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On the Bernstein-Gel'fand-Gel'fand correspondence and a result of Eisenbud, Fløystad, and Schreyer

By

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Abstract

We show that a combination between a remark of I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand [2] and the idea, systematically investigated by D. Eisenbud, G. Fløystad and F.-O. Schreyer [3], of taking Tate resolutions over exterior algebras leads to quick proofs of the main results of [2] and [3] (Theorems 7 and 10 below). This combination is expressed by Lemma 6 from the text, a result which can be seen as a formula for computing hyperext groups on projective spaces in terms of linear algebra. We prove it directly, using only the cohomology of invertible sheaves on a projective space and a few basic facts about derived categories.

Since the above abstract may serve as an introduction as well, we begin by recalling (in (0)–(4)) some definitions and facts. We use the Chapter I of [5] as our main reference for homological algebra (except that we denote mapping cones by "Con").

0. Definition. Let k be a field, V an $(n+1)$-dimensional k-vector space, $e_0, \dots, e_n$ a k-basis of V and $X_0, \dots, X_n$ the dual basis of V^*. Let $\Lambda = \wedge(V)$ be the exterior algebra of V. Λ is a (positively) graded k-algebra : $\Lambda = \Lambda_0 \oplus \cdots \oplus \Lambda_{n+1}$ with $\Lambda_i = \wedge^i(V)$. Let $\Lambda_+ := \Lambda_1 \oplus \cdots \oplus \Lambda_{n+1}$ and $\underline{k} := \Lambda/\Lambda_+$. We denote by mod-$\Lambda$ the category of finitely generated, graded, right Λ-modules (with morphisms of degree 0).

Let $\mathbb{P} = \mathbb{P}(V)$ be the projective space of 1-dimensional k-vector subspaces of V (such that $\mathrm{H}^0\mathcal{O}_{\mathbb{P}}(1) = V^*$). If $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ one defines a bounded complex $\mathrm{L}(N)$ of coherent sheaves on $\mathbb{P}(V)$ by $\mathrm{L}(N)^p := \mathcal{O}_{\mathbb{P}}(p) \otimes_k N_p$ and $d_{\mathrm{L}(N)} := \sum_{i=0}^n (X_i \cdot -) \otimes (- \cdot e_i)$. In this way one obtains *the BGG functor* $\mathrm{L} : \text{mod-}\Lambda \to \mathrm{C}^b(\mathrm{Coh}\mathbb{P}(V))$. It can be extended to a functor $\mathrm{L} : \mathrm{C}(\text{mod-}\Lambda) \to \mathrm{C}(\mathrm{Qcoh}\mathbb{P}(V))$ as it follows : if $K^\bullet$ is a complex in mod-Λ one considers the double complex $X^{\bullet\bullet}$ in $\mathrm{Coh}\mathbb{P}(V)$ with $X^{p,\bullet} := \mathrm{L}(K^p)$ and with $d'^p : X^{p,\bullet} \to X^{p+1,\bullet}$

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equal to $\mathrm{L}(d_K^p)$ and one takes $\mathrm{L}(K^\bullet) := \mathrm{s}(X^{\bullet\bullet})$ (the simple complex associated to $X^{\bullet\bullet}$).

The (extended) functor L is exact, commutes with the translation functor T and with mapping cones and maps morphisms homotopically equivalent to 0 to morphisms with the same property (see [3], Remark after (2.5) for a nice argument) hence it induces a functor $\mathrm{L} : \mathrm{K}(\text{mod-}\Lambda) \to \mathrm{K}(\mathrm{Qcoh}\mathbb{P}(V))$. L also maps quasi-isomorphisms in $\mathrm{K}^+(\text{mod-}\Lambda)$ to quasi-isomorphisms in $\mathrm{K}(\mathrm{Qcoh}\mathbb{P}(V))$, hence it induces a functor $\mathrm{L} : \mathrm{D}^+(\text{mod-}\Lambda) \to \mathrm{D}(\mathrm{Qcoh}\mathbb{P}(V))$.

We shall often use the following shorter notations : $\mathrm{K}(\Lambda) := \mathrm{K}(\text{mod-}\Lambda)$, $\mathrm{D}(\Lambda) := \mathrm{D}(\text{mod-}\Lambda)$, $\mathrm{D}(\mathbb{P}) := \mathrm{D}(\mathrm{Qcoh}\mathbb{P}(V))$ and $\mathrm{D}^b(\mathbb{P}) := \mathrm{D}^b(\mathrm{Coh}\mathbb{P}(V))$.

1. Definition. (i) If $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ and $a \in \mathbb{Z}$ one defines a new object $N(a)$ of mod-Λ by : $N(a)_p := N_{a+p}$ and $(y\cdot v)_{N(a)} := (-1)^a(y\cdot v)_N$, $\forall y \in N$, $\forall v \in V$. With this convention, if $\omega \in \Lambda_b$ then $(-\cdot\omega)_N$ defines a morphism in mod-Λ : $N(a) \to N(a+b)$. If $u : N' \to N$ is a morphism then $u(a) : N'(a) \to N(a)$ is just u if one forgets the gradings.

One has : $\mathrm{L}(N(a)) = \mathrm{T}^a\mathrm{L}(N)(-a)$. If $K^\bullet$ is a complex in mod-Λ, let $K^\bullet((a))$ be the complex which coincides with $K^\bullet(a)$ term by term but with $d_{K((a))} := (-1)^a d_{K(a)}$. Then $\mathrm{L}(K^\bullet((a))) = \mathrm{T}^a\mathrm{L}(K^\bullet)(-a)$ and if one applies L to the isomorphism $((-1)^{ap}\cdot \mathrm{id}_{K^p(a)})_{p\in\mathbb{Z}} : K^\bullet(a) \xrightarrow{\sim} K^\bullet((a))$ one gets a functorial isomorphism $\mathrm{L}(K^\bullet(a)) \simeq \mathrm{T}^a\mathrm{L}(K^\bullet)(-a)$.

(ii) If $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ let $N^\vee$ denote the graded k-vector space $\mathrm{Hom}_k(N, k)$ endowed with the following right Λ-module structure : for $v \in V$, the multiplication $(-\cdot v)_{N^\vee} : (N^\vee)_p \to (N^\vee)_{p+1}$ is, by definition, $(-1)^{p+1}\cdot$ the dual of the multiplication $(-\cdot v)_N : N_{-p-1} \to N_{-p}$. With this definition, $\mathrm{L}(N^\vee) = \mathcal{H}om^\bullet_{\mathcal{O}_\mathbb{P}}(\mathrm{L}(N), \mathcal{O}_\mathbb{P})$.

The canonical isomorphism of k-vector spaces $\mu : N \to (N^\vee)^\vee$ is not a morphism in mod-Λ : $\mu(y\cdot v) = -\mu(y)\cdot v$, $\forall y \in N$, $\forall v \in V$. However, $\mu' := ((-1)^p\mu_p)_{p\in\mathbb{Z}}$ defines an isomorphism in mod-Λ : $N \xrightarrow{\sim} (N^\vee)^\vee$.

(iii) Of a particular importance is the object $\Lambda^\vee$ of mod-Λ. One has $(\Lambda^\vee)_{-p} = \wedge^p V^*$, $\forall p \in \mathbb{Z}$ and, for $v \in V$, the multiplication $(-\cdot v)_{\Lambda^\vee} : (\Lambda^\vee)_{-p} \to (\Lambda^\vee)_{-p+1}$ is the contraction by v : $(f_1\wedge\cdots\wedge f_p\cdot v)_{\Lambda^\vee} = \sum_{i=1}^p(-1)^{i-1}f_i(v)\cdot f_1\wedge\cdots\wedge\widehat{f_i}\wedge\cdots\wedge f_p$ for $f_1,\dots,f_p \in V^*$. It follows that $\mathrm{L}(\Lambda^\vee)$ is the tautological Koszul complex on $\mathbb{P}(V)$:

$$0 \to \mathcal{O}_\mathbb{P}(-n-1)\otimes_k\wedge^{n+1}V^* \to \cdots \to \mathcal{O}_\mathbb{P}(-1)\otimes_k V^* \to \mathcal{O}_\mathbb{P} \to 0.$$

(iv) If $N \in \mathrm{Ob}(\text{mod-}\Lambda)$, $\mathrm{soc}(N)$ consists of the elements of N annihilated by Λ_+. In particular, $\mathrm{soc}(\Lambda) = \Lambda_{n+1}$ and $\mathrm{soc}(\Lambda^\vee) = (\Lambda^\vee)_0$.

2. Remark. (i) Let $\mathcal{A}$ be an abelian category. Consider a short exact sequence:

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

in the category $\mathrm{C}(\mathcal{A})$ of complexes in $\mathcal{A}$. Let $w : Z^\bullet \to \mathrm{T}X^\bullet$ be the morphism in the derived category $\mathrm{D}(\mathcal{A})$ defined by the diagram:

$$Z^\bullet \xleftarrow[\text{qis}]{(0,v)} \mathrm{Con}(u) \xrightarrow{(\mathrm{id}_{\mathrm{T}X},0)} \mathrm{T}X^\bullet$$

(recall that $\mathrm{Con}(u) = \mathrm{T}X^\bullet \oplus Y^\bullet$ term by term, not as complexes). Then $(X^\bullet, Y^\bullet, Z^\bullet, u, v, w)$ is a *distinguished triangle* in $\mathrm{D}(\mathcal{A})$ hence $(Y^\bullet, Z^\bullet, \mathrm{T}X^\bullet, v, w, -\mathrm{T}u)$ and $(\mathrm{T}^{-1}Z^\bullet, X^\bullet, Y^\bullet, -\mathrm{T}^{-1}w, u, v)$ are distinguished triangles too. One gets a "long" complex in $\mathrm{D}(\mathcal{A})$:

$$\cdots \longrightarrow \mathrm{T}^{-1}Z^\bullet \xrightarrow{-\mathrm{T}^{-1}w} X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} \mathrm{T}X^\bullet \xrightarrow{-\mathrm{T}u} \mathrm{T}Y^\bullet \longrightarrow \cdots$$

and for every $W^\bullet \in \mathrm{Ob}\mathrm{C}(\mathcal{A})$ if one applies $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(W^\bullet, -)$ or $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(-, W^\bullet)$ to this long complex one gets long exact sequences in the category $\mathcal{A}b$ of abelian groups.

(ii) Assume that the short exact sequence of complexes from (i) is *semi-split*, i.e., that

$$0 \longrightarrow X^p \xrightarrow{u^p} Y^p \xrightarrow{v^p} Z^p \longrightarrow 0$$

is split-exact $\forall p \in \mathbb{Z}$, i.e., there exist morphisms $s^p : Z^p \to Y^p$ and $t^p : Y^p \to X^p$ such that $t^p \circ u^p = \mathrm{id}_{X^p}$, $v^p \circ s^p = \mathrm{id}_{Z^p}$ and $u^p \circ t^p + s^p \circ v^p = \mathrm{id}_{Y^p}$ (hence $t^p \circ s^p = 0$). Then $\delta := (t^{p+1} \circ d_Y^p \circ s^p)_{p\in\mathbb{Z}}$ is a morphism of complexes (i.e., in $\mathrm{C}(\mathcal{A})$) : $Z^\bullet \to \mathrm{T}X^\bullet$ and $w = -\delta$ in $\mathrm{D}(\mathcal{A})$ (in fact, $({}^t(-\delta^p, s^p))_{p\in\mathbb{Z}} : Z^\bullet \to \mathrm{Con}(u)$ is an inverse of $(0, v)$ in $\mathrm{K}(\mathcal{A})$).

(iii) Assume that $X^\bullet, Y^\bullet, Z^\bullet \in \mathrm{Ob}\mathrm{C}^+(\mathcal{A})$ and consider a short exact sequence as in (i). Let $I^\bullet \in \mathrm{Ob}\mathrm{C}(\mathcal{A})$ be a complex consisting of *injective* objects of $\mathcal{A}$. Then the functor $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(-, I^\bullet)$ maps quasi-isomorphisms in $\mathrm{K}^+(\mathcal{A})$ to isomorphisms in $\mathcal{A}b$, hence it induces a (contravariant) functor : $\mathrm{D}^+(\mathcal{A})^\circ \to \mathcal{A}b$ and if one applies this functor to the "long" complex in $\mathrm{D}^+(\mathcal{A})$ defined in (i) one gets a long exact sequence in $\mathcal{A}b$ (because $(X^\bullet, Y^\bullet, \mathrm{Con}(u), u, {}^t(0, \mathrm{id}_Y), (\mathrm{id}_{\mathrm{T}X}, 0))$ is a distinguished triangle in $\mathrm{K}(\mathcal{A})$).

(iv) We also recall that if $I^\bullet \in \mathrm{Ob}\mathrm{K}^+(\mathcal{A})$ consists of injective objects of $\mathcal{A}$ then, for every $X^\bullet \in \mathrm{Ob}\mathrm{K}(\mathcal{A})$, the canonical map $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X^\bullet, I^\bullet)$ is bijective.

(v) One can deduce from (iv) the following generalization of it: assume that $\mathcal{A}$ has enough injectives and let $X^\bullet \in \mathrm{Ob}\mathrm{C}^-(\mathcal{A})$, $Y^\bullet \in \mathrm{Ob}\mathrm{C}^+(\mathcal{A})$ be such that $\mathrm{Ext}^{p-q}(X^p, Y^q) = 0$ $\forall p > q$. Then the map $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$ is surjective. There is a similar statement for injectivity. We shall not use this generalization, except for a comment after the proof of (7).

3. Example. (a) Consider (as in [3], par.3) the short exact sequence in mod-Λ:

$$0 \longrightarrow \underline{k}\otimes_k V \longrightarrow (\Lambda/(\Lambda_+)^2)(1) \longrightarrow \underline{k}(1) \longrightarrow 0.$$

Let $w : \underline{k}(1) \to \mathrm{T}(\underline{k}\otimes_k V)$ be the morphism in $\mathrm{D}^b(\text{mod-}\Lambda)$ defined in (2) (i) and let $\nu = \mathrm{T}^{-1}w : \mathrm{T}^{-1}\underline{k}(1) \to \underline{k}\otimes_k V$. If one applies L to the short exact sequence one gets a *semi-split* short exact sequence in $\mathrm{C}(\mathrm{Coh}\mathbb{P}(V))$. Applying (2) (ii) one derives easily that $\mathrm{L}(\nu)$ is the canonical injection: $\mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}\otimes_k V$ (recall that the module structure of $(\Lambda/(\Lambda_+)^2)(1)$ differs by sign from the module structure of $\Lambda/(\Lambda_+)^2$).

(b) Dually, consider the short exact sequence in mod-Λ:

$$0 \longrightarrow \underline{k}(-1) \longrightarrow (\Lambda/(\Lambda_+)^2)^\vee(-1) \longrightarrow \underline{k}\otimes_k V^* \longrightarrow 0$$

and let $\varepsilon : \underline{k}\otimes_k V^* \to \mathrm{T}\underline{k}(-1)$ be the morphism in $\mathrm{D}^b(\text{mod-}\Lambda)$ defined in (2) (i). Then $\mathrm{L}(\varepsilon)$ is the canonical epimorphism: $\mathscr{O}_{\mathbb{P}}\otimes_k V^* \to \mathscr{O}_{\mathbb{P}}(1)$.

In the next proposition we gather some well-known properties of the category mod-Λ, stated in [2]. We include a sketch of proof for the reader's convenience.

4. Proposition. (i) *If* $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ *and* $a \in \mathbb{Z}$ *then the map*:

$$\mathrm{Hom}_{\mathrm{mod}-\Lambda}(N, \Lambda^\vee(a)) \longrightarrow \mathrm{Hom}_k(N_{-a}, \Lambda^\vee(a)_{-a}) = (N_{-a})^*, \quad f \mapsto f_{-a}$$

is bijective. In particular, $\Lambda^\vee(a)$ *is an injective object of* mod-Λ.

(ii) mod-Λ *has enough injective objects.*

(iii) *In* mod-Λ: *free* $\Rightarrow$ *injective.*

(iv) *Every* $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ *has a decomposition*:

$$N \simeq \Lambda(a_1) \oplus \cdots \oplus \Lambda(a_m) \oplus N^0$$

with $m \in \mathbb{N}$, $a_1 \geq \cdots \geq a_m$ *integers and* N^0 *annihilated by* $\mathrm{soc}(\Lambda) = \Lambda_{n+1}$. *Moreover,* $m, a_1, \ldots, a_m$ *and* N^0 (*up to isomorphism*) *are unique.*

(v) *In* mod-Λ: *projective* $\Leftrightarrow$ *free* $\Leftrightarrow$ *finite direct sum of* Λ*-modules of the form* $\Lambda^\vee(a)$ $\Leftrightarrow$ *injective.*

Proof. (i) Let $f \in \mathrm{Hom}_{\text{mod-}\Lambda}(N, \Lambda^\vee(a))$. If $b > a$, $y \in N_{-b}$ and $\omega \in \Lambda_{b-a}$ then $f_{-b}(y) \cdot \omega = f_{-a}(y \cdot \omega)$. One can use now the fact that the pairing: $\Lambda^\vee(a)_{-b} \times \Lambda_{b-a} \to \Lambda^\vee(a)_{-a} = k$ is perfect.

(ii) N can be embedded into: $\oplus_a N_{-a}\otimes_k \Lambda^\vee(a)$.

(iii) One can easily show that: $\Lambda \simeq \Lambda^\vee(-n-1)$.

(iv) For the existence of the decomposition, let $y \in N$ be a homogeneous element (let's say, of degree $-a$) not annihilated by $\mathrm{soc}(\Lambda)$. Then $y\Lambda \simeq \Lambda(a)$. By (ii), $y\Lambda$ is injective in mod-Λ hence it is a direct sumand of N. One concludes by induction on $\dim_k N$.

For the uniqueness, observe firstly that $N \cdot \mathrm{soc}(\Lambda) \simeq \underline{k}(a_1 - n - 1) \oplus \cdots \oplus \underline{k}(a_m - n - 1)$. This proves the uniqueness of m and $a_1, \ldots, a_m$. Assume, now, that one has an isomorphism:

$$\varphi : \Lambda(b_1)^{r_1} \oplus \cdots \oplus \Lambda(b_p)^{r_p} \oplus N^0 \xrightarrow{\sim} \Lambda(b_1)^{r_1} \oplus \cdots \oplus \Lambda(b_p)^{r_p} \oplus N^1$$

with $b_1 > \cdots > b_p$ and N^0, N^1 annihilated by $\mathrm{soc}(\Lambda)$. Applying $- \cdot \mathrm{soc}(\Lambda)$ one derives that the component of $\varphi : \Lambda(b_1)^{r_1} \to \Lambda(b_1)^{r_1}$ is an isomorphism. By a well known trick (about matrices of $2 \times 2 = 4$ blocks with invertible left upper block) it follows that:

$$\Lambda(b_2)^{r_2} \oplus \cdots \oplus \Lambda(b_p)^{r_p} \oplus N^0 \simeq \Lambda(b_2)^{r_2} \oplus \cdots \oplus \Lambda(b_p)^{r_p} \oplus N^1$$

and one concludes by induction on $\dim_k N$.

(v) Every projective or injective object of mod-Λ is a direct sumand of a free object (for injective by the proof of (ii)). Now one can apply (iv). □

5. Lemma. *Let* $P^\bullet \in \mathrm{Ob}\,\mathrm{C}^-(\text{mod-}\Lambda)$ *be a complex bounded to the right of free objects of* mod-Λ. *Then the complex* $\mathrm{L}(P^\bullet)$ *is acyclic.*

Proof. By definition, $\mathrm{L}(P^\bullet) = \mathrm{s}(X^{\bullet\bullet})$ for a double complex $X^{\bullet\bullet}$ with $X^{p,\bullet} = \mathrm{L}(P^p)$. By (1) (iii), the columns of $X^{\bullet\bullet}$ are acyclic bounded complexes. Now, $\mathrm{s}(X^{\bullet\bullet})$ is the direct limit of the complexes $\mathrm{s}(\sigma_I^{\geq -p}X^{\bullet\bullet})$, $p \geq 0$, where $(\sigma_I^{\geq -p}X^{\bullet\bullet})^{ij} = X^{ij}$ for $i \geq -p$ and $= 0$ for $i < -p$. $\sigma_I^{\geq -p}X^{\bullet\bullet}$ is a "first quadrant" type double complex (i.e., $\exists i_0,\ j_0$ such that its (i,j)-component is 0 for $i < i_0$ and, also, for $j < j_0$) with acyclic columns, hence $\mathrm{s}(\sigma_I^{\geq -p}X^{\bullet\bullet})$ is acyclic. □

The next result, which is the key point of this paper, is a generalization of the Remark 3 after Theorem 2 in [2]. Its proof can be easily reduced to the particular case $K^\bullet = \underline{k}$ of the remark in [2]. In [2], the remark is a consequence of the main result. Here we reverse the order: we prove directly the (general version of the) remark and then we show that it immediately implies the main result of [2].

6. Lemma. *Let* $I^\bullet \in \mathrm{Ob}\,\mathrm{C}(\text{mod-}\Lambda)$ *be an acyclic complex of injective* ($\Leftrightarrow$ *free*) *objects of* mod-Λ. *For* $p \in \mathbb{Z}$, *let* $Z^p := \mathrm{Ker}\,d_I^p$. *Then*:

(a) $\forall p \in \mathbb{Z}$, *the canonical morphism* $\mathrm{T}^{-p}Z^p \to I^\bullet$ *induces a quasi-isomorphism*: $\mathrm{L}(\mathrm{T}^{-p}Z^p) \to \mathrm{L}(I^\bullet)$.

(b) $\forall K^\bullet \in \mathrm{Ob}\,\mathrm{C}^b(\text{mod-}\Lambda)$, *the canonical map*:

$$\mathrm{Hom}_{\mathrm{K}(\Lambda)}(K^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(K^\bullet), \mathrm{L}(I^\bullet))$$

is an isomorphism of k*-vector spaces.*

Proof. (a) Let $\sigma^{\geq p}I^\bullet$ be the "stupid" truncation of $I^\bullet$ defined by $(\sigma^{\geq p}I^\bullet)^i = I^i$ for $i \geq p$ and $= 0$ for $i < p$. The morphism $\mathrm{T}^{-p}Z^p \to I^\bullet$ factorizes as $\mathrm{T}^{-p}Z^p \xrightarrow{\text{qis}} \sigma^{\geq p}I^\bullet \to I^\bullet$. One has an exact sequence of complexes:

$$0 \longrightarrow \sigma^{\geq p}I^\bullet \longrightarrow I^\bullet \longrightarrow \sigma^{<p}I^\bullet \longrightarrow 0.$$

By (5), $\mathrm{L}(\sigma^{<p}I^\bullet)$ is acyclic. It follows that $\mathrm{L}(\sigma^{\geq p}I^\bullet) \to \mathrm{L}(I^\bullet)$ is a quasi-isomorphism.

(b) Let $a := \min\{i \in \mathbb{Z} \mid K^i \neq 0\}$ and $b := \min\{j \in \mathbb{Z} \mid K_j^a \neq 0\}$. Then one has a short exact sequence:

$$0 \longrightarrow K'^\bullet \longrightarrow K^\bullet \longrightarrow K''^\bullet \longrightarrow 0$$

with $K''^\bullet = \mathrm{T}^{-a}(K_b^a \otimes_k \underline{k}(-b))$. Using (2) (iii) and (i) and the Five Lemma one can easily reduce the proof, by induction on $\sum_i \dim_k K^i$, to the case $K^\bullet = \mathrm{T}^p\underline{k}(q)$, $p, q \in \mathbb{Z}$, and this case reduces immediately to the case $p = q = 0$.

In the case $K^\bullet = \underline{k}$, using (2) (iv) and the fact that $\mathrm{T}Z^{-1} \to \sigma^{\geq -1}I^\bullet$ and $\mathrm{L}(\mathrm{T}Z^{-1}) \to \mathrm{L}(I^\bullet)$ are quasi-isomorphisms one gets isomorphisms:

$$\mathrm{Hom}_{\mathrm{K}(\Lambda)}(\underline{k}, I^\bullet) = \mathrm{Hom}_{\mathrm{K}(\Lambda)}(\underline{k}, \sigma^{\geq -1}I^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \sigma^{\geq -1}I^\bullet)$$
$$\xleftarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}Z^{-1}),$$
$$\mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\underline{k}), \mathrm{L}(I^\bullet)) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\underline{k}), \mathrm{L}(\mathrm{T}Z^{-1})).$$

It follows that it suffices to prove that the map:

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}Z^{-1}) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\underline{k}), \mathrm{L}(\mathrm{T}Z^{-1}))$$

is an isomorphism of k-vector spaces. We shall prove that, $\forall N \in \mathrm{Ob}(\text{mod-}\Lambda)$:

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}^p N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\underline{k}), \mathrm{L}(\mathrm{T}^p N)), \quad \forall p \geq 1. \tag{6.1}$$

The proof of (6.1) is based on the following :

Claim. $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}^p \underline{k}(a)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\underline{k}), \mathrm{L}(\mathrm{T}^p \underline{k}(a)))$, $\forall p \geq 0$, $\forall a \in \mathbb{Z}$.

Assuming the Claim, for the moment, we prove (6.1) by induction on $\dim_k N$. The initial case $\dim_k N = 1$ follows from the Claim. For the induction step, let $a := \min\{i \mid N_i \neq 0\}$. One has an exact sequence: $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$, with $N'' = N_a \otimes_k \underline{k}(-a)$. Using the considerations from (2) (i), the induction hypothesis for N', the Claim for N'' and the Five Lemma one gets immediately (6.1).

Finally, let us prove the Claim. One has:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}^p \underline{k}(a)) &\simeq \mathrm{Ext}^p_{\text{mod-}\Lambda}(\underline{k}, \underline{k}(a)), \\
\mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\underline{k}), \mathrm{L}(\mathrm{T}^p \underline{k}(a))) &= \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}, \mathrm{T}^{p+a} \mathcal{O}_{\mathbb{P}}(-a)) \\
&\simeq \mathrm{Ext}^{p+a}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(-a)) \simeq \mathrm{H}^{p+a} \mathcal{O}_{\mathbb{P}}(-a).
\end{aligned}$$

$\underline{k}$ has an injective resolution in mod-Λ:

$$0 \longrightarrow \underline{k} \longrightarrow \Lambda^\vee \longrightarrow V^* \otimes_k \Lambda^\vee(1) \longrightarrow \cdots \longrightarrow S^i V^* \otimes_k \Lambda^\vee(i) \longrightarrow \cdots$$

with differential $d = \sum_{i=0}^n (X_i \cdot -) \otimes (- \cdot e_i)_{\Lambda^\vee}$. It follows that both sides of the Claim are 0 for $p + a \neq 0$ (assuming, of course, $p \geq 0$). It remains to show that:

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}^p \underline{k}(-p)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(p)), \quad \forall p \geq 0. \tag{6.2}$$

Consider the morphism (in $\mathrm{D}^b(\Lambda)$) $\varepsilon : \underline{k} \otimes_k V^* \to \mathrm{T}\underline{k}(-1)$ from (3) (b). Since $\mathrm{L}(\varepsilon)$ is the canonical morphism $\mathcal{O}_{\mathbb{P}} \otimes_k V^* \to \mathcal{O}_{\mathbb{P}}(1)$, $\mathrm{L}(\mathrm{T}^{p-1}\varepsilon(-p+1))$ is the canonical morphism $\mathcal{O}_{\mathbb{P}}(p-1) \otimes_k V^* \to \mathcal{O}_{\mathbb{P}}(p)$. Using the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}^{p-1}\underline{k}(-p+1) \otimes_k V^*) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(p-1) \otimes_k V^*) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\underline{k}, \mathrm{T}^p \underline{k}(-p)) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(p))
\end{array}$$

one proves easily, by induction on $p \geq 0$, that the morphism in (6.2) is surjective, hence it is an isomorphism since both sides are isomorphic over k to $S^p V^*$. $\square$

7. Theorem (Bernstein-Gel'fand-Gel'fand)**.**
(a) *The functor* $\mathrm{L}\colon \text{mod-}\Lambda \to \mathrm{D}^b(\mathrm{Coh}\mathbb{P}(V))$ *is essentially surjective.*
(b) *If* $N, N' \in \mathrm{Ob}(\text{mod-}\Lambda)$ *then the map*:

$$\mathrm{Hom}_{\mathrm{mod}-\Lambda}(N', N) \longrightarrow \mathrm{Hom}_{\mathrm{D}^b(\mathbb{P})}(\mathrm{L}(N'), \mathrm{L}(N))$$

is surjective and its kernel consists of the morphisms factorizing through a free ($\Leftrightarrow$ *injective*) *object of* mod-Λ.

Proof. We firstly prove the second assertion.

(b) Let $0 \to N \to I^0 \to I^1 \to \cdots$ be an injective resolution of N in mod-Λ and $\cdots \to I^{-2} \to I^{-1} \to N \to 0$ a free resolution. Glue them in order to get an acyclic complex $I^\bullet$ consisting of injective ($\Leftrightarrow$ free) objects. By (6) (a), $\mathrm{L}(N) \to \mathrm{L}(I^\bullet)$ is a quasi-isomorphism and by (6) (b) the bottom horizontal arrow of the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\text{mod-}\Lambda}(N', N) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^b(\mathbb{P})}(\mathrm{L}(N'), \mathrm{L}(N)) \\ \big\downarrow & & \big\downarrow \wr \\ \mathrm{Hom}_{\mathrm{K}(\Lambda)}(N', I^\bullet) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(N'), \mathrm{L}(I^\bullet)) \end{array}$$

is an isomorphism. The left vertical arrow of the diagram is surjective and its kernel consists of the morphisms factorizing through I^{-1}.

(a) We observe, firstly, that if $K^\bullet \in \mathrm{ObC}^b(\text{mod-}\Lambda)$ then $\exists N \in \mathrm{Ob}(\text{mod-}\Lambda)$ such that $\mathrm{L}(K^\bullet) \simeq \mathrm{L}(N)$ in $\mathrm{D}^b(\mathbb{P})$. Indeed, consider a quasi-isomorphism $u : K^\bullet \to J^\bullet$ (resp., $v : P^\bullet \to K^\bullet$) with $J^\bullet \in \mathrm{ObC}^+(\text{mod-}\Lambda)$ (resp., $P^\bullet \in \mathrm{ObC}^-(\text{mod-}\Lambda)$) consisting of injective (resp., free) objects. Then $I^\bullet := \mathrm{Con}(u \circ v)$ is an acyclic complex consisting of injective ($\Leftrightarrow$ free) objects of mod-Λ. Using the short exact sequence:

$$0 \longrightarrow J^\bullet \longrightarrow I^\bullet \longrightarrow \mathrm{T}P^\bullet \longrightarrow 0$$

and applying (5) to $\mathrm{T}P^\bullet$ one derives that $\mathrm{L}(J^\bullet) \to \mathrm{L}(I^\bullet)$ is a quasi-isomorphism hence $\mathrm{L}(K^\bullet) \to \mathrm{L}(I^\bullet)$ is a quasi-isomorphism. On the other hand, by (6) (a), one has a quasi-isomorphism $\mathrm{L}(Z^0) \to \mathrm{L}(I^\bullet)$. Consequently, $\mathrm{L}(K^\bullet) \simeq \mathrm{L}(Z^0)$ in $\mathrm{D}(\mathbb{P})$.

Let now $\mathcal{F}^\bullet \in \mathrm{ObC}^b(\mathrm{Coh}\mathbb{P}(V))$. Let $p := \max\{i \in \mathbb{Z} \mid \mathcal{F}^i \neq 0\}$ and let $u : \sigma^{<p}\mathcal{F}^\bullet \to \mathrm{T}^{-p+1}\mathcal{F}^p$ be the morphism defined by $d_{\mathcal{F}}^{p-1} : \mathcal{F}^{p-1} \to \mathcal{F}^p$. Then $\mathcal{F}^\bullet = \mathrm{Con}(u)$. Assume there exist $N', N'' \in \mathrm{Ob}(\text{mod-}\Lambda)$ and isomorphisms in $\mathrm{D}^b(\mathbb{P})$ $\psi : \mathrm{L}(N'') \xrightarrow{\sim} \sigma^{<p}\mathcal{F}^\bullet$, $\varphi : \mathrm{L}(N') \xrightarrow{\sim} \mathrm{T}^{-p+1}\mathcal{F}^p$. By (b), $\exists f \in \mathrm{Hom}_{\text{mod-}\Lambda}(N'', N')$ such that $\mathrm{L}(f) = \varphi^{-1} \circ u \circ \psi$. Then $\mathcal{F}^\bullet \simeq \mathrm{L}(\mathrm{Con}(f))$ in $\mathrm{D}^b(\mathbb{P})$, hence, by the above observation, $\exists N \in \mathrm{Ob}(\text{mod-}\Lambda)$ such that $\mathcal{F}^\bullet \simeq \mathrm{L}(N)$.

By induction on the length of $\mathcal{F}^\bullet$, one can now reduce the proof to the case when $\mathcal{F}^\bullet$ has only one non-zero term. Using the result of Serre [6] asserting that any coherent sheaf on $\mathbb{P}(V)$ is the sheafification of a finitely generated graded module over the symmetric algebra $S(V^*)$ and Hilbert's syzygy theorem, one deduces that any such sheaf has a finite resolution with finite direct sums of invertible sheaves $\mathcal{O}_{\mathbb{P}}(a)$. By induction on the length of this resolution, one reduces the proof, as above, to the case when $\mathcal{F}^\bullet = \mathrm{T}^p\mathcal{O}_{\mathbb{P}}(a)$. But $\mathrm{T}^p\mathcal{O}_{\mathbb{P}}(a) = \mathrm{L}(\mathrm{T}^{p+a}\underline{k}(-a))$. □

One can deduce the surjectivity assertion of (7) (b) from the general fact (2) (v). The above proof of (7) (a) is based on this surjectivity asertion and on (5). The difficult part of the BGG correspondence is, therefore, the description of the kernel in (7) (b). The proof of the next corollary is based on this description.

8. Corollary ([2], Remark 3 after Theorem 1). *For every* $\mathcal{F}^\bullet \in \mathrm{Ob}\mathrm{C}^b(\mathrm{Coh}(\mathbb{P}(V))$ *there exists* $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ *annihilated by* $\mathrm{soc}(\Lambda)$ *such that* $\mathcal{F}^\bullet \simeq \mathrm{L}(N)$ *in* $\mathrm{D}^b(\mathrm{Coh}\mathbb{P}(V))$. *Moreover, N is unique up to isomorphism.*

Proof. The existence of N follows from (7) (a) and (4) (iv). Let N' be another such Λ-module. By (7) (b), there exists a morphism $u : N' \to N$ in mod-Λ such that $\mathrm{L}(u) : \mathrm{L}(N') \to \mathrm{L}(N)$ is an isomorphism in $\mathrm{D}^b(\mathbb{P})$ (i.e., it is a quasi-isomorphism). By (7) (b) again, there exists $v : N \to N'$ such that $\mathrm{L}(v)$ is the inverse of $\mathrm{L}(u)$ in $\mathrm{D}^b(\mathbb{P})$. By the last part of (7) (b), there exists a free object P of mod-Λ such that $\mathrm{id}_N - u \circ v$ factorizes as $N \xrightarrow{f} P \xrightarrow{g} N$.

The submodule of P consisting of the elements annihilated by $\mathrm{soc}(\Lambda)$ is $P \cdot \Lambda_+$, hence $f(N) \subseteq P \cdot \Lambda_+$, hence $\mathrm{Im}(\mathrm{id}_N - u \circ v) \subseteq N \cdot \Lambda_+$. Using the exterior algebra version of the graded NAK, one derives that $u \circ v$ is surjective, hence it is an isomorphism because N is a finite dimensional k-vector space. Similarly, $v \circ u$ is an isomorphism. Consequently, u is an isomorphism. □

9. Definition. Let $N \in \mathrm{Ob}(\text{mod-}\Lambda)$ *annihilated by* $\mathrm{soc}(\Lambda)$. Consider a minimal free resolution of N in mod-Λ: $\cdots \to I^{-2} \to I^{-1} \to N \to 0$. Minimality is equivalent to the condition: $\mathrm{Im}(I^{-p-1} \to I^{-p}) \subseteq I^{-p} \cdot \Lambda_+$, $\forall p \geq 1$. Consider also an injective resolution of N in mod-Λ: $0 \to N \to I^0 \to I^1 \to \cdots$ such that $\mathrm{Im}(I^p \to I^{p+1}) \subseteq I^{p+1} \cdot \Lambda_+$, $\forall p \geq 0$. To get such a resolution, take a minimal free resolution of $N^\vee$ and dualize it. Glueing the two resolutions one gets an acyclic complex $I^\bullet$ consisting of injective ($\Leftrightarrow$ free) objects of mod-Λ such that $\mathrm{Im} d_I^p \subseteq I^{p+1} \cdot \Lambda_+$, $\forall p \in \mathbb{Z}$ (for $p = -1$ this follows from the fact that $N \cdot \mathrm{soc}(\Lambda) = (0)$).

Such a complex $I^\bullet$ is called a *Tate resolution* of N.

10. Theorem (Eisenbud-Fløystad-Schreyer). *Let* $\mathcal{F}^\bullet \in \mathrm{Ob}\mathrm{C}^b(\mathrm{Coh}\mathbb{P}(V))$, *let N be the unique* (*up to isomorphism*) *object of* mod-Λ *annihilated by* $\mathrm{soc}(\Lambda)$ *with* $\mathcal{F}^\bullet \simeq \mathrm{L}(N)$ *in* $\mathrm{D}^b(\mathrm{Coh}\mathbb{P}(V))$ *and let $I^\bullet$ be a Tate resolution of N. Then*:

(a) $I^p \simeq \bigoplus_i \mathbb{H}^{p-i}\mathcal{F}^\bullet(i) \otimes_k \Lambda^\vee(i)$, $\forall p \in \mathbb{Z}$ (*where $\mathbb{H}$ denotes hypercohomology*),

(b) $d_I^p : I^p \to I^{p+1}$ *maps* $\mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k\Lambda^\vee(i)$ *to* $\bigoplus_{j>i} \mathbb{H}^{p+1-j}\mathcal{F}^\bullet(j) \otimes_k \Lambda^\vee(j)$ *and the component*: $\mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k\Lambda^\vee(i) \to \mathbb{H}^{p-i}\mathcal{F}^\bullet(i+1)\otimes_k\Lambda^\vee(i+1)$ *of d_I^p is defined* (*see* (4) (i)) *by the multiplication map*: $\mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k V^* \to \mathbb{H}^{p-i}\mathcal{F}^\bullet(i+1)$ (*up to sign*).

Proof of (10) (a) (according to Remark 3 after Theorem 2 in [2]). If $I^p \simeq \oplus_i \Lambda^\vee(i)^{\gamma_{pi}}$ then $\mathrm{soc}(I^p) \simeq \oplus_i \underline{k}(i)^{\gamma_{pi}}$. Taking into account that $\mathrm{Im} d_I^q \subseteq I^{q+1} \cdot \Lambda_+$, $\forall q \in \mathbb{Z}$, one gets that:

$$\mathrm{soc}(I^p)_{-i} \simeq \mathrm{Hom}_{\text{mod-}\Lambda}(\underline{k}, I^p(-i)) \simeq \mathrm{Hom}_{\mathrm{K}(\Lambda)}(\underline{k}, \mathrm{T}^p I^\bullet(-i)).$$

On the other hand, by (6):

$$\begin{aligned}\mathrm{Hom}_{\mathrm{K}(\Lambda)}(\underline{k}, \mathrm{T}^p I^\bullet(-i)) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}, \mathrm{T}^{p-i}\mathcal{F}^\bullet(i)) \\ &\simeq \mathrm{Ext}^{p-i}(\mathcal{O}_{\mathbb{P}}, \mathcal{F}^\bullet(i)) \simeq \mathbb{H}^{p-i}\mathcal{F}^\bullet(i).\end{aligned}$$

□

For the proof of (10) (b) we need the following addendum to (2) (iii):

11. Remark. Under the assumptions of (2) (iii), let $w : Z^\bullet \to \mathrm{T}X^\bullet$ be the morphism in $\mathrm{D}^+(\mathcal{A})$ defined in (2) (i). Then:

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\mathrm{T}^{-1}w, \mathrm{id}_{\mathrm{T}^p I}) : \ \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X^\bullet, \mathrm{T}^p I^\bullet) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\mathrm{T}^{-1}Z^\bullet, \mathrm{T}^p I^\bullet)$$

equals $(-1)^p \partial^p$ where $\partial^p : \ \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X^\bullet, \mathrm{T}^p I^\bullet) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(Z^\bullet, \mathrm{T}^{p+1} I^\bullet)$ is the "classical" connecting morphism associated to the short exact sequence of complexes of abelian groups:

$$0 \longrightarrow \mathrm{Hom}^\bullet(Z^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}^\bullet(Y^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}^\bullet(X^\bullet, I^\bullet) \longrightarrow 0.$$

Proof. ∂^p is defined as follows: let $f : X^\bullet \to \mathrm{T}^p I^\bullet$ be a morphism of complexes. Lift every $f^i : X^i \to I^{i+p}$ to a morphism $g^i : Y^i \to I^{i+p}$. Then the morphism of complexes $(d_I^{i+p} \circ g^i - (-1)^p g^{i+1} \circ d_Y^i)_{i \in \mathbb{Z}} : Y^\bullet \to \mathrm{T}^{p+1} I^\bullet$ vanishes on $X^\bullet$ hence induces a morphism of complexes $\partial^p(f) : Z^\bullet \to \mathrm{T}^{p+1} I^\bullet$ (in fact, to be rigorous, one has to take homotopy classes).

We have to prove that the diagram:

$$\begin{array}{ccc} \mathrm{T}^{-1}\mathrm{Con}(u) & \xrightarrow{(\mathrm{id}_X, 0)} & X^\bullet \\ {\scriptstyle (0, \mathrm{T}^{-1}v)}\downarrow & & \downarrow {\scriptstyle f} \\ \mathrm{T}^{-1}Z^\bullet & \xrightarrow{(-1)^p \mathrm{T}^{-1}\partial^p(f)} & \mathrm{T}^p I^\bullet \end{array}$$

is homotopically commutative. One can use the homotopy operators $h^i := (0, g^{i-1}) : (\mathrm{T}^{-1}\mathrm{Con}(u))^i = X^i \oplus Y^{i-1} \to I^{i+p-1} = (\mathrm{T}^p I^\bullet)^{i-1}$. □

Proof of (10) (b). The first assertion follows from the fact that $\mathrm{Im} d_I^p \subseteq I^{p+1} \cdot \Lambda_+$. For the second assertion we consider the morphism (in $\mathrm{D}^b(\Lambda)$) $\nu : \mathrm{T}^{-1}\underline{k}(1) \to \underline{k} \otimes_k V$ from (3) (a). By (6), the map:

$$\mathrm{Hom}_{\mathrm{K}(\Lambda)}(\nu, \mathrm{id}) : \ \mathrm{Hom}_{\mathrm{K}(\Lambda)}(\underline{k} \otimes_k V, \mathrm{T}^p I^\bullet(-i)) \longrightarrow \mathrm{Hom}_{\mathrm{K}(\Lambda)}(\mathrm{T}^{-1}\underline{k}(1), \mathrm{T}^p I^\bullet(-i))$$

can be identified with the map:

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathrm{L}(\nu), \mathrm{id}) : \ \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}\otimes_k V, \mathrm{T}^{p-i}\mathcal{F}^\bullet(i)) \\ \to \mathrm{Hom}_{\mathrm{D}(\mathbb{P})}(\mathcal{O}_{\mathbb{P}}(-1), \mathrm{T}^{p-i}\mathcal{F}^\bullet(i))$$

and this one can be identified with the multiplication map: $\mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k V^* \to \mathbb{H}^{p-i}\mathcal{F}^\bullet(i+1)$.

We want now to explicitate $\mathrm{Hom}_{\mathrm{K}(\Lambda)}(\nu, \mathrm{id})$. Let $\xi \in \mathbb{H}^{p-i}\mathcal{F}^\bullet(i)$, $\lambda \in V^*$ and let $f\colon \underline{k}\otimes_k V \to I^p(-i)$ be the morphism defined by $\xi \otimes \lambda : \ \underline{k}\otimes_k V \to \mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k(\Lambda^\vee)_0$. f can be lifted to the morphism $g : (\Lambda/(\Lambda_+)^2)(1) \to I^p(-i)$ sending $\hat{1} \in (\Lambda/(\Lambda_+)^2)(1)_{-1}$ to $-\xi \otimes \lambda \in \mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k(\Lambda^\vee)_{-1}$.

From (11), $\mathrm{Hom}_{\mathrm{K}(\Lambda)}(\nu, \mathrm{id})(f) = (-1)^p\partial^p(f)$ where $\partial^p(f)$ is defined at the beginning of the proof of (11) (in our case, the complexes $X^\bullet, Y^\bullet, Z^\bullet$ are concentrated in cohomological degree 0). One derives that $\mathrm{Hom}_{\mathrm{K}(\Lambda)}(\nu, \mathrm{id})$ can be identified with

$$(-1)^{p-1}d_I^p \mid \ \mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k\Lambda^\vee(i)_{-i-1} \longrightarrow \mathbb{H}^{p-i}\mathcal{F}^\bullet(i+1)\otimes_k\Lambda^\vee(i+1)_{-i-1}.$$

□

One can easily deduce from (10) the Lemma of Castelnuovo-Mumford. More important, Eisenbud et al. [3] show that (10) implies the results of A. A. Beilinson [1]. We close the paper by briefly explaining this, in terms of the present approach.

12. Theorem (Beilinson). *Let $\mathcal{F}^\bullet \in \mathrm{Ob}\mathrm{C}^b(\mathrm{Coh}\mathbb{P}(V))$. Then $\mathcal{F}^\bullet$ is isomorphic in $\mathrm{D}^b(\mathrm{Coh}\mathbb{P}(V))$ to a complex $C^\bullet$ with $C^p = \bigoplus_i \mathbb{H}^{p+i}\mathcal{F}^\bullet(-i)\otimes_k\Omega^i_{\mathbb{P}}(i)$, $\forall p \in \mathbb{Z}$ and also to a complex $C'^\bullet$ with $C'^p = \bigoplus_i \mathcal{O}_{\mathbb{P}}(-i)\otimes_k\mathbb{H}^{p+i}(\mathcal{F}^\bullet \otimes \Omega^i_{\mathbb{P}}(i))$, $\forall p \in \mathbb{Z}$.*

Proof (according to [3], (6.1) and (8.11)). Let N and $I^\bullet$ be as in the statement of (10). Recall, from (6) (a), that $\mathrm{L}(N) \to \mathrm{L}(I^\bullet)$ is a quasi-isomorphism. By definition, $\mathrm{L}(I^\bullet) = \mathrm{s}(X^{\bullet\bullet})$ for a certain double complex $X^{\bullet\bullet}$ with $X^{pq} = \mathcal{O}_{\mathbb{P}}(q)\otimes_k I^p_q$.

(I) In order to prove the first assertion, one takes $C^\bullet := \mathrm{Ker}(X^{\bullet,0} \to X^{\bullet,1})$. Taking into account that $\mathrm{Ker}(\mathrm{L}(\Lambda^\vee(-i))^0 \to \mathrm{L}(\Lambda^\vee(-i))^1) = \Omega^i_{\mathbb{P}}(i)$, the formula for C^p follows from (10) (a).

It remains to show that $C^\bullet \to \mathrm{s}(X^{\bullet\bullet})$ is a quasi-isomorphism. It decomposes as $C^\bullet \to \mathrm{s}(\sigma^{\geq 0}_{II}X^{\bullet\bullet}) \to \mathrm{s}(X^{\bullet\bullet})$, where $(\sigma^{\geq 0}_{II}X^{\bullet\bullet})^{ij} := X^{ij}$ for $j \geq 0$ and $= 0$ for $j < 0$. Since the columns of $X^{\bullet\bullet}$ are acyclic, $C^\bullet \to \mathrm{s}(\sigma^{\geq 0}_{II}X^{\bullet\bullet})$ is a quasi-isomorphism. On the other hand, one has a short exact sequence:

$$0 \longrightarrow \mathrm{s}(\sigma^{\geq 0}_{II}X^{\bullet\bullet}) \longrightarrow \mathrm{s}(X^{\bullet\bullet}) \longrightarrow \mathrm{s}(\sigma^{<0}_{II}X^{\bullet\bullet}) \longrightarrow 0$$

hence it suffices to prove that $\mathrm{s}(\sigma^{<0}_{II}X^{\bullet\bullet})$ is acyclic.

For $p \in \mathbb{Z}$, let $a(p) := \min\{q \in \mathbb{Z} \mid I^p_q \neq 0\}$. Since $I^{p-1} \to I^p \to Z^{p+1} \to 0$ is a minimal free presentation, it follows that $a(p-1) > a(p)$, $\forall p \in \mathbb{Z}$. One deduces that *the rows of $X^{\bullet\bullet}$ are bounded to the left.*

Now, $\mathrm{s}(\sigma_{II}^{<0}X^{\bullet\bullet})$ is the direct limit of the complexes $\mathrm{s}(\sigma_{II}^{\geq -p}\sigma_{II}^{<0}X^{\bullet\bullet})$ for $p \geq 1$. But $\sigma_{II}^{\geq -p}\sigma_{II}^{<0}X^{\bullet\bullet}$ is a "first quadrant" type double complex with acyclic rows hence its associated simple complex is acyclic.

(II) Let us prove the second assertion. One takes the subcomplex $J^\bullet$ of $I^\bullet$ defined by $J^p := \bigoplus_{i\geq 0}\mathbb{H}^{p-i}\mathcal{F}^\bullet(i)\otimes_k\Lambda^\vee(i)$. One has $J^p = 0$ for $p << 0$ and $J^p = I^p$ for $p >> 0$ hence, by (5), $\mathrm{L}(J^\bullet) \to \mathrm{L}(I^\bullet)$ is a quasi-isomorphism.

Now, $\mathrm{L}(J^\bullet) = \mathrm{s}(Y^{\bullet\bullet})$ for a certain double complex $Y^{\bullet\bullet}$ with splitting rows $Y^{\bullet,q}$, $q \in \mathbb{Z}$ and with columns $Y^{p,\bullet} = \mathrm{L}(J^p) = 0$, for $p << 0$. According to a general lemma about such double complexes (see [3], (3.5)), $\mathrm{s}(Y^{\bullet\bullet})$ is homotopically equivalent to a complex $C'^\bullet$ whose "linear part" is $\mathrm{L}(\mathrm{H}^\bullet(J^\bullet))$ (where $\mathrm{H}^\bullet(J^\bullet)$ is the complex with pth term $\mathrm{H}^p(J^\bullet)$ and with all the differentials equal to 0). In particular: $C'^p \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}}(-i)\otimes_k\mathrm{H}^{p+i}(J^\bullet)_{-i}$. But $J^\bullet_{-i} = 0$ for $i < 0$ and $J^\bullet_{-i} = I^\bullet_{-i}$ for $i > n$ hence (since $I^\bullet$ is acyclic) $\mathrm{H}^q(J^\bullet)_{-i} = 0$ for $i < 0$ and for $i > n$, $\forall q \in \mathbb{Z}$. On the other hand, for $0 \leq i \leq n$ and $q \in \mathbb{Z}$: $J^q_{-i} \simeq \mathrm{Hom}_{\text{mod-}\Lambda}((\Lambda/(\Lambda_+)^{i+1})(i), I^q)$ hence $\mathrm{H}^q(J^\bullet)_{-i} \simeq \mathrm{Hom}_{\mathrm{K}(\Lambda)}((\Lambda/(\Lambda_+)^{i+1})(i), \mathrm{T}^q I^\bullet)$. One can now apply (6), taking into account that $\mathrm{L}((\Lambda/(\Lambda_+)^{i+1})(i)) \simeq (\Omega^i_{\mathbb{P}}(i))^*$ in $\mathrm{D}^b(\mathbb{P})$. □

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