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A generalization of the Buckdahn-Föllmer formula for composite transformations defined by finite dimensional substitution

By

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Abstract

A generalization of the Buckdahn-Föllmer formula is obtained by considering a composite transformation $\xi(x, F(x))$ in the framework of the Ramer-Kusuoka formula where $F(x)$ takes values in a finite dimensional space. The point is to establish the chain rule for composite Wiener functionals through the continuity of the substitution. The localization argument makes it possible to deal in our framework with the transformations studied by C. Donati-Martin, H. Matsumoto and M. Yor [5]. Our formula gives a new approach to the study of quadratic Wiener functionals.

1. Introduction

The change of variables formula on Wiener spaces has been studied since by R. H. Cameron and W. T. Martin [2], [3]. For adapted transformations on the classical Wiener space the formula was established by G. Maruyama [13] and I. V. Girsanov [7]. For anticipative transformations the formula was obtained by R. Ramer [17] and S. Kusuoka [11] and generalized by A. S. Üstünel and M. Zakai [19] in the framework of the Malliavin calculus. The density in the Ramer-Kusuoka formula is given by the product of two factors: one is the modified Girsanov density in the sense that the Itô integral is replaced by the Skorohod integral and the other is the Carleman-Fredholm determinant. Besides, there is an intermediate formula obtained by R. Buckdahn and H. Föllmer [1]. They deal with anticipative transformations on the classical Wiener space of the form $\xi(x, x_T)$ where $\xi(\cdot, y)$ for fixed y is the solution to a certain stochastic differential equation. The density in their formula is given by the Girsanov density of $\xi(\cdot, y)$ evaluated at $y = x_T$ multiplied by an extra factor.

In this paper we consider the Buckdahn-Föllmer type transformations in the framework of the Ramer-Kusuoka formula and give a generalization of the Buckdahn-Föllmer formula. We study composite transformations on an abstract Wiener space (X, H, μ) of the form $\xi(x, F(x))$ where $\xi(x, y) = x +$

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$u(x,y)$ and $u(x,y)$ takes values in H. We assume that $F(x)$ takes values in a finite dimensional space and belongs to a wide class of Wiener functionals including x_T. Then we obtain another factorization of the Ramer-Kusuoka density. The density in our formula is given by the Ramer-Kusuoka density of $\xi(\cdot,y)$ evaluated at $y=F$ multiplied by an extra factor which is expressed as a finite dimensional determinant. The point is the factorization of the Carleman-Fredholm determinant of the gradient of $\xi(x,F(x))$, which we carry out by applying the chain rule.

C. Donati-Martin, H. Matsumoto and M. Yor studied in their paper [5] a certain class of transformations and found that their transformations can be considered to be an example of the Buckdahn-Föllmer formula. We show that their transformations can be dealt with in our framework by the localization argument. We can also deal in our framework with the class of quadratic Wiener functionals studied by N. Ikeda, S. Kusuoka and S. Manabe [8], which means that their functionals can be characterized by some anticipative transformations of the form $\xi(x,F(x))$.

The essential part of our proof is to establish the chain rule for composite Wiener functionals of the form $u(x,F(x))$. If u is a polynomial functional then the chain rule is obvious. In order to make the polynomial approximation work we need the continuity of the substitution $u(x,y)\mapsto u(x,F(x))$ on suitable functional spaces. We regard $u(x,y)$ as a Wiener functional on the extended Wiener space $(X\oplus\mathbb{R}^m, H\oplus\mathbb{R}^m, \mu\times\nu)$ where ν is the standard Gaussian measure on $\mathbb{R}^m$. We introduce the partial Sobolev spaces $\mathbb{D}^{(m,k),p}(X\oplus\mathbb{R}^m;H)$ and prove that the substitution is $\mathbb{D}^{(0,m),p}/L^p$-continuous (Proposition 4.1). The key to the proof of the continuity is to regard the evaluation of $u(x,y)$ at $y=F(x)$ as the product of $u(x,y)$ and $\delta_0(y-F(x))$ (Lemma 4.1). Here we define the pullback $\delta_0(y-F(x))$ as a distribution in $\mathbb{D}^{(0,-m),\infty-}(X\oplus\mathbb{R}^m;\mathbb{R})$ following S. Watanabe's original idea [21].

The paper is organized as follows. In Section 2 we state our main theorems. We also refer to the relation between our result and the Buckdahn-Föllmer formula. In Section 3 we use the partial Malliavin calculus to define the Sobolev spaces associated with the partial derivatives. The theory of the partial Malliavin calculus was first introduced in [12] and developed in [9], [16]. In this paper we confine ourselves to the product Wiener spaces and define the pullback $\delta_0(y-F(x))$. In Section 4 we prove the continuity of the substitution $f\mapsto f_F$ on the partial Sobolev spaces. In Section 5 we establish the chain rule for a composite functional u_F. In these two sections we impose boundedness on F and ∇F. In Section 6 we relax the boundedness condition of F and ∇F for the chain rule by the localization argument. Section 7 is devoted to the proof of Theorem 2.1. In Section 8 we consider Buckdahn-Föllmer type transformations for general Wiener functionals F. In Section 9 we apply our theorems to the transformations of C. Donati-Martin, H. Matsumoto and M. Yor and to the quadratic Wiener functionals of N. Ikeda, S. Kusuoka and S. Manabe.

2. Main theorems

To begin with, we introduce some notations which are used in what follows. The subscript F stands for the substitution $y = F$: for example,

$$u_F(x) = u(x, F(x)), \quad \xi_F(x) = \xi(x, F(x)) \quad \text{and} \quad \Lambda_F(x) = \Lambda(x, F(x)) \tag{2.1}$$

where $u(x, y)$, $\xi(x, y)$ and $\Lambda(x, y)$ are functionals on $X \oplus \mathbb{R}^m$. We say F belongs to the class $\mathbb{D}^{1,\infty}_{\text{loc}}(X; \mathbb{R}^m)$ if F is an $\mathbb{R}^m$-valued functional on X such that F is locally differentiable and both F and ∇F are locally bounded. The precise definition of $\mathbb{D}^{k,p}_{\text{loc}}$ will be given in Section 6. The class $\mathbb{D}^{(n,k),p}(X \oplus \mathbb{R}^m; H)$ consists of H-valued functionals on the extended Wiener space $X \oplus \mathbb{R}^m$ which are n and k-times partially differentiable in the direction of X and $\mathbb{R}^m$ respectively, and all the partial derivatives are in L^p. The partial gradient operators in the direction of X and $\mathbb{R}^m$ will be denoted by ∇_X and ∇_Y respectively. The partial divergence operator in the direction of X will be denoted by ∇^*_X. The precise definitions will be given in Section 3. By $\mathbb{E}_x$ we denote the expectation in x with respect to the measure μ. We denote the Carleman-Fredholm determinant of $I_H + B$ by $\text{Det}_2(I_H + B)$ where B is a Hilbert-Schmidt operator on the Hilbert space H. The definition and the properties which we use in this paper are found in [6], [18], [22]. We denote the trace of a trace class operator B on H by $\text{Trace}\, B$. In contrast, we denote the finite dimensional determinant and trace of a matrix M on $\mathbb{R}^m$ by $\det M$ and $\text{trace}\, M$ respectively.

What we deal with is the composite transformation ξ_F. Before we state the main theorem we list the assumptions.

(A0) $F(x)$ belongs to $\mathbb{D}^{1,\infty}_{\text{loc}}(X; \mathbb{R}^m)$.

It is immediately seen by definition given in Section 6 that the class $\mathbb{D}^{1,\infty}_{\text{loc}}(X; \mathbb{R}^m)$ includes such functionals as x_T and $\int_0^T x_s ds$. More generally, it includes the class H-$C^1_{\text{loc}}(X; \mathbb{R}^m)$ introduced in [11]. The proof is given in [11] and also found in [20]. For $\xi(x, y)$ we assume the following four.

(A1) For almost every $y \in \mathbb{R}^m$, the transformation $\xi(\cdot, y)$ has an inverse $\eta(\cdot, y)$:

$$\xi(\eta(x, y), y) = \eta(\xi(x, y), y) = x$$

for almost every $x \in X$.

(A2) There are functionals

$$u(x, y) \quad \text{and} \quad v(x, y) \in \mathbb{D}^{(1,m+1),\infty-}(X \oplus \mathbb{R}^m; H)$$

such that

$$\xi(x, y) = x + u(x, y) \quad \text{and} \quad \eta(x, y) = x + v(x, y).$$

(A3) For almost every $y \in \mathbb{R}^m$, the transformation $\xi(\cdot, y)$ satisfies the Ramer-Kusuoka formula:

$$\mathbb{E}[\varphi] = \mathbb{E}_x[\varphi(\xi(x, y))|\Lambda(x, y)|]$$

for any non-negative functional φ where the density $\Lambda(x,y)$ is given by

$$\begin{aligned}\Lambda(x,y) &= \mathrm{Det}_2(I_H + (\nabla_X u)(x,y)) \\ &\qquad \times \exp\left(-(\nabla_X^* u)(x,y) - \frac{1}{2}|u(x,y)|_H^2\right).\end{aligned} \tag{2.2}$$

(A4) The density $\Lambda(x,y)$ belongs to $L^{1+\varepsilon}(X \oplus \mathbb{R}^m; \mathbb{R})$ for some $\varepsilon > 0$.

Now we state the main theorem. The proof will be given in Section 7. We denote by $\widetilde{\Lambda}$ the Ramer-Kusuoka density of the composite transformation ξ_F. That is, $\widetilde{\Lambda}$ is factorized as follows:

$$\widetilde{\Lambda} = \mathrm{Det}_2(I_H + \nabla u_F) \exp\left(-\nabla^* u_F - \frac{1}{2}|u_F|_H^2\right). \tag{2.3}$$

We define

$$G_Y(x,y) = (\nabla_Y v) \circ (\xi(x,y), y). \tag{2.4}$$

Theorem 2.1. *Suppose* (A0)–(A4). *Then the Ramer-Kusuoka density* $\widetilde{\Lambda}$ *admits another factorization*:

$$\widetilde{\Lambda} = \Lambda_F \cdot \det(I_{\mathbb{R}^m} - (\nabla F)(G_Y)_F). \tag{2.5}$$

Theorem 2.1 is immediately derived from the following.

Proposition 2.1. *Under the assumptions of Theorem* 2.1 *the following factorizations hold*:

$$\begin{aligned}\mathrm{Det}_2(I_H + \nabla u_F) &= \mathrm{Det}_2(I_H + (\nabla_X u)_F) \det(I_{\mathbb{R}^m} - (\nabla F)(G_Y)_F) \\ &\qquad \times \exp(\mathrm{trace}((\nabla F)(I_H + (\nabla_X u)_F)(G_Y)_F))\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\exp\left(-\nabla^* u_F - \frac{1}{2}|u_F|_H^2\right) &= \exp\left(-(\nabla_X^* u)_F - \frac{1}{2}|u_F|_H^2\right) \\ &\qquad \times \exp(-\,\mathrm{trace}((\nabla F)(I_H + (\nabla_X u)_F)(G_Y)_F)).\end{aligned} \tag{2.7}$$

The key to the proof of Theorem 2.1 is to establish the chain rule for ∇u_F and $\nabla^* u_F$. The proof of the following theorem will be given by Theorems 6.1 and 6.2 in Section 6.

Theorem 2.2. *Suppose that* $F(x)$ *belongs to* $\mathbb{D}^{1,\infty}_{\mathrm{loc}}(X; \mathbb{R}^m)$. *Let* $u(x,y)$ *be in* $\mathbb{D}^{(1,m+1),p}_{\mathrm{loc}}(X \oplus \mathbb{R}^m; H)$ *for some* $1 < p < \infty$. *Then*

$$u_F \in \mathbb{D}^{1,p}_{\mathrm{loc}}(X; H)$$

and

$$\begin{aligned}\nabla u_F &= (\nabla_X u)_F + (\nabla_Y u)_F(\nabla F), \\ \nabla^* u_F &= (\nabla_X^* u)_F - \mathrm{trace}((\nabla F)(\nabla_Y u)_F).\end{aligned} \tag{2.8}$$

From Theorem 2.2 we can derive the following substitution formula (Example 6.1):

$$\nabla^* \left(\int_0^{\cdot} (\alpha_s)_F ds \right) = \int_0^T \langle \alpha_s(x,y), dx_s \rangle_{\mathbb{R}^m} \Big|_{y=F} - \int_0^T \operatorname{trace}((D_s F)(\nabla_Y \alpha_s)_F) ds. \tag{6.3}$$

Finally we recall the Buckdahn-Föllmer formula to compare it with our formula (2.5). Let $X = C_0([0,T];\mathbb{R})$ be the 1-dimensional classical Wiener space. R. Buckdahn and H. Föllmer considered the transformation defined by the strong solution to a stochastic differential equation of the form

$$d\tilde{\xi}_t = dx_t + \alpha_t(\tilde{\xi}, x_T)dt, \qquad \tilde{\xi}_0 = 0 \tag{2.9}$$

where $(\alpha_t(\cdot, y); t \in [0,T])$ is adapted. By involving the terminal value x_T, the drift $(\alpha_t(\tilde{\xi}, x_T); t \in [0,T])$ is anticipative. Let $\xi(x,y)$ be the strong solution to the stochastic differential equation

$$d\xi_t(x,y) = dx_t + \alpha_t(\xi(x,y), y)dt, \qquad \xi_0(x,y) = 0.$$

Then $\tilde{\xi}$ is expressed as $\tilde{\xi}(x) = \xi(x, x_T)$. They imposed some integrability condition on $\alpha_t(\cdot, y)$ so that for fixed y the Maruyama-Girsanov formula

$$\mathbb{E}[\varphi(x)] = \mathbb{E}_x[\varphi(\xi(x,y))\Lambda(x,y)]$$

holds where the density $\Lambda(x,y)$ is given by

$$\Lambda(x,y) = \exp\left(-\int_0^T \alpha_s(\xi(x,y),y)dx_s - \frac{1}{2}\int_0^T \alpha_s^2(\xi(x,y),y)ds \right). \tag{2.10}$$

Moreover, they imposed some additional regularity conditions for the quasi-sure analysis in order that the conditional expectation $\mathbb{E}[f|x_T = y]$ has a smooth version in y. They split $X = X^0 \oplus \mathbb{R}$ where X^0 is the pinned Wiener space and then used the co-area formula on the Wiener space. Then they obtained a change of variables formula (Corollary 3.36 in [1])

$$\mathbb{E}[\varphi(x)] = \mathbb{E}[\varphi(\tilde{\xi}(x))|\Lambda'(x)|] \tag{2.11}$$

where the density Λ' is given by

$$\Lambda'(x) = \Lambda(x, x_T) \cdot \left(1 + \int_0^T \frac{\partial}{\partial y} \alpha_s(\tilde{\xi}(x), x_T) ds \right). \tag{2.12}$$

In this setting the factorization (2.5) coincides with (2.12). To put it more precisely, Λ_F and the finite dimensional determinant in (2.5) turn out to be $\Lambda(x, x_T)$ and the remaining factor in (2.12) respectively. This will be verified in Section 8.

3. Partial Sobolev spaces and partially non-degenerate functionals

Let (X, H, μ) be an abstract Wiener space. That is, X is a real separable Banach space, H is a real separable Hilbert space which is continuously and densely embedded into X and μ is a probability measure on X such that

$$\int_X \exp(\sqrt{-1}_X\langle x, h\rangle_{X^*})\mu(dx) = \exp\left(-\frac{1}{2}|h|_H^2\right)$$

for all $h \in X^*$. Here the H-norm $|h|_H$ of an element h in X^* is measured through the injection: $X^* \hookrightarrow H^* \simeq H$. Let (Y, H', ν) be another abstract Wiener space. Then the product space $(X \oplus Y, H \oplus H', \mu \times \nu)$ is again an abstract Wiener space. Let E be a separable Hilbert space. We call $f(x, y)$ an E-valued polynomial functional on $X \oplus Y$ if $f(x, y)$ is of the form

$$f(x,y) = \sum_{k:\text{finite}} p^k(\langle x, h_1\rangle, \ldots, \langle x, h_n\rangle, \langle y, h'_1\rangle, \ldots, \langle y, h'_n\rangle)e_k \tag{3.1}$$

where n is a positive integer, $p^k(t_1, \ldots, t_{2n})$ is an $\mathbb{R}$-valued polynomial in $2n$ variables, (h_j) and (h'_j) are orthonormal systems of H and H' which are taken from X^* and Y^* respectively, and e_k's are elements of E. We denote by $\mathcal{P}(X \oplus Y; E)$ the set of E-valued polynomial functionals on $X \oplus Y$.

Define the partial derivative of $f(x, y) \in \mathcal{P}(X \oplus Y; E)$ given by (3.1) in the direction of X as the Hilbert-Schmidt operator from H to E such that

$$\nabla_X f(x,y)[h] = \frac{d}{d\lambda}(f(x+\lambda h, y))|_{\lambda=0}$$

for any $h \in H$. For two real separable Hilbert spaces E_1 and E_2 we identify the Hilbert space of the tensor product $E_1 \otimes E_2$ with that of Hilbert-Schmidt operators from E_2 to E_1 equipped with the Hilbert-Schmidt norm. Under this identification $\nabla_X f(x, y)$ is an element of $\mathcal{P}(X \oplus Y; E \otimes H)$ such that

$$\nabla_X f(x,y) = \sum_{\substack{k:\text{finite}\\ j=0,\ldots,n}} \frac{\partial p^k}{\partial t_j}(\langle x, h_1\rangle, \ldots, \langle x, h_n\rangle, \langle y, h'_1\rangle, \ldots, \langle y, h'_n\rangle)e_k \otimes h_j.$$

In the same way the partial derivative of $f(x, y)$ in the direction of Y is defined as an $E \otimes H'$-valued functional and denoted by $\nabla_Y f(x, y)$.

It is immediate to see that the partial gradient operators ∇_X and ∇_Y are closable on $L^p(X \oplus Y; E)$ for any $1 < p < \infty$. So we can define the Sobolev spaces associated with the partial derivatives.

Definition 3.1. For $n, k \in \mathbb{Z}_{\geq 0}$, $1 < p < \infty$, the partial Sobolev space $\mathbb{D}^{(n,k),p}(X \oplus Y; E)$ is defined by the completion of $\mathcal{P}(X \oplus Y; E)$ with respect to the norm

$$\sum_{\substack{i=0,\ldots,n\\ j=0,\ldots,k}} \|\nabla_X^i \nabla_Y^j f\|_p.$$

Its dual space will be denoted by $\mathbb{D}^{(-n,-k),q}(X \oplus Y; E)$ where $p^{-1} + q^{-1} = 1$.

The partial gradient operators ∇_X and ∇_Y defined on polynomials are extended to those on the partial Sobolev spaces, which are denoted by the same symbols. The partial divergence operators ∇_X^* and ∇_Y^* are defined as the dual operators of ∇_X and ∇_Y respectively.

Remark 1. The partial derivatives do not depend on the order of the partial differentiation up to the change of the order of the tensor products. For example, let

$$\iota : E \otimes H \otimes H' \to E \otimes H' \otimes H$$

be the isomorphism defined by

$$\iota(e \otimes h \otimes h') = e \otimes h' \otimes h.$$

Then it follows that

$$\iota(\nabla_Y \nabla_X f(x, y)) = \nabla_X \nabla_Y f(x, y)$$

for any polynomial functional $f(x, y)$.

Remark 2. The following relations hold between the partial Sobolev spaces $\mathbb{D}^{(n,k),p}$ and the "total" Sobolev spaces $\mathbb{D}^{k,p}$: for any $n, k \in \mathbb{Z}_{\geq 0}$ and $1 < p < \infty$,

$$\mathbb{D}^{n+k,p}(X \oplus Y; E) \subset \mathbb{D}^{(n,k),p}(X \oplus Y; E) \subset \mathbb{D}^{\min\{n,k\},p}(X \oplus Y; E).$$

In order to consider the pullbacks of tempered distributions, we need the following.

Definition 3.2. A functional $\widetilde{F}(x, y)$ in $\mathbb{D}^{(0,\infty),\infty-}(X \oplus Y; \mathbb{R}^m)$ is called partially non-degenerate in the direction of Y if

$$1/\det((\nabla_Y \widetilde{F})(\nabla_Y \widetilde{F})^*) \in L^{\infty-}(X \oplus Y; \mathbb{R}).$$

Here the superscript $*$ stands for the adjoint of a Hilbert-Schmidt operator.

Now let us introduce the notations of the Schwartz distribution spaces. Let $\mathcal{S}(\mathbb{R}^m; \mathbb{R})$ and $\mathcal{S}'(\mathbb{R}^m; \mathbb{R})$ be the class of rapidly decreasing functions and tempered distributions respectively. Let $A\varphi(y) = (1 - \Delta_{\mathbb{R}^m} + |y|^2_{\mathbb{R}^m})\varphi(y)$ for $\varphi \in \mathcal{S}(\mathbb{R}^m; \mathbb{R})$ and define S_{2k} by the completion of $\mathcal{S}(\mathbb{R}^m; \mathbb{R})$ with respect to the norm $\| \cdot \|_{2k} := \|A^k \cdot \|_{\sup}$. Then it holds that $\mathcal{S}(\mathbb{R}^m; \mathbb{R}) = \cap_{k \in \mathbb{Z}} S_{2k}$ and $\mathcal{S}'(\mathbb{R}^m; \mathbb{R}) = \cup_{k \in \mathbb{Z}} S_{2k}$. The key fact to the definition of the pullbacks of tempered distributions is the continuity of the substitution $T \mapsto T \circ F$ on the Schwartz distribution space.

Lemma 3.1. *Suppose $\widetilde{F}$ in $\mathbb{D}^{(0,\infty),\infty-}(X \oplus Y; \mathbb{R}^m)$ is partially non-degenerate in the direction of Y. Then for any $1 < p < \infty$ the linear map*

$$\varphi \mapsto \varphi \circ \widetilde{F}$$

is $\mathcal{S}'(\mathbb{R}^m;\mathbb{R})/\mathbb{D}^{(0,-\infty),p}(X\oplus Y;\mathbb{R})$*-continuous on* $\mathcal{S}(\mathbb{R}^m;\mathbb{R})$*. More precisely, for any* $1<p<\infty$ *and for any* $k\in\mathbb{N}$ *there exists a constant* $c_{k,p}<\infty$ *such that*

$$\|\varphi\circ\widetilde{F}\|_{(0,-2k),p}\leq c_{k,p}\|\varphi\|_{-2k}$$

holds for any $\varphi\in\mathcal{S}(\mathbb{R}^m;\mathbb{R})$*.*

The proof can be given through a parallel procedure to that in [21]. So we omit the proof of Lemma 3.1.

By Lemma 3.1 we can define the pullbacks as follows. Let T be in $\mathcal{S}'(\mathbb{R}^m;\mathbb{R})$ and $\widetilde{F}$ be in $\mathbb{D}^{(0,\infty),\infty-}(X\oplus Y;\mathbb{R}^m)$ and suppose $\widetilde{F}$ is partially non-degenerate in the direction of Y. Take $\varphi_n\in\mathcal{S}(\mathbb{R}^m;\mathbb{R})$ as an approximating sequence of $T\in S_{-2k}$ such that $\varphi_n\to T$ in S_{-2k}. Then the pullback $T\circ\widetilde{F}$ is defined by the limit of the sequence $\varphi_n\circ\widetilde{F}$ which converges in $\mathbb{D}^{(0,-2k),p}(X;\mathbb{R})$ for any $1<p<\infty$.

In what follows we deal with the case where

$$Y=\mathbb{R}^m \text{ and } \nu \text{ is the standard Gaussian measure on } \mathbb{R}^m.$$

In other words, we extend an abstract Wiener space (X,H,μ) to the space

$$(X\oplus\mathbb{R}^m, H\oplus\mathbb{R}^m, \mu\times\nu)$$

by adding a finite dimensional standard Gaussian space $(\mathbb{R}^m,\mathbb{R}^m,\nu)$. We denote the gradient (resp. divergence) operator on (X,H,μ) by ∇ (resp. ∇^*).

In what follows we need the pullback of the Dirac delta δ_0 on $\mathbb{R}^m$ concentrated at 0 by a functional

$$\widetilde{F}(x,y)=y-F(x).$$

We suppose that $F(x)\in L^{\infty-}(X;\mathbb{R}^m)$. Then

$$\widetilde{F}(x,y)\in\mathbb{D}^{(0,\infty),\infty-}(X\oplus\mathbb{R}^m;\mathbb{R}^m).$$

Since

$$\det((\nabla_Y\widetilde{F})(\nabla_Y\widetilde{F})^*)=1,$$

the functional $\widetilde{F}(x,y)$ is partially non-degenerate in the direction of Y. Thus the pullback $\delta_0(y-F(x))$ is well-defined. Note that $F(x)$ itself may possibly be degenerate. The Dirac delta is expressed as $\delta_0(y)=(D_1\cdots D_mH)(y)$ where $H(y)=\prod_{i=1}^m 1_{[0,\infty)}(y_i)$ is the Heviside function on $\mathbb{R}^m$. So we find that

$$\delta_0(y-F(x))\in\mathbb{D}^{(0,-m),\infty-}(X\oplus\mathbb{R}^m;\mathbb{R}). \tag{3.2}$$

4. Continuity of the substitution

In this section we prove the continuity of the substitution $f\mapsto f_F$. We have referred to the notation of the subscript F in (2.1).

Proposition 4.1. *Suppose $F : X \to \mathbb{R}^m$ is measurable and bounded. Then for any $1 < p < \infty$ the linear map*

$$f \mapsto f_F$$

is $\mathbb{D}^{(0,m),p}/L^p$-continuous.

Remark 3. We impose on $f(x,y)$ the m-times differentiability in the direction of the finite dimensional subspace Y. Then by Sobolev's lemma $f(x,y)$ has a version $\widetilde{f}(x,y)$ which is continuous in y for almost every x. We can define the composite functional $f(x,F(x))$ as the composite functional $\widetilde{f}(x,F(x))$ obtained by taking such a version.

The key to the proof of Proposition 4.1 is the following.

Lemma 4.1. *Suppose $F : X \to \mathbb{R}^m$ is measurable and bounded. Let $f(x,y)$ be in $\mathbb{D}^{(0,m),p}(X \oplus \mathbb{R}^m; E)$ for $1 < p < \infty$ and ρ be a smooth $\mathbb{R}$-valued function on $\mathbb{R}^m$ with compact support such that $\rho(0) = 1$. Then for any test functional $\varphi \in \mathbb{D}^{\infty,\infty-}(X;E)$*

$$\begin{aligned}\mathbb{E}[\langle f_F, \varphi\rangle] = \mathbb{E}_{(x,y)} \Bigg[& \langle f(x,y), \varphi(x)\rangle_E \rho(y - F(x)) c\delta_0(y - F(x)) \\ & \times \exp\left(\langle y, F(x)\rangle_{\mathbb{R}^m} - \frac{1}{2}|F(x)|^2_{\mathbb{R}^m}\right)\Bigg],\end{aligned} \tag{4.1}$$

where $c = (2\pi)^{m/2}$. In the right hand side the expectation on $X \oplus \mathbb{R}^m$ is understood as the dual paring of a smooth functional and a distribution $\delta_0(y - F(x))$ (see (3.2)).

Proof. Note that the expectation of the right hand side of (4.1) makes sense because

$$\langle f(x,y), \varphi(x)\rangle_E \rho(y - F(x)) \exp\left(\langle y, F(x)\rangle_{\mathbb{R}^m} - \frac{1}{2}|F(x)|^2_{\mathbb{R}^m}\right)$$

belongs to $\mathbb{D}^{(0,m),p-}(X \oplus \mathbb{R}^m; \mathbb{R})$. Then by direct computation we have

$$\begin{aligned} f(x,F(x)) &= \int_{\mathbb{R}^m} f(x, y + F(x))\rho(y)\delta_0(y) e^{-\frac{1}{2}|y|^2} dy \\ &= \int_{\mathbb{R}^m} f(x, y + F(x))\rho(y) c\delta_0(y)\nu(dy) \\ &= \int_{\mathbb{R}^m} f(x,y)\rho(y - F(x)) c\delta_0(y - F(x)) \\ &\qquad \times \exp\left(\langle y, F(x)\rangle_{\mathbb{R}^m} - \frac{1}{2}|F(x)|^2_{\mathbb{R}^m}\right)\nu(dy). \end{aligned}$$

□

Now we give the proof to the continuity of the substitution.

Proof of Proposition 4.1. Since $\delta_0(y) = (D_1 \cdots D_m H)(y)$ we have

$$\delta_0(y - F(x)) = \nabla_Y^m (H(y - F(x)))[r_1 \otimes \cdots \otimes r_m]$$

where $\{r_1, \ldots, r_m\}$ denotes the standard basis of $\mathbb{R}^m$. By this and Lemma 4.1 we have

$$\begin{aligned} \mathbb{E}[\langle f_F, \varphi \rangle] &= \mathbb{E}_{(x,y)} \Bigg[\langle f(x,y), \varphi(x) \rangle_E \exp\left(\langle y, F(x) \rangle_{\mathbb{R}^m} - \frac{1}{2} |F(x)|^2_{\mathbb{R}^m} \right) \\ &\qquad \times \rho(y - F(x)) c \nabla_Y^m (H(y - F(x)))[r_1 \otimes \cdots \otimes r_m] \ \Bigg] \end{aligned} \tag{4.2}$$

for any $\varphi(x) \in \mathbb{D}^{\infty,\infty-}(X; E)$. Applying the integration by parts formula

$$\nabla_Y^* G = \langle G, y \rangle_{\mathbb{R}^m} - \operatorname{trace}(\nabla_Y G)$$

to the right hand side of (4.2), we obtain

$$\begin{aligned} \mathbb{E}[\langle f_F, \varphi \rangle] &= \mathbb{E}_{(x,y)} \Bigg[\left\langle \varphi(x),\ \sum_{i=0}^m (\nabla_Y^i f)(x,y)[P_i(x,y)] \right\rangle_E H(y - F(x)) \\ &\qquad \times 1\{y - F(x) \in \operatorname{supp}(\rho)\} \exp\left(\langle y, F(x) \rangle_{\mathbb{R}^m} - \frac{1}{2} |F(x)|^2_{\mathbb{R}^m} \right) \Bigg]. \end{aligned}$$

Here for $i = 0, 1, \ldots, m$, $P_i(x,y)$ is an $(\mathbb{R}^m)^{\otimes i}$-valued polynomial in y, $F(x)$ and $\rho^{(k)}(y - F(x))$ for $k = 0, \ldots, m$. Since $F(x)$ is bounded and the support of ρ is compact, we find that there is a constant $M < \infty$ independent of φ and f such that

$$|\mathbb{E}[\langle f_F, \varphi \rangle]| \le M \|\varphi\|_{L^q(X;E)} \|f\|_{\mathbb{D}^{(0,m),p}(X \oplus \mathbb{R}^m; E)}$$

where $p^{-1} + q^{-1} = 1$. Thus we obtain

$$\|f_F\|_{L^p(X;E)} \le M \|f\|_{\mathbb{D}^{(0,m),p}(X \oplus \mathbb{R}^m; E)},$$

which completes the proof. □

5. Chain rule for composite functionals

For simplicity we say that a functional f belongs to $\mathbb{D}^{1,\infty}$ if f belongs to $\mathbb{D}^{1,\infty-}$ and both f and ∇f are bounded.

Lemma 5.1. *Suppose $F(x)$ belongs to $\mathbb{D}^{1,\infty}(X; \mathbb{R}^m)$. Let $f(x,y)$ be in $\mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; E)$ for $1 < p < \infty$. Then*

$$f_F \in \mathbb{D}^{1,p}(X; E) \tag{5.1}$$

and the following chain rule holds:

$$\nabla f_F = (\nabla_X f)_F + (\nabla_Y f)_F (\nabla F). \tag{5.2}$$

Proof. The assertion is trivial when $f \in \mathcal{P}(X \oplus \mathbb{R}^m; E)$. For a general f in $\mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; E)$ we take an approximating sequence $\{f^n\}$ such that $f^n \in \mathcal{P}(X \oplus \mathbb{R}^m; E)$ and $f^n \to f$ in $\mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; E)$. Then each f^n satisfies the chain rule (5.2). By Proposition 4.1 we have

$$(f^n)_F \to f_F \qquad \text{in} \quad L^p(X; E),$$
$$(\nabla_X f^n)_F \to (\nabla_X f)_F \qquad \text{in} \quad L^p(X; E \otimes H)$$

and

$$(\nabla_Y f^n)_F \to (\nabla_Y f)_F \qquad \text{in} \quad L^p(X; E \otimes \mathbb{R}^m).$$

By the completeness of $\mathbb{D}^{1,p}(X; E)$ we have (5.1) and

$$(f^n)_F \to f_F \qquad \text{in} \quad \mathbb{D}^{1,p}(X; E).$$

Thus we obtain the chain rule (5.2) for any $f \in \mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; E)$, which completes the proof. □

The chain rule for ∇^* is also derived by the polynomial approximation as follows.

Lemma 5.2. *Suppose $F(x)$ belongs to $\mathbb{D}^{1,\infty}(X; \mathbb{R}^m)$. Let $u(x,y)$ be in $\mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; H)$ for $1 < p < \infty$. Then the following chain rule holds*:

$$\nabla^* u_F = (\nabla_X^* u)_F - \operatorname{trace}((\nabla F)(\nabla_Y u)_F). \tag{5.3}$$

Proof. First we prove the chain rule (5.3) for H-valued polynomials. It is sufficient to consider the polynomials of the form

$$u(x,y) = p(\langle x, h_1\rangle, \ldots, \langle x, h_n\rangle, y_1, \ldots, y_m)h$$

for some $n \in \mathbb{N}$, some $\mathbb{R}$-valued polynomial $p(t_1, \ldots, t_{n+m})$ of $n+m$ variables, some orthonormal system $(h_j; j = 1, \ldots, n)$ of H which is taken from X^* and some $h \in H$. Since for a smooth $\mathbb{R}$-valued functional φ we have

$$\nabla^*(\varphi(x)h) = \varphi(x)\nabla^* h(x) - \langle \nabla\varphi(x), h\rangle_H, \tag{5.4}$$

the left hand side of (5.3) is

$$\begin{aligned}
\nabla^* u_F = {} & p(\langle x, h_1\rangle, \ldots, \langle x, h_n\rangle, F_1(x), \ldots, F_m(x))\nabla^* h(x) \\
& - \sum_{j=1}^{n} \frac{\partial p}{\partial t_j}(\langle x, h_1\rangle, \ldots, \langle x, h_n\rangle, F_1(x), \ldots, F_m(x))\langle h_j, h\rangle_H \\
& - \sum_{j=n+1}^{n+m} \frac{\partial p}{\partial t_j}(\langle x, h_1\rangle, \ldots, \langle x, h_n\rangle, F_1(x), \ldots, F_m(x))\langle \nabla F_j(x), h\rangle_H.
\end{aligned}$$

It is clear that the first two terms amount to $(\nabla_X^* u)_F$ and that the last term is equal to trace$((\nabla F)(\nabla_Y u)_F)$. Thus we obtain (5.3) when u is an H-valued polynomial functional.

For a general $u \in \mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; H)$ we take an approximating sequence $\{u^n\}$ such that $u^n \in \mathcal{P}(X \oplus \mathbb{R}^m; H)$ and $u^n \to u$ in $\mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; H)$. Then each u_n satisfies the chain rule (5.3). By the continuity of ∇^* we have

$$\nabla^*(u^n_F) \to \nabla^*(u_F) \qquad \text{in} \quad L^p(X;\mathbb{R}).$$

On the other hand, in the right hand side of (5.3) we have

$$(\nabla_X^* u^n)_F \to (\nabla_X^* u)_F \qquad \text{in} \quad L^p(X;\mathbb{R})$$

and

$$\mathrm{trace}((\nabla F)(\nabla u^n)_F) \to \mathrm{trace}((\nabla F)(\nabla u)_F) \qquad \text{in} \quad L^p(X;\mathbb{R})$$

by Proposition 4.1. Thus we obtain the chain rule (5.3) for any $u \in \mathbb{D}^{(1,m+1),p}(X \oplus \mathbb{R}^m; H)$, which completes the proof. □

6. Localization

We begin with the precise definition of the class $\mathbb{D}^{k,p}_{\mathrm{loc}}(X;E)$ of locally differentialble functionals. In accordance with the definition of $\mathbb{D}^{1,\infty}_{\mathrm{loc}}$ in Section 5, we say that $f \in \mathbb{D}^{k,\infty}(X;E)$ if $f \in \mathbb{D}^{k,\infty-}(X;E)$ and $\nabla^j f$ for $j = 0, \dots, k$ are all bounded.

Definition 6.1. Let k be a nonnegative integer and $1 < p \le \infty$. An E-valued functional $f(x)$ is said to be in $\mathbb{D}^{k,p}_{\mathrm{loc}}(X;E)$ if there exists a sequence $(\mathcal{A}_n, f_n)$ such that

(1) $(\mathcal{A}_n)$ is a sequence of measurable sets whose union covers X almost surely,

(2) (f_n) is a sequence of functionals such that $f_n(x) \in \mathbb{D}^{k,p}(X;E)$,

(3) $f_n(x) = f(x)$ on $\mathcal{A}_n$ for each n.

Such a sequence $(\mathcal{A}_n, f_n)$ is called a *localizing sequence* for f.

The local derivative of f which is denoted by $\nabla_{\mathrm{loc}} f$ is defined by

$$\nabla_{\mathrm{loc}} f(x) = \nabla f_n(x) \qquad \text{on} \quad \mathcal{A}_n$$

for each n.

The definition of the local derivative $\nabla_{\mathrm{loc}} f(x)$ does not depend on the choice of a localizing sequence $(\mathcal{A}_n, f_n)$ because the gradient operator ∇ has a local property in the following sense.

Lemma 6.1 ([15], [14]). *Suppose that $f(x) \in \mathbb{D}^{1,p}(X;E)$ for some $1 < p < \infty$. If $f(x)$ vanishes almost surely on some measurable set A, then so does ∇f.*

The divergence operator ∇^* also has a local property in the following sense.

Lemma 6.2 ([14]). *Suppose that $u(x) \in \mathbb{D}^{1,p}(X; H)$ for some $1 < p < \infty$. If $u(x)$ vanishes almost surely on some measurable set A, then so does $\nabla^* u$.*

So we can define the local divergence $\nabla^*_{\mathrm{loc}} u$ for a functional u in $\mathbb{D}^{1,p}_{\mathrm{loc}}(X; H)$.

Definition 6.2. Let u be in $\mathbb{D}^{1,p}_{\mathrm{loc}}(X; H)$. The local divergence $\nabla^*_{\mathrm{loc}} u$ is defined as a functional such that

$$\nabla^*_{\mathrm{loc}} u(x) = \nabla^* u_n(x) \qquad \text{on} \quad \mathcal{A}_n$$

for each n where $(\mathcal{A}_n, u_n)$ is any localizing sequence for u.

If $f(x) \in \mathbb{D}^{1,p}$ then $f(x) \in \mathbb{D}^{1,p}_{\mathrm{loc}}$ and its local derivative $\nabla_{\mathrm{loc}} f(x)$ coincides with the usual derivative $\nabla f(x)$. The same is true for the local divergence. Therefore we will omit the subscript "loc" in the local gradient and divergence without any confusion.

The classes $\mathbb{D}^{(n,k),p}_{\mathrm{loc}}(X \oplus \mathbb{R}^m; E)$ of locally partially differentiable functionals and the local partial gradient and divergence are defined in the same way by replacing $\mathbb{D}^{k,p}$ by $\mathbb{D}^{(n,k),p}$. (The definition of the partial Sobolev spaces $\mathbb{D}^{(n,k),p}$ is given in Definition 3.1.)

Now we can consider the composite functionals for locally Sobolev differentiable functionals.

Theorem 6.1. *Suppose that $F(x)$ belongs to $\mathbb{D}^{1,\infty}_{\mathrm{loc}}(X; \mathbb{R}^m)$. Let $f(x, y)$ be in $\mathbb{D}^{(1,m+1),p}_{\mathrm{loc}}(X \oplus \mathbb{R}^m; E)$ for some $1 < p < \infty$. Then*

$$f_F \in \mathbb{D}^{1,p}_{\mathrm{loc}}(X; E) \tag{6.1}$$

and the following chain rule holds:

$$\nabla f_F = (\nabla_X f)_F + (\nabla_Y f)_F(\nabla F). \tag{6.2}$$

Proof. Take localizing sequences $(\mathcal{A}_n, F_n)$ for F and $(\mathcal{B}_k, f_k)$ for f. By Lemma 5.1 each $(f_k)_{F_n}$ satisfies the chain rule (6.2). Set

$$\mathcal{C}_{n,k} := \mathcal{A}_n \cap \{x \in X; (x, F_n(x)) \in \mathcal{B}_k\}.$$

Then each $\mathcal{C}_{n,k}$ is a measurable subset of X and the union of the countable family $\{\mathcal{C}_{n,k}\}$ covers the whole X almost surely. By the definition of $\mathcal{C}_{n,k}$ we find that

$$f_F = (f_k)_{F_n} \qquad \text{on} \quad \mathcal{C}_{n,k}.$$

Thus we have (6.1) and the chain rule (6.2). Therefore we obtained the desired conclusion. □

The chain rule for the divergence is also derived as follows.

Theorem 6.2. *Suppose that $F(x)$ belongs to $\mathbb{D}^{1,\infty}_{\mathrm{loc}}(X;\mathbb{R}^m)$. Let $u(x,y)$ be in $\mathbb{D}^{(1,m+1),p}_{\mathrm{loc}}(X\oplus\mathbb{R}^m;H)$ for some $1<p<\infty$. Then the following chain rule holds*:

$$\nabla^* u_F = (\nabla^*_X u)_F - \mathrm{trace}((\nabla F)(\nabla_Y u)_F).$$

This follows from Lemma 5.2 in the same way as in Theorem 6.1. So we omit the proof of Theorem 6.2.

Example 6.1 (Substitution formula)**.** Consider the case of the classical Wiener space $X = C_0([0,1];\mathbb{R}^m)$. Let $F(x)$ be in $\mathbb{D}^{1,\infty}_{\mathrm{loc}}(X;\mathbb{R}^m)$. Let $(\alpha_s(x,y); s\in[0,1])$ be an $\mathbb{R}^m$-valued measurable process on $X\oplus\mathbb{R}^m$. Suppose that $(\alpha_s(\cdot,y); s\in[0,1])$ is adapted for almost every $y\in\mathbb{R}$ and that

$$\int_0^{\cdot} \alpha_s(x,y)ds \in \mathbb{D}^{(1,m+1),p}_{\mathrm{loc}}(X\oplus\mathbb{R}^m;H)$$

for some $1<p<\infty$. Then Theorem 6.2 says that

$$\nabla^*\left(\int_0^{\cdot}(\alpha_s)_F ds\right) = \int_0^T \langle\alpha_s(x,y), dx_s\rangle_{\mathbb{R}^m}\bigg|_{y=F} - \int_0^T \mathrm{trace}((D_sF)(\nabla_Y\alpha_s)_F)ds \tag{6.3}$$

where D_sF is defined by

$$\nabla F(t) = \int_0^t D_sF ds.$$

This formula is derived in [14] by direct computation of the Skorohod integral.

7. Proof of Theorem 2.1

Theorem 2.1 immediately follows from Proposition 2.1. So it is sufficient to prove Proposition 2.1. The key to the proof is the following lemma.

Lemma 7.1. *Under the assumptions of Theorem* 2.1*,*

$$(\nabla_Y u)(x,y) = -(I_H + (\nabla_X u)(x,y))(G_Y) \tag{7.1}$$

and

$$I_H + \nabla u_F = (I_H + (\nabla_X u)_F)(I_H - (G_Y)_F(\nabla F)). \tag{7.2}$$

Here, as is defined in (2.4)*,*

$$G_Y(x,y) = (\nabla_Y v)\circ(\xi(x,y),y).$$

Proof. Since $\eta(\cdot, y)$ is the inverse of $\xi(\cdot, y)$, we get

$$v(x, y) = -u(\eta(x, y), y).$$

Differentiating both sides in the direction of Y, we obtain

$$(\nabla_Y v)(x, y) = -(\nabla_X u)(\eta(x, y), y)(\nabla_Y v)(x, y) - (\nabla_Y u)(\eta(x, y), y). \tag{7.3}$$

The chain rule used in the right hand side will be justified in Lemma 7.2. By (7.3) we have

$$(\nabla_Y u)(\eta(x, y), y) = -(I_H + (\nabla_X u)(\eta(x, y), y))(\nabla_Y v)(x, y).$$

Thus we obtain (7.1). Combining (7.1) with the chain rule (2.8), we obtain (7.2). □

Lemma 7.2. *Under the assumptions of Theorem* 2.1*,*

$$\begin{aligned} \nabla_Y (f(\eta(x, y), y)) = (\nabla_X f)(\eta(x, y), y)(\nabla_Y v)(x, y) \\ + (\nabla_Y f)(\eta(x, y), y) \end{aligned} \tag{7.4}$$

for any $f(x, y) \in \mathbb{D}^{1,\infty-}(X \oplus \mathbb{R}^m; E)$*.*

Proof. Take an approximating sequence $\{f_n; n \in \mathbb{N}\}$ of polynomial functionals on $X \oplus \mathbb{R}^m$ such that $f_k \to f$ in $\mathbb{D}^{1,\infty-}(X \oplus \mathbb{R}^m; E)$. Then the identity (7.4) clearly holds for each polynomial functional f_k. Thus it suffices to show

$$f_k(\eta(x, y), y) \to f(\eta(x, y), y) \qquad \text{in} \quad L^{\infty-}(X \oplus \mathbb{R}^m; E), \tag{7.5}$$

$$(\nabla_X f_k)(\eta(x, y), y) \to (\nabla_X f)(\eta(x, y), y) \qquad \text{in} \quad L^{\infty-}(X \oplus \mathbb{R}^m; E \otimes H) \tag{7.6}$$

and

$$(\nabla_Y f_k)(\eta(x, y), y) \to (\nabla_Y f)(\eta(x, y), y) \qquad \text{in} \quad L^{\infty-}(X \oplus \mathbb{R}^m; E \otimes \mathbb{R}^m). \tag{7.7}$$

In fact, the above three and (7.4) for f_k will imply that

$$f_k(\eta(x, y), y) \to f(\eta(x, y), y) \qquad \text{in} \quad \mathbb{D}^{(0,1),\infty-}(X \oplus \mathbb{R}^m; E)$$

and that (7.4) for f holds.

Let us prove (7.5). By the assumption (A3) we have for any $1 < p < \infty$

$$\begin{aligned} &\mathbb{E}_{(x,y)}[|f_k(\eta(x, y), y) - f(\eta(x, y), y)|_E^p] \\ &\quad = \int_Y \nu(dy) \int_X |f_k(\eta(x, y), y) - f(\eta(x, y), y)|_E^p \mu(dx) \\ &\quad = \int_Y \nu(dy) \int_X |f_k(x, y) - f(x, y)|_E^p \cdot |\Lambda(x, y)| \mu(dx) \\ &\quad \le \mathbb{E}_{(x,y)}[|f_k - f|_E^{pq}]^{1/q} \cdot \mathbb{E}_{(x,y)}[|\Lambda(x, y)|^{1+\varepsilon}]^{1/(1+\varepsilon)} \end{aligned}$$

where $q^{-1}+(1+\varepsilon)^{-1}=1$. The right hand side converges to zero by the assumption (A4). Thus we obtain (7.5). By a similar argument we have (7.6) and (7.7). Thus the proof is completed. $\square$

Now we proceed to the proof of Proposition 2.1. The equality (2.7) follows from (2.8) and (7.1). So it suffices to prove (2.6). We use the formula

$$\begin{aligned}&\mathrm{Det}_2((I_H+A)(I_H-B))\\&\quad=\mathrm{Det}_2(I_H+A)\,\mathrm{Det}_2(I_H-B)\exp(\mathrm{Trace}(AB))\end{aligned}$$

for two Hilbert-Schmidt operators A, B on H. Then by (7.2) we have

$$\begin{aligned}\mathrm{Det}_2(I_H+\nabla u_F)=&\ \mathrm{Det}_2(I_H+(\nabla_X u)_F)\,\mathrm{Det}_2(I_H-(G_Y)_F(\nabla F))\\&\times\exp(\mathrm{Trace}((\nabla_X u)_F(G_Y)_F(\nabla F))).\end{aligned}$$

Since

$$\mathrm{Det}_2(I_H+AB)=\det{}_2(I_{\mathbb{R}^m}+BA), \tag{7.8}$$

$$\mathrm{Trace}(AB)=\mathrm{trace}(BA) \tag{7.9}$$

for $A\in\mathrm{HS}(\mathbb{R}^m;H)$, $B\in\mathrm{HS}(H;\mathbb{R}^m)$, we have

$$\begin{aligned}\mathrm{Det}_2(I_H+\nabla u_F)=&\ \mathrm{Det}_2(I_H+(\nabla_X u)_F)\,\det{}_2(I_H-(\nabla F)(G_Y)_F)\\&\times\exp(\mathrm{trace}((\nabla F)(\nabla_X u)_F(G_Y)_F)).\end{aligned}$$

Therefore we obtain (2.6), which completes the proof. $\square$

8. The case of the Buckdahn-Föllmer type transformations

In this section we give a sufficient condition for the Buckdahn-Föllmer type transformations to satisfy the assumptions (A1)–(A4) of Theorem 2.1.

Let (X,H,μ) be the m-dimensional classical Wiener space. That is, the Banach space X is $C_0([0,T];\mathbb{R}^m)$ equipped with the norm

$$\|\zeta\|_X=\sup_{t\in[0,T]}|\zeta_t|_{\mathbb{R}^m}.$$

The Cameron-Martin subspace H is given by the Hilbert space

$$H=\left\{h=\int_0^\cdot \dot h_s ds\ ;\ \int_0^T|\dot h_s|^2_{\mathbb{R}^m}ds<\infty\right\}$$

equipped with an inner product

$$\langle h,k\rangle_H=\int_0^T\langle\dot h_s,\dot k_s\rangle_{\mathbb{R}^m}ds.$$

The measure μ is the Wiener measure on X under which the coordinate process $x=(x_s;s\in[0,T])$ is an m-dimensional Brownian motion.

We consider the transformation ξ_F where for fixed y the transformation $\xi(\cdot, y)$ is the solution to the stochastic differential equation

$$d\xi_t(x,y) = dx_t + \alpha_t(\xi, y)dt, \qquad \xi_0 = 0. \tag{8.1}$$

We impose the following assumptions on α so that the stochastic differential equation (8.1) has a unique strong solution.

(B1) The process $(\alpha_t(\cdot, y); t \in [0,T])$ is adapted for any fixed $y \in \mathbb{R}^m$.

(B2) The mapping $\zeta \mapsto \alpha_t(\zeta, y)$ is of class H-C^1 for any t and y. That is, the mapping $h \mapsto \alpha_t(\zeta + h, y)$ is Fréchet differentiable on H for any ζ with H-continuous derivative $(\nabla_X \alpha_t)(\zeta, y)$.

(B3) For any fixed ζ and y the operator $h \mapsto \int_0^{\cdot} (\nabla_X \alpha_t) \circ (\zeta, y)[h] dt$ on H is a Hilbert-Schmidt operator with integral kernel $\beta_{t,s}(\zeta, y)$:

$$\int_0^{\cdot} (\nabla_X \alpha_t) \circ (\zeta, y)[h] dt = \int_0^{\cdot} \int_0^{\cdot} \beta_{t,s}(\zeta, y) \dot{h}_s ds dt.$$

(B4) The mappings $y \mapsto \alpha_t(\zeta, y)$ and $y \mapsto \beta_{t,s}(\zeta, y)$ are of class C^{m+1} for any t, s and ζ.

(B5) There is a constant $K < \infty$ such that

$$|(\nabla_Y^k \alpha_t)(\zeta, y)|_{(\mathbb{R}^m)^{\otimes (k+1)}} \leq K,$$
$$|(\nabla_Y^k \beta_{t,s})(\zeta, y)|_{(\mathbb{R}^m)^{\otimes (k+2)}} \leq K$$

and

$$\sup_{h \in H} \frac{|(\nabla_Y^k \beta_{t,s})(\zeta + h, y) - (\nabla_Y^k \beta_{t,s})(\zeta, y)|}{|h|_H} \leq K$$

for any t, s, ζ, y and $k = 0, 1, \ldots, m$.

Remark 4.

(a) By the assumption (B1) the integral operator $\int_0^{\cdot} (\nabla_X \alpha_t)(\zeta, y) dt$ is of Volterra type:

$$\beta_{t,s}(\zeta, y) = 0 \qquad \text{for} \quad s > t.$$

(b) All of the assumptions (B1)–(B5) are satisfied if

$$\alpha_t(\zeta, y) = f\left(y, \zeta_t, \int_0^t \zeta_s ds\right)$$

for some smooth function $f(t_1, t_2, t_3)$ with compact support.

Now we state the main theorem in this section.

Theorem 8.1. *Suppose* (B1)–(B5). *Then for any fixed x and y the ordinary integral equation*

$$\zeta_t = x_t + \int_0^t \alpha_s(\zeta, y) ds \tag{8.2}$$

has a unique solution $\zeta = \xi(x, y)$. *For any fixed* $y \in \mathbb{R}^m$ *the mapping* $\xi(\cdot, y)$ *defines a transformation on* X *and it satisfies all the assumptions* (A1)–(A4) *of Theorem* 2.1. *Moreover, for any* $F(x) \in \mathbb{D}^{(1,m+1),\infty}_{\mathrm{loc}}(X; \mathbb{R}^m)$, u_F *belongs to* $\mathbb{D}^{1,\infty}_{\mathrm{loc}}(X; H)$ *and the following holds*:

$$\begin{aligned} &\mathrm{Det}_2(I_H + \nabla u_F) \exp\left(-\nabla^* u_F - \frac{1}{2}|u_F|^2_H\right) \\ &\quad = \det\left(I_{\mathbb{R}^m} + \int_0^T (D_t F)(\nabla_Y \alpha_t) \circ (\xi_F, F) dt\right) \\ &\quad\quad \times \exp\left(-\int_0^T \langle \alpha_t(\xi_F, F), dx_t \rangle_{\mathbb{R}^m} - \frac{1}{2}\int_0^T |\alpha_t(\xi_F, F)|^2_{\mathbb{R}^m} dt\right). \end{aligned} \tag{8.3}$$

For the proof of Theorem 8.1 we prepare a lemma for certain ordinary integral equations. Define the H-norm up to the time t of an element $h \in H$ as

$$|h|^2_{H_t} = \int_0^t |\dot{h}_s|^2 ds.$$

Clearly, $|\cdot|_{H_T} = |\cdot|_H$.

Lemma 8.1. *Suppose that the mapping*

$$\alpha : [0, T] \times X \to \mathbb{R}^m$$

satisfies the following conditions:

(1) *The process* $(\alpha_t(\cdot); t \in [0, T])$ *is adapted.*

(2) *The mapping* $\zeta \mapsto \alpha_t(\zeta)$ *is* H*-Lipschitz continuous uniformly in* $t \in [0, T]$, *that is, there exists a constant* $K < \infty$ *such that*

$$|\alpha_t(\zeta + h) - \alpha_t(\zeta)| \le K|h|_{H_t}$$

for any $t \in [0, T]$, $\zeta \in X$, $h \in H$. (*The right hand side can be estimated by the* H*-norm up to the time* t *because of adaptedness of* α.)

(3) *There exists a constant* $\Phi(x) < \infty$ *independent of* t *such that*

$$|\alpha_t(x)| \le \Phi(x).$$

Let $x : [0, T] \to \mathbb{R}^m$ *be a fixed continuous function. Then, the ordinary integral equation*

$$\zeta_t = x_t + \int_0^t \alpha_s(\zeta) dt \tag{8.4}$$

has a unique solution.

Lemma 8.1 is easily shown by using a standard argument but we give a proof for the completeness of the paper.

Proof of Lemma 8.1. First we prove the uniqueness. Suppose ζ and ζ' are two solutions to the equation (8.4). Noting that $\zeta - \zeta'$ belongs to H, we have

$$\begin{aligned} |\zeta - \zeta'|^2_{H_t} &= \int_0^t |\alpha_s(\zeta) - \alpha_s(\zeta')|^2 ds \\ &\le K^2 \int_0^t |\zeta - \zeta'|^2_{H_s} ds. \end{aligned}$$

By Gronwall's inequality we have $|\zeta - \zeta'|^2_{H_t} = 0$ for all $t \in [0,T]$. This implies $\zeta_t = \zeta'_t$ for all $t \in [0,T]$.

Next we construct a solution by Picard's successive approximation method. Define a sequence $\{\zeta^n\}$ of X by the iteration

$$\begin{cases} \zeta_t^0 = x_t, \\ \zeta_t^{n+1} = x_t + \int_0^t \alpha_s(\zeta^n) ds \qquad \text{for} \quad n = 0, 1, \dots . \end{cases}$$

By induction we have

$$|\zeta^{n+1} - \zeta^n|^2_{H_t} \le \Phi(x)^2 \frac{(K^2 t)^n}{n!},$$

which implies that

$$\sum_{n=0}^\infty |\zeta^{n+1} - \zeta^n|_H \le \Phi(x) \sum_{n=0}^\infty \frac{(K\sqrt{T})^n}{\sqrt{n!}} < \infty.$$

Thus the sum $\sum_{n=0}^\infty (\zeta^{n+1} - \zeta^n)$ converges in H and the sequence $\zeta^n = \zeta^0 + \sum_{k=0}^n (\zeta^{k+1} - \zeta^k)$ converges in X. Denote the limit $\lim_n \zeta^n$ by ζ. Since

$$\begin{aligned} \left| \int_0^\cdot \alpha_s(\zeta) ds - \int_0^\cdot \alpha_s(\zeta^n) ds \right|^2_H &\le K^2 \int_0^T |\zeta - \zeta^n|^2_{H_s} ds \\ &\le K^2 T |\zeta - \zeta^n|^2_H \to 0 \end{aligned}$$

as n tends to ∞, it follows that ζ satisfies (8.4). □

Now we turn to the proof of Theorem 8.1.

Proof of Theorem 8.1. Since

$$\begin{aligned} |\alpha_t(\zeta + h, y) - \alpha_t(\zeta, y)|_{\mathbb{R}^m} &\le \int_0^1 \left| \frac{d}{d\lambda} (\alpha_t(\zeta + \lambda h, y)) \right| d\lambda \\ &= \int_0^1 |(\nabla_X \alpha_t) \circ (\zeta + \lambda h, y)[h]| d\lambda \\ &\le K |h|_{H_t}, \end{aligned}$$

the mapping $\zeta \mapsto \alpha_t(\zeta, y)$ is H-Lipschitz continuous uniformly in $t \in [0,T]$. Thus the pathwise uniqueness and existence of the strong solution to the ordinary integral equation (8.2) is an immediate consequence of Lemma 8.1.

Now we verify the assumptions of Theorem 2.1. We devide the proof into several steps.

Step 1. We verify the assumption (A1).
It suffices to show that if we set

$$\eta_t(x,y) = x_t - \int_0^t \alpha_s(x,y)ds$$

then the transformation $\eta(\cdot,y)$ gives the inverse of $\xi(\cdot,y)$. Clearly, $\eta(\xi(x,y),y) = x$ by definition. On the other hand, if we write $\zeta(x,y) = \xi(\eta(x,y),y)$ we have

$$\zeta_t(x,y) - x_t = \int_0^t (\alpha_s(\zeta(x,y),y) - \alpha_s(x,y))ds.$$

Taking the H-norm up to the time t, we have

$$|\zeta(x,y) - x|_{H_t}^2 \le K^2 \int_0^t |\zeta(x,y) - x|_{H_s}^2 ds$$

by the adaptedness and the H-Lipschitz continuity of α_s. By Gronwall's inequality we have $\zeta(x,y) = x$, which means that $\xi(\eta(x,y),y) = x$.

Step 2. We verify the assumption (A2).
It is easy to see that

$$v(x,y) := -\int_0^{\cdot} \alpha_s(x,y)ds \in \mathbb{D}^{(1,m+1),\infty}(X \oplus \mathbb{R}^m; H).$$

So it only remains to show that

$$u(x,y) := \int_0^{\cdot} \alpha_s(\xi,y)ds \in \mathbb{D}^{(1,m+1),\infty}(X \oplus \mathbb{R}^m; H).$$

Recall that $\xi(x,y)$ is obtained as the limit of the sequence $\{\xi^n(x,y)\}$ which is defined by the iteration

$$\begin{cases} \xi_t^0(x,y) = x_t, \\ \xi_t^{n+1}(x,y) = x_t + \int_0^t \alpha_s(\xi^n,y)ds \qquad \text{for} \quad n = 0,1,\dots. \end{cases} \tag{8.5}$$

Let

$$u_t^n(x,y) = \int_0^t \alpha_s(\xi^{n-1},y)ds \tag{8.6}$$

and let $U_{t,s}^n(x,y)$ be the integral kernel of the Volterra type Hilbert-Schmidt operator $\nabla_X u^n(x,y)$ on H:

$$\nabla_X u_t^n(x,y)[h] = \int_0^t \int_0^s U_{s,r}^n(x,y)\dot{h}_r dr ds.$$

Differentiating both sides of (8.6) in the direction of X we have

$$\nabla_X u_t^n(x,y) = \int_0^t (\nabla_X \alpha_s) \circ (\xi^{n-1},y)(I_H + \nabla_X u^{n-1}(x,y))ds. \tag{8.7}$$

In terms of the integral kernels we can express (8.7) as

$$U^n_{t,r}(x,y) = \beta_{t,r}(\xi^{n-1}, y) + \int_r^t \beta_{t,s}(\xi^{n-1}, y) U^{n-1}_{s,r}(x,y) ds. \tag{8.8}$$

First we show that $U^n_{t,r}(x,y)$ is uniformly bounded. By (8.8) we have

$$|U^n_{t,r}(x,y)|^2 \le C_1 \left(1 + \int_r^t |U^{n-1}_{s,r}(x,y)|^2 ds\right)$$

for some constant $C_1 < \infty$. By induction we get

$$|U^n_{t,r}(x,y)|^2 \le C_1 \sum_{i=0}^{n-1} \frac{(C_1 t)^i}{i!} \le C_2$$

where $C_2 = C_1 \sum_{i=0}^{\infty} (C_1 T)^i / i! < \infty$.

Next we estimate the difference $U^{n+1}_{t,r}(x,y) - U^n_{t,r}(x,y)$. By (8.8) we have

$$\begin{aligned} U^{n+1}_{t,r}(x,y) - U^n_{t,r}(x,y) &= \beta_{t,r}(\xi^n, y) - \beta_{t,r}(\xi^{n-1}, y) \\ &\quad + \int_r^t (\beta_{t,s}(\xi^n, y) - \beta_{t,s}(\xi^{n-1}, y)) U^n_{s,r}(x,y) ds \\ &\quad + \int_r^t \beta_{t,s}(\xi^{n-1}, y)(U^n_{s,r}(x,y) - U^{n-1}_{s,r}(x,y)) ds. \end{aligned}$$

The H-Lipschitz continuity of $\beta_{t,s}$ implies that

$$|\beta_{t,s}(\xi^n, y) - \beta_{t,s}(\xi^{n-1}, y)|^2 \le K^2 |\xi^n(x,y) - \xi^{n-1}(x,y)|^2_{H_t} \le \frac{(C_3 t)^{n-1}}{(n-1)!}.$$

By the uniform boundedness of $U^n_{t,r}$ we have

$$\begin{aligned} &|U^{n+1}_{t,r}(x,y) - U^n_{t,r}(x,y)|^2 \\ &\qquad \le C_4 \left(\frac{(C_4 t)^{n-1}}{(n-1)!} + \int_r^t |U^n_{s,r}(x,y) - U^{n-1}_{s,r}(x,y)|^2 ds \right). \end{aligned}$$

By induction we obtain the following estimate:

$$|U^{n+1}_{t,r}(x,y) - U^n_{t,r}(x,y)|^2 \le \frac{(C_5)^n}{n!}.$$

Thus we can estimate $\nabla_X u^{n+1} - \nabla_X u^n$ as

$$\begin{aligned} &|\nabla_X u^{n+1}(x,y) - \nabla_X u^n(x,y)|_{H\otimes H} \\ &\qquad = \left(\int_0^T \int_0^T |U^{n+1}_{t,r}(x,y) - U^n_{t,r}(x,y)|^2 dr dt \right)^{1/2} \le T \frac{(\sqrt{C_5})^n}{\sqrt{n!}}. \end{aligned}$$

Since C_5 depends only on K and T, the sequence $\{u^n(x,y)\}$ is Cauchy in $\mathbb{D}^{(1,0),\infty}(X \oplus \mathbb{R}^m; H)$. Thus the limit $u(x,y)$ proves to be in $\mathbb{D}^{(1,0),\infty}(X \oplus \mathbb{R}^m; H)$.

We can show $u(x,y) \in \mathbb{D}^{(i,j),\infty}(X \oplus \mathbb{R}^m; H)$ for $i = 0, 1$ and $j = 0, 1, \dots, m + 1$ through the same procedure. So we omit the proof.

Step 3. We verify the assumption (A3).
We fix $y \in \mathbb{R}^m$. Each process $(\xi_t^n(\cdot, y); t \in [0,T])$ is adapted by the construction (8.5), and so is the limit process $(\xi_t(\cdot, y); t \in [0,T])$. Since it is bounded, $u(x,y)$ satisfies Novikov's condition:

$$\mathbb{E}_x \left[\exp\left(\frac{1}{2} |u(x,y)|_H^2 \right) \right] < \infty.$$

Thus the transformation $\xi(\cdot, y)$ on X satisfies the Maruyama-Girsanov formula:

$$\mathbb{E}[\varphi] = \mathbb{E}_x[\varphi(\xi(x,y))\Lambda(x,y)]$$

where the density $\Lambda(x,y)$ is given by

$$\Lambda(x,y) = \exp\left(-(\nabla_X^* u)(x,y) - \frac{1}{2} |u(x,y)|_H^2 \right).$$

Note that

$$\mathrm{Det}_2(I_H + (\nabla_X u)(x,y)) = 1 \tag{8.9}$$

because $\nabla_X u(x,y)$ is a Hilbert-Schmidt operator of Volterra type.

Now let us express the density $\Lambda(x,y)$ in terms of α. The transformation $\eta(\cdot, y)$ also satisfies the Maruyama-Girsanov formula:

$$\mathbb{E}[\varphi] = \mathbb{E}_x[\varphi(\eta(x,y))\mathcal{L}(x,y)]$$

where the density $\mathcal{L}(x,y)$ is given by

$$\mathcal{L}(x,y) = \exp\left(\int_0^T \alpha_t(x,y) dx_t - \frac{1}{2} \int_0^T |\alpha_t(x,y)|^2 dt \right).$$

Since $\eta(\cdot, y)$ is the inverse transformation of $\xi(\cdot, y)$ we have

$$\Lambda(x,y) = 1/\mathcal{L}(\xi(x,y), y) \tag{8.10}$$

$$= \exp\left(-\int_0^T \alpha_t(\xi(x,y), y) dx_t - \frac{1}{2} \int_0^T |\alpha_t(\xi(x,y), y)|^2 dt \right). \tag{8.11}$$

Step 4. We verify the assumption (A4).
More strongly, we can show that

$$\Lambda(x,y) \in L^p(X \oplus \mathbb{R}^m; \mathbb{R})$$

for any $1 < p < \infty$. By (8.10) we have

$$\mathbb{E}_{(x,y)}[\Lambda(x,y)^p] = \mathbb{E}_{(x,y)}[\mathcal{L}(\xi(x,y),y)^{-p}]. \tag{8.12}$$

Changing the variable x by $\eta(x,y)$ in the right hand side of (8.12) we have

$$\begin{aligned}
&\mathbb{E}_{(x,y)}[\Lambda(x,y)^p] \\
&\quad = \mathbb{E}_{(x,y)}[\mathcal{L}(x,y)^{-(p-1)}] \\
&\quad = \mathbb{E}_{(x,y)}\left[\exp\left(-(p-1)\int_0^T \alpha_t(x,y)dx_t + \frac{1}{2}(p-1)\int_0^T |\alpha_t(x,y)|^2 dt\right)\right] \\
&\quad \le \exp\left(\frac{1}{2}p(p-1)K^2T\right) < \infty.
\end{aligned}$$

The last inequality is valid because

$$\mathbb{E}_{(x,y)}\left[\exp\left(-\int_0^T (p-1)\alpha_t(x,y)dx_t - \frac{1}{2}\int_0^T |(p-1)\alpha_t(x,y)|^2 dt\right)\right] = 1.$$

Therefore we have verified all the assumptions of Theorem 2.1. Thus we obtain (8.3) by applying to the right hand side of (2.5) the identity (8.11) and the following:

$$I_{\mathbb{R}^m} - (\nabla F)(G_Y) = I_{\mathbb{R}^m} + \int_0^T (D_sF)(\nabla_Y \alpha_t) \circ (\xi(x,y),y)ds.$$

□

9. Applications

We prepare a lemma for later use. We keep the notations of Theorem 2.1. In accordance with the notation of G_Y given in (2.4), we define

$$G_X(x,y) = (\nabla_X v) \circ (\xi(x,y),y).$$

Lemma 9.1. *Suppose the assumptions* (A1)–(A4) *of Theorem* 2.1. *Then*

$$I_H + \nabla_X u = (I_H + G_X)^{-1} = I_H + \sum_{n=1}^{\infty}(-G_X)^n. \tag{9.1}$$

and

$$(I_H + \nabla u_F)^{-1} = (I_H + (G_Y)_F(I_{\mathbb{R}^m} - (\nabla F)(G_Y)_F)^{-1}(\nabla F))(I_H + (G_X)_F). \tag{9.2}$$

Proof. Since $\eta(\xi(x,y),y) = x$,

$$(I_H + G_X)(I_H + \nabla_X u) = I_H. \tag{9.3}$$

Thus we obtain (9.1) because G_X is a Volterra type integral operator.

The identity (7.2) implies that

$$(I_H + \nabla u_F)^{-1} = (I_H - (G_Y)_F(\nabla F))^{-1}(I_H + (\nabla_X u)_F)^{-1}. \tag{9.4}$$

Then (9.2) immediately follows from (9.4) by applying the formula $(I - AB)^{-1} = I - A(I - BA)^{-1}B$ and (9.3). □

Example 1 (C. Donati-Martin, H. Matsumoto and M. Yor [5])**.** Let us compute the Carleman-Fredholm determinant induced by the transformation dealt with in [5].

Consider the 1-dimensional classical Wiener space: $X = C_0([0,T];\mathbb{R})$. In [5] the authors considered an anticipative transformation $\tilde{\xi}$ on X given by

$$\tilde{\xi}_t(x) = x_t - \int_0^t \frac{c\exp(2x_s)}{e^{x_T} + cA_s(x)} ds \tag{9.5}$$

where c is a positive constant and

$$A_t(\zeta) = \int_0^t \exp(2\zeta_s) ds.$$

Let $Z_t(\zeta) = A_t(\zeta)\exp(-\zeta_t)$. Then they found that the process Z is an invariant for the transformation $\tilde{\xi}$: $Z_t(\tilde{\xi}) = Z_t(x)$. By marvelous direct computations they succeeded in obtaining the following change of variables formula (Corollary 1.2 in [5]):

$$\mathbb{E}[\varphi] = \mathbb{E}[\varphi(\tilde{\xi})\widetilde{\Lambda}] \tag{9.6}$$

where

$$\widetilde{\Lambda}(x) = \exp\left(\frac{c}{2}\left(\frac{\exp(x_T)}{1 + cZ_T(x)} - \frac{1}{\exp(x_T)}\right)\right). \tag{9.7}$$

Comparing (9.7) with the Ramer-Kusuoka density they derived that the Carleman-Fredholm determinant of the gradient of the transformation $\tilde{\xi}$ is equal to

$$(1 + cZ_T(x))\exp\left(-\frac{cZ_T(x)}{1 + cZ_T(x)}\right).$$

Let us verify this in our context.

It is shown in [5] that the transformation $\tilde{\xi}$ satisfies the following Buckdahn-Föllmer type stochastic differential equation:

$$d\tilde{\xi}_t = dx_t + \alpha_t(\tilde{\xi}, x_T)dt, \qquad \tilde{\xi}_0 = 0 \tag{9.8}$$

where

$$\alpha_t(\zeta, y) = -\frac{c\exp(2\zeta_t)}{e^y - cA_t(\zeta)}. \tag{9.9}$$

It is also shown in [5] that

$$cA_t(\tilde{\xi}) < e^{x_T} \qquad \text{for any} \quad t \in [0,T]. \tag{9.10}$$

By (9.10) the denominator does not vanish in the right hand side of (9.9) for $\zeta = \tilde{\xi}$ and $y = x_T$. What we deal with are the functional $F(x) = x_T$ and the transformation $\xi(x,y)$ induced by the solution to the ordinary integral equation

$$\xi_t(x,y) = x_t + \int_0^t \alpha_s(\xi(x,y),y)ds. \tag{9.11}$$

Then we can express $\tilde{\xi} = \xi_F$. Since $e^F = e^{\xi_F} + cA_T(\xi_F)$ as is shown in [5], it suffices to show that

$$\mathrm{Det}_2(I_H + \nabla u_F) = \frac{e^F}{e^F - cA_T(\xi_F)} \exp\left(-\frac{cA_T(\xi_F)}{e^F}\right). \tag{9.12}$$

Now we appeal to Theorem 8.1. Clearly, $F(x)$ belongs to $\mathbb{D}^{1,\infty}_{\mathrm{loc}}(X;\mathbb{R})$. Note that the process α itself does not satisfy the assumptions of Theorem 8.1. So we need a cutoff. For each n, let ρ^n be a smooth function with compact support such that

$$\rho^n(y) = 1 \qquad \text{on} \quad [1/n, n]$$

and set

$$\alpha^n_t(\zeta,y) = \alpha_t(\zeta,y)\rho^n(e^y)\rho^n(c\exp(2\zeta_t))\rho^n(1/\{e^y - cA_t(\zeta)\}). \tag{9.13}$$

Then by Remark 4 (b) the process α^n satisfies all of the assumptions (B1)–(B5) of Theorem 8.1. Thus for each n, x and y, the ordinary integral equation

$$\zeta_t = x_t + \int_0^t \alpha^n_s(\zeta,y)ds \tag{9.14}$$

has a unique solution $\zeta = \xi^n(x,y)$. Moreover,

$$u^n(x,y) := \xi^n(x,y) - x = \int_0^{\cdot} \alpha^n_s(\xi^n,y)ds \in \mathbb{D}^{(1,2),\infty}(X \oplus \mathbb{R}; H).$$

Then we can construct the solution $\xi(x,y)$ to the ordinary integral equation (9.11) on a certain subset of $X \oplus \mathbb{R}$ as follows. Set

$$\begin{aligned}\mathcal{A}_n = \{&(x,y) \in X \oplus \mathbb{R}^m \;; \\ &e^y,\ c\exp(2\xi^n_t) \text{ and } e^y - cA_t(\xi^n) \in [1/n, n] \qquad \text{for any} \quad t \in [0,T]\}\end{aligned}$$

and $\mathcal{B}_n = \{x \in X \;;\; (x, F(x)) \in \mathcal{A}_n\}$. For each n the sets $\mathcal{A}_n$ and $\mathcal{B}_n$ are measurable because the map $t \mapsto \xi^n_t$ is continuous. By the uniqueness of the solution to (9.14), the sequence $\{\xi^n, \mathcal{A}_n\}$ has the following compatibility: if $n > m$, then $\mathcal{A}_n \supset \mathcal{A}_m$ and $\xi^n = \xi^m$ on $\mathcal{A}_m$. Then it follows that $\alpha^n_t(\xi^n, y) =$

$\alpha_t(\xi^n, y)$ for any $t \in [0,T]$ on each $\mathcal{A}_n$ and that $\xi^n(x,y)$ solves the equation (9.11) on each $\mathcal{A}_n$. Thus we can construct the solution $\xi(x,y)$ on the set $\cup_n \mathcal{A}_n$ by setting

$$\xi(x,y) = \xi^n(x,y) \qquad \text{on each} \quad \mathcal{A}_n. \tag{9.15}$$

For each fixed $x \in \mathcal{B}_n$, the point $(x, F(x))$ in $X \oplus \mathbb{R}$ belongs to $\mathcal{A}_n$. Then by (9.15) for $y = F(x)$ we have

$$\xi_F(x) = \xi_F^n(x) \quad \text{and} \quad u_F(x) = u_F^n(x) \qquad \text{on} \quad \mathcal{B}_n$$

where $u(x,y) = \xi(x,y) - x$. By (9.10) the union of the increasing sequence $\{\mathcal{B}_n\}$ covers the whole X. This fact owes to the global solution (9.5) in [5] to the stochastic differential equation (9.8). Thus it suffices to compute the right hand side of (2.6) for u on each $\mathcal{A}_n$ and u_F on each $\mathcal{B}_n$.

Since

$$v_t = \int_0^t \frac{c\exp(2x_s)}{e^y - cA_s} ds = \log \frac{e^y}{e^y - cA_t}, \tag{9.16}$$

we have

$$\nabla_Y v_t = \frac{-cA_t}{e^y - cA_t}.$$

Thus the 1-dimensional determinant of the right hand side of (2.6) turns out to be

$$\frac{e^F}{e^F - cA_T(\xi_F)}. \tag{9.17}$$

Next we calculate the exponential factor of the right hand side of (2.6). By (9.16) the Volterra type integral operator $\nabla_X v$ is given by

$$\nabla_X v_t[h] = \frac{1}{e^y - cA_t} \int_0^t c\exp(2x_s)(2h_s) ds.$$

The integral kernel of G_X is written as

$$K(t,s) = \frac{-2g'(s)}{g(t)} \cdot 1_{\{t>s\}}$$

where

$$g(t) = e^y - cA_t(\xi).$$

Then we can show by induction that the kernel of G_X^n is expressed as

$$K^n(t,s) = \frac{-2g'(s)}{g(t)} \frac{(-2)^{n-1}}{(n-1)!} \{\log g(t) - \log g(s)\}^{n-1} \cdot 1_{\{t>s\}}.$$

Applying this to (9.1) we have

$$(I_H + \nabla_X u)[h](t) = h_t + \int_0^t \frac{2g(t)g'(s)}{g(s)^2} h_s ds.$$

Since $\nabla F[h] = h_T$ and $G_Y(t) = \{g(t) - g(0)\}/g(t)$ we have

$$\begin{aligned}
&(\nabla F)(I_H + \nabla_X u)(G_Y) \\
&\quad = \frac{g(T) - g(0)}{g(T)} + \int_0^T \frac{2g(T)g'(t)}{g(t)^2} \frac{g(t) - g(0)}{g(t)} dt \\
&\quad = \frac{g(T) - g(0)}{g(0)} \\
&\quad = -\frac{cA_T(\xi)}{e^y}.
\end{aligned}$$

Thus we obtain

$$(\nabla F)(I_H + (\nabla_X u)_F)(G_Y)_F = -\frac{cA_T(\xi_F)}{e^F}. \tag{9.18}$$

By applying (8.9), (9.17) and (9.18) to (2.6), we obtain (9.12).

Remark 5. We can directly compute the Carleman-Fredholm determinant of the gradient of $\tilde{\xi}$ given by (9.5) by using the following expansion: for any Hilbert-Schmidt operator A on H,

$$\mathrm{Det}_2(I_H + A) = 1 + \sum_{n=2}^{\infty} \int_{0 \le s_1 \le \cdots \le s_n \le T} \det(\hat{A}(s_i, s_j)) ds_1 \cdots ds_n \tag{9.19}$$

where A is identified with the associated integral kernel on $L^2([0,T])$ and $\hat{A}(s_i, s_j) = A(s_i, s_j)$ if $i \neq j$ and $\hat{A}(s_i, s_i) = 0$. In this case $A = \nabla(\tilde{\xi}(x) - x)$ and the integral kernel is given by

$$A(t, s) = \begin{cases} a(t)b(s) & ;\ \text{for } t > s \\ a(t) & ;\ \text{for } t < s \end{cases}$$

where

$$a(t) = \frac{ce^{2x_t + x_T}}{(e^{x_T} + cA_t)^2} \quad \text{and} \quad b(s) = -\frac{e^{x_T} + 2cA_s}{e^{x_T}}.$$

The key fact to the computation is the following expansion of the finite dimensional determinant:

$$\det(\hat{A}(s_i, s_j)) = (-1)^{n-1} a(s_1) \cdots a(s_n) \left(\sum_{m=1}^{n-1} b(s_1) \cdots b(s_m) \right).$$

Then the right hand side of (9.19) is immediately computed and turns out to be

$$1+\int_0^T a(t)dt\left\{\exp\left(-\int_0^t a(s)b(s)ds - \int_t^T a(s)ds\right) - \exp\left(-\int_t^T a(s)ds\right)\right\}.$$

Example 2 (Quadratic Wiener functional). We show that the quadratic Wiener functionals discussed in [8] can be characterized by a certain anticipative transformation.

Let us consider an abstract Wiener space (X,H,μ) and a quadratic functional

$$S(x)=\nabla^{*2}B(x)$$

where B is some symmetric Hilbert-Schmidt operator on H. We consider the following transformation on the Wiener space:

$$\tilde{\xi}(x)=x+\nabla S(x)=x+2\nabla^* B(x),$$

which plays an essential role in studying the (conditonal) Laplace transform of the functional $S(x)$ as is stated below. Following N. Ikeda, S. Kusuoka and S. Manabe [8] we suppose that the operator B has the following decomposition:

$$B=B_V+B_F$$

where B_V is of Volterra type and of Hilbert-Schmidt type and B_F of finite dimensional range. Let $\{h_1,\dots,h_m\}$ be a basis of the range of B_F. Then we can write

$$B_F=\sum_{j,k}^m b_{j,k}h_j\otimes h_k.$$

If we choose

$$\xi(x,y)=x+2\left\{\nabla^* B_V(x)+\sum_{j,k}^m b_{j,k}y_k h_j\right\}$$

and

$$F(x)=(\nabla^* h_1,\dots,\nabla^* h_m)^\top,$$

then we can express

$$\xi_F(x)=\tilde{\xi}(x).$$

Now we apply to a concrete example. We consider the following quadratic Wiener functional on the 2-dimensional classical Wiener space $X = C_0([0,T];\mathbb{R}^2)$:

$$S(x) = \int_0^T |x_t|^2 dt - T^2$$

where T^2 is subtracted to make the expectation vanish. We give another proof to the following known formula concerning the conditional Laplace transform of $S(x)$:

$$\mathbb{E}[e^{-zS(x)}\delta_a(x_T)] = \frac{1}{2\pi T}\frac{\sqrt{2z}T}{\sinh(\sqrt{2z}T)}\exp\left(zT^2 - \frac{\sqrt{2z}T}{\tanh(\sqrt{2z}T)}\frac{|a|^2}{2T}\right). \tag{9.20}$$

This formula has been proved in several ways, for example, in [10] and [8].

The functional $S(x)$ is expressed as $S(x) = \nabla^{*2}B$ for some Hilbert-Schmidt operator B. Under the transformation $x + 2z\nabla^* B(x)$, the conditional Laplace transform of the quadratic Wiener functional $\nabla^{*2}B$ is changed as follows (see [4]):

$$\begin{aligned} &\mathbb{E}[\exp(-z\nabla^{*2}B)\delta_{a_1}(\nabla^* h_1)\cdots\delta_{a_n}(\nabla^* h_n)] \\ &\quad = \{\mathrm{Det}_2(I+2zB)\}^{-1/2}q(a_1,\dots,a_n) \end{aligned} \tag{9.21}$$

for $2|z|\|B\|_{\mathrm{op}} < 1$ where $q(a_1,\dots,a_n)$ is given by

$$q(a_1,\dots,a_n) = \{\det(2\pi V)\}^{-1/2}\exp\left(-\frac{1}{2}\langle V^{-1}a, a\rangle\right)$$

and for $i,j = 1,\dots,n$,

$$V_{i,j} = \langle (I+2zB)^{-1}h_i, h_j\rangle_H.$$

Let us compute the right hand side of (9.21) without using the eigenexpansion of the operator B.

Set

$$\xi_t = x_t + 2zyt - 2z\mathcal{I}_t^2[x] \quad \text{and} \quad F(x) = \mathcal{I}_T[x]$$

where

$$\mathcal{I}_t[\zeta] = \int_0^t \zeta_s ds.$$

Then we have

$$\begin{aligned} u_F &= z\nabla S = 2z\nabla^* B, \\ \nabla u_F &= z\nabla^2 S = 2zB. \end{aligned} \tag{9.22}$$

The inverse transformation $\eta(\cdot, y)$ of $\xi(\cdot, y)$ is defined as the solution to the integral equation

$$\eta_t = x_t - 2zyt + 2z\mathcal{I}_t^2[\eta]. \tag{9.23}$$

The integral equaiton (9.23) can be solved by the iterated approximation as follows:

$$\eta_t = x_t + \sum_{n=1}^{\infty}(2z)^n \mathcal{I}_t^{2n}[x] - 2z\left(\sum_{n=0}^{\infty}\frac{(2z)^n}{(2n+1)!}t^{2n+1}\right)y.$$

Then we have

$$\begin{aligned}\nabla F[h] &= \mathcal{I}_T[h], \\ (G_Y)(t) &= -(\sqrt{2z}\sinh\sqrt{2z}t)I_{\mathbb{R}^2}, \\ (I_H + \nabla_X u)[h](t) &= h_t - 2z\mathcal{I}_t^2[h].\end{aligned}$$

Thus by (2.6) together with (8.9) we obtain

$$\{\mathrm{Det}_2(I_H + 2zB)\}^{-1/2} = \frac{\exp(zT^2)}{\cosh(\sqrt{2z}T)}. \tag{9.24}$$

Next we compute the covariance matrix V for $h_1(t) = tr_1$ and $h_2(t) = tr_2$ where $r_1 = (1, 0)$ and $r_2 = (0, 1)$. By (9.22) we have

$$(I_H + 2zB)^{-1} = (I_H + \nabla u_F)^{-1}.$$

Combining

$$\begin{aligned}I_{\mathbb{R}^2} - (\nabla F)(G_Y) &= (\cosh(\sqrt{2z}T))I_{\mathbb{R}^2}, \\ (I_H + G_X)[h](t) &= \sum_{n=0}^{\infty}(2z)^n\mathcal{I}_t^{2n}[h],\end{aligned}$$

with (9.2), we have for $i = 1, 2$,

$$(I_H + \nabla u_F)^{-1}[h_i](t) = \frac{1}{\sqrt{2z}}\frac{\sinh(\sqrt{2z}t)}{\cosh(\sqrt{2z}T)}r_i.$$

Thus we obtain

$$\begin{aligned}V_{i,j} &= \int_0^T \frac{d}{dt}(I_H + \nabla u_F)^{-1}h_i(t)\cdot\frac{d}{dt}h_j(t)dt \\ &= \frac{\tanh(\sqrt{2z}T)}{\sqrt{2z}}r_i\cdot r_j.\end{aligned} \tag{9.25}$$

Applying (9.24) and (9.25) to (9.21), we obtain (9.20).

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