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Classification of spherical nilpotent orbits for $U(p,p)$

By

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Abstract

We consider the symmetric pair $(G,K) = (U(p,q), U(p) \times U(q))$. For this pair, we classify spherical nilpotent $K_{\mathbb{C}}$-orbits which are theta lift in the stable range. For the pair $(G,K) = (U(p,p), U(p) \times U(p))$ where $p = q$, we prove that a spherical nilpotent $K_{\mathbb{C}}$-orbit must be a theta lift. As a consequence, we get a complete classification of the spherical nilpotent $K_{\mathbb{C}}$-orbits for the symmetric pair $(U(p,p), U(p) \times U(p))$.

Introduction

Let G be a reductive Lie group and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ (respectively $\mathfrak{k}$) the *complexified* Lie algebra of G (respectively K). Then the choice of K determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, and the complexification $K_{\mathbb{C}}$ acts on $\mathfrak{s}$ by the restriction of the adjoint action. Let $\mathcal{N}(\mathfrak{s})$ be the nilpotent variety consisting of all nilpotent elements in $\mathfrak{s}$. The action of $K_{\mathbb{C}}$ preserves $\mathcal{N}(\mathfrak{s})$ and it has a finite number of $K_{\mathbb{C}}$-orbits ([6]). We call them *nilpotent $K_{\mathbb{C}}$-orbits for the symmetric pair* (G,K), or just for G.

In this article, we investigate *spherical* nilpotent $K_{\mathbb{C}}$-orbits for the pair $(G,K) = (U(p,q), U(p) \times U(q))$. A nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}$ is called spherical if a Borel subgroup of $K_{\mathbb{C}}$ has an open dense orbit in $\mathbb{O}$. These orbits are relatively small and play a fundamental role in the representation theory of G (see, for example, [4], [8], [9], etc.). For a complex simple algebraic group $G_{\mathbb{C}}$, there is a complete classification of spherical nilpotent $G_{\mathbb{C}}$-orbits by Panyushev ([13]). Recently D. R. King [5] has announced a classification of the spherical nilpotent orbits for irreducible symmetric pairs by the method completely different from ours. Our method here provides an explicit formula of the $K_{\mathbb{C}}$-type decomposition of the regular function ring of the spherical orbits (see Equation (2.1)), and at the same time establishes a relation to the theta correspondence (or Howe correspondence) of irreducible unitary representations of G.

The main tool of our classification is the notion of theta lifting of nilpotent orbits in the stable range (see Section 1; we refer [10] for detail). In fact, we

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classify all the spherical nilpotent $K_{\mathbb{C}}$-orbits for $G = U(p,q)$ which are theta lift in the stable range.

Theorem A (Theorem 2.1). *Let $(G,K) = (U(p,q), U(p) \times U(q))$ as above. Let D be a signed Young diagram of signature (p,q) and $\mathbb{O}_D$ be the corresponding nilpotent $K_{\mathbb{C}}$-orbit for $G = U(p,q)$. Then $\mathbb{O}_D$ is a theta lift in the stable range if and only if $\ell(D) \geq \max\{p,q\}$, where $\ell(D)$ denotes the length of D.*

If $\mathbb{O}_D$ is a theta lift, then it is spherical if and only if the shape of D is of the following form:

$$[3^{\varepsilon} \cdot 2^k \cdot 1^l] \qquad \varepsilon = 0,1; \quad k,l \geq 0; \quad 3\varepsilon + 2k + l = p + q.$$

Since the length of $[3^{\varepsilon} \cdot 2^k \cdot 1^l]$ is $\varepsilon + k + l$, the above theorem tells us that the orbit corresponding to $[3^{\varepsilon} \cdot 2^k \cdot 1^l]$ with $\varepsilon + k + l \geq \max\{p,q\}$ is spherical. According to the classification by Panyushev, spherical $G_{\mathbb{C}}$-orbits for type A is given by Young diagrams $[2^k \cdot 1^l]$. Therefore, Theorem A provides an example of a spherical nilpotent $K_{\mathbb{C}}$-orbit whose $G_{\mathbb{C}}$-hull is not spherical.

Unfortunately, a spherical nilpotent orbit need not be a theta lift in the stable range in general. However, it is so in the case of $p = q$. We prove the following theorem.

Theorem B (Theorem 3.1). *Let $(G,K) = (U(p,p), U(p) \times U(p))$ and D a signed Young diagram of signature (p,p). The corresponding nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}_D$ is spherical if and only if the shape of D is of the following form:*

$$[3^{\varepsilon} \cdot 2^k \cdot 1^l] \qquad \varepsilon = 0,1; \quad k,l \geq 0; \quad 3\varepsilon + 2k + l = 2p \text{ and } \varepsilon + k + l \geq p.$$

In particular, spherical nilpotent $K_{\mathbb{C}}$-orbits are obtained by the theta lifting in the stable range.

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1. Theta lifting of nilpotent orbits

In this section, we review the notion of the theta lift of nilpotent orbits for symmetric pairs in the case of indefinite unitary groups. For general and detailed discussion, see [10].

We denote $U(p,q)$ and its maximal compact subgroup $U(p)\times U(q)$ by G and K respectively. Let us begin with the description of the nilpotent orbits for the symmetric pair $(G,K) = (U(p,q), U(p)\times U(q))$. Let $\mathfrak{g}$ be the complexification of the Lie algebra $\mathrm{Lie}\,(G)$, and similarly $\mathfrak{k}$ denotes the complexification of $\mathrm{Lie}\,(K)$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the corresponding Cartan decomposition over $\mathbb{C}$. We realize it as

$$\mathfrak{g} = \mathfrak{gl}_{p+q}(\mathbb{C}) = \left(\begin{array}{c|c} M_p(\mathbb{C}) & 0 \\ \hline 0 & M_q(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & M_{p,q}(\mathbb{C}) \\ \hline M_{q,p}(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k} \oplus \mathfrak{s}.$$

For a subset $S \subset \mathfrak{g}$, we denote by $\mathcal{N}(S)$ the set of all nilpotent elements in S. Then, we call $\mathcal{N}(\mathfrak{s})$ the *nilpotent variety* for the symmetric pair (G, K). Let us denote by the complexified algebraic group of K by $K_{\mathbb{C}}$. Then $K_{\mathbb{C}}$ acts on $\mathfrak{s}$ via the restriction of the adjoint action, and preserves the nilpotent variety $\mathcal{N}(\mathfrak{s})$. It is well known that the $K_{\mathbb{C}}$-orbits in $\mathcal{N}(\mathfrak{s})$ is finite, and that they are classified by the signed Young diagrams. Namely, the $K_{\mathbb{C}}$-orbits in $\mathcal{N}(\mathfrak{s})$ are classified by the Young diagrams of size $p+q$, whose boxes are occupied by p plus signs and q minus signs. The signs must appear mutually in rows. For details, see [1, Th. 9.3.3] for example.

Take another unitary group and put $G' = U(m,n)$. We *assume the stable range condition* $m+n \le p, q$. The pair (G, G') forms a reductive dual pair in a large symplectic group $\mathbb{G} = Sp(2(p+q)(m+n), \mathbb{R})$. In the symplectic group, there are four reductive dual pairs related to (G, G'), which is called diamond pairs [2]. Let us denote:

$$\begin{array}{ccc} M = U(p,q)^2 & \overset{\Delta}{\supset} & G = U(p,q) \\ \cup & & \cup \\ L = L^+ \times L^- = U(p)^2 \times U(q)^2 & \overset{\Delta}{\supset} & K = K^+ \times K^- = U(p) \times U(q), \end{array}$$

where Δ denotes the diagonal embedding. Similarly, we denote M', G', L', K' replacing p and q by m and n respectively. Then, the four dual pairs in $\mathbb{G}$ are $(G, G'), (M, K'), (L, L'), (K, M')$. Put

$$\begin{aligned} W &= M_{p+q,m+n}(\mathbb{C}) \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A \in M_{p,m}, B \in M_{p,n}, C \in M_{q,m}, D \in M_{q,n} \right\} \\ &= W^+ \oplus W^- = M_{p,m+n}(\mathbb{C}) \oplus M_{q,m+n}(\mathbb{C}). \end{aligned}$$

Then $K_{\mathbb{C}} = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ and $K'_{\mathbb{C}} = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ act on W as

$$((g_1, g_2), (h_1, h_2)) \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} g_1 A\, {}^t h_1 & {}^t g_1{}^{-1} B h_2^{-1} \\ {}^t g_2{}^{-1} C h_1^{-1} & g_2 D\, {}^t h_2 \end{pmatrix},$$

$$(g_1, g_2) \in GL_p(\mathbb{C}) \times GL_q(\mathbb{C}), \quad (h_1, h_2) \in GL_m(\mathbb{C}) \times GL_n(\mathbb{C}).$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ as above, and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ is chosen similarly. We identify

$$\mathfrak{s} = M_{p,q}(\mathbb{C}) \oplus M_{q,p}(\mathbb{C}), \qquad \mathfrak{s}' = M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C}).$$

Then there is a natural double fibration of W by $\mathfrak{s}$ and $\mathfrak{s}'$, which are explicitly given as follows.

$$\varphi : W \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto (A\, {}^t C, D\, {}^t B) \in \mathfrak{s},$$

$$\psi : W \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto ({}^t A B, {}^t D C) \in \mathfrak{s}'.$$

These double fibration maps are called moment maps, and they are $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant with the trivial $K_{\mathbb{C}}$-action on $\mathfrak{s}'$, and the trivial $K'_{\mathbb{C}}$-action on $\mathfrak{s}$ respectively. The maps φ and ψ are almost the same. Therefore we will treat only the map ψ in the following.

The map ψ induces an algebra homomorphism $\psi^* : \mathbb{C}[\mathfrak{s}'] \to \mathbb{C}[W]$ by

$$\psi^*(x_{i,j}) = \sum_{k=1}^{p} a_{k,i} b_{k,j}, \qquad X = (x_{i,j}) \in M_{m,n}(\mathbb{C}),$$
$$\psi^*(y_{i,j}) = \sum_{l=1}^{q} c_{l,j} d_{l,i}, \qquad Y = (y_{i,j}) \in M_{n,m}(\mathbb{C}).$$

GL_p-invariants (respectively GL_q-invariants) on $\mathbb{C}[W^+]$ (respectively $\mathbb{C}[W^-]$) are generated by $\psi^*(x_{i,j})$'s (respectively $\psi^*(y_{i,j})$'s). Hence $\psi : W \to \mathfrak{s}'$ is an affine quotient map by $K_{\mathbb{C}}$, which is surjective under the condition of the stable range. Notice that φ is *not* surjective in general.

Note that $\varphi(\psi^{-1}(\mathcal{N}(\mathfrak{s}'))) \subset \mathcal{N}(\mathfrak{s})$, i.e., $\varphi \circ \psi^{-1}$ carries nilpotent elements to nilpotent elements. This easily follows from the fact that $(A, B) \in M_{p,q} \oplus M_{q,p} = \mathfrak{s}$ belongs to $\mathcal{N}(\mathfrak{s})$ if and only if AB is a nilpotent matrix.

Take a nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}'$. Since φ and ψ are $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant, $\varphi(\psi^{-1}(\overline{\mathbb{O}'})) \subset \mathcal{N}(\mathfrak{s})$ is a union of $K_{\mathbb{C}}$-orbits. However, it has much better properties.

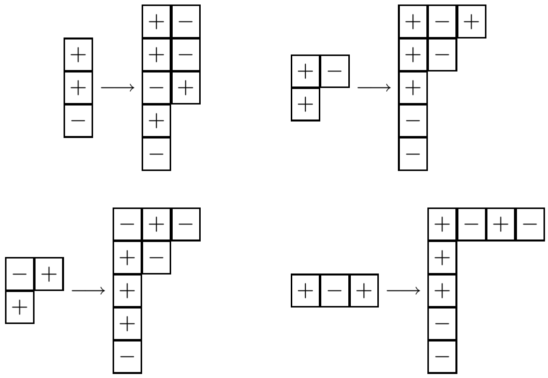
Theorem 1.1 ([10]). *Assume the stable range condition $m + n \le p, q$. Then the scheme theoretic inverse image $\psi^{-1}(\overline{\mathbb{O}'}) = \overline{\mathbb{O}'} \times_{\mathfrak{s}'} W$ is reduced and irreducible. Its image $\varphi(\psi^{-1}(\overline{\mathbb{O}'}))$ is the closure of a single nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}$: $\varphi(\psi^{-1}(\overline{\mathbb{O}'})) = \overline{\mathbb{O}}$.*

Thus we have a correspondence

$$\theta : \mathcal{N}(\mathfrak{s}')/K'_{\mathbb{C}} \ni \mathbb{O}' \longmapsto \mathbb{O} \in \mathcal{N}(\mathfrak{s})/K_{\mathbb{C}},$$

which is called the *theta lifting*. Let D' be a signed Young diagram corresponding to $\mathbb{O}'$, and we denote it by $\mathbb{O}' = \mathbb{O}'_{D'}$. Then the Young diagram D corresponding to the theta lift $\mathbb{O}_D = \theta(\mathbb{O}'_{D'})$ is obtained by adding an extra box to the end of each row of D' ; if the row is empty, we place one box and make the total number of boxes equal to $p + q$. The signs in the added boxes are automatically determined by those in D'. For more detailed description of the correspondence $D' \to D$, see [12].

Example 1.2. Here we give an example for the pair $U(4,4) \times U(2,1)$.



If we take various $G' = U(m,n)$ satisfying $m+n \leq p, q$, we obtain a number of lifted nilpotent $K_{\mathbb{C}}$-orbits. However, they do not exhaust all the nilpotent orbits. Let $\mathbb{O}_D \subset \mathcal{N}(\mathfrak{s})$ be a nilpotent $K_{\mathbb{C}}$-orbit corresponding to the signed Young diagram D. By the length $\ell = \ell(D)$ of D, we mean the number of non-empty rows in D (in other words, the number of boxes in the first column). Then we have the following

Lemma 1.3. *A nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}_D$ is a theta lift from certain nilpotent orbit in the stable range if and only if $\ell(D) \geq \max\{p,q\}$ holds.*

Remark 1.4. We make a convention that the trivial orbit $\{0\}$ is lifted from the (ideal) trivial orbit of the trivial group $U(0,0)$. This enables us to state our results in a uniform fashion.

Proof. The original D' should be the diagram obtainable by deleting the last box in each row of D. So, we delete $\ell(D)$ boxes. Therefore the condition of the stable range becomes $m+n = p+q-\ell(D) \leq p, q$, which is equivalent to the condition given in the lemma. $\square$

Since $K_{\mathbb{C}}$ acts on the closure $\overline{\mathbb{O}}$, it naturally acts on the regular function ring $\mathbb{C}[\overline{\mathbb{O}}]$. Let us describe the $K_{\mathbb{C}}$-module structure of the regular function ring of the closure of the lifted orbit $\mathbb{O} = \theta(\mathbb{O}')$. To state it, we need some notations.

Let $\mathcal{P}_m$ be the set of all partitions of length $\leq m$ and consider $\alpha \in \mathcal{P}_m$ as a dominant integral weight for GL_m as usual. Then $\tau_\alpha^{(m)}$ denotes the irreducible finite dimensional representation of GL_m with highest weight α, and $\tau_\alpha^{(m)*}$ its contragredient. For $\alpha \in \mathcal{P}_m$ and $\beta \in \mathcal{P}_n$, we put

$$\alpha \odot_p \beta = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0, -\beta_n, \ldots, -\beta_1) = (\alpha, 0, \ldots, 0, \beta^*) \in \mathbb{Z}^p$$

where there appear $(p-(m+n))$-times of zeroes between α and β^*. This is also a dominant integral weight for GL_p.

The following theorem is proved in [10] (also see [3, Th. 2.5.4] for the structure of harmonics), and it plays a key role in determining the spherical nilpotent orbits among the lifted ones.

Theorem 1.5. *Let $\mathbb{O}' \subset \mathcal{N}(\mathfrak{s}')$ be a nilpotent $K'_{\mathbb{C}}$-orbit and $\mathbb{O} = \theta(\mathbb{O}') \subset \mathcal{N}(\mathfrak{s})$ its theta lift. Then the $K_{\mathbb{C}}$-module structure of the regular function ring of $\overline{\mathbb{O}}$ is completely described via that of $\overline{\mathbb{O}'}$. Namely, as a $K_{\mathbb{C}}$-module, we have*

$$\mathbb{C}[\overline{\mathbb{O}}] \simeq \sum_{\substack{\alpha,\gamma \in \mathcal{P}_m \\ \beta,\delta \in \mathcal{P}_n}}^{\oplus} m(\alpha, \beta, \gamma, \delta, \mathbb{O}') \otimes (\tau^{(p)}_{\alpha \odot_p \beta}{}^{*} \boxtimes \tau^{(q)}_{\gamma \odot_q \delta}),$$

where

$$m(\alpha, \beta, \gamma, \delta; \mathbb{O}') = \mathrm{Hom}_{K'_{\mathbb{C}}}((\tau^{(m)*}_{\alpha} \otimes \tau^{(m)}_{\gamma}) \boxtimes (\tau^{(n)}_{\beta} \otimes \tau^{(n)*}_{\delta}), \mathbb{C}[\overline{\mathbb{O}'}])$$

is the space of multiplicity on which $K_{\mathbb{C}} = GL_p \times GL_q$ acts trivially.

2. Spherical nilpotent orbits

Let X be a variety on which $K_{\mathbb{C}}$ acts regularly. Then X is called a *spherical variety* (or more specifically, $K_{\mathbb{C}}$-spherical variety), if there exists a Borel subgroup $B_{K_{\mathbb{C}}}$ of $K_{\mathbb{C}}$ which has an open dense orbit in X. If a nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}$ is a spherical variety, it is called a *spherical nilpotent orbit*. By definition, $\mathbb{O}$ is a spherical orbit if and only if $\overline{\mathbb{O}}$ is spherical. Since $\overline{\mathbb{O}}$ is a closed affine cone, it is spherical if and only if the regular function ring $\mathbb{C}[\overline{\mathbb{O}}]$ decomposes without multiplicity as a $K_{\mathbb{C}}$-module, i.e., the action of $K_{\mathbb{C}}$ on $\overline{\mathbb{O}}$ is multiplicity-free.

Let D be a signed Young diagram of signature (p, q). We denote by $\lambda(D) = \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ the partition of $p + q$ corresponding to D, i.e., λ_i is the number of boxes in the i-th row of D. We call λ the *shape* of D.

Theorem 2.1. *Let $\mathbb{O}_D$ be a nilpotent $K_{\mathbb{C}}$-orbit for $G = U(p, q)$ corresponding to the signed Young diagram D. Assume that $\ell(D) \geq \max\{p, q\}$, where $\ell(D)$ is the length of D. Then $\mathbb{O}_D$ is spherical if and only if its shape $\lambda(D)$ is given by*

$$\lambda(D) = [3^{\varepsilon} \cdot 2^k \cdot 1^l] \qquad \varepsilon = 0, 1; \quad k, l \geq 0; \quad 3\varepsilon + 2k + l = p + q.$$

Remark 2.2.

(1) The shape $\lambda(D)$ determines the Jordan type of nilpotent elements from $\mathbb{O}_D$. This means that the sphericality does not depend on $\mathbb{O}_D$ but on its complex hull $G_{\mathbb{C}} \cdot \mathbb{O}_D$.

(2) Panyushev [13] completely classified the spherical nilpotent orbit for complex simple Lie algebra. According to his classification, only the orbits with Jordan types $[2^k \cdot 1^l]$ are $G_{\mathbb{C}}$-spherical. So, if $\varepsilon = 1$, the complex hull of the orbit above is *not* spherical under the action of $G_{\mathbb{C}}$.

The rest of this section is devoted to the proof of Theorem 2.1.

2.1. Spherical lifted orbits

Let us check that the nilpotent orbits listed in Theorem 2.1 are in fact spherical.

If the shape of a signed Young diagram is $[2^k \cdot 1^l]$, then it is a theta lift from the trivial orbit of $U(m,n)$ $(m+n=k)$, where m (respectively n) is the number of rows $\boxed{+}\boxed{-}$ (respectively $\boxed{-}\boxed{+}$) in D. By Corollary 3.2 in [10], the theta lift from the trivial orbit is spherical.

Next, we consider a signed Young diagram D of shape $[3 \cdot 2^k \cdot 1^l]$. Let $\mathbb{O}_D$ be the corresponding nilpotent $K_{\mathbb{C}}$-orbit which is lifted from a nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}'_{D'}$. Without loss of generality, we can assume that D and D' are of the following form.

$$D = \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & - & \multicolumn{1}{c}{} \\ \cline{1-2} \vdots & \vdots & \multicolumn{1}{c}{} \\ \cline{1-2} - & + & \multicolumn{1}{c}{} \\ \cline{1-2} \vdots & \vdots & \multicolumn{1}{c}{} \\ \cline{1-2} + & \multicolumn{2}{c}{} \\ \cline{1-1} \vdots & \multicolumn{2}{c}{} \\ \cline{1-1} - & \multicolumn{2}{c}{} \\ \cline{1-1} \vdots & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad D' = \begin{array}{|c|c|} \hline + & - \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} - & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

where, in D, the row $\boxed{+}\boxed{-}$ appears $(m-1)$-times, and the row $\boxed{-}\boxed{+}$ appears $(n-1)$-times $(m,n \geq 1)$. Then, $\mathbb{O}'_{D'}$ is a nilpotent $K'_{\mathbb{C}}$-orbit for $G' = U(m,n)$. Note that $\mathbb{O}' = \mathbb{O}'_{D'}$ is also a theta lift from the trivial orbit for $U(1) = U(1,0)$. Thus we have

$$\mathbb{C}[\overline{\mathbb{O}'}] = \sum_{\mu \in P_1}^{\oplus} \tau^{(m)}_{(\mu,0,\ldots,0)}{}^* \boxtimes \tau^{(n)}_{(\mu,0,\ldots,0)},$$

where $P_1 = \mathbb{Z}_{\geq 0}$ is the set of partitions of length 1 or 0. Substituting $\mathbb{C}[\overline{\mathbb{O}'}]$ in Theorem 1.5 by the above formula, we get a decomposition

$$\mathbb{C}[\overline{\mathbb{O}_D}] = \sum_{\substack{\alpha,\gamma \in \mathcal{P}_m \\ \beta,\delta \in \mathcal{P}_n}}^{\oplus} \left(\sum_{\mu \geq 0}^{\oplus} \operatorname{Hom}_{GL_m}\big(\tau^{(m)*}_{\alpha} \otimes \tau^{(m)}_{\gamma}, \tau^{(m)}_{(\mu,0,\ldots,0)}{}^*\big) \otimes \operatorname{Hom}_{GL_n}\big(\tau^{(n)}_{\beta} \otimes \tau^{(n)*}_{\delta}, \tau^{(n)}_{(\mu,0,\ldots,0)}\big) \right) \otimes \big(\tau^{(p)}_{\alpha \odot_p \beta}{}^* \boxtimes \tau^{(q)}_{\gamma \odot_q \delta}\big).$$

If the multiplicity of $\tau^{(p)}_{\alpha \odot_p \beta}{}^* \boxtimes \tau^{(q)}_{\gamma \odot_q \delta}$ does not vanish, we must have $|\alpha| - |\gamma| = \mu$ and $|\beta| - |\delta| = \mu$, where $|\alpha| = \alpha_1 + \cdots + \alpha_m$ is the size of a partition α. Thus μ is uniquely determined by the pair (α,γ) or (β,δ). Let us assume that $\mu = |\alpha| - |\gamma|$ and consider

$$\operatorname{Hom}_{GL_m}\big(\tau^{(m)*}_{\alpha} \otimes \tau^{(m)}_{\gamma}, \tau^{(m)}_{(\mu,0,\ldots,0)}{}^*\big) = \operatorname{Hom}_{GL_m}\big(\tau^{(m)}_{\alpha}, \tau^{(m)}_{\gamma} \otimes \tau^{(m)}_{(\mu,0,\ldots,0)}\big).$$

By Pieri formula, the last expression does not vanish if and only if the skew diagram $\alpha - \gamma$ is a horizontal μ-strip (see [7, (5.16)] for Pieri formula and the terminologies used here), and in that case $\tau_\alpha^{(m)}$ appears in $\tau_\gamma^{(m)} \otimes \tau_{(\mu,0,\ldots,0)}^{(m)}$ with multiplicity one. The similar assertion holds for β and δ.

Thus we get a multiplicity-free decomposition

$$\mathbb{C}[\overline{\mathbb{O}_D}] = \sum_{\substack{\alpha,\gamma\in\mathcal{P}_m \\ \beta,\delta\in\mathcal{P}_n}}^{\oplus} \tau_{\alpha\odot_p\beta}^{(p)}{}^{*} \boxtimes \tau_{\gamma\odot_q\delta}^{(q)}, \tag{2.1}$$

where the summation is taken over $\alpha, \beta, \gamma, \delta$ satisfying

$\mu = |\alpha| - |\gamma| = |\beta| - |\delta|$; and, $\alpha - \gamma$ and $\beta - \delta$ are horizontal μ-strips.

This proves that $\mathbb{O}_D$ is spherical.

2.2. Non-spherical lifted orbits

We are to prove that the rest of lifted orbits are not spherical. We prepare two lemmas.

Lemma 2.3. *Let $\mathbb{O}$ be a theta lift of $\mathbb{O}'$ in the stable range. if $\mathbb{O}$ is spherical, then $\mathbb{O}'$ is spherical too. In other words, theta lifting does not produce any spherical orbit from non-spherical ones.*

Proof. We assume that $\mathbb{O}'$ is not spherical so that $\mathbb{C}[\overline{\mathbb{O}'}]$ contains some representation $\tau_\mu^{(m)} \otimes \tau_\eta^{(n)}$ with multiplicity ≥ 2. Let us denote the multiplicity by $M_{\mu,\eta}$. Then we have

$$\begin{aligned} &\operatorname{Hom}_{GL_m\times GL_n}((\tau_\alpha^{(m)*} \otimes \tau_\gamma^{(m)}) \boxtimes (\tau_\beta^{(n)} \otimes \tau_\delta^{(n)*}), \mathbb{C}[\overline{\mathbb{O}'}]) \\ &\supset M_{\mu,\eta} \operatorname{Hom}_{GL_m}(\tau_\alpha^{(m)*} \otimes \tau_\gamma^{(m)}, \tau_\mu^{(m)}) \otimes \operatorname{Hom}_{GL_n}(\tau_\beta^{(n)} \otimes \tau_\delta^{(n)*}, \tau_\eta^{(n)}). \end{aligned}$$

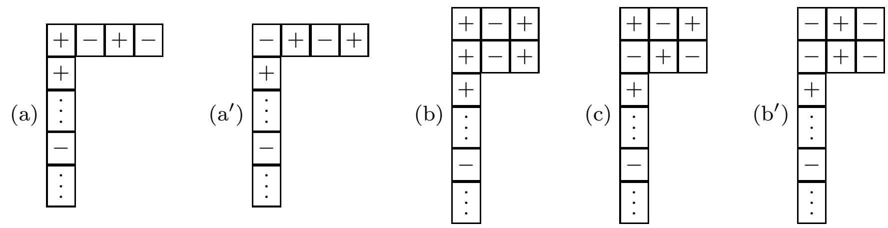
Clearly we can choose $\alpha, \beta, \gamma, \delta$ so that the above tensor product survives. Then, by Theorem 1.5, $\mathbb{C}[\overline{\mathbb{O}}]$ is not multiplicity-free, hence $\mathbb{O}$ is not spherical. □

Lemma 2.4. *Let $\mathbb{O}_1$ and $\mathbb{O}_2$ be two nilpotent $K_\mathbb{C}$-orbits, and suppose that $\mathbb{O}_1$ is adherent to the closure $\overline{\mathbb{O}_2}$. If $\mathbb{O}_2$ is spherical, then $\mathbb{O}_1$ is also spherical. In other words, if $\mathbb{O}_1$ is not spherical, $\mathbb{O}_2$ cannot be spherical.*

Proof. Since $\overline{\mathbb{O}_1} \subset \overline{\mathbb{O}_2}$ is a closed subvariety, the restriction is a $K_\mathbb{C}$-equivariant surjection $\mathbb{C}[\overline{\mathbb{O}_2}] \to \mathbb{C}[\overline{\mathbb{O}_1}]$. Thus, if $\mathbb{C}[\overline{\mathbb{O}_2}]$ is multiplicity-free, $\mathbb{C}[\overline{\mathbb{O}_1}]$ is multiplicity-free, which is what we wanted to show. □

This lemma tells that the set of spherical nilpotent orbits has a hereditary property with respect to the closure relation.

The closure relation of nilpotent $K_\mathbb{C}$-orbits is known by [11]. Among the signed Young diagrams which are *not* listed in Theorem 2.1, the minimal orbits with respect to the closure relation are the following form:



Thus it is enough to check the orbits corresponding to (a), (a′), (b), (b′), (c) are *not* spherical. Since (a) and (a′) (respectively (b) and (b′)) are treated in the similar manner, we only consider cases (a), (b) and (c).

Case (a)

Let D be the Young diagram (a). Then $\mathbb{O}_D$ is lifted from $\mathbb{O}'_{D'}$ where $D' = \boxed{+}\boxed{-}\boxed{+}$. $\mathbb{O}'_{D'}$ is the largest nilpotent $K'_{\mathbb{C}}$-orbit of $G' = U(2,1)$, and its closure coincides with the whole nilpotent variety $\mathcal{N}(\mathfrak{s}')$. Therefore, by Kostant-Rallis theorem [6, Theorem 18], we get

$$\mathbb{C}[\overline{\mathbb{O}'_{D'}}] = \mathbb{C}[\mathcal{N}(\mathfrak{s}')] \simeq \operatorname{Ind}_{M'_{\mathbb{C}}}^{K'_{\mathbb{C}}} \mathbb{C} \qquad (\text{as } K'_{\mathbb{C}}\text{-module}),$$

where $K'_{\mathbb{C}} = GL_2 \times GL_1$ and $\mathbb{C}$ denotes the trivial representation of

$$M'_{\mathbb{C}} = \{\operatorname{diag}(t, s, s) \mid t, s \in \mathbb{C}^\times\}.$$

Frobenius reciprocity law tells us that the above formula becomes

$$\begin{aligned}\mathbb{C}[\overline{\mathbb{O}'_{D'}}] &\simeq \sum_{\mu,\nu}^{\oplus} \operatorname{Hom}_{M'_{\mathbb{C}}}(\tau_\mu^{(2)} \boxtimes \tau_\nu^{(1)}, \mathbb{C}) \otimes (\tau_\mu^{(2)} \boxtimes \tau_\nu^{(1)}) \\ &\simeq \sum_{\mu_1 \geq 0 \geq \mu_2}^{\oplus} \tau_\mu^{(2)} \boxtimes \tau_{\mu_1+\mu_2}^{(1)}{}^*,\end{aligned}$$

where $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ is a dominant integral weight for GL_2. This shows that $\mathbb{O}'_{D'}$ is spherical, and it gives an example of a spherical nilpotent orbit which is not a theta lift in the stable range.

Now by Theorem 1.5, we obtain the decomposition of $\mathbb{C}[\overline{\mathbb{O}_D}]$ as a $K_{\mathbb{C}}$-module:

$$\sum_{\substack{\alpha,\gamma \in \mathcal{P}_2 \\ \beta,\delta \in \mathcal{P}_1}}^{\oplus} \left(\sum_{\mu_1 \geq 0 \geq \mu_2}^{\oplus} \operatorname{Hom}_{GL_2 \times GL_1}((\tau_\alpha^{(2)*} \otimes \tau_\gamma^{(2)}) \boxtimes (\tau_\beta^{(1)} \otimes \tau_\delta^{(1)*}), \right.$$
$$\left. \tau_\mu^{(2)} \boxtimes \tau_{\mu_1+\mu_2}^{(1)}{}^*) \right) \otimes (\tau_{\alpha \odot_p \beta}^{(p)}{}^* \boxtimes \tau_{\gamma \odot_q \delta}^{(q)}).$$

The multiplicity in the above formula can be rewritten as

$$\sum_{\mu_1 \geq 0 \geq \mu_2}^{\oplus} \operatorname{Hom}_{GL_2}(\tau_\alpha^{(2)*} \otimes \tau_\gamma^{(2)}, \tau_\mu^{(2)}) \otimes \operatorname{Hom}_{GL_1}(\tau_\delta^{(1)}, \tau_\beta^{(1)} \otimes \tau_{\mu_1+\mu_2}^{(1)}) \tag{2.2}$$

Take integers $l \geq k \geq 1$ and put $\alpha = (k, 0)$, $\gamma = (l, 0)$. Then, by Pieri formula, we get

$$\tau_\alpha^{(2)*} \otimes \tau_\gamma^{(2)} \simeq \sum_{k_1+k_2=k}^{\oplus} \tau_{(l-k_1, -k_2)}^{(2)},$$

where the summation is taken over non-negative integers $k_1, k_2 \geq 0$ such that $k = k_1 + k_2$. Therefore $\mu = (l - k_1, -k_2)$ gives a non-zero term in (2.2) if and only if $\delta = \beta + \mu_1 + \mu_2 = \beta + l - k$. Let us take $\beta = k$ and $\delta = l$. Thus the multiplicity of ${\tau^{(p)}_{\alpha \odot \beta}}^* \boxtimes \tau^{(q)}_{\gamma \odot \delta} = \tau^{(p)}_{(k,0,\dots,0,-k)} \boxtimes \tau^{(q)}_{(l,0,\dots 0,-l)}$ is equal to $k + 1 \geq 2$. This shows that $\mathbb{C}[\overline{\mathbb{O}_D}]$ is not multiplicity-free.

Case (b)

As above, we put $D' = \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array}$, which corresponds to the anti-holomorphic orbit for $G' = U(2,2)$. Since $\mathbb{O}'_{D'}$ is a theta lift from the trivial orbit of $U(2)$, we have

$$\mathbb{C}[\overline{\mathbb{O}'_{D'}}] \simeq \sum_{\mu \in \mathcal{P}_2}^{\oplus} \tau^{(2)}_\mu \boxtimes \tau^{(2)*}_\mu .$$

By Theorem 1.5, we obtain

$$\mathbb{C}[\overline{\mathbb{O}_D}] \simeq \sum_{\alpha,\beta,\gamma,\delta \in \mathcal{P}_2}^{\oplus} \left(\sum_{\mu \in \mathcal{P}_2}^{\oplus} \mathrm{Hom}_{GL_2 \times GL_2}\big((\tau^{(2)*}_\alpha \otimes \tau^{(2)}_\gamma) \boxtimes (\tau^{(2)}_\beta \otimes \tau^{(2)*}_\delta), \tau^{(2)}_\mu \boxtimes \tau^{(2)*}_\mu \big) \right) \otimes ({\tau^{(p)}_{\alpha \odot_p \beta}}^* \boxtimes \tau^{(q)}_{\gamma \odot_q \delta}).$$

Therefore the multiplicity becomes

$$\sum_{\mu \in \mathcal{P}_2}^{\oplus} \mathrm{Hom}_{GL_2}(\tau^{(2)}_\mu, \tau^{(2)*}_\alpha \otimes \tau^{(2)}_\gamma) \otimes \mathrm{Hom}_{GL_2}(\tau^{(2)}_\mu, \tau^{(2)*}_\beta \otimes \tau^{(2)}_\delta). \tag{2.3}$$

We take integers $l \geq k \geq 1$, and put $\alpha = \beta = (k, 0)$ and $\gamma = \delta = (k + l, k)$. Pieri formula tells us that

$$\tau^{(2)*}_\alpha \otimes \tau^{(2)}_\gamma \simeq \tau^{(2)*}_\beta \otimes \tau^{(2)}_\delta \simeq \sum_{k_1 + k_2 = k}^{\oplus} \tau^{(2)}_{(l+k_1, k_2)}.$$

Thus, the above multiplicity (2.3) is equal to $k + 1 \geq 2$. Hence $\mathbb{C}[\overline{\mathbb{O}_D}]$ is not multiplicity-free.

Case (c)

We put $D' = \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline\end{array}$. Then $\mathbb{O}'_{D'}$ is a theta lift from the trivial orbit of $U(1,1)$. So we obtain

$$\mathbb{C}[\overline{\mathbb{O}'_{D'}}] \simeq \sum_{\mu_1 \geq 0 \geq \mu_2}^{\oplus} \tau^{(2)*}_\mu \boxtimes \tau^{(2)}_\mu ,$$

and

$$\mathbb{C}[\overline{\mathbb{O}_D}] \simeq \sum^{\oplus}_{\alpha,\beta,\gamma,\delta\in\mathcal{P}_2} \left(\sum^{\oplus}_{\mu_1\geq 0\geq\mu_2} \mathrm{Hom}_{GL_2\times GL_2} \left((\tau^{(2)*}_{\alpha}\otimes\tau^{(2)}_{\gamma})\boxtimes(\tau^{(2)}_{\beta}\otimes\tau^{(2)*}_{\delta}), \right.\right.$$
$$\left.\left. \tau^{(2)*}_{\mu}\boxtimes\tau^{(2)}_{\mu}\right) \right) \otimes (\tau^{(p)}_{\alpha\odot_p\beta}{}^{*}\boxtimes\tau^{(q)}_{\gamma\odot_q\delta}).$$

The multiplicity becomes

$$\sum^{\oplus}_{\mu_1\geq 0\geq\mu_2} \mathrm{Hom}_{GL_2}(\tau^{(2)}_{\mu},\tau^{(2)}_{\alpha}\otimes\tau^{(2)*}_{\gamma})\otimes\mathrm{Hom}_{GL_2}(\tau^{(2)}_{\mu},\tau^{(2)}_{\beta}\otimes\tau^{(2)*}_{\delta}). \tag{2.4}$$

For integers $l\geq k\geq 1$, we put $\alpha=\beta=(l,0)$ and $\gamma=\delta=(k,0)$. Then, by Pieri formula, we have

$$\tau^{(2)}_{\alpha}\otimes\tau^{(2)*}_{\gamma}\simeq\tau^{(2)}_{\beta}\otimes\tau^{(2)*}_{\delta}\simeq\sum^{\oplus}_{k_1+k_2=k}\tau^{(2)}_{(l-k_1,-k_2)}.$$

Thus, the multiplicity (2.4) is equal to $k+1\geq 2$ for $\tau^{(p)}_{\alpha\odot_p\beta}{}^{*}\boxtimes\tau^{(q)}_{\gamma\odot_q\delta}=\tau^{(p)}_{(l,0,\ldots,0,-l)}\boxtimes\tau^{(q)}_{(k,0,\ldots,0,-k)}$. Hence $\mathbb{O}_D$ is not spherical.

Now Lemma 2.4 tells us that the nilpotent orbit $\mathbb{O}$ whose closure contains the diagrams (a) – (c) cannot be spherical. This completes the proof of Theorem 2.1.

3. Classification of spherical nilpotent orbits

In Section 2, we have identified the spherical nilpotent orbits which are theta lifts in the stable range. In this section, we obtain a sufficient condition for non-sphericality. As a consequence, in the case of $U(p,p)$ where $p=q$, we prove there is no spherical nilpotent orbit other than lifted ones. This gives a complete classification of spherical nilpotent orbits for $G=U(p,p)$.

Theorem 3.1. *Let $\mathbb{O}=\mathbb{O}_D$ be a nilpotent $K_{\mathbb{C}}$-orbit for $U(p,p)$. It is spherical if and only if the shape $\lambda(D)$ of the signed Young diagram D is of form*:

$$\lambda(D)=[3^{\varepsilon}\cdot 2^{k}\cdot 1^{l}] \qquad \varepsilon=0,1;\quad k,l\geq 0;\quad 3\varepsilon+2k+l=2p,$$

and the length satisfies an inequality $\ell(D)=\varepsilon+k+l\geq p$.

As explained above, Theorem 3.1 follows from Theorem 2.1 and the following

Lemma 3.2. *Let $\mathbb{O}=\mathbb{O}_D$ be a nilpotent $K_{\mathbb{C}}$-orbit for $U(p,q)$. If $\ell(D)<\min\{p,q\}$, it is not spherical. In particular, if $p=q$, a spherical orbit $\mathbb{O}$ is a theta lift from certain nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}'$ in the stable range.*

The rest of the section is devoted to the proof of the above lemma.

Let $B = B_{K_{\mathbb{C}}}$ be a Borel subgroup of $K_{\mathbb{C}}$. The following lemma is almost trivial.

Lemma 3.3. *Let $\mathbb{O}$ be a nilpotent $K_{\mathbb{C}}$-orbit. If $\dim \mathbb{O} \geq \dim B_{K_{\mathbb{C}}}$, it is not spherical.*

Proof. Since $\mathbb{O}$ is a cone and B contains a dilation, the dimension of a B-orbit in $\mathbb{O}$ cannot exceed $\dim B - 1$. □

Take a signed Young diagram D of shape $\lambda = \lambda(D)$ with signature (p, q). We put ${}^t\lambda = \mu = (\mu_1, \ldots, \mu_k)$, the transposed partition of λ.

Lemma 3.4. *With the above notation, $\dim \mathbb{O}_D \geq \dim B_{K_{\mathbb{C}}}$ if and only if*

$$2pq - (p+q) \geq \sum_{i=1}^{k} \mu_i^2. \tag{3.1}$$

Proof. It is well known that the dimension of the $G_{\mathbb{C}}$-hull of $\mathbb{O}_D$ is given by

$$\dim G_{\mathbb{C}} \cdot \mathbb{O}_D = \dim \mathfrak{g} - \sum_{i=1}^{k} \mu_i^2 = 2 \dim \mathbb{O}_D.$$

For this, see [1, Cor. 6.1.4, Remark 9.5.2] for example. Since $K_{\mathbb{C}} = GL_p \times GL_q$, we have $\dim B_{K_{\mathbb{C}}} = p(p+1)/2 + q(q+1)/2$. Thus we get

$$\begin{aligned} \dim \mathbb{O}_D - \dim B_{K_{\mathbb{C}}} &= \frac{1}{2}\left\{(p+q)^2 - \sum_{i=1}^{k} \mu_i^2\right\} - \left\{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}\right\} \\ &= \frac{1}{2}\left\{2pq - (p+q) - \sum_{i=1}^{k} \mu_i^2\right\}, \end{aligned}$$

which proves the lemma. □

Let us return to the proof of Lemma 3.2. Without loss of generality, we can assume that $\mu_1 = \ell(D) < p \leq q$. Since $\mu = (\mu_1, \ldots, \mu_k)$ is a partition of $p+q$, we have $\sum_{i=1}^{k} \mu_i = p+q$. Therefore we have

$$\begin{aligned} 2pq - (p+q) - \sum_{i=1}^{k} \mu_i^2 &\geq (p-1)(p+q) - \sum_{i=1}^{k} \mu_i^2 = (p-1)\sum_{i=1}^{k} \mu_i - \sum_{i=1}^{k} \mu_i^2 \\ &= \sum_{i=1}^{k} \mu_i(p-1-\mu_i) \geq \sum_{i=1}^{k} \mu_i(p-\mu_1-1) \geq 0. \end{aligned}$$

By Lemma 3.4, the above inequality assures $\dim \mathbb{O}_D \geq \dim B_{K_{\mathbb{C}}}$, hence $\mathbb{O}_D$ cannot be spherical. This completes the proof of Lemma 3.2.

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References

[1] D. H. Collingwood and W. M. McGovern, *Nilpotent Orbits in semisimple Lie algebras*, Van Nostrand Reinhold, 1993.

[2] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2**-3 (1989), 535–552.

[3] ———, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. in *The Schur lectures* (1992) *(Tel Aviv)*, 1–182, Bar-Ilan Univ., Ramat Gan, 1995.

[4] J.-S. Huang and J.-S. Li, *Unipotent representations attached to spherical nilpotent orbits*, Amer. J. Math. **121**-3 (1999), 497–517.

[5] D. R. King, *Classification of spherical nilpotent orbits in complex symmetric space*, preprint.

[6] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[7] I. G. Macdonald, *Symmetric functions and Hall polynomials. Second edition*. Oxford Mathematical Monographs, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[8] K. Nishiyama, H. Ochiai and K. Taniguchi, *Bernstein degree and associated cycles of Harish-Chandra modules—Hermitian symmetric case*, Astérisque **273** (2001), 13–80.

[9] K. Nishiyama and C.-B. Zhu, *Theta lifting of holomorphic discrete series*, (The case of $U(p,q) \times U(n,n)$), Trans. AMer. Math. Soc. **353** (2001), 3327–3345.

[10] K. Nishiyama, H. Ochiai and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, **math.RT/0312453**, 2003.

[11] T. Ohta, *The closures of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. (2) **43** (1991), 161–211.

[12] ———, *Nilpotent orbits of Z_4-graded Lie algebra and geometry of the moment maps associated to the dual pair* $(U(p,q), U(r,s))$, preprint.

[13] D. I. Panyushev, *On spherical nilpotent orbits and beyond*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1453–1476.