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Finite homological dimension and primes associated to integrally closed ideals, II

By

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Abstract

Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Assume that R contains ideals I and J satisfying the conditions (1) $I \subseteq J$, (2) $I : \mathfrak{m} \nsubseteq J$, and (3) J is $\mathfrak{m}$-full, that is $\mathfrak{m}J : x = J$ for some $x \in \mathfrak{m}$. Then the theorem says that R is a regular local ring, if the projective dimension $\mathrm{pd}_R I$ of I is finite. Let $\mathfrak{q} = (a_1, a_2, \ldots, a_t)R$ be an ideal in a Noetherian local ring R generated by a maximal R-regular sequence $a_1, a_2, \ldots, a_t$ and let $\overline{\mathfrak{q}}$ denote the integral closure of $\mathfrak{q}$. Then, thanks to the theorem applied to the ideals $I = \mathfrak{q} : \mathfrak{m}$ and $J = \overline{\mathfrak{q}}$, it follows that $I^2 = \mathfrak{q}I$, unless R is a regular local ring. Consequences are discussed.

1. Introduction

Let R be a commutative Noetherian ring. For each ideal I in R let $\mathrm{pd}_R I$ and $\overline{I}$ denote respectively the projective dimension and the integral closure of I. This paper is a continuation of [GH] and the present purpose is to prove the following theorem.

Theorem 1.1. *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Assume that R contains ideals I and J satisfying the following three conditions*:

(1) $I \subseteq J$,

(2) $I : \mathfrak{m} \nsubseteq J$, *and*

(3) J *is* $\mathfrak{m}$*-full, that is* $\mathfrak{m}J : x = J$ *for some* $x \in \mathfrak{m}$.

Then R is a regular local ring, if $\mathrm{pd}_R I < \infty$.

Theorem 1.1 is closely related to [B, p. 947, Corollaries 1 (ii) and 2], although the proof is totally different. Our method of proof is applicable to

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explore the primes associated to integrally closed ideals of finite homological dimension of many different kinds (cf. Theorem 2.2 and [GHI, Theorem 2.1]. For example, in [GH] the authors studied the primes associated to integrally closed ideals I in R and showed that the local ring $R_\mathfrak{p}$ is regular (resp. Gorenstein) for all $\mathfrak{p} \in \mathrm{Ass}_R R/I$ if $\mathrm{pd}_R I < \infty$ (resp. $\text{G-dim}_A I < \infty$ and I contains a non-zerodivisor in R, where G-dim stands for the Gorenstein dimension). The results can be deduced also from Theorem 1.1. In fact, in order to show the regularity, passing to the local ring $R_\mathfrak{p}$, we may assume that $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m} = \mathfrak{p} \in \mathrm{Ass}_R R/I$. Then, since integrally closed ideals are $\mathfrak{m}$-full ([G, Theorem 2.4]), applying Theorem 1.1 to the case where $I = J$, we get that R is regular (Corollary 2.4).

Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Let $\mathfrak{q} = (a_1, a_2, \ldots, a_t)R$ be an ideal in R generated by a maximal R-regular sequence $a_1, a_2, \ldots, a_t$. Let us apply Theorem 1.1 to the specific ideals $I = \mathfrak{q} : \mathfrak{m}$ and $J = \overline{\mathfrak{q}}$. Then we have $I = \mathfrak{q} : \mathfrak{m} \subseteq \overline{\mathfrak{q}}$, if R is not a regular local ring, whence the ideal $\mathfrak{q}$ is a minimal reduction of I with $I^2 = \mathfrak{q}I$ (Proposition 3.1). This observation will lead us to the following.

Theorem 1.2. *Let R be a Noetherian ring and let $\mathfrak{q}$ be an ideal in R generated by an R-regular sequence. Let $\mathfrak{p} \in \mathrm{Ass}_R R/\mathfrak{q}$ and assume that $\mathfrak{p}$ is maximal in $\mathrm{Ass}_R R/\mathfrak{q}$ with respect to inclusion. We put $I = \mathfrak{q} : \mathfrak{p}$. Then the local ring $R_\mathfrak{p}$ is regular and $\overline{\mathfrak{q}R_\mathfrak{p}} = \mathfrak{q}R_\mathfrak{p}$, if $I^2 \neq \mathfrak{q}I$.*

Theorem 1.2 is a generalization of [CP, Theorem 2.3] and is originated in [CPV, Theorem 2.1]. Our proof of Theorem 1.2 is based on Theorem 1.1 and is rather different from that of [CP], although we will follow in the steps developed by [CP]. Here we note that the equality $I^2 = \mathfrak{q}I$ with $I = \mathfrak{q} : \mathfrak{m}$ holds true for many parameter ideals $\mathfrak{q}$ in a huge class of (not necessarily Cohen-Macaulay) local rings R, which we shall study in the forthcoming papers [GS1], [GS2].

Theorem 1.1 will be proven in Section 2. We shall prove Theorem 1.2 in Section 3. Let us confirm in Section 3 that, unless $R_\mathfrak{p}$ is a regular local ring with $\overline{\mathfrak{q}R_\mathfrak{p}} = \mathfrak{q}R_\mathfrak{p}$, as for the ideal $I = \mathfrak{q} : \mathfrak{p}$ the graded ring $\mathcal{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is Gorenstein if and only if $R/\mathfrak{p}$ is a Cohen-Macaulay ring with $\mathfrak{p} = \mathfrak{q} : \mathfrak{p}$, provided that R is a Gorenstein local ring, $\mathfrak{q}$ is a complete intersection in R, and $\mathfrak{p} \in \mathrm{Ass}_R R/\mathfrak{q}$ (Corollary 3.5 (3)). This is a consequence of [GI, Theorem 5.3, Corollary 5.4] and will lead to the analysis of *good* prime ideals in the sense of [GIK], which we shall perform in Section 4.

2. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. We begin with the following.

Lemma 2.1. *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Let I, J be ideals in R and $x \in \mathfrak{m}$. Assume that*

(1) *$I \subseteq J$,*

(2) *$I : \mathfrak{m} \nsubseteq J$, and*

(3) $\mathfrak{m}J : x = J$.

Then $R/\mathfrak{m}$ is a direct summand of I/xI.

Proof. Let $V = ([I : \mathfrak{m}] + J)/J$ and $\ell = \ell_R(V)$, where $\ell_R(*)$ denotes the length. Then $\ell > 0$ since $I : \mathfrak{m} \nsubseteq J$. Choose elements $y_1, y_2, \ldots, y_\ell \in I : \mathfrak{m}$ so that $\{y_i \mod J\}_{1 \le i \le \ell}$ forms an $R/\mathfrak{m}$-basis of V. The elements $\{xy_i\}_{1 \le i \le \ell}$ are a part of a minimal basis of the ideal I. In fact, let $a_i \in R$ and assume that $\sum_{i=1}^{\ell} a_i(xy_i) \in \mathfrak{m}I$. Then since $x \cdot \sum_{i=1}^{\ell} a_i y_i = \sum_{i=1}^{\ell} a_i(xy_i) \in \mathfrak{m}J$, we get $\sum_{i=1}^{\ell} a_i y_i \in \mathfrak{m}J : x = J$ so that $a_i \in \mathfrak{m}$ for all $1 \le i \le \ell$. We write $I = (xy_i \mid 1 \le i \le \ell) + (z_j \mid 1 \le j \le m)$ with $z_j \in I$ and $\mu_R(I) = \ell + m$, where $\mu_R(*)$ denotes the number of generators. We will show that

$$I/xI = \sum_{i=1}^{\ell} R \cdot \overline{xy_i} \ \oplus \sum_{j=1}^{\mathfrak{m}} R \cdot \overline{z_j}.$$

Let $\{a_i\}_{1 \le i \le \ell}$ and $\{b_j\}_{1 \le j \le m}$ be elements in R and assume that

$$\sum_{i=1}^{\ell} a_i(xy_i) + \sum_{j=1}^{m} b_j z_j \in xI.$$

Then since $\{xy_i\}_{1 \le i \le \ell}$ and $\{z_j\}_{1 \le j \le m}$ form a minimal basis of I, we have $a_i, b_j \in \mathfrak{m}$ for all i and j, so that $a_i y_i \in I$ for $1 \le i \le \ell$. Consequently $\sum_{i=1}^{\ell} a_i(xy_i) \in xI$, whence $\sum_{j=1}^{m} b_j z_j \in xI$ too. Thus $I/xI = \sum_{i=1}^{\ell} R \cdot \overline{xy_i} \ \oplus \sum_{j=1}^{\mathfrak{m}} R \cdot \overline{z_j}$. Since $\mathfrak{m} \cdot (xy_i) \subseteq xI$, this argument also shows that $\mathfrak{m} \cdot \sum_{i=1}^{\ell} R \cdot \overline{xy_i} = (0)$ and $\ell_R(\sum_{i=1}^{\ell} R \cdot \overline{xy_i}) = \ell > 0$. Hence $R/\mathfrak{m}$ is a direct summand of I/xI. $\square$

Proof of Theorem 1.1. We firstly notice that $I \neq (0)$ because $(0) : \mathfrak{m} \subseteq \mathfrak{m}J : x = J$; hence $(0) \neq I \subseteq J \subsetneq R$. Therefore, since $\mathrm{pd}_R I < \infty$, the ideal I contains a regular element, say f, and so we may choose the element x so that x is a non-zerodivisor in R. In fact, since $f^2 \in (x) + \mathfrak{m}J$, we have $(x) + \mathfrak{m}J \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, R} \mathfrak{p}$. Choose $y \in \mathfrak{m}J$ so that $x + y \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, R} \mathfrak{p}$. Then $x + y \in \mathfrak{m}$ and $\mathfrak{m}J : (x+y) = \mathfrak{m}J : x$. Hence we may assume that x is R-regular. Consequently, $\mathrm{pd}_R I/xI < \infty$, whence by Lemma 2.1 $\mathrm{pd}_R R/\mathfrak{m} < \infty$ too. Thus R is a regular local ring. $\square$

Let $\mathrm{id}_R(*)$ (resp. $\text{G-dim}_R(*)$) denote the injective (resp. Gorenstein) dimension. Then the argument in the proof of Theorem 1.1 works also to show the following. Let us leave the details to the readers.

Theorem 2.2. *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Assume that R contains two ideals I and J satisfying the following three conditions* (1) *$I \subseteq J$,* (2) *$I : \mathfrak{m} \nsubseteq J$, and* (3) *$J$ is $\mathfrak{m}$-full. Then R is a regular* (*resp. Gorenstein*) *local ring, if $\mathrm{id}_R I < \infty$* (*resp. $G\text{-dim}_R\, I < \infty$ and I contains a non-zerodivisor in R*).

We now summarize a few consequences of Theorem 1.1.

Corollary 2.3 ([GH, Remark 3.3]). *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Then R is a regular local ring, if R contains an $\mathfrak{m}$-full ideal I such that $\mathrm{pd}_R I < \infty$ and $\mathfrak{m} \in \mathrm{Ass}_R\, R/I$.*

Proof. Take $I = J$ and apply Theorem 1.1. $\square$

Corollary 2.4 ([B, p. 947, Corollary 3], [GH, Theorem 1.1 (1)]). *Let R be a Noetherian ring and let I be an integrally closed ideal in R. Then the local ring $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \mathrm{Ass}_R\, R/I$ if $\mathrm{pd}_R I < \infty$.*

Proof. Passing to the local ring $R_\mathfrak{p}$, we may assume that $(R, \mathfrak{m})$ is local and $\mathfrak{m} \in \mathrm{Ass}_R R/I$. In order to see R is regular, enlarging the residue class field, we may also assume that $R/\mathfrak{m}$ is infinite. If $I = (0)$, then $\sqrt{(0)} = (0)$ since $\overline{I} = I$. Thus $\mathfrak{m} = (0)$ since $\mathfrak{m} \in \mathrm{Ass}\, R$. Let $I \neq (0)$. Then I contains a non-zerodivisor of R, since $\mathrm{pd}_R I < \infty$. Consequently, by [G, Theorem 2.4] I is $\mathfrak{m}$-full, so that by Corollary 2.3 R is a regular local ring. $\square$

3. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Let us begin with the following.

Proposition 3.1. *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $t = \mathrm{depth}\, R$. Let $a_1, a_2, \ldots, a_t$ be a maximal R-regular sequence. We put $\mathfrak{q} = (a_1, a_2, \ldots, a_t)$ and $I = \mathfrak{q} : \mathfrak{m}$. Then $I^2 = \mathfrak{q}I$ if R is not a regular local ring.*

Proof. Enlarging the residue class field, we may assume that $R/\mathfrak{m}$ is infinite. If $t = 0$, then $I = (0) : \mathfrak{m} \neq R$ since R is not regular. Thus $I^2 = (0)$. Suppose $t > 0$. Then $\overline{\mathfrak{q}}$ is $\mathfrak{m}$-full ([G, Theorem 2.4]), so that by Theorem 1.1 we get $I \subseteq \overline{\mathfrak{q}}$. Consequently, $\mathfrak{q}$ is a minimal reduction of I, whence $\mathfrak{m}I \cap \mathfrak{q} = \mathfrak{m}\mathfrak{q}$. Therefore $\mathfrak{m}I^n = \mathfrak{m}\mathfrak{q}^n$ for all $n \in \mathbb{Z}$, since $\mathfrak{m}I \subseteq \mathfrak{q}$. Let $x \in I^2$ and write $x = \sum_{i=1}^t a_i x_i$ with $x_i \in R$. Let $\alpha \in \mathfrak{m}$. Then $\alpha x = \sum_{i=1}^t a_i(\alpha x_i)$ and $\alpha x \in \mathfrak{m}I^2 = \mathfrak{m}\mathfrak{q}^2 \subseteq \mathfrak{q}^2$. Hence $\alpha x_i \in \mathfrak{q}$ for all $1 \leq i \leq t$, because the sequence $a_1, a_2, \ldots, a_t$ is R-regular. Thus $x_i \in I = \mathfrak{q} : \mathfrak{m}$ whence $I^2 = \mathfrak{q}I$. $\square$

The following result is known (cf. [G, Theorem 3.1] and [CHV, Corollary 3.12]). Let us note a brief proof in our context.

Proposition 3.2. *Let R be a regular local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim R \geq 1$. Let I be an $\mathfrak{m}$-primary ideal in R. Then the following three conditions are equivalent.*

(1) *$\overline{I} = I$ and R/I is a Gorenstein ring.*

(2) *I is a parameter ideal of R with $\overline{I} = I$.*

(3) $\mu_R(\mathfrak{m}/I) \leq 1$.

When this is the case, the equality $I = (a_1, a_2, \ldots, a_{d-1}, a_d^q)$ holds true for some $0 < q \in \mathbb{Z}$ and some regular system $a_1, a_2, \ldots, a_d$ of parameters in R.

Proof. (1) $\Rightarrow$ (3) Enlarging the residue class field if necessary, we may assume that $R/\mathfrak{m}$ is infinite. Thus I is $\mathfrak{m}$-full ([G, Theorem 2.4]). Choose $x \in \mathfrak{m}$ so that $\mathfrak{m}I : x = I$. Then since $I : \mathfrak{m} = I : x$ ([W, Lemma 1]), considering the exact sequence

$$0 \to (I : x)/I \to R/I \xrightarrow{x} R/I \to R/(I + (x)) \to 0,$$

we see that $\ell_R(R/(I+(x)) = \ell_R((I : \mathfrak{m})/I)$. Hence $\ell_R(R/(I+(x)) = 1$ because R/I is a Gorenstein ring. Thus $\mathfrak{m} = I + (x)$ and $\mu_R(\mathfrak{m}/I) \leq 1$.

(3) $\Rightarrow$ (2) Because $\mu_R(\mathfrak{m}/I) \leq 1$, we have $I = (a_1, a_2, \ldots, a_{d-1}, a_d^q)$ for some $0 < q \in \mathbb{Z}$ and some regular system $a_1, a_2, \ldots, a_d$ of parameters in R. Hence I is integrally closed ([G, Theorem 3.1]) and the ring R/I is Gorenstein.

The last assertion and the implication (2) $\Rightarrow$ (1) are clear. $\square$

The following result generalizes [CHV, Corollaries 3.2 and 3.12].

Corollary 3.3. *Let I be an ideal in a Noetherian ring R.*

(1) *Suppose that R/I is a Gorenstein ring, $\mathrm{pd}_R I < \infty$, and $\overline{I} = I$. Then $R_\mathfrak{p}$ is a regular local ring and $\mu_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}/IR_\mathfrak{p}) \leq 1$ for all $\mathfrak{p} \in \mathrm{Ass}_R\, R/I$.*

(2) *Suppose that $R/\overline{I}$ is a Gorenstein ring and $\mathrm{pd}_R \overline{I} < \infty$. Then $\overline{I} = I$ if $\mathrm{Ass}_R\, R/I = \mathrm{Min}_R R/I$.*

Proof. (1) The local ring $R_\mathfrak{p}$ is regular by Corollary 2.4. Hence $\mu_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}/IR_\mathfrak{p}) \leq 1$ by Proposition 3.2.

(2) Assume that $\overline{I} \neq I$ and choose $\mathfrak{p} \in \mathrm{Ass}_R\, R/I$ so that $\overline{I}R_\mathfrak{p} \neq IR_\mathfrak{p}$. Then since $\mathfrak{p} \in \mathrm{Min}_R R/I$, we have $\mathfrak{p} \in \mathrm{Min}_R R/\overline{I}$ too, so that by Corollary 2.4 the local ring $R_\mathfrak{p}$ is regular. Because $R_\mathfrak{p}/\overline{I}R_\mathfrak{p}$ is Gorenstein, by Proposition 3.2 the ideal $\overline{IR_\mathfrak{p}} = \overline{I}R_\mathfrak{p}$ is a parameter ideal in $R_\mathfrak{p}$, so that $\overline{I}R_\mathfrak{p}$ is a unique reduction for itself. Hence $\overline{I}R_\mathfrak{p} = IR_\mathfrak{p}$, which is absurd. $\square$

The following result is the key for our proof of Theorem 1.2.

Proposition 3.4 ([CP, Theorem 2.2])**.** *Let R be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and let $\mathfrak{q}$ be a parameter ideal in R. Let $I = \mathfrak{q} : \mathfrak{m}$ and assume that $I^2 \neq \mathfrak{q}I$. Then R is a regular local ring, $\mu_R(\mathfrak{m}/\mathfrak{q}) \leq 1$, and $\overline{\mathfrak{q}} = \mathfrak{q}$.*

Proof. By Proposition 3.1 the ring R is regular. Hence by Proposition 3.2 we see that $\mu_R(\mathfrak{m}/\mathfrak{q}) \leq 1$ if and only if $\overline{\mathfrak{q}} = \mathfrak{q}$. Since the ring $R/\mathfrak{q}$ is Gorenstein and $I^2 \neq \mathfrak{q}I$, the equality $\overline{\mathfrak{q}} = \mathfrak{q}$ readily follows from [CHV, Theorem 3.7]. However, because the proof of [CHV, Theorem 3.7] contains some confusion, let us note a brief proof for completeness (cf. Proof of [CHV, Lemma 3.6]). We may assume $d = \dim R \geq 1$. Let $n = \mu_R(\mathfrak{m}/\mathfrak{q})$ and write $\mathfrak{m} = (x_1, x_2, \ldots, x_n) + \mathfrak{q}$ with $x_i \in \mathfrak{m}$. Let $\Delta \in I$ such that $I = (\Delta) + \mathfrak{q}$ and choose the elements

$\{y_j\}_{1\le j\le n}$ of R so that $x_iy_j \equiv \delta_{ij}\Delta \bmod \mathfrak{q}$ for all $1 \le i,j \le n$ (this choice is possible, thanks to [CHV, Lemma 3.5]). Hence $x_iy_i \in I$ and $x_iy_j \in \mathfrak{q}$ if $i \ne j$. Let $\Delta = x_iy_i + a_i$ with $a_i \in \mathfrak{q}$. Suppose now that $n \ge 2$. Then, because

$$\Delta^2 = (x_1y_1 + a_1)(x_2y_2 + a_2) = (x_1y_2)(x_2y_1) + a_1{\cdot}x_2y_2 + a_2{\cdot}x_1y_1 + a_1a_2,$$

we get $\Delta^2 \in \mathfrak{q}I$ whence $I^2 = \mathfrak{q}I$, which is impossible. Thus $n = \mathfrak{m}_R(\mathfrak{m}/\mathfrak{q}) \le 1$. $\square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathfrak{q} = (a_1, a_2, \ldots, a_t)$ with the sequence $a_1, a_2, \ldots, a_t$ regular. Hence $\operatorname{depth} R_\mathfrak{p} = t$. Let $\mathcal{F} = \operatorname{Ass}_R R/\mathfrak{q}$. Then $\operatorname{Ass}_R R/I \subseteq \mathcal{F}$, since $I = \mathfrak{q} : \mathfrak{p}$. Because $\mathfrak{q}/\mathfrak{q}I = R/I \otimes_R \mathfrak{q}/\mathfrak{q}^2 \cong (R/I)^t$, the R/I-module $\mathfrak{q}/\mathfrak{q}I$ is free. Hence, thanks to the exact sequence

$$0 \to \mathfrak{q}/\mathfrak{q}I \to R/\mathfrak{q}I \to R/\mathfrak{q} \to 0$$

of R-modules, we get $\operatorname{Ass}_R R/\mathfrak{q}I \subseteq \mathcal{F}$ too. We claim that $\operatorname{Ass}_R I^2/\mathfrak{q}I \subseteq \{\mathfrak{p}\}$. To see this, let $P \in \operatorname{Ass}_R I^2/\mathfrak{q}I$. Then since $P \in \mathcal{F}$ and $I^2R_P \ne \mathfrak{q}R_P{\cdot}IR_P$, we have $IR_P \ne \mathfrak{q}R_P$ so that $P \supseteq \mathfrak{p}$ (recall that $I = \mathfrak{q} : \mathfrak{p}$). Thus $P = \mathfrak{p}$, since $\mathfrak{p}$ is maximal in $\mathcal{F}$. Consequently, $I^2R_\mathfrak{p} \ne \mathfrak{q}R_\mathfrak{p}{\cdot}IR_\mathfrak{p}$ because $I^2 \ne \mathfrak{q}I$, so that by Proposition 3.1 the local ring $R_\mathfrak{p}$ must be regular. We have $\overline{\mathfrak{q}R_\mathfrak{p}} = \mathfrak{q}R_\mathfrak{p}$ by Proposition 3.4. $\square$

For a given ideal I in a Noetherian local ring R let $\mathcal{G}(I) = \bigoplus_{n\ge 0} I^n/I^{n+1}$ be the associated graded ring of I.

Corollary 3.5. *Let R be a Cohen-Macaulay local ring and let $\mathfrak{q}$ be an ideal of R generated by a subsystem of parameters of R. Let $\mathfrak{p} \in \operatorname{Ass}_R R/\mathfrak{q}$ and put $I = \mathfrak{q} : \mathfrak{p}$. Assume that* (1) *$R_\mathfrak{p}$ is not a regular local ring, or* (2) *$R_\mathfrak{p}$ is a regular local ring but $\overline{\mathfrak{q}R_\mathfrak{p}} \ne \mathfrak{q}R_\mathfrak{p}$. Then the following assertions hold true.*

(1) ([CP, Theorem 2.3]) *$I^2 = \mathfrak{q}I$ but $I \ne \mathfrak{q}$.*

(2) ([CP, Corollary 2.4]) *$\mathcal{G}(I)$ is a Cohen-Macaulay ring if and only if R/I is Cohen-Macaulay. Consequently, $\mathcal{G}(I)$ is a Cohen-Macaulay ring, if R is Gorenstein and $R/\mathfrak{p}$ is Cohen-Macaulay.*

(3) *$\mathcal{G}(I)$ is a Gorenstein ring if and only if R is Gorenstein, $R/\mathfrak{p}$ is Cohen-Macaulay, and $\mathfrak{p} = \mathfrak{q} : \mathfrak{p}$.*

Proof. (1) See Theorem 1.2.

(2) This is well-known. Recall that I is equi-multiple.

(3) The ring $\mathcal{G}(I)$ is Gorenstein if and only if R is Gorenstein, R/I is Cohen-Macaulay, and $I = \mathfrak{q} : I$ ([GI, Theorem 5.3 and Corollary 5.4]). When R is a Gorenstein local ring, the condition that R/I is Cohen-Macaulay and $I = \mathfrak{q} : I$ is equivalent to saying that $R/\mathfrak{p}$ is Cohen-Macaulay and $\mathfrak{q} : \mathfrak{p} = \mathfrak{p}$ (notice that $\mathfrak{q} : (\mathfrak{q} : \mathfrak{p}) = \mathfrak{p}$). $\square$

4. Good prime ideals

Let us note in this section a few remarks on good prime ideals in the sense of [GIK]. Let R be a Gorenstein local ring and let I be an ideal in R of height t. Then, following [GIK], we say that I is a *good* ideal in R if I contains a reduction $\mathfrak{q}$ generated by t elements in R and the associated graded ring $\mathcal{G}(I)$ of I is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I)) = 1 - t$, where $\mathrm{a}(\mathcal{G}(I))$ denotes the a-invariant of $\mathcal{G}(I)$ ([GW, Definition 3.1.4]). We denote by $\mathcal{X}_R^t$ the set of good ideals I in R of height t.

Theorem 4.1. *Let R be a Gorenstein local ring and let $\mathfrak{p}$ be a prime ideal in R with $\mathrm{ht}_R\mathfrak{p} = t$. Assume that $\mathfrak{p}$ contains a reduction $\mathfrak{q}$ generated by t elements. Then the following two conditions are equivalent.*

(1) $\mathfrak{p} \in \mathcal{X}_R^t$.

(2) $R/\mathfrak{p}$ *is a Cohen-Macaulay ring and* $\mathfrak{p} = \mathfrak{q} : \mathfrak{p}$

When this is the case, $\mathfrak{p}^2 = \mathfrak{q}\mathfrak{p}$ *and the local ring* $R_\mathfrak{p}$ *is an abstract hypersurface of multiplicity* 2.

Proof. (1) $\Rightarrow$ (2) See [GIK, Proposition 2.3].

(2) $\Rightarrow$ (1) It suffices to show $\mathfrak{p}^2 = \mathfrak{q}\mathfrak{p}$ (cf. [GIK, Proposition 2.3]). Assume that $\mathfrak{p}^2 \neq \mathfrak{q}\mathfrak{p}$. Then by Corollary 3.5 (1) the local ring $R_\mathfrak{p}$ is regular and $\overline{\mathfrak{q}R_\mathfrak{p}} = \mathfrak{q}R_\mathfrak{p}$. Since $\mathfrak{q}$ is a reduction of $\mathfrak{p}$, this forces $\mathfrak{p}R_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}$, which is impossible (recall that $\mathfrak{p} = \mathfrak{q} : \mathfrak{p}$). Hence $\mathfrak{p}^2 = \mathfrak{q}\mathfrak{p}$.

To see the last assertion, notice that $\mathfrak{p}^2 R_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}{\cdot}\mathfrak{p}R_\mathfrak{p}$ but $\mathfrak{p}R_\mathfrak{p} \neq \mathfrak{q}R_\mathfrak{p}$. Hence $\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) = 2$ because $R_\mathfrak{p}$ is a Gorenstein ring, so that $R_\mathfrak{p}$ is an abstract hypersurface of multiplicity 2. $\square$

Corollary 4.2. *Let R be a regular local ring. Then $\mathcal{X}_R^t \cap \operatorname{Spec} R = \emptyset$ for all $t \in \mathbb{Z}$.*

This assertion (4.2) is no longer true, unless R is regular. The simplest example is as follows. Let $P = k[[X, Y, Z]]$ be the formal power series ring in three variables over a field k. Let $R = P/(Z^2 - XY)$ and $\mathfrak{m}$ the maximal ideal in R. Then R is a 2-dimensional rational singularity and $\mathcal{G}(\mathfrak{m}) = k[X, Y, Z]/(Z^2 - XY)$. Hence $\mathfrak{m} \in \mathcal{X}_R^2$, because $\mathcal{G}(\mathfrak{m})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(\mathfrak{m})) = -1$ ([GW, Remark 3.1.6]).

In order to produce more examples, let A be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{n}$ and assume that A possess the canonical module K_A. Let $R = A \ltimes \mathrm{K}_A$ be the idealization of K_A. Hence $R = A \oplus \mathrm{K}_A$ as A-modules, the multiplication in R is given by $(a, x) \cdot (b, y) = (ab, ay + bx)$, and R is a Gorenstein local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times \mathrm{K}_R$ ([R]). The structure of good ideals in $R = A \ltimes \mathrm{K}_A$ is completely known ([GIK]). Here let us add the following.

Proposition 4.3. *Let $0 \le t \le \dim A$. Then*

$$\mathcal{X}_R^t \cap \operatorname{Spec} R = \{\mathfrak{p} \times \mathrm{K}_A \mid \mathfrak{p} \in \operatorname{Spec} A \text{ such that } \mathrm{ht}_A\mathfrak{p} = \mu_A(\mathfrak{p}) = t\}.$$

Proof. Let $P \in \operatorname{Spec} R$ and write $P = \mathfrak{p} \times \mathrm{K}_A$ with $\mathfrak{p} \in \operatorname{Spec} A$. Then by [GIK, Theorem 1.1] we see that $P \in \mathcal{X}_R^t$ if and only if (i) $\mathrm{ht}_A \mathfrak{p} = t$, (ii) $\mathfrak{p}$ contains a reduction $\mathfrak{q} = (a_1, a_2, \dots, a_t)A$ generated by t elements such that $\mathfrak{p}^2 = \mathfrak{q}\mathfrak{p}$ and $\mathrm{K}_A = \mathfrak{q}\mathrm{K}_A :_{\mathrm{K}_A} \mathfrak{p}$, and (iii) $A/\mathfrak{p}$ is a Cohen-Macaulay ring. If $\mathrm{ht}_A \mathfrak{p} = \mu_A(\mathfrak{p}) = t$, then the ideal $\mathfrak{p}$ certainly satisfies conditions (i), (ii) and (iii), whence $P \in \mathcal{X}_R^t$. Suppose that conditions (i) and (ii) are satisfied. Then by (i) the sequence $a_1, a_2, \dots, a_t$ is A-regular. Consequently, because $K_A/\mathfrak{q}K_A = \mathrm{K}_{A/\mathfrak{q}}$ is a faithful $A/\mathfrak{q}$-module ([BH, Proposition 3.3.11]), by ((ii) we get $\mathfrak{p} = \mathfrak{q}$, so that $\mathrm{ht}_A \mathfrak{p} = \mu_A(\mathfrak{p}) = t$. $\square$

Corollary 4.4. *Let A be a regular local ring and put $R = A \ltimes A$. Then $\mathcal{X}_R^t \cap \operatorname{Spec} R \neq \emptyset$ for any $0 \leq t \leq d = \dim A$.*

We close this paper with the following.

Example 4.5. Let $P = k[[X, Y, Z]]$ be the formal power series ring in three variables over a field k. Let $1 \leq \ell \in \mathbb{Z}$. Let $n = 2\ell + 1$ and $R = P/(Z^2 - X^nY^n)$. We denote by x, y and z the reduction of X, Y and Z mod $(Z^2 - X^nY^n)$ respectively. Then the good prime ideals in R are exactly (x, z), (y, z) and $\mathfrak{m} = (x, y, z)$.

Proof. Notice that R is an integral domain of $\dim R = 2$ and $\mathcal{G}(\mathfrak{m}) = k[X, Y, Z]/(Z^2)$. Hence $\mathcal{G}(\mathfrak{m})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(\mathfrak{m})) = -1$ ([GW, Remark 3.1.6]), so that $\mathfrak{m} \in \mathcal{X}_R^2$. Let $\mathfrak{p}$ be a good prime ideal in R with $\mathrm{ht}_R \mathfrak{p} = 1$. Then since $\mathcal{X}_A^1 = \{(x^iy^j, z) \mid 0 \leq i, j \leq \ell \text{ and } i + j \geq 1\}$ ([GK, Example 3.2]), we get either $\mathfrak{p} = (x, z)$ or $\mathfrak{p} = (y, z)$ as was claimed. $\square$

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