J. Math. Kyoto Univ. (JMKYAZ)
44-3 (2004), 669–674

KO-theory of complex Stiefel manifolds

By

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1. Introduction

The purpose of this paper is to determine the KO^*-group of the complex Stiefel manifold $V_{n,q}$ which is q-frames in $\mathbf{C}^n$. We compute it by making use of the Atiyah-Hirzebruch spectral sequence of $KO^*(V_{n,q})$ and obtain the following:

Theorem. *Let a_k, b_k and c_k be as follows, then we have*

$$KO^{-i}(V_{n,q}) \cong a_k\mathbf{Z} \oplus c_k\mathbf{Z}_2$$

for $n = 2k$, $q \le 4$ and $n = 2k+1$, $q \le 3$, and

$$KO^{-i}(V_{n,q}) \cong KO^{-i-4}(V_{n,q}) \cong a_k\mathbf{Z} \oplus b_k\mathbf{Z}_2$$

for otherwise.

q	a_0	a_{-1}	a_{-2}	a_{-3}
$2l$	$2^{2l-2}+2^{l-1}$	2^{2l-2}	$2^{2l-2}-2^{l-1}$	2^{2l-2}
$2l+1$	$2^{2l-2}+2^{l-1}$	$2^{2l-2}-(-1)^n2^{l-1}$	$2^{2l-2}-2^{l-1}$	$2^{2l-2}+(-1)^n2^{l-1}$

(n,q)	b_0	b_{-1}	b_{-2}	b_{-3}
$(2k,2l)$	2^{q-4}	$3\cdot 2^{q-4}$	$3\cdot 2^{q-4}$	2^{q-4}
$(2k,2l+1)$	0	2^{q-3}	2^{q-2}	2^{q-3}
$(2k+1,2l)$	0	2^{q-2}	2^{q-2}	0
$(2k+1,2l+1)$	2^{q-3}	2^{q-2}	2^{q-3}	0

Received June 24, 2004

(n,q)	c_0 / c_{-4}	c_{-1} / c_{-5}	c_{-2} / c_{-6}	c_{-3} / c_{-7}
$(2k,2)$	$\frac{1}{2}(1-(-1)^k)$	$\frac{1}{2}(3-(-1)^k)$	$\frac{1}{2}(3+(-1)^k)$	$\frac{1}{2}(1+(-1)^k)$
	$\frac{1}{2}(1+(-1)^k)$	$\frac{1}{2}(3+(-1)^k)$	$\frac{1}{2}(3-(-1)^k)$	$\frac{1}{2}(1-(-1)^k)$
$(2k,3)$	0	2	$3+(-1)^k$	$1+(-1)^k$
	0	0	$1-(-1)^k$	$1-(-1)^k$
$(2k,4)$	$1-(-1)^k$	$3-(-1)^k$	$3+(-1)^k$	$1+(-1)^k$
	$1+(-1)^k$	$3+(-1)^k$	$3-(-1)^k$	$1-(-1)^k$
$(2k+1,2)$	0	2	2	0
	0	0	0	0
$(2k+1,3)$	$1-(-1)^k$	$3-(-1)^k$	2	0
	$1+(-1)^k$	$1+(-1)^k$	0	0

2. The Atiyah-Hirzebruch spectral sequence

First we recall that the coefficient ring of KO-theory is that

$$KO^* = \mathbf{Z}[\alpha, x, \beta, \beta^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2 - 4\beta),$$

where $|\alpha| = -1$, $|x| = -4$ and $|\beta| = -8$.

Let X be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is the spectral sequence with $E_2^{p,q} \cong H^p(X; KO^q)$ converging to $KO^*(X)$. It is well known that the differential d_2 of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is given by the following ([1]).

$$d_2^{*,q} = \begin{cases} Sq^2\,\pi_2, & q \equiv 0\ (8), \\ Sq^2, & q \equiv -1\ (8), \\ 0, & \text{otherwise}, \end{cases}$$

where π_2 is the modulo 2 reduction.

In this paper we compute the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ with X in two special classes of CW-complexes.

Let $\mathcal{E}$ be the class of CW-complexes of only even cells and $\mathcal{O}$ be those of only odd cells and 0-cells. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ for X in $\mathcal{E}$ is considered in [2]. It is easily seen that Proposition 1 in [2] is valid for CW-complexes in $\mathcal{O}$ and we have the following.

Proposition 2.1. *Let X be a finite CW-complex in either $\mathcal{E}$ or $\mathcal{O}$ and $E_r(X)$ be the r-th term of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$. Then we have the following.*

1. $E_3^{p,-1}(X) \cong H^p(H^*(X;\mathbf{Z}_2);Sq^2)$.
2. *Let d_r be the first non-trivial differential for $r \geq 3$.*
   (a) $r \equiv 2\ (8)$.
   (b) *There exists $x \in E_r^{*,0}(X)$ such that $\alpha x \neq 0$ and $\alpha d_r x \neq 0$.*

Corollary 2.1. *Let X and X_i $(i = 1, 2, \ldots, k)$ be pointed finite CW-complexes such that X_i is in $\mathcal{E}$ or $\mathcal{O}$ and there exists a homotopy equivalence $X \underset{s}{\simeq} \bigvee_{i=1}^k X_i$. Then we have:*

1. $E_3^{p,-1}(X) \cong H^p(H^*(X;\mathbf{Z}_2);Sq^2)$.
2. *Let d_r be the first non-trivial differential for $r \geq 3$.*
   (a) $r \equiv 2\ (8)$.
   (b) *There exists $x \in E_r^{*,0}(X)$ such that $\alpha x \neq 0$ and $\alpha d_r x \neq 0$.*

Proof. Since X_i is in $\mathcal{E}$ or $\mathcal{O}$ and $X \underset{s}{\simeq} \bigvee_{i=1}^k X_i$, $E_r(X)$ splits into $E_r(X_i)$. Then we can apply Proposition 2.1 to $E_r(X_i)$ for each i. $\square$

3. Convergence of $E_r(V_{n,q})$

Let $G_{q,k}$ denote the complex Grassmannian which is the homogeneous space $U(q)/U(k) \times U(q-k)$. In [4] it is shown that there exists a stable homotopy equivalence :

$$(*) \qquad V_{n,q} \underset{s}{\simeq} \bigvee_{k=1}^{q} G_{q,k}^{E_k},$$

where $G_{q,k}^{E_k}$ is the Thom space of the real vector bundle $E_k \to G_{q,k}$. Note that $G_{q,k}^{E_k}$ is in $\mathcal{E}$ or $\mathcal{O}$, then we can apply Corollary 2.1 to $V_{n,q}$. Then we compute the Sq^2-cohomology of $V_{n,q}$, which is $H^*(H^*(V_{n,q};\mathbf{Z}_2);Sq^2)$, to consider the convergence of $E_r(V_{n,q})$ by Corollary 2.1.

It is well known that $V_{n,q} \simeq U(n)/U(n-q)$ and

$$H^*(V_{n,q};\mathbf{Z}) \cong \bigwedge(e_{2(n-q)+1}, e_{2(n-q)+3}, \ldots, e_{2n-1}),$$

where $U(k)$ is the k-dimensional unitary group and $|e_i| = i$. Since

$$Sq^2 e_{4i-1} = e_{4i+1},$$

we have the following.

Proposition 3.1. $H^*(H^*(V_{n,q};\mathbf{Z}_2);Sq^2)$ *is the exterior algebra generated by the elements in the table below.*

(n,q)	
$(2k,2l)$	$[e_{2(n-q)+1}],[e_{2(n-q)+3}e_{2(n-q)+5}],\ldots,[e_{2n-5}e_{2n-3}],[e_{2n-1}]$
$(2k+1,2l)$	$[e_{2(n-q)+1}e_{2(n-q)+3}],\ldots,[e_{2n-3}e_{2n-1}]$
$(2k,2l+1)$	$[e_{2(n-q)+1}e_{2(n-q)+3}],\ldots,[e_{2n-5}e_{2n-3}],[e_{2n-1}]$
$(2k+1,2l+1)$	$[e_{2(n-q)+1}],[e_{2(n-q)+3}e_{2(n-q)+5}],\ldots,[e_{2n-3}e_{2n-1}]$

Lemma 3.1. *The Atiyah-Hirzebruch spectral sequence $E_r(V_{n,q})$ collapses at the third term.*

Proof. By Corollary 2.1 we only have to show that all generators of $E_3^{*,-1}(V_{n,q}) \cong H^*(H^*(V_{n,q};\mathbf{Z}_2);Sq^2)$ in the table of Proposition 3.1 are the permanent cycles.

For $(n,k)=(2k+1,2l)$, $H^*(H^*(V_{n,q}\mathbf{Z}_2);Sq^2)$ has elements of degree $8i$ only. Then the elements in the table of Proposition 3.1 are the permanent cycles by Corollary 2.1 and the degree reason.

For $(n,k)=(2k+1,2l+1)$, $[e_{4i-1}e_{4i+1}]$ $(k-l+2 \le i \le k)$ are the permanent cycles by Corollary 2.1 and the degree reason as the above. Then $[e_{2(n-q)+1}]$ is also the permanent cycle by Corollary 2.1 and the degree reason.

For $(n,k)=(2k,2l+1)$, it is similar to the case of $(n,k)=(2k+1,2l+1)$.

For $(n,k)=(2k,2l)$ consider the homomorphism $E_r(V_{n+1,q+1}) \to E_r(V_{n,q})$ induced from the inclusion $V_{n,q} \to V_{n+1,q+1}$. Then we see that $d_r[e_{2(n-q)+1}]=0$ and $d_r[e_{4i-1}e_{4i+1}]=0$ $(k-l+2 \le i \le k-1)$ for $r \ge 3$. Then we also see that $d_r[e_{2n-1}]=0$ by Corollary 2.1 and the degree reason. □

4. Proof of Theorem

It is easily seen that $K^n(X)$ is torsion free and concentrated in even (odd) dimension, if X is in $\mathcal{E}$ $(\mathcal{O})$. Consider the Bott sequence

$$\cdots \to K^n(X) \to KO^{n+2}(X) \to KO^{n+1}(X) \xrightarrow{\mathbf{c}} K^{n+1}(X) \to \cdots,$$

where $\mathbf{c}: KO^i(X) \to K^i(X)$ is the complexification map. Since $\mathbf{rc}=2$ we have the following, where $\mathbf{r}: K^i(X) \to KO^i(X)$ is the realization map ([3]).

Proposition 4.1. *If X is in $\mathcal{E}$, we have*

$$\begin{aligned} KO^{2i+1}(X) &\cong s\mathbf{Z}_2, \\ KO^{2i}(X) &\cong r\mathbf{Z} \oplus s\mathbf{Z}_2. \end{aligned}$$

If X is in $\mathcal{O}$, we have

$$\begin{aligned} KO^{2i}(X) &\cong s\mathbf{Z}_2, \\ KO^{2i-1}(X) &\cong r\mathbf{Z} \oplus s\mathbf{Z}_2. \end{aligned}$$

Proof of Theorem. By Proposition 4.1 we have

$$\bigoplus_{p+q=2n-1} E_\infty^{p,q}(X) \cong \bigoplus_i E_\infty^{2n+8i,-1} \cong KO^{2n-1}(X) \text{ for } X \text{ in } \mathcal{E},$$

$$\bigoplus_{p+q=2n} E_\infty^{p,q}(X) \cong \bigoplus_i E_\infty^{2n+8i+1,-1} \cong KO^{2n}(X) \text{ for } X \text{ in } \mathcal{O}.$$

Note that the Thom space $G_{q,k}^{E_k}$ in the stable splitting $(*)$ is either in $\mathcal{E}$ or $\mathcal{O}$, then we obtain that

$$KO^i(V_{n,q}) \cong r\mathbf{Z} \oplus s\mathbf{Z}_2$$

for (r,s) below by the argument above, where $s_k = \sum_i \mathrm{rank} H^{4i+k}(V_{n,q};\mathbf{Z}), t_k = \sum_i \dim_{\mathbf{Z}_2} E_\infty^{8i+k+1,-1}(V_{n,q})$.

i	0 / -4	-1 / -5	-2 / -6	-3 / -7
(r,s)	$(s_0, t_{-7}+t_0)$ / $(s_0, t_{-3}+t_{-4})$	(s_{-1}, t_0+t_{-1}) / $(s_{-1}, t_{-4}+t_{-5})$	$(s_{-2}, t_{-1}+t_{-2})$ / $(s_{-2}, t_{-5}+t_{-6})$	$(s_{-3}, t_{-2}+t_{-3})$ / $(s_{-3}, t_{-6}+t_{-7})$

Let $P(t)$ and $Q(t)$ be the polynomials

$$\sum_i \mathrm{rank} H^*(V_{n,q};\mathbf{Z})t^i$$

and

$$\sum_i \dim_{\mathbf{Z}_2} E_\infty^{i,-1}(V_{n,q})t^i.$$

Then we have the following by Proposition 3.1 and Lemma 3.1.

$$P(t) = \prod_{i=n-q+1}^{n} (1+t^{2i-1}).$$

(n,q)	$Q(t)$
$(2k,2l)$	$(1+t^{2(n-q)+1})(1+t^{2n-1})\prod_{i=n-q+2}^{n-2}(1+t^{4i})$
$(2k+1,2l)$	$\prod_{i=n-q+1}^{n-1}(1+t^{4i})$
$(2k,2l+1)$	$(1+t^{2n-1})\prod_{i=n-q+1}^{n-2}(1+t^{4i})$
$(2k+1,2l+1)$	$(1+t^{2(n-q)+1})\prod_{i=n-q+2}^{n-1}(1+t^{4i})$

By setting $t = \pm 1, \sqrt{-1}$ for $P(t)$ and $t = \pm 1, \sqrt{-1}, e^{\sqrt{-1}\pi/4}$ for $Q(t)$, we obtain s_i and t_j. Therefore the proof is completed. $\square$

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