J. Math. Kyoto Univ. (JMKYAZ)
44-1 (2004), 193–201

Global L^∞ solutions of the compressible Euler equations with damping and spherical symmetry

By

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Abstract

We study the Euler equations of compressible isentropic gas dynamics with damping and spherical symmetry. For spherically symmetric flow, the global existence of the weak entropy L^∞ solutions with damping isn't still obtained. In this paper, we prove the existence of global solutions with small L^∞ data. We construct the approximate solutions by using modified Godunov scheme. A L^∞ bound for the approximate solutions can be obtained with the aid of the presence of the damping term.

1. Introduction

Let us consider the Euler equations in $\mathbf{R}^3$ with spherically initial data and damping,

$$\begin{cases} \rho_t + \nabla \cdot \vec{m} = 0, \\ \vec{m}_t + \nabla \cdot \left(\dfrac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p = -\alpha \vec{m}, \quad \vec{x} \in \mathbf{R}^3, \end{cases} \tag{1.1}$$

where ρ, $\vec{m}$ and p are the density, the momentum and the pressure of the gas, respectively, while $\alpha > 0$ is the friction constant. On the non-vacuum state $\rho > 0$, $\vec{u} = \vec{m}/\rho$ is the velocity. For polytropic gas, $p(\rho) = \rho^\gamma/\gamma$, where $\gamma \in (1, 5/3]$ is the adiabatic exponent for usual gases.

The one-dimensional case of the Cauchy problem of (1.1) has studied in [D] and [HL]. For spherically symmetric flow, the global existence of the weak entropy solutions without damping was studied in [CG] by using compensated compactness framework. However the proofs of this result are incorrect (see Section 8 in [T]). Therefore the global existence theorem isn't still obtained except the special cases (for example [C1]). The global existence of BV solutions with damping has obtained when $\gamma = 1$ in [HLY]. In this paper, we prove the

Received August 5, 2003
Revised December 5, 2003

global existence of solutions with damping and small L^∞ data in general case $1 < \gamma \leq 5/3$. In Section 4, the damping term enable one to get L^∞ estimates with the difficulty caused by inhomogeneous terms.

Consider the initial value problem (1.1) and

$$(\rho, \vec{m})|_{t=0} = (\rho_0(\vec{x}), \vec{m}_0(\vec{x})), \tag{1.2}$$

with following geometric structure

$$(\rho_0(\vec{x}), \vec{m}_0(\vec{x})) = \left(\rho_0(|\vec{x}|), m_0(|\vec{x}|)\frac{\vec{x}}{|\vec{x}|}\right), \tag{1.3}$$

where $m_0(x)$ is a scalar function of $x = |\vec{x}| \geq 1$. We look for the solutions of the form

$$(\rho(\vec{x}), \vec{m}(\vec{x})) = \left(\rho(|\vec{x}|, t), m(|\vec{x}|, t)\frac{\vec{x}}{|\vec{x}|}\right). \tag{1.4}$$

We rewrite (1.1) as

$$\begin{cases} \rho_t + m_x = -\dfrac{2}{x}m, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = -\dfrac{2}{x}\dfrac{m^2}{\rho} - \alpha m, \quad p(\rho) = \rho^\gamma/\gamma, \end{cases} \tag{1.5}$$

where $\rho(x,t)$ and $m(x,t)$, $x = |\vec{x}| \geq 1$ are the scalar functions. This equation can be written as

$$\begin{cases} v_t + f(v)_x = g(x,v), \quad x \geq 1, \\ v|_{t=0} = v_0(x), \end{cases} \tag{1.6}$$

where $v = (\rho, m)^\top$, $u = m/\rho$, $f(v) = (m, m^2/\rho + p(\rho))^\top$, $g(x,v) = (-\frac{2}{x}m, -\frac{2}{x}\frac{m^2}{\rho} -\alpha m)^\top$.

We consider the initial-boundary value problem

$$\begin{cases} v|_{t=0} = v_0(x), \\ m|_{x=1} = 0, \end{cases} \tag{1.7}$$

with the initial data $v_0 \in L^\infty(x \geq 1)$.

A pair of mapping $(\eta, q) : \mathbf{R}^2 \to \mathbf{R}^2$ is called an entropy-entropy flux pair, if it satisfies an identity

$$\nabla q = \nabla \eta \nabla f. \tag{1.8}$$

Furthermore, if, for any fixed $m/\rho \in (-\infty, \infty)$, η vanishes on the vacuum $\rho = 0$, then η is called a weak entropy. For example, the mechanical energy-energy flux pair

$$\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \frac{1}{\gamma(\gamma-1)}\rho^\gamma, \quad q_* = m\left(\frac{1}{2}\frac{m^2}{\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right) \tag{1.9}$$

is a strictly convex weak entropy-entropy flux pair.

Our main result is as follows.

Theorem 1.1. *We assume that initial velocity and nonnegative density data* $(\rho_0, m_0) \in L^\infty(x \geq 1)$ *satisfy*

$$\alpha \geq 2\theta \max\left(\sup_x w(v_0(x)), -\inf_x z(v_0(x))\right), \tag{1.10}$$

where w, z *are Riemann invariants defined as*

$$w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta} = u + \frac{\rho^\theta}{\theta}, \quad z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta} = u - \frac{\rho^\theta}{\theta}.$$

Then initial-boundary value problem (1.6)–(1.7) *has a global weak entropy solution* $(\rho(x,t), m(x,t))$.

2. Preliminary

In this section, we first review some results of Riemann solutions for the homogeneous system of gas dynamics. Consider the homogeneous system

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = 0, \quad p(\rho) = \rho^\gamma/\gamma. \end{cases} \tag{2.1}$$

The eigenvalues of the system are

$$\lambda_1 = \frac{m}{\rho} - c, \quad \lambda_2 = \frac{m}{\rho} + c.$$

Any discontinuity in the weak solutions to (2.1) must satisfy the Rankine-Hugoniot condition

$$\sigma(v - v_0) = f(v) - f(v_0),$$

where σ is the propagation speed of the discontinuity, $v_0 = (\rho_0, m_0)$ and $v = (\rho, m)$ are the corresponding left state and right state. This means that

$$\begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho - \rho_0) \pm \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}(\rho - \rho_0), \\ \sigma = \dfrac{m - m_0}{\rho - \rho_0} = \dfrac{m_0}{\rho_0} \pm \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}. \end{cases}$$

A discontinuity is called a shock if it satisfies the entropy condition

$$\sigma(\eta(v) - \eta(v_0)) - (q(v) - q(v_0)) \geq 0$$

for any convex entropy pair (η, q).

Consider the Riemann problem of (2.1) with initial data

$$v|_{t=0} = \begin{cases} v_-, & x < x_0, \\ v_+, & x > x_0, \end{cases} \tag{2.2}$$

and the Riemann initial boundary problem of (2.1) with data

$$v|_{t=0} = v_+, \quad m|_{x=1} = 0, \tag{2.3}$$

where $x_0 \in (-\infty, \infty)$, $\rho_\pm \geq 0$ and $m_\pm$ are constants satisfying $|m_\pm/\rho_\pm| < \infty$. For the problem (2.1) and (2.3), we draw a diagram the inverse wave curve of the second family and the vacuum for given right state v_+ and boundary condition $m = 0$ as follows.

1. If $\rho_+ > 0$ and $u_+ \leq 0$, there exists v_- with $u_- = 0$ from which v_+ is connected by a 2-shock curve.

2. If $u_+ \geq 0$ and $z(v_+) \leq 0$, then there exists v_- with $u_- = 0$ from which v_+ is connected by a 2-rarefaction curve.

3. If $u_+ \geq 0$ and $z(v_+) \geq 0$, then there exists v_* with $\rho_* = 0$ from which v_+ is connected by a 2-rarefaction, and v_* and v_- with $\rho_- = u_- = 0$ are connected by the vacuum.

4. If $u_+ \leq 0$ and $\rho_+ = 0$, then v_- with $\rho_- = u_- = 0$ is connected from v_+ by the vacuum.

Then the following theorem and lemma hold.

Theorem 2.1. *There exists a unique piecewise entropy solution* $(\rho(x,t), m(x,t))$ *containing the vacuum state* ($\rho = 0$) *on the upper plane* $t > 0$ *for each problem of* (2.2) *and* (2.3) *satisfying*

(1) *For the Riemann problem* (2.2),

$$\begin{cases} w(\rho(x,t), m(x,t)) \leq \max(w(\rho_-, m_-), w(\rho_+, m_+)), \\ z(\rho(x,t), m(x,t)) \geq \min(z(\rho_-, m_-), z(\rho_+, m_+)), \\ w(\rho(x,t), m(x,t)) - z(\rho(x,t), m(x,t)) \geq 0. \end{cases}$$

(2) *For the Riemann initial boundary problem* (2.3),

$$\begin{cases} w(\rho(x,t), m(x,t)) \leq \max(w(\rho_-, m_-), -z(\rho_+, m_+)), \\ z(\rho(x,t), m(x,t)) \geq \min(z(\rho_+, m_+), 0), \\ w(\rho(x,t), m(x,t)) - z(\rho(x,t), m(x,t)) \geq 0. \end{cases}$$

Such solutions have the following properties.

Lemma 2.2. *For* $\mathrm{B}_+ \geq \mathrm{B}_-$, *the region* $\sum(\mathrm{B}_+, \mathrm{B}_-) = \{(\rho, \rho u) \in \mathbf{R}^2 : w = u + \rho^\theta/\theta,\ z = u - \rho^\theta/\theta,\ w \leq \mathrm{B}_+,\ z \geq \mathrm{B}_-,\ w - z \geq 0\}$ *is invariant with respect to both of the Riemann problem* (2.2) *and the average of the Riemann solutions in* x. *More preciously, if the Riemann date lie in* $\sum(\mathrm{B}_+, \mathrm{B}_-)$, *the corresponding Riemann solutions* $(\rho(x,t), m(x,t)) = (\rho(x,t), \rho(x,t)u(x,t))$ *lie in* $\sum(\mathrm{B}_+, \mathrm{B}_-)$, *and their corresponding averages in* x *also in* $\sum(\mathrm{B}_+, \mathrm{B}_-)$, *that is,*

$$\left(\frac{1}{b-a} \int_a^b \rho(x,t)dx, \frac{1}{b-a} \int_a^b m(x,t)dx \right) \in \sum(\mathrm{B}_+, \mathrm{B}_-).$$

Furthermore, for $\mathrm{B}_- \leq 0 \leq (\mathrm{B}_+ + \mathrm{B}_-)/2$, *the region* $\sum(\mathrm{B}_+, \mathrm{B}_-)$ *is invariant with respect to both of the Riemann initial-boundary problem* (2.3) *and the average of the corresponding Riemann solution in* x.

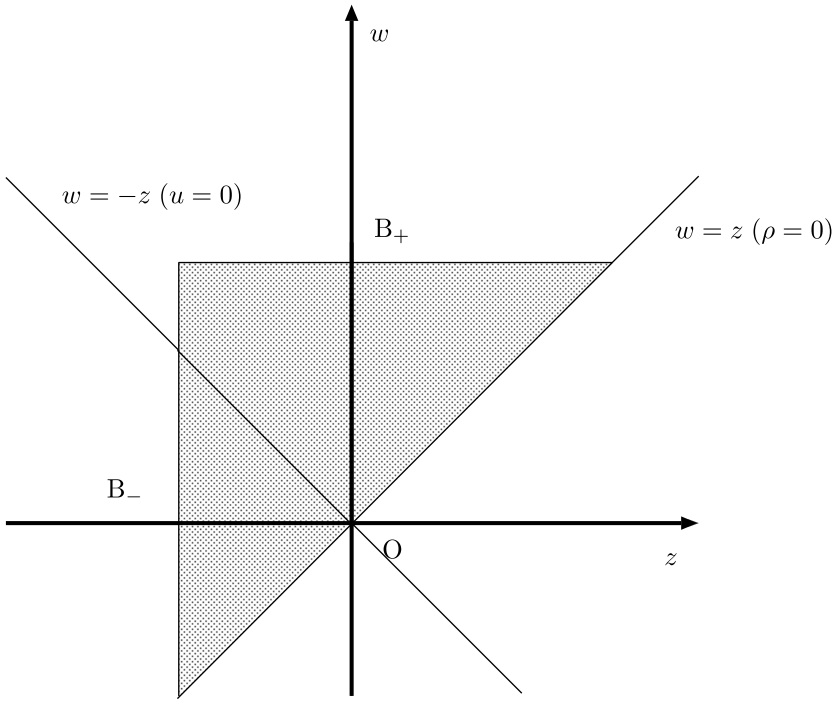

Figure 2.1. The invariant region in (w, z)-plane.

The proof of Lemma 2.2 can be found in [C3].

3. Approximate solutions

In this section we construct approximate solutions $v^h = (\rho^h, m^h) = (\rho^h, \rho^h u^h)$ in the strip $0 \le t \le T$ for any fixed $T \in (0, \infty)$, where h is the space mesh length, together with the time mesh length Δt, satisfying the following Courant-Friedrichs-Lewy condition

$$4\Lambda \equiv 4 \max_{i=1,2} \left(\sup_{0 \le t \le T} |\lambda_i(\rho^h, m^h)| \right) \le \frac{h}{\Delta t} \le 6\Lambda. \tag{3.1}$$

We will prove that the approximate solutions are bounded uniformly in the mesh length $h > 0$ and $\rho^h(x, t) \ge 0$ to guarantee the construction of (ρ^h, m^h).

We construct the approximate solutions (ρ^h, m^h). Let

$$x_j = jh, \quad t_n = n\Delta t, \quad (j, n) \in \mathbf{Z}_{\ge 0} \times \mathbf{Z}_+.$$

Assume that $v^h(x, t)$ is defined for $t < n\Delta t$. Then we define $v_j^n \equiv (\rho_j^n, m_j^n)$ as, for $j \geq 1$,

$$\begin{cases} \rho_j^n \equiv \dfrac{1}{h}\displaystyle\int_{(j-1)h+1}^{jh+1} \rho^h(x, n\Delta t - 0)dx, & (j-1)h+1 \leq x \leq jh+1, \\ m_j^n \equiv \dfrac{1}{h}\displaystyle\int_{(j-1)h+1}^{jh+1} m^h(x, n\Delta t - 0)dx, & (j-1)h+1 \leq x \leq jh+1. \end{cases}$$

Then, in the strip $n\Delta t \leq t < (n+1)\Delta t$, $v_0^h(x,t)$ is defined as, for $1 \leq x \leq 1+\frac{1}{2}h$, the solution of the Riemann initial-boundary problem at $x = 1$,

$$\begin{cases} v_t + f(v)_x = 0, \quad 1 \leq x < 1 + \frac{1}{2}h, \\ v|_{t=n\Delta t} = v_1^n, \quad m_{x=1} = 0, \end{cases}$$

and for $(j-1/2)h+1 \leq x < (j+1/2)h+1$ $(j = 1, 2, \ldots)$, the solution of the Riemann problem at $x = jh+1$

$$\begin{cases} v_t + f(v)_x = 0, \quad (j-\frac{1}{2})h+1 \leq x < (j+\frac{1}{2})h+1, \\ v|_{t=n\Delta t} = \begin{cases} v_j^n, & x < jh+1, \\ v_{j+1}^n, & x > jh+1. \end{cases} \end{cases}$$

Finally we define $v^h(x,t)$ in the strip $n\Delta t \leq t < (n+1)\Delta t$ by the fractional step procedure:

$$v^h(x,t) = v_0^h(x,t) + g(x, v_0^h(x,t))(t - n\Delta t). \tag{3.2}$$

4. L^∞ estimates

First, we notice the following properties of Riemann Invariants.

Remark 4.1.

$$|w| \geq |z|, \ w \geq 0, \ \text{when } u \geq 0.$$
$$|w| \leq |z|, \ z \leq 0, \ \text{when } u \leq 0.$$

We now derive a L^∞ bound for the approximate solutions $v^h(x,t)$ of the initial-boundary value problem (1.6) and (1.7).

Theorem 4.1. *Assume that the initial velocity and nonnegative density data* $(\rho_0, u_0) \in L^\infty(x \geq 1)$ *satisfy* (1.10). *Then the difference approximate solutions of the initial-boundary value problem* (1.6) *and* (1.7) *are uniformly bounded. That is, there exists a constant* $C > 0$ *such that*

$$|u^h(x,t)| \leq C, \quad 0 \leq \rho^h(x,t) \leq C, \quad (x,t) \in \{x \geq 1\} \times \mathbf{R}_+. \tag{4.1}$$

Proof. We assume that, for $t < n\Delta t$,

$$\alpha - \varepsilon \geq 2\theta \max\left(\sup_x w(v^h(x,t)), -\inf_x z(v^h(x,t))\right) \tag{4.2}$$

for any fixed $\varepsilon > 0$. Then we apply Lemma 2.2 with $\mathrm{B}_+ = -\mathrm{B}_- = \dfrac{\alpha-\varepsilon}{2\theta}$ and the construction of v_0^h to get

$$\alpha - \varepsilon \geq 2\theta \max\left(\sup_x w(v_0^h(x,t)), -\inf_x z(v_0^h(x,t))\right) \tag{4.3}$$

for $n\Delta t \leq t < (n+1)\Delta t$.

Then, from (3.2), we have

$$\rho^h = \rho_0^h \left\{1 - \frac{2}{x}u_0^h(t-n\Delta t)\right\},$$

$$\begin{aligned} u^h &= u_0^h \frac{1-\left(\alpha+\frac{2}{x}u_0^h\right)(t-n\Delta t)}{1-\frac{2}{x}u_0^h(t-n\Delta t)} \\ &= u_0^h\left[1-\alpha(t-n\Delta t)+\frac{\frac{2\alpha}{x}u_0^h}{\left\{1-\frac{2\tau_1}{x}u_0^h(t-n\Delta t)\right\}^2}(t-n\Delta t)^2\right], \end{aligned} \tag{4.4}$$

$$\begin{aligned} (\rho^h)^\theta = (\rho_0^h)^\theta \Bigg[&1-\frac{2\theta}{x}u_0^h(t-n\Delta t) \\ &+\theta(\theta-1)\frac{2}{x^2}(u_0^h)^2\left\{1-\frac{2\tau_2}{x}u_0^h(t-n\Delta t)\right\}^{\theta-2}(t-n\Delta t)^2\Bigg], \end{aligned}$$

where τ_1 and τ_2 are constants satisfying $0<\tau_1<1$ and $0<\tau_2<1$ respectively.

For $n\Delta t \leq t < (n+1)\Delta t$ and $u_0^h \geq 0$, we have

$$\begin{aligned} w(v^h) = u^h + \frac{(\rho^h)^\theta}{\theta} &= w(v_0^h) - u_0^h\left\{\alpha+\frac{2}{x}(\rho_0^h)^\theta+\mathcal{O}(\Delta t)\right\}(t-n\Delta t) \\ &\leq w(v_0^h), \end{aligned}$$

choosing Δt enough small, where Landau symbol $\mathcal{O}(\Delta t)$ is a constant depending only on the uniform bound of v_0^h.

Observing (4.4), since $u^h = (1+\mathcal{O}(\Delta t))u_0^h \geq 0$, we have

$$z(v^h) = -w(v^h) + 2u^h \geq -w(v^h) \geq -w(v_0^h).$$

Similarly, for $u_0^h \leq 0$, from (4.3), we have

$$\begin{aligned} z(v^h) = u^h - \frac{(\rho^h)^\theta}{\theta} &= z(v_0^h) - u_0^h\left\{\alpha-\frac{2}{x}(\rho_0^h)^\theta+\mathcal{O}(\Delta t)\right\}(t-n\Delta t) \\ &\geq z(v_0^h). \end{aligned}$$

Observing (4.4), since $u^h = (1+\mathcal{O}(\Delta t))u_0^h \leq 0$, we have

$$w(v^h) = -z(v^h) + 2u^h \leq -z(v^h) \leq -z(v_0^h).$$

Therefore, it follows that (4.2) for $n\Delta t \leq t < (n+1)\Delta t$, that is, there is a constant $C > 0$ such that

$$|u^h(x,t)| = \left|\frac{m^h(x,t)}{\rho^h(x,t)}\right| \leq C, \quad 0 \leq \rho^h(x,t) \leq C,$$

by choosing Δt enough small. Since ε is arbitrary, by induction, we prove the theorem. □

The following proposition and theorem can be proved in the same manner to [HM] and [MT].

Proposition 4.2. *The measure sequence*

$$\eta(v^h)_t + q(v^h)_x$$

lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$ *for all weak entropy pair* (η, q)*, where* $\Omega \subset \{x \geq 1\} \times \mathbf{R}_+$ *is any bounded and open set.*

Theorem 4.3. *Assume that the approximate solution* (ρ^h, m^h) *satisfy Theorem* 4.1 *and Proposition* 4.2*. Then there is a convergent subsequence in the approximate solutions* $(\rho^h(x,t), m^h(x,t))$ *such that*

$$(\rho^{h_n}(x,t), m^{h_n}(x,t)) \to (\rho(x,t), m(x,t)), \quad \text{a.e.} \tag{4.5}$$

The pair of the functions $(\rho(x,t), m(x,t))$ *is a global entropy solution of the initial-boundary value problem* (1.6)–(1.7) *satisfying*

$$0 \leq \rho(x,t) \leq C, \quad \left|\frac{m(x,t)}{\rho(x,t)}\right| \leq C, \tag{4.6}$$

for some C *in the region* $\{x \geq 1\} \times \mathbf{R}_+$.

Acknowledgements. The author would like to thank the referee for useful comments.

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