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An example of non-uniqueness for a hyperbolic equation with non-Lipschitz-continuous coefficients

By

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1. Introduction and main result

Let Ω be an open neighborhood of the origin in $\mathbf{R}^{n+1}$ and let P be a second order operator of the form

$$P = \partial_t^2 - \sum_{j,k=1}^{n} a_{jk}(t,x)\partial_{x_j}\partial_{x_k} + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} + c(t,x)\partial_t + d(t,x), \tag{1.1}$$

with bounded complex valued coefficients defined in Ω.

We say that the operator P has the uniqueness in the Cauchy problem with respect to $\{t=0\}$ at the origin if there exists Ω' open neighborhood of the origin, $\Omega' \subseteq \Omega$, such that if $u \in \mathcal{C}^2(\Omega)$, $\operatorname{supp} u \subseteq \{(t,x) \in \Omega \,:\, t \geq 0\}$ and $Pu = 0$ in Ω then $u = 0$ in Ω'.

Suppose that the operator P is strictly hyperbolic (with respect to $\{t = c\}$ in Ω) i.e. the coefficients a_{jk} are real valued, $a_{jk} = a_{kj}$ and there exists $\lambda_0 > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k \geq \lambda_0|\xi|^2 \tag{1.2}$$

for all $(t,x) \in \Omega$ and for all $\xi \in \mathbf{R}^n$; the question we are interested in is the following: *how the uniqueness in the Cauchy problem for the operator P is related with the regularity of the coefficients of the principal part of P?*

It is well known that if the coefficients a_{jk} are Lipschitz-continuous then P has the uniqueness in the Cauchy problem. Conversely Colombini, Jannelli and Spagnolo proved that there exists a hyperbolic operator $\tilde{P}$ of the form (1.1) such that for all $\alpha < 1$ the coefficients of the principal part of $\tilde{P}$ are Hölder-continuous of exponent α and $\tilde{P}$ does not have the cited uniqueness property (see [2]).

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In the present note we improve the result of [2] showing that there is a quite precise relation between the modulus of continuity of the coefficients of the principal part and the possibility of constructing a non-uniqueness example.

Let μ be a modulus of continuity, i.e. μ is a non-negative function defined on $[0, r]$ for some $r \in (0, 1)$, continuous, strictly increasing, concave and such that $\mu(0) = 0$. We say that the function f defined on Ω is μ-continuous (and we will write $f \in \mathcal{C}^\mu(\Omega)$) if for all K compact set in Ω there exists $\varepsilon > 0$ such that

$$\sup_{y,z\in K,\ 0<|y-z|<\varepsilon} \frac{|f(y) - f(z)|}{\mu(|y - z|)} < +\infty.$$

Our result is the following.

Theorem 1. *Suppose that*

$$\int_0^r \frac{1}{\mu(s)}\, ds < +\infty \tag{1.3}$$

and

$$\textit{the function} \quad s \mapsto -\frac{\mu(s)}{s \log(s)} \quad \textit{is decreasing in } (0, r]. \tag{1.4}$$

Then there exist a real valued function $a(t)$ and two complex valued functions $d(t, x)$ and $u(t, x)$ such that

$$\begin{gathered}
a \in \mathcal{C}^\mu(\mathbf{R}) \quad \textit{and} \quad 1/2 \le a(t) \le 3/2 \qquad \textit{for all} \quad t \in \mathbf{R};\\
d \in \mathcal{C}^\infty(\mathbf{R}^2) \quad \textit{and} \quad \operatorname{supp} d \subseteq \{(t, x) \in \mathbf{R}^2 \,:\, t \ge 0\};\\
u \in \mathcal{C}^\infty(\mathbf{R}^2) \quad \textit{and} \quad \operatorname{supp} u = \{(t, x) \in \mathbf{R}^2 \,:\, t \ge 0\};\\
u_{tt}(t, x) - a(t)u_{xx}(t, x) + d(t, x)u(t, x) = 0 \qquad \textit{for all} \quad (t, x) \in \mathbf{R}^2.
\end{gathered}$$

It is worthy to compare the result of Theorem 1 with the similar results known in the case of second order elliptic operators with real principal part. Consider the operator

$$Q = \partial_t^2 + \sum_{j,k=1}^n a_{jk}(t, x)\partial_{x_j}\partial_{x_k} + \sum_{j=1}^n b_j(t, x)\partial_{x_j} + c(t, x)\partial_t + d(t, x),$$

under the condition (1.2). Let μ be a modulus of continuity such that

$$\lim_{s\to 0^+} \frac{\mu(s)}{s^\alpha} = 0 \qquad \text{for all} \quad \alpha \in [0, 1) \tag{1.5}$$

and

$$\int_0^r \frac{1}{\mu(s)}\, ds = +\infty. \tag{1.6}$$

If the coefficients of the principal part of Q are in $\mathcal{C}^\mu(\Omega)$ then Q has the uniqueness in the Cauchy problem with respect to $\{t = 0\}$ at the origin (see [6]). On the other hand if the modulus of continuity satisfies the conditions (1.5) and (1.3) it is possible to construct a non-uniqueness example for an elliptic operator like Q with the coefficients of the principal part in $\mathcal{C}^\mu(\Omega)$ (see [5] and [4]).

It would be very interesting to prove a result similar to that one of [6] in the case of hyperbolic operators. We think that the condition (1.6) is related to the uniqueness in the Cauchy problem also for hyperbolic operators, but unfortunately we are not able to prove it.

Let us finally recall what is known for a similar subject, namely the relation between the well-posedness of the Cauchy problem for a second order hyperbolic operator and the regularity of the coefficients of its principal part. Also in this case the crucial condition is given in terms of the modulus of continuity of the coefficients of the principal part. Consider the operator

$$P_2 = \partial_t^2 - \sum_{j,k=1}^{n} a_{jk}(t,x)\partial_{x_j}\partial_{x_k},$$

under the condition (1.2). Suppose that the coefficients a_{jk} are $\mathcal{C}^\infty$ in the x variables for all fixed t, the a_{jk}'s and its first and second derivatives in the x variables are bounded in $\mathbf{R}^{n+1}$ and

$$\sup_{0<|t-s|<1/2,\, x\in\mathbf{R}^n} \frac{|a_{jk}(t,x) - a_{jk}(s,x)|}{\mu(|t-s|)} < +\infty,$$

where $\mu(\tau) = \tau|\log\tau|$; then the Cauchy problem for P_2 is $\mathcal{C}^\infty$-well-posed. This result is sharp in the sense that if a modulus of continuity is of the type $\mu(\tau) = \tau|\log\tau|\psi(|\log\tau|)$ with ψ increasing, concave and such that $\lim_{\sigma\to+\infty}\psi(\sigma) = +\infty$, then there exists a function $a \in \mathcal{C}^\mu$ with $1/2 \le a(t) \le 3/2$ such that for the operator

$$\partial_t^2 - a(t)\partial_x^2$$

the Cauchy problem is not $\mathcal{C}^\infty$-well-posed (see [1] and [3]).

2. Proof of Theorem 1

The main step of the proof of Theorem 1 is the following refinement of the 'change of phase' Lemma (see [2, Lemma 2]). A detailed proof of this result can be found in the Appendix.

Lemma 1. *There exists a positive constant $M > 1$ such that for all positive integers h_1, h_2, for all positive constants ε_1, ε_2, ρ, η_1, η_2 and for all $t_1 \in \mathbf{R}$, if*

$$0 < \varepsilon_2 \le \varepsilon_1 \le \frac{1}{2M}, \tag{2.1}$$

$$\frac{\varepsilon_2 h_2}{\varepsilon_1 h_1} \geq 2M^2, \tag{2.2}$$

and

$$4e^{-\varepsilon_1 h_1 \rho} \leq \frac{\eta_2}{\eta_1} \leq \frac{1}{4} e^{\varepsilon_2 h_2 \rho}, \tag{2.3}$$

then there exist $t_2 > t_1$, a $\mathcal{C}^\infty$ real valued function $a(t)$ defined on $I = [t_1, t_2]$ and two complex valued functions $d(t,x)$, $u(t,x)$ defined on $I \times \mathbf{R}$, $\mathcal{C}^\infty$ in t and 2π-periodic and analytic in x, such that

$$t_2 - t_1 \leq 12M\rho + \frac{4\pi}{h_1}, \tag{2.4}$$

$$u_{tt}(t,x) - a(t)u_{xx}(t,x) + d(t,x)u(t,x) = 0 \qquad \textit{for all} \quad (t,x) \in I \times \mathbf{R}, \tag{2.5}$$

$$\operatorname{supp}(u) = I \times \mathbf{R}, \tag{2.6}$$

$$a(t) = 1 \quad \textit{and} \quad d(t,x) = 0 \qquad \textit{for} \quad t \textit{ near } t_j, j = 1, 2, \tag{2.7}$$

$$u(t,x) = \eta_j \cos(h_j |t - t_j|) e^{i h_j x} \qquad \textit{for} \quad t \textit{ near } t_j, j = 1, 2. \tag{2.8}$$

Moreover

$$\sup_{t \in I} |1 - a(t)| \leq \frac{1}{2}, \tag{2.9}$$

$$|a|_{\mathcal{C}^\mu(I)} = \sup_{t,\tau \in I,\ t \neq \tau} \frac{|a(t) - a(\tau)|}{\mu(|t - \tau|)} \leq C_0 \max\left\{ \frac{\varepsilon_1}{\mu(1/h_1)}, \frac{\varepsilon_2}{\mu(1/h_2)} \right\}, \tag{2.10}$$

and finally

$$\sup_{(t,x) \in I \times \mathbf{R}} \left| \left(\frac{\partial}{\partial t}\right)^p \left(\frac{\partial}{\partial x}\right)^q u \right| \leq C_p h_2^{p+q} \max\{\eta_1, \eta_2\} \qquad \textit{for all} \quad p, q \in \mathbf{N}, \tag{2.11}$$

$$\begin{aligned} &\sup_{(t,x) \in I \times \mathbf{R}} \left| \left(\frac{\partial}{\partial t}\right)^p \left(\frac{\partial}{\partial x}\right)^q d \right| \\ &\qquad \leq C_p h_2^{p+q+2} \sum_{n=1}^{+\infty} n^{p+q} e^{-n \varepsilon_1 h_1 \rho} \qquad \textit{for all} \quad p, q \in \mathbf{N}, \end{aligned} \tag{2.12}$$

where C_0, C_p do not depend on h_1, h_2, ε_1, ε_2, ρ, η_1, η_2 and t_1.

In what follows we find $T > 0$ and we construct $a(t)$, $d(t,x)$, $u(t,x)$ such that

$$\begin{gathered} a \in \mathcal{C}^\mu(\mathbf{R}) \quad \text{and} \quad 1/2 \leq a(t) \leq 3/2 \qquad \text{for all} \quad t \in \mathbf{R}, \\ a(t) = 1 \qquad \text{for all} \quad t \leq 0 \quad \text{and} \quad t \geq T, \\ d \in \mathcal{C}^\infty(\mathbf{R}^2) \quad \text{and} \quad \operatorname{supp}(d) \subseteq [0,T] \times \mathbf{R}, \end{gathered}$$

$$u \in \mathcal{C}^\infty(\mathbf{R}^2) \quad \text{and} \quad \operatorname{supp}(u) \subseteq (-\infty, T] \times \mathbf{R},$$

and

$$u_{tt}(t,x) - a(t)u_{xx}(t,x) + d(t,x)u(t,x) = 0 \qquad \text{for all} \quad (t,x) \in \mathbf{R}^2.$$

The conclusion of the proof will be easily obtained by a reflection respect to $t = T/2$.

Let us consider four sequences $\{h_k\}$, $\{\varepsilon_k\}$, $\{\rho_k\}$, $\{\eta_k\}$ of real positive numbers such that

$$h_k \in \mathbf{N} \qquad \text{for all} \quad k \in \mathbf{N} \quad \text{and} \quad \lim_{k\to+\infty} h_k = +\infty, \tag{2.13}$$

$$\{\varepsilon_k\} \quad \text{is decreasing} \quad \text{and} \quad \lim_{k\to+\infty} \varepsilon_k = \lim_{k\to+\infty} \rho_k = \lim_{k\to+\infty} \eta_k = 0, \tag{2.14}$$

$$\varepsilon_k \leq \frac{1}{2M} \qquad \text{for all} \quad k \in \mathbf{N}, \tag{2.15}$$

$$\frac{\varepsilon_{k+1}h_{k+1}}{\varepsilon_k h_k} \geq 2M^2 \qquad \text{for all} \quad k \in \mathbf{N}, \tag{2.16}$$

and

$$4e^{-\varepsilon_k h_k \rho_k} \leq \frac{\eta_{k+1}}{\eta_k} \leq \frac{1}{4} e^{\varepsilon_{k+1}h_{k+1}\rho_k} \qquad \text{for all} \quad k \in \mathbf{N}. \tag{2.17}$$

Using Lemma 1 we construct an increasing sequence of positive real numbers $\{t_k\}$, with $t_1 = 0$, such that the functions $a(t)$, $d(t,x)$, $u(t,x)$ are defined on each strip $[t_k,\, t_{k+1}] \times \mathbf{R}$ and satisfy (2.5), ... , (2.12) with h_1, h_2, ε_1, ε_2, ρ, η_1, η_2 and t_1 replaced by h_k, h_{k+1}, ε_k, ε_{k+1}, ρ_k, η_k, η_{k+1} and t_k respectively.

Since $|t_{k+1} - t_k| \leq 12M\rho_k + 4\pi/h_k$, if

$$\sum_{k=1}^{+\infty} \rho_k < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{1}{h_k} < +\infty, \tag{2.18}$$

then the sequence $\{t_k\}$ is convergent. We define $\lim_{k\to+\infty} t_k = T$. We set

$$a(t) = 1 \quad \text{and} \quad d(t,x) = 0 \qquad \text{for} \quad t \leq 0 \quad \text{and} \quad t \geq T,$$
$$u(t,x) = \eta_1 \cos(h_1 t) e^{ih_1 x} \quad \text{for } t \leq 0, \quad u(t,x) = 0 \quad \text{for } t \geq T.$$

In view of (2.6) through (2.8) we have that $a \in \mathcal{C}^\infty(\mathbf{R} \setminus \{T\})$, d, $u \in \mathcal{C}^\infty(\mathbf{R}^2 \setminus \{(t,x) : t = T\})$ and $\operatorname{supp}(u) = (-\infty,\, T] \times \mathbf{R}$. From (2.10) we easily deduce that if there exists $C > 0$ such that

$$\frac{\varepsilon_k}{\mu(1/h_k)} \leq C \qquad \text{for all} \quad k \in \mathbf{N}, \tag{2.19}$$

then $a \in \mathcal{C}^\mu(\mathbf{R})$. Finally (2.11) and (2.12) will imply the $\mathcal{C}^\infty$-regularity for u and d on $\mathbf{R}^2$ provided the following conditions hold

$$\lim_{k\to+\infty} h_{k+1}^p \max\{\eta_k, \eta_{k+1}\} = 0 \qquad \text{for all} \quad p \in \mathbf{N}, \tag{2.20}$$

$$\lim_{k\to+\infty} h_{k+1}^{p+2}\sum_{n=1}^{+\infty} n^p e^{-n\rho_k\varepsilon_k h_k} = 0 \qquad \text{for all} \quad p\in\mathbf{N}. \tag{2.21}$$

To end the proof it will be sufficient to choose the sequences $\{h_k\}$, $\{\varepsilon_k\}$, $\{\rho_k\}$ and $\{\eta_k\}$ in such a way that (2.13), ... , (2.21) are verified.

We remark that (1.3) and (1.4) imply that for any positive integer N the function

$$s \mapsto \frac{2^{-2^{Ns}}2^{Ns}}{\mu(2^{-2^{Ns}})}$$

is decreasing in $[1, +\infty[$ and

$$\int_1^{+\infty} \frac{2^{-2^{Ns}}2^{Ns}}{\mu(2^{-2^{Ns}})}\, ds < +\infty.$$

Consequently

$$\sum_{k=1}^{+\infty} \frac{2^{-2^{Nk}}2^{Nk}}{\mu(2^{-2^{Nk}})} < +\infty.$$

Moreover it is possible to find a function $\tilde\mu : [0\,, r] \to [0\,, +\infty)$ such that

$$\sum_{k=1}^{+\infty} \frac{2^{-2^{Nk}}2^{Nk}}{\tilde\mu(2^{-2^{Nk}})} < +\infty, \tag{2.22}$$

and

$$\lim_{s\to 0}\frac{\mu(s)}{\tilde\mu(s)} = +\infty. \tag{2.23}$$

Let $N\in\mathbf{N}$, $N\geq 1$. We define

$$h_k = 2^{2^{Nk}}, \quad \varepsilon_k = \mu(2^{-2^{Nk}}), \quad \rho_k = \frac{2^{-2^{Nk}}2^{Nk}}{\tilde\mu(2^{-2^{Nk}})} \qquad \text{for all} \quad k\in\mathbf{N}$$

and

$$\eta_k = \begin{cases} 1 & \text{for} \quad k=1, \\ \exp\left(-\dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1}\varepsilon_j h_j \rho_j\right) & \text{for} \quad k\geq 2. \end{cases}$$

With these choices the condition (2.13) holds; easily, using also (2.22) and remarking that (2.23) implies

$$\lim_{k\to+\infty}\varepsilon_k h_k \rho_k = \lim_{k\to+\infty}\frac{2^{Nk}\mu(2^{-2^{Nk}})}{\tilde\mu(2^{-2^{Nk}})} = +\infty, \tag{2.24}$$

we obtain (2.14). If N is sufficiently large (2.15) is verified. Since the function $s \mapsto 2^{-2^{Ns}} 2^{Ns}/\mu(2^{-2^{Ns}})$ is decreasing on $[1, +\infty)$ we have

$$\frac{\mu(2^{-2^{N(k+1)}})}{2^{-2^{N(k+1)}} 2^{N(k+1)}} \geq \frac{\mu(2^{-2^{Nk}})}{2^{-2^{Nk}} 2^{Nk}}$$

so that $\varepsilon_{k+1} h_{k+1} 2^{-N(k+1)} \geq \varepsilon_k h_k 2^{-Nk}$, consequently

$$\frac{\varepsilon_{k+1} h_{k+1}}{\varepsilon_k h_k} \geq 2^N$$

and (2.16) follows for N is sufficiently large. For $k \geq 1$ we have $\eta_{k+1}/\eta_k = e^{-\varepsilon_k h_k \rho_k/2}$ and then (2.17) is a consequence of (2.24). The first part of (2.18) is deduced by (2.22) while the second is trivial. Let us finally come to (2.20) and (2.21). We have

$$\left| \sum_{n=1}^{+\infty} n^p e^{-n\varepsilon_k h_k \rho_k} \right| \leq C_p e^{-\varepsilon_k h_k \rho_k},$$

then

$$\left| h_{k+1}^{p+2} \sum_{n=1}^{+\infty} n^p e^{-n\varepsilon_k h_k \rho_k} \right| \leq C_p 2^{(p+2)2^{N(k+1)}} \exp\left(-2^{Nk} \frac{\mu(2^{-2^{Nk}})}{\tilde{\mu}(2^{-2^{Nk}})} \right).$$

Since, by (2.23),

$$\lim_{k \to +\infty} (\log 2)(p+2) 2^{N(k+1)} - 2^{Nk} \frac{\mu(2^{-2^{Nk}})}{\tilde{\mu}(2^{-2^{Nk}})} = -\infty \qquad \text{for all} \quad p \in \mathbf{N},$$

we obtain (2.21). Similarly, since the sequence $\{\eta_k\}$ is decreasing and $\eta_k \leq e^{-\varepsilon_{k-1} h_{k-1} \rho_{k-1}/2}$, we have

$$h_{k+1}^p \max\{\eta_k, \eta_{k+1}\} \leq 2^{p 2^{N(k+1)}} \exp\left(-2^{(N(k-1)-1)} \frac{\mu(2^{-2^{N(k-1)}})}{\tilde{\mu}(2^{-2^{N(k-1)}})} \right)$$

and (2.20) follows. The proof is complete.

A. Appendix

In this Appendix we prove Lemma 1. We will follow closely the proof of [2, Lemma 2] and for the reader's convenience we will point out the different parts. We need first the following lemma. The proof of this result can be found in [2, p. 502].

Lemma 2 ([2, Lemma 1]). *For all $\varepsilon \in (0, 1]$ there exist two real valued functions $\alpha_\varepsilon(\tau)$, $w_\varepsilon(\tau)$ satisfying the following properties*

$$\begin{cases} w_\varepsilon''(\tau) + \alpha_\varepsilon(\tau) w_\varepsilon(\tau) = 0 & \text{on} \quad \mathbf{R}, \\ w_\varepsilon(0) = 1, \ w_\varepsilon'(0) = 0, \end{cases} \tag{A.1}$$

$$\alpha_\varepsilon(\tau) \quad is \quad 2\pi - periodic, \tag{A.2}$$

$$\alpha_\varepsilon(\tau) = 1 \qquad for\ all \quad \tau \in \left[\frac{-\pi}{3}, \frac{\pi}{3}\right], \tag{A.3}$$

$$w_\varepsilon(\tau) = e^{-\varepsilon\tau}\tilde{w}_\varepsilon(\tau) \qquad with \quad \tilde{w}_\varepsilon(\tau) \quad 2\pi - periodic, \tag{A.4}$$

$$|\alpha_\varepsilon(\tau) - 1| \le M_0\varepsilon \qquad for\ all \quad \tau \in \mathbf{R}, \tag{A.5}$$

$$|\alpha'_\varepsilon(\tau)| \le \frac{1}{2}M_0\varepsilon \qquad for\ all \quad \tau \in \mathbf{R}, \tag{A.6}$$

where M_0 does not depend on ε. Moreover

$$|w_\varepsilon(\tau)| \le 1 \qquad for\ all \quad \tau \ge 0, \tag{A.7}$$

$$|\alpha_\varepsilon^{(p)}(\tau)| \le M_p \qquad for\ all \quad \tau \in \mathbf{R} \qquad and\ for\ all \quad p \in \mathbf{N}, \tag{A.8}$$

$$|w_\varepsilon^{(p)}(\tau)| \le M_p e^{-\varepsilon\tau} \qquad for\ all \quad \tau \in \mathbf{R} \qquad and\ for\ all \quad p \in \mathbf{N}, \tag{A.9}$$

where M_p does not depend on ε for all $p \in \mathbf{N}$.

Let $M = M_0$, where M_0 is the constant which appears in Lemma 2. Let then h_1, h_2, ε_1, ε_2, ρ, η_1, η_2 satisfying the conditions (2.1) through (2.3). We claim that there exist two positive numbers ρ_1, ρ_2 such that

$$\frac{\rho_j h_j}{2\pi} \quad \text{is a positive integer for} \quad j = 1,\, 2, \tag{A.10}$$

$$\rho_j \ge 4\rho \qquad \text{for} \quad j = 1, 2, \tag{A.11}$$

$$M \le \rho_1/\rho_2 \le 2M, \tag{A.12}$$

and

$$\rho_1 + \rho_2 \le 12M\rho + 4\pi/h_1. \tag{A.13}$$

In fact, from (2.1) and (2.2), we have that

$$h_2 \ge 2M^2 h_1\varepsilon_1/\varepsilon_2 \ge 2M^2 h_1. \tag{A.14}$$

We take $\rho_1 = 8M\rho + \theta_1$ with $\theta_1 \in [0,\, 2\pi/h_1]$ and θ_1 such that

$$\frac{\rho_1 h_1}{2\pi} = \frac{8M\rho h_1 + \theta_1 h_1}{2\pi} \quad \text{is a positive integer.}$$

Consequently $\rho_1 \ge 4\rho$. Then we take $\rho_2 = \rho_1/(2M) + \theta_2$ with $\theta_2 \in [0,\, 2\pi/h_2]$ and θ_2 such that

$$\frac{\rho_2 h_2}{2\pi} = \frac{\rho_1 h_2}{4M\pi} + \frac{\theta_2 h_2}{2\pi} = \frac{8M\rho h_2 + \theta_1 h_2 + 2\theta_2 M h_2}{4M\pi} \quad \text{is a positive integer.}$$

As a consequence $\rho_2 \ge 4\rho$. Moreover $\rho_2/\rho_1 = 1/(2M) + \theta_2/\rho_1$ and from (A.10) and (A.14) we deduce that

$$\frac{\theta_2}{\rho_1} \le \frac{2\pi}{\rho_1 h_2} \le \frac{2\pi}{2M^2\rho_1 h_1} \le \frac{1}{2M^2}.$$

Recalling that $M \geq 1$ we obtain (A.12). Finally, again using the fact that $M \geq 1$ and $h_1 \leq h_2/2$, we have

$$\rho_1 + \rho_2 = 8M\rho + \theta_1 + 4\rho + \theta_1/(2M) + \theta_2 \leq 12M\rho + 4\pi/h_1.$$

We set

$$\bar{t} = t_1 + \rho_1, \qquad t_2 = \bar{t} + \rho_2 = t_1 + \rho_1 + \rho_2. \tag{A.15}$$

We denote by I_1, I_2, I the intervals $[t_1,\, \bar{t}]$, $[\bar{t},\, t_2]$, $[t_1,\, t_2]$ respectively. We define

$$a(t) = \begin{cases} \alpha_{\varepsilon_1}(h_1(t - t_1)) & \text{for} \quad t \in I_1, \\ \alpha_{\varepsilon_2}(h_2(t_2 - t)) & \text{for} \quad t \in I_2, \end{cases} \tag{A.16}$$

where α_ε is the function constructed in Lemma 2. By (A.2), (A.3), (A.10) and (A.15) we deduce that $a(t) = 1$ for t in a neighborhood of t_1, $\bar{t}$, t_2; consequently $a \in \mathcal{C}^\infty(I)$ and the first part of (2.7) holds. More precisely (A.3) implies that

$$a(t) = 1 \qquad \text{for} \quad t \in J_1 \cup J_2, \tag{A.17}$$

where

$$J_1 = \left[t_1, t_1 + \frac{\pi}{3h_1}\right], \qquad J_2 = \left[t_2 - \frac{\pi}{3h_2}, t_2\right] \tag{A.18}$$

and a consequence of (A.10) is that $J_j \subseteq I_j$ for $j = 1,\, 2$.

Let us verify (2.9) and (2.10). (2.9) is a trivial consequence of (A.5) and (2.1), while from (A.6) and the concavity of μ we have that

$$\begin{aligned} |a|_{\mathcal{C}^\mu(I_j)} &= \sup_{t,\tau \in I_j,\ t \neq \tau} \frac{|a(t) - a(\tau)|}{\mu(|t - \tau|)} \\ &= \sup_{t,\tau \in I_j,\ 0 < |t-\tau| < 2\pi/h_j} \frac{|a(t) - a(\tau)|}{\mu(|t - \tau|)} \\ &= \sup_{t,\tau \in I_j,\ 0 < |t-\tau| < 2\pi/h_j} \frac{|a(t) - a(\tau)|}{|t - \tau|} \frac{|t - \tau|}{\mu(|t - \tau|)} \\ &\leq \frac{1}{2} M \varepsilon_j h_j \sup_{t,\tau \in I_j,\ 0 < |t-\tau| < 2\pi/h_j} \frac{|t - \tau|}{\mu(|t - \tau|)} \\ &\leq M\pi \frac{\varepsilon_j}{\mu(1/h_j)} \qquad \text{for} \quad j = 1,\, 2 \end{aligned}$$

and (2.10) follows. Moreover

$$\left| \frac{a'(t)}{a(t)} \right| \leq M \varepsilon_j h_j \qquad \text{for all} \quad t \in I_j,\ j = 1, 2. \tag{A.19}$$

Consider now a $\mathcal{C}^\infty$ real valued function β defined on $\mathbf{R}$ such that, for all $t \in \mathbf{R}$, $0 \leq \beta(t) \leq 1$ and

$$\beta(t) = \begin{cases} 0 & \text{for} \quad t \leq \dfrac{1}{4}, \\ 1 & \text{for} \quad t \geq \dfrac{3}{4}. \end{cases}$$

We set

$$\beta_1(t) = \beta\left(\frac{3}{\pi}(h_2(t_2 - t))\right), \quad \beta_2(t) = \beta\left(\frac{3}{\pi}(h_1(t - t_1))\right). \tag{A.20}$$

Let ψ_1, ψ_2 be the solutions of

$$\begin{cases} \psi_j''(t) + h_j^2 a(t)\psi_j(t) = 0 \qquad \text{on} \quad I, \\ \psi_j(t_j) = \eta_j, \ \psi_j'(t_j) = 0, \end{cases} \tag{A.21}$$

for $j = 1$, 2. We define

$$u(x,t) = \beta_1\psi_1 e^{ih_1 x} + \beta_2\psi_2 e^{ih_2 x}. \tag{A.22}$$

It is immediate to verify that u is $\mathcal{C}^\infty$ in t and 2π-periodic and analytic in x. Moreover u does not vanish identically on any open set of $I \times \mathbf{R}$, i.e. (2.6) holds. Since $a(t) = 1$ for all $t \in J_1 \cup J_2$ then

$$\psi_j(t) = \eta_j \cos(h_j(t - t_j)) \qquad \text{for} \quad t \in J_j, \ j = 1, 2, \tag{A.23}$$

and (2.8) follows.

We claim now that

$$u(t,x) \neq 0 \qquad \text{for} \quad (t,x) \in (J_1 \cup J_2) \times \mathbf{R}. \tag{A.24}$$

(A.24) will be deduced by the following facts:

$$\psi_j(t) \geq \frac{\eta_j}{2} \qquad \text{for} \quad t \in J_j, \ j = 1, 2, \tag{A.25}$$

and

$$|\psi_1(t)| \leq \frac{\eta_2}{4} \qquad \text{for} \quad t \in I_2, \qquad\qquad |\psi_2(t)| \leq \frac{\sqrt{2}}{4}\eta_1 \qquad \text{for} \quad t \in I_1. \tag{A.26}$$

In fact (A.25) and (A.26) will give

$$|u(t,x)| \geq \frac{\eta_1}{8} \qquad \text{for} \quad (t,x) \in J_1 \times \mathbf{R}$$

and

$$|u(t,x)| \geq \frac{\eta_2}{4} \qquad \text{for} \quad (t,x) \in J_2 \times \mathbf{R}.$$

The inequalities of (A.25) are a consequence of (A.23). Let's estimate ψ_1 on I_2 and ψ_2 on I_1. From (A.4) we have that

$$\begin{aligned} \psi_j(t) &= \eta_j w_{\varepsilon_j}(h_j|t - t_j|) \\ &= \eta_j e^{-\varepsilon_j h_j |t - t_j|}\tilde{w}_{\varepsilon_j}(h_j|t - t_j|) \qquad \text{for} \quad t \in I_j \end{aligned} \tag{A.27}$$

and, recalling (A.10), (A.15) and the properties of the function w_ε as stated in Lemma 2, we deduce that

$$\psi_j(\bar{t}) = \eta_j e^{-\varepsilon_j h_j \rho_j}, \qquad \psi_j'(\bar{t}) = 0, \quad j = 1,\, 2.$$

Let's introduce the quantities

$$e_j(t) = h_j^2 \psi_j^2(t) + (\psi_j'(t))^2, \quad E_j(t) = h_j^2 a(t) \psi_j^2(t) + (\psi_j'(t))^2.$$

Since $a(\bar{t}) = 1$ we have

$$e_j(\bar{t}) = E_j(\bar{t}) = h_j^2 \eta_j^2 e^{-2\varepsilon_j h_j \rho_j}, \quad j = 1,\, 2. \tag{A.28}$$

Using the fact that ψ_j solves the Cauchy problem (A.21) it is easy to obtain via differentiation and Gronwall's lemma that

$$e_1(t) \le e_1(\bar{t}) e^{h_1 \int_{\bar{t}}^{t} |1-a(s)|\, ds} \qquad \text{for} \quad t \ge \bar{t} \tag{A.29}$$

and

$$E_2(t) \le E_2(\bar{t}) e^{h_1 \int_{t}^{\bar{t}} \frac{|a'(s)|}{a(s)} ds} \qquad \text{for} \quad t \le \bar{t}. \tag{A.30}$$

Let now $t \in I_2$. Then from (A.28) and (A.29) we deduce

$$e_1(t) \le h_1^2 \eta_1^2 e^{-2\varepsilon_1 h_1 \rho_1} e^{M h_1 \varepsilon_2 \rho_2} = h_1^2 \eta_1^2 e^{-h_1 (2\varepsilon_1 \rho_1 - M \varepsilon_2 \rho_2)}.$$

By (2.1) and (A.12) we have $\varepsilon_1 \rho_1 \ge M \varepsilon_2 \rho_2$ and then

$$e_1(t) \le h_1^2 \eta_1^2 e^{-\varepsilon_1 h_1 \rho_1} \qquad \text{for all} \quad t \in I_2. \tag{A.31}$$

On the other hand if $t \in I_1$ we obtain from (A.19), (A.28) and (A.30)

$$E_2(t) \le h_2^2 \eta_2^2 e^{-2\varepsilon_2 h_2 \rho_2 + M h_1 \varepsilon_1 \rho_1}.$$

From (2.2) and (A.12) we have $\varepsilon_2 h_2 \rho_2 \ge M h_1 \varepsilon_1 \rho_1$ and hence

$$E_2(t) \le h_2^2 \eta_2^2 e^{-\varepsilon_2 h_2 \rho_2} \qquad \text{for all} \quad t \in I_1. \tag{A.32}$$

Recalling (A.11), the inequality (A.31) gives

$$|\psi_1(t)| \le \frac{\sqrt{e_1(t)}}{h_1} \le \eta_1 e^{-h_1 \varepsilon_1 \rho_1 / 2} \le \eta_1 e^{-2 h_1 \varepsilon_1 \rho} \qquad \text{for} \quad t \in I_2$$

but (2.3) implies that $4\eta_1 \le \eta_2 e^{h_1 \varepsilon_1 \rho}$, and then the first part of (A.26) follows. Similarly by (A.32)

$$|\psi_2(t)| \le \frac{\sqrt{2E_2(t)}}{h_2} \le \sqrt{2}\eta_2 e^{-h_2 \varepsilon_2 \rho_2 / 2} \le \sqrt{2}\eta_2 e^{-2 h_2 \varepsilon_2 \rho} \qquad \text{for} \quad t \in I_1,$$

again by (2.3) we have $4\eta_2 \le \eta_1 e^{h_2 \varepsilon_2 \rho}$ and from this we obtain the second part of (A.26).

We finally define

$$d(t,x) = \begin{cases} -\dfrac{u_{tt}(t,x) - a(t)u_{xx}(t,x)}{u(t,x)} & \text{for} \quad (t,x) \in (J_1 \cup J_2) \times \mathbf{R}, \\ 0 & \text{for} \quad (t,x) \in (I \setminus (J_1 \cup J_2)) \times \mathbf{R}. \end{cases}$$

Since $u_{tt}(t,x) - a(t)u_{xx}(t,x)$ is identically 0 in a neighborhood of $(I \setminus (J_1 \cup J_2)) \times \mathbf{R}$ and $u(t,x)$ is never 0 in $(J_1 \cup J_2) \times \mathbf{R}$ the function d is $\mathcal{C}^\infty$ in $I \times \mathbf{R}$.

To end the proof of the lemma it remains to show (2.11) and (2.12). For $p = 0$ (2.11) is a consequence of (A.7), (A.26) and (A.27). To prove (2.11) for $p \geq 1$ we argue as in [2, p. 508]. In particular for $j = 1, 2$ we have

$$\left|\left(\frac{d}{dt}\right)^p \beta_j(t)\right| \leq K_p h_2^p \qquad \text{for} \quad t \in I, \tag{A.33}$$

$$\left|\left(\frac{d}{dt}\right)^p \psi_j(t)\right| \leq \tilde{K}_p \eta_j h_2^p \qquad \text{for} \quad t \in I, \tag{A.34}$$

where K_p, $\tilde{K}_p$ does not depend on h_1, h_2, ε_1, ε_2, ρ, η_1, η_2 and t_1. (A.33) is trivial in view of (A.20). (A.34) is obtained from the following inequalities via [2, Lemma 3]:

$$\left|\left(\frac{d}{dt}\right)^p \psi_1(t)\right| \leq \begin{cases} L_p \eta_1 h_1^p & \text{for} \quad t \in I_1, \\ L_p \eta_1 h_1^p e^{\varepsilon_1 h_1 \rho_1/2} & \text{for} \quad t \in I_2, \end{cases} \tag{A.35}$$

$$\left|\left(\frac{d}{dt}\right)^p \psi_2(t)\right| \leq \begin{cases} \sqrt{2} L_p \eta_2 h_2^p e^{\varepsilon_2 h_2 \rho_2/2} & \text{for} \quad t \in I_1, \\ L_p \eta_2 h_2^p & \text{for} \quad t \in I_2, \end{cases} \tag{A.36}$$

where $L_0 = 1$ and L_p does not depend on h_1, h_2, ε_1, ε_2, ρ, η_1, η_2, t_1. (A.35) and (A.36) can be obtained by induction on p (see [2, p. 509]).

Let us finally show (2.12). Recalling that $d(t,x) = 0$ for t in a neighborhood of $I \setminus (J_1 \cup J_2)$, it will be sufficient to estimate the derivatives of d for $t \in J_1 \cup J_2$. Suppose first that $t \in J_2$. Setting

$$f_1(t) = 2\beta_1'(t)\psi_1'(t) + \beta_1''(t)\psi_1(t), \tag{A.37}$$

we have

$$d(t,x) = \frac{f_1(t)e^{ih_1 x}}{\beta_1(t)\psi_1(t)e^{ih_1 x} + \psi_2(t)e^{ih_2 x}}.$$

Since $|\psi_1(t)/\psi_2(t)| \leq 1/2$ for $t \in J_2$, we deduce from (A.37) that

$$d(t,x) = \sum_{n=1}^{+\infty} f_1(t)(-\beta_1(t)\psi_1(t))^{n-1}(\psi_2(t))^{-n} e^{-in\tilde{h}x}, \tag{A.38}$$

where $\tilde{h} = h_2 - h_1$. By (A.33), (A.35) and (A.37) we have

$$\left|\left(\frac{d}{dt}\right)^p f_1\right| \leq \tilde{L}_p C \eta_1 h_2^{p+2} e^{-\varepsilon_1 h_1 \rho_1/2},$$

where $\tilde{L}_0 = 1$. Again by (A.33) and (A.35) using [2, Lemma 3], we deduce that

$$\left|\left(\frac{d}{dt}\right)^p \beta_1\psi_1\right| \leq K_p\eta_1 h_2^p e^{-\varepsilon_1 h_1\rho_1/2}$$

with $K_0 = 1$. Arguing similarly we have

$$\left|\left(\frac{d}{dt}\right)^p (\beta_1\psi_1)^{n-1}\right| \leq \tilde{K}_p(n-1)^p\eta_1^{n-1}h_2^p e^{-(n-1)\varepsilon_1 h_1\rho_1/2},$$

$$\left|\left(\frac{d}{dt}\right)^p \psi_2^{-n}\right| \leq \tilde{K}_p n^p \left(\frac{2}{\eta_2}\right)^n h_2^p,$$

$$\left|\left(\frac{d}{dt}\right)^p (f_1(\beta_1\psi_1)^{n-1})\right| \leq \tilde{K}_p C n^p \eta_1^n h_2^{p+2} e^{-n\varepsilon_1 h_1\rho_1/2}$$

with $\tilde{K}_0 = 1$; finally

$$\left|\left(\frac{d}{dt}\right)^p (f_1(\beta_1\psi_1)^{n-1}\psi_2^{-n})\right| \leq \tilde{C}_p n^p \left(\frac{2\eta_1}{\eta_2}\right)^n h_2^{p+2} e^{-n\varepsilon_1 h_1\rho_1/2}. \tag{A.39}$$

By using (2.3) and (A.11) we obtain from (A.39) that

$$\left|\left(\frac{d}{dt}\right)^p (f_1(\beta_1\psi_1)^{n-1}\psi_2^{-n})\right| \leq \tilde{C}_p n^p h_2^{p+2} e^{-n\varepsilon_1 h_1\rho}$$

and since $|\tilde{h}| \leq h_2$ the inequality (2.12) follows from (A.38). We let to the interested reader to verify the similar estimate for $t \in J_1$. The proof of Lemma 1 is concluded.

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References

[1] F. Colombini, E. De Giorgi and S. Spagnolo, Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temp, Ann. Sc. Norm. Sup. Pisa, **6** (1979), 511–559.

[2] F. Colombini, E. Jannelli and S. Spagnolo, Non-uniqueness in hyperbolic Cauchy problems, Ann. of Math., **126** (1987), 495–524.

[3] F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, Duke Math. J., **77** (1995), 657–698.

[4] D. Del Santo, A remark on non-uniqueness in the Cauchy problem for elliptic operators having non-Lipschitz coefficients, to appear in "Differential Operators and Mathematical Physics", V. Ancona and J. Vaillant eds., Marcel Dekker.

[5] A. Pliś, On non-uniqueness in Cauchy problem for an elliptic second order differential equation, Bull. Acad. Pol. Sci., **11** (1963), 95–100.

[6] S. Tarama, Local uniqueness in the Cauchy problem for second order elliptic equations with non-Lipschitzian coefficients, Publ. Res. Inst. Math. Sci., **33** (1997), 167–188.