which can be also assumed to be $\mathcal{G}$-invariant, having similar properties to those of A; in particular, A'^* is also finitely generated and its minimal number of generators is bounded in terms of k, n. Then by using the fact regarding Picard group mentioned above, we will get, by inverting an element $a \in A'$, a $\mathcal{G}$-stable, regular, commutative $\mathbf{Z}$-algebra A'' with fraction field K and trivial Picard group. Still, A'' has minimal number of generators bounded in terms of k, n. Now we consider the exact sequence of $\mathcal{G}$-modules

$$0 \to A''^* \to K^* \to \mathrm{Div}(A'') \to \mathrm{Pic}(A'') \to 0,$$

or the same $0 \to A''^* \to K^* \to \mathrm{Div}(A'') \to 0$ by the construction of A''. By the definition of S, $\hat{S}$ is a flasque $\mathcal{G}$-module, hence the dual module $\hat{S}^0 := \mathrm{Hom}_{\mathbf{Z}}(\hat{S}, \mathbf{Z})$ is co-flasque. Hence by Lemme 1 of [CTS], from above exact sequence we get the following exact sequence (by applying $\otimes_{\mathbf{Z}} \hat{S}^0$)

$$0 \to A''^* \otimes \hat{S}^0 \to K^* \otimes \hat{S}^0 \to \mathrm{Div}(A'') \otimes \hat{S}^0 \to 0.$$

Since K splits S, we have $\mathrm{H}^1(k, S) = \mathrm{H}^1(\mathcal{G}, S(K)) = \mathrm{H}^1(\mathcal{G}, K^* \otimes \hat{S}^0)$ and we derived an exact sequence of cohomology

$$\mathrm{H}^1(\mathcal{G}, A''^* \otimes \hat{S}^0) \to \mathrm{H}^1(\mathcal{G}, K^* \otimes \hat{S}^0) \to \mathrm{H}^1(\mathcal{G}, \mathrm{Div}(A'') \otimes \hat{S}^0.$$

Since $\mathrm{Div}(A'')$ is an induced $\mathcal{G}$-module and $\hat{S}^0$ is co-flasque, we have

$$\mathrm{H}^1(\mathcal{G}, \mathrm{Div}(A'') \otimes \hat{S}^0) = 0$$

by [CTS, Lemme 1]. Since A''^* is finitely generated, we know that the cohomology $\mathrm{H}^1(\mathcal{G}, A''^* \otimes \hat{S}^0)$ is finite, the cardinality of which is bounded by a constant depending only on $rk_{\mathbf{Z}}(\hat{S}^0)$ and the minimal number of generators of A''^*. Since $Card(\mathrm{H}^1(\mathcal{G}, S)) \leq Card(\mathrm{H}^1(\mathcal{G}, A''^* \otimes \hat{S}^0))$ these facts show that $Card(\mathrm{H}^1(k, S))$ is bounded by a constant depending only on k, n, hence the lemma. □

The following proposition shows the boundedness of arithmetic invariants $\mathrm{A}(G)$ and $\mathrm{III}(G)$.

Proposition 3.3. *Let H be a connected k-subgroup of a linear algebraic group G defined over a number field k, S a finite set of valuations of k. Then $Card(\mathrm{A}(S, H))$ is bounded by a constant depending only on G, k, S. In particular, when H runs over connected k-subgroups of G, $Card(\mathrm{A}(H))$ and $Card(\mathrm{III}(H))$ are bounded by a constant depending only on G and k.*

Proof. We take the smallest natural number n such that $G \hookrightarrow \mathrm{GL}_n$. Therefore $H \hookrightarrow \mathrm{GL}_n$, so we may assume $G = \mathrm{GL}_n$. We may assume also that H is reductive. The constant we are going to produce therefore depends only on k, S, n. Let $1 \to Z \to H_1 \to H \to 1$ be a z-extension of H. From the proof of Lemma 3.8 and Theorem 4.2.3 of [T2], we derive the following result.

3.4. *Let the notation be as above, and let S be a finite set of valuations of k, containing the set of all archimedean ones. Let $T = H_1/H_1'$ be the torus quotient of H_1. Then there are canonical isomorphisms of finite abelian groups*

$$\mathrm{A}(S,H) \simeq \mathrm{A}(S,H_1) \simeq \mathrm{A}(S,T),$$

$$\mathrm{A}(H) \simeq \mathrm{A}(H_1) \simeq \mathrm{A}(T),\ \mathrm{III}(H) \simeq \mathrm{III}(H_1) \simeq \mathrm{III}(T).$$

We may also bound the dimension of such T appearing in the above exact sequence. From the very construction of z-extensions it follows that we can bound the number N where $H_1 \hookrightarrow \mathrm{GL}_N$. Therefore it is sufficient to prove our assertion for H_1. Due to the above isomorphisms, we can bound the number M such that $T \hookrightarrow \mathrm{GL}_M$, thus we are reduced to proving the assertion for T. We may also bound the dimension of such T appearing in the above exact sequence. Hence it suffices to prove the following.

Let T be a subtorus of GL_n *over a global field k. Then the cardinality of* $\mathrm{A}(T)$ (*resp.* $\mathrm{III}(T)$) *is bounded by a constant depending only on k, n.*

Proof. We make use of the following exact sequence (due to Voskresenskiǐ [V])

$$1 \to \mathrm{A}(T) \to \mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(T)))^* \to \mathrm{III}(T) \to 1.$$

Therefore it suffices to show the boundedness of the middle term in the exact sequence above. But we know that for a fixed flasque resolution of T

$$1 \to S \to N \to T \to 1,$$

in which all three groups are split by a finite Galois extension K/k with Galois group $\mathcal{G}$ as in the proof of Lemma 3.2, we can bound the minimal number of generators of $\hat{S}$ in terms of $k, n = \dim(T)$. Since $\hat{S}$ and $\mathrm{Pic}(\bar{V}(T))$ are similar $\mathcal{G}$-modules (see [V1]), we have

$$\mathrm{H}^1(k, \mathrm{Pic}(\bar{V}(T))) = \mathrm{H}^1(\mathcal{G}, \mathrm{Pic}(\bar{V}(T))) = \mathrm{H}^1(\mathcal{G}, \hat{S}) = \mathrm{H}^1(k, \hat{S}).$$

In the same way as in the proof of Lemma 3.2, and we can bound the cardinality of $\mathrm{H}^1(\mathcal{G}, \hat{S})$ for such S. and the proposition is proved. □

In the case of a global field we have the following result.

Proposition 3.5. *Let k be a global number field and G a connected linear algebraic group over k. Then for all connected k-subgroups H of G* (*resp. for those containing no anisotropic almost simple factors of type* D_4 *nor* E_6), *Card*$(H(k)/Br)$ (*resp. Card*$(H(k)/R)$) *are bounded by a constant depending only on k and G.*

Proof. First we use the following canonical isomorphism deduced from Theorems 3.6 and 3.7 of [T2]:

$$H(k)/Br \simeq T(k)/Br$$

for local or global field k of characteristic 0. Therefore to prove the assertion regarding Br-part, as above, one is reduced to proving the same thing for T since we can bound the minimal dimension of T. We make use again of the following exact sequence

$$1 \to \mathrm{A}(H) \to \mathrm{H}^1(k, \mathrm{Pic}\bar{V}(H))^* \to \mathrm{III}(H) \to 1.$$

Let $1 \to Z \to H \to H_L \to 1$ be a z-extension of a Levi subgroup H_L of H, $T = H_L/[H_L, H_L]$ the torus quotient of H_L. By [T2, Proposition 4.2.3], there are canonical isomorphisms of finite commutative groups

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{A}(T) & \to & \mathrm{H}^1(k, \mathrm{Pic}\bar{V}(T))^* & \to & \mathrm{III}(T) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{A}(H) & \to & \mathrm{H}^1(k, \mathrm{Pic}\bar{V}(H))^* & \to & \mathrm{III}(H) & \to & 1. \end{array}$$

By previous results we can bound the dimension of T, therefore we can bound the cardinality of $T(k)/R$ (and a fortiori that of $T(k)/Br$) and that of $\mathrm{A}(T)$ and $\mathrm{III}(T)$. By [T2, Theorem 3.4], there exists an exact sequence of finite abelian groups

$$1 \to T(k)/Br \to \prod_v T(k_v)/Br \to \mathrm{A}(T) \to 1.$$

Thus we can bound also the cardinality of $\prod_v T(k_v)/Br = \prod_v T(k_v)/R$ by combining with Corolary 1 to Proposition 17 of [CTS]. We now make use of the following exact sequence of finite abelian groups

$$1 \to \mathrm{III}(S) \to H(k)/R \to \prod_v H(k_v)/R \to \mathrm{A}(H) \to 1,$$

proved in [T2], Theorem 4.12, in the case where the semisimple part of a Levi k-subgroup of H does not contain anisotropic almost simple factors of types D_4 nor types E_6. (There it was also proved that the exactness of above sequence is equivalent to the triviality of the group of R-equivalence classes of simply connected semisimple groups over k.) Since $H(k_v)/R = H(k_v)/Br \simeq T(k_v)/Br$, we have also the following exact sequence

$$1 \to \mathrm{III}(S) \to H(k)/R \to \prod_v T(k_v)/Br \to \mathrm{A}(H) \to 1.$$

Since we can bound the cardinality of $\mathrm{III}(S)$, $\prod_v T(k_v)/Br$ (see above), it follows that the same is true for $H(k)/R$ as required. □

Remark. We take a chance to make some changes in the proof of Proposition 3.6, 1) of [T2]. There should be added the adjective *rational* to the k-section $i : G \to H$. So i is defined on a Zariski open subset $U \subset G$, which can be taken sufficiently small, so that the projection π defines a k-morphism of quasi-projective k-varieties $\pi^{-1}(U) \to U$, i.e., i is a regular k-section of π restricted to $\pi^{-1}(U)$. Since $\pi(H(k)) = G(k)$, in order to prove that the natural homomorphism $\pi' : H(k)/Br \to G(k)/Br$ induced by π is an isomorphism $H(k)/Br \simeq G(k)/Br$, it suffices to show that π' is injective. The argument given in the proof of (loc. cit) shows that π induces a bijection $\pi'' : \pi^{-1}(U)(k)/Br \simeq U(k)/Br$. From the birational invariant property of Brauer equivalence (see Section 7 of [CTS]) it follows that we have the following commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U)(k)/Br & \stackrel{\pi''}{\to} & U(k)/Br \\ \simeq \big\downarrow j_H & & \simeq \big\downarrow j_G \\ H(k)/Br & \stackrel{\pi'}{\to} & G(k)/Br, \end{array}$$

where j_H, j_G are bijections (induced from corresponding inclusions $\pi^{-1}(U) \hookrightarrow H, U \hookrightarrow G$). It follows immediately that π' is also bijection as required, hence due to the surjectivity above, we have a canonical isomorphism $H(k)/Br \simeq G(k)/Br$. (Also, in the case 2) of loc. cit we may combine the surjectivity with the bijection $H(k)/Br \to G(k)/Br$ to get the above isomorphism.)

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