J. Math. Kyoto Univ. (JMKYAZ)
46-1 (2006), 187–229

On the construction problem for uni-instantaneous bilateral birth-death processes

By

Qing-Ping Liu

Abstract

Let $E = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Given a uni-instantaneous bilateral birth-death pre-generator matrix $Q = (q_{ij})$ defined on $E \times E$, we investigate the existence conditions for the corresponding birth-death Q-processes, and furthermore, construct all the processes when the existence conditions are satisfied. In addition, the uniqueness of the honest Q-processes are investigated as well.

1. Introduction

In this paper, we consider the construction problem for *uni-instantaneous bilateral birth-death processes*, which is one specific kind of *denumerable continuous-time homogeneous Markov processes*. A denumerable continuous-time homogeneous Markov process, which, throughout this paper, will be simply called a (*Markov*) *process*, is a time-homogeneous Markov process with a continuous-time parameter and a countable state space. The time interval is usually taken as $[0, \infty)$ and the countable state space is denoted by E.

A matrix $Q = (q_{ij};\, i, j \in E)$, where q_{ij} are real numbers, is called a *pre-generator* if $\forall\, i, j \in E$, $i \neq j$, $-\infty \leq q_{ii} \leq 0 \leq q_{ij} < \infty$, and $\sum_{j \neq i} q_{ij} \leq -q_{ii}$. Furthermore, if there exists a Markov process such that its transition function $P(t) = (p_{ij}(t);\, i, j \in E), t \geq 0$ satisfies

$$Q = P'(0) = \lim_{t \downarrow 0} \frac{P(t) - I}{t} \tag{1.1}$$

(where I is the unit matrix and the derivative $P'(0)$ is taken in componentwise), then Q is called a *generator*, or more precisely, the (*infinitesimal*) *generator* of the corresponding Markov process, while the process itself is called a *Q-process* to emphasize the relation (1.1) between $P(t)$ and Q. Following Reuter [22],

Received November 17, 2005
Revised January 10, 2006

the transition function $P(t)$ (and the *resolvent*—the Laplace transform $\psi(\lambda)$ of $P(t)$, as well), will be also called a Markov process (Q-process). When $P(t)$ satisfies $P(t)\mathbf{1} = \mathbf{1}, \forall\, t \geq 0$ (or equivalently, $\psi(\lambda)$ satisfies $\lambda\psi(\lambda)\mathbf{1} = \mathbf{1}, \forall\, \lambda \geq 0$), the process is called an *honest process.* Here and elsewhere, the bold $\mathbf{1}$ represents the column vector whose components are 1, while the bold $\mathbf{0}$ will represent the zero matrix or sometimes the zero column (row) vector.

For a pre-generator $Q = (q_{ij};\, i, j \in E)$, denote $q_i \equiv -q_{ii} \leq \infty$. We call a state $i \in E$ *stable* if $q_i < \infty$ and, *instantaneous* if $q_i = \infty$. If all the states are stable (instantaneous), then Q is said to be *totally stable* (*totally instantaneous*). A stable state i is called *conservative* (*non-conservative*) if

$$\sum_{j\neq i} q_{ij} = q_i \qquad \text{(correspondingly, } \sum_{j\neq i} q_{ij} < q_i).$$

A totally stable pre-generator Q is said to be conservative if all its states are conservative. A non-totally stable pre-generator is said to be *almost conservative* if all its stable states are conservative.

For a non-totally stable pre-generator $Q = (q_{ij})$, if $\sum_{j\neq i} q_{ij} < \infty$, for each instantaneous state i, then Q is called a *summable pre-generator.*

It is obvious that a generator must be a pre-generator, but the converse is not always true. Therefore the following three basic and important questions arise.

• **Existence** For a given pre-generator Q, under what conditions does it become a generator, i.e. there exists a Markov process $P(t)$ satisfying (1.1)?

• **Uniqueness** If the corresponding Markov process exists, is it unique?

• **Construction** How do we construct all the processes when they exist?

These questions are of particular significance since in most cases we only know the infinitesimal behavior—the pre-generator.

In the totally stable case, when Q is a birth-death pre-generator:

$$Q = \begin{pmatrix} -(a_0+b_0) & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

or a bilateral birth-death pre-generator:

$$Q = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-1} & -(a_{-1}+b_{-1}) & b_{-1} & 0 & 0 & \cdots \\ \cdots & 0 & a_0 & -(a_1+b_0) & b_0 & 0 & \cdots \\ \cdots & 0 & 0 & a_1 & -(a_1+b_1) & b_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

the above three questions were perfectly solved by Feller [13], [15], Reuter [22], Hou [16], Wang and Yang [30], [31], [32], [39], [40]. In deed, the existence and uniqueness are nothing but the special cases of some more general results. Feller [13] showed that any totally stable pre-generator must be a generator and constructed a Q-process $F(t) = (f_{ij}(t))$, $t \geq 0$ for a given Q. He further showed $F(t)$ enjoys the minimal property in the sense that for any Q-process $P(t) = (p_{ij}(t))$, $p_{ij}(t) \geq f_{ij}(t)$, $i, j \in E$, $t \geq 0$. The process $F(t)$ satisfies both the Kolmogorov backward equation system

$$\frac{d\,f_{ij}(t)}{dt} = \sum_{k \in E} q_{ik}\, f_{kj}(t), \qquad \forall\, i, j \in E$$

and the Kolmogorov forward equation system

$$\frac{d\,f_{ij}(t)}{dt} = \sum_{k \in E} f_{ik}(t)\, q_{kj}, \qquad \forall\, i, j \in E.$$

This process $F(t)$ and its resolvent, denoted by $\phi(\lambda) = (\phi_{ij}(\lambda))$, are known as the *Feller minimal process*.

For the uniqueness, Reuter [22] gave, for a conservative totally stable generator Q, the uniqueness criterion, which states that the Q-process is unique if and only if the equation $(\lambda I - Q)u = \mathbf{0}$ has no bounded nontrivial solution for some, and therefore for all, $\lambda > 0$. While for a general non-conservative Q, Hou [16] showed that the Feller minimal process $\phi(\lambda)$ is the unique Q-process if and only if the following two conditions hold simultaneously: (1) For some (and therefore for all) $\lambda > 0$, there exists a number $c_\lambda > 0$ such that $\lambda\phi(\lambda)\mathbf{1} > c_\lambda \mathbf{1}$; (2) The equation $v(\lambda I - Q) = \mathbf{0}$, $\mathbf{0} \leq v \in l$ has no nontrivial solution for some, and therefore for all, $\lambda > 0$. Where l stands for the space of all absolutely summable vectors on E.

Concerning the construction of processes, in the birth and death case, Feller [15] obtained all the processes which satisfy simultaneously both the Kolmogorov backward and forward equation systems, Wang [30] derived all the honest processes, Wang and Yang [31], [32] and Yang [40] constructed all the birth and death processes; in the bilateral birth and death case, all the processes were constructed by Yang [39].

In the non-totally stable case, when Q is a uni-instantaneous birth-death pre-generator:

$$Q = \begin{pmatrix} -\infty & q_{01} & q_{02} & q_{03} & \cdots \\ a_1 & -(a_1 + b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2 + b_2) & b_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Tang [29], using Chen's resolvent decomposition theorem [1] (see also [2]), got the following results for the existence.

z is regular	z is exit	z is entrance or natural
Process exists iff $\alpha\phi(\lambda) \in l$	Process exists iff $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l$	Process does not exist

Where z is the boundary point of Q_N (Q_N is the pre-generator by eliminating in Q the first row and the first column); $\phi_N(\lambda)$ is the minimal Q_N-process;

$\alpha = (q_{01}, q_{02}, q_{03}, \cdots)$. Tang constructed all the processes when the existence conditions are satisfied. He also investigated the uniqueness for the honest processes.

In the double-infinite birth-death case, i.e. both infinitely many instantaneous and stable states exist, where the pre-generator Q has the following form

$$Q = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & q_{-2,-3} & -\infty & q_{-2,-1} & q_{-2,0} & q_{-2,1} & q_{-2,2} & \cdots \\ \cdots & q_{-1,-3} & q_{-1,-2} & -\infty & q_{-1,0} & q_{-1,1} & q_{-1,2} & \cdots \\ \cdots & 0 & 0 & a_0 & -(a_0+b_0) & b_0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & a_1 & -(a_1+b_1) & b_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Liu [19] showed by an approximating method that when the pre-generator is summable, it must be a generator.

In this paper, we study the construction problem for the uni-instantaneous bilateral birth-death processes, where the pre-generator is

$$Q = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-1} & -(a_{-1}+b_{-1}) & b_{-1} & 0 & 0 & \cdots \\ \cdots & q_{0,-2} & q_{0,-1} & -\infty & q_{01} & q_{02} & \cdots \\ \cdots & 0 & 0 & a_1 & -(a_1+b_1) & b_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We have the following conclusions for the existence.

z^1 \ z^2	Regular	Exit	Entrance or Natural
Regular	Process exists iff $\alpha\phi(\lambda) \in l_N$ (Theorem 3.4)	Process exists iff $\alpha\phi(\lambda) \in l_N$ (Theorem 3.6)	Process exists iff $\alpha\phi(\lambda) \in l_N$ (Theorem 3.3)
Exit	Process exists iff $\alpha\phi(\lambda) \in l_N$ (Theorem 3.6)	Process exists iff $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l_N$ (Theorem 3.5)	Process exists iff $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l_N$ (Theorem 3.3)
Entrance or Natural	Process exists iff $\alpha\phi(\lambda) \in l_N$ (Theorem 3.3)	Process exists iff $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l_N$ (Theorem 3.3)	Process does not exist (Theorem 3.1)

Where z^1, z^2 are two boundary points of Q_N (see Definition 2.2 below), Q_N is the generator in (2.2) below, $\phi(\lambda)$ is the minimal Q_N process, and $\alpha = (\cdots, q_{0,-2}, q_{0,-1}, q_{01}, q_{02}, \cdots)$.

When the existence conditions are satisfied, all the corresponding Q-processes are constructed in Section 3. In addition, the uniqueness for the honest Q-processes is considered there as well.

We note that the conclusions for the uni-instantaneous bilateral case can not be directly derived from the known results for the unilateral cases. In fact, no matter the bilateral pre-generator Q is decomposed into two uni-instantaneous unilateral ones Q_1 and Q_2, where

$$Q_1 = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-2} & -(a_{-2}+b_{-2}) & b_{-2} & 0 \\ \cdots & 0 & a_{-1} & -(a_{-1}+b_{-1}) & b_{-1} \\ \cdots & q_{0,-3} & q_{0,-2} & q_{0,-1} & -\infty \end{pmatrix} \quad \text{and}$$

$$Q_2 = \begin{pmatrix} -\infty & q_{01} & q_{02} & q_{03} & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

or is decomposed into two unilateral ones with one being totally stable and one uni-instantaneous, it is difficult to combine two unilateral processes, both having infinitely many states, into a bilateral process, since we do not know how the process from the states in one part transits to the states in the other part. Moreover, it seems no hope to construct all the bilateral processes by combining two unilateral ones, as well. For example, as the existence is concerned, if one applies Tang's results for the former decomposition, then when one of the two boundary points, say z^1, is entrance or natural, the other one z^2 is regular, it is seen that the Q_1-process does not exist, while when $\alpha_2\phi_2(\lambda)\mathbf{1} < \infty$, the Q_2-process exists ($\alpha_2 = (q_{01}, q_{02}, q_{03}, \cdots)$, $\phi_2(\lambda)$ is the minimal Q_{2N}-process, where Q_{2N} is the pre-generator by eliminating in Q_2 the first row and the first column). Therefore one may possibly conclude incorrectly that the Q-process does not exist! However, from the last table above, we see that the Q-process does exist when $\alpha\phi(\lambda)\mathbf{1} < \infty$. So we shall not divide the bilateral pre-generator into two unilateral ones. Instead, we shall adopt the method that starts from the totally stable Q_N-processes (where Q_N is the totally stable part of Q), then extend them into the uni-instantaneous Q-processes.

In our argument, Chen's resolvent decomposition theorem plays a key role. To state Chen's theorem, let $Q_E = (q_{ij};\, i,j \in E)$ be a pre-generator defined on $E \times E$. Let $b \in E$ be a single state and $N = E \setminus \{b\}$. Let $Q_N = (q_{ij};\, i,j \in N)$ denote the restriction of Q_E on $N \times N$. Denote $\alpha = (q_{bj};\, j \in N)$ and $\beta = (q_{ib};\, i \in N)$. l_N represents the set of all absolutely summable column vectors on N.

Theorem 1.1 (Chen's resolvent decomposition theorem [1], [2])**.** *Suppose $R(\lambda) = (r_{ij}(\lambda);\, i,j \in E)$, $\lambda > 0$ is a Q_E-process defined on $E \times E$ where $E = N \cup \{b\}$ and the generator $Q = (q_{ij};\, i,j \in E) = \begin{pmatrix} -q_b & \alpha \\ \beta & Q_N \end{pmatrix}$, where $q_b = -q_{bb}$. Then $R(\lambda)$ can be uniquely decomposed into*

$$R(\lambda) = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \psi(\lambda) \end{pmatrix} + r(\lambda) \begin{pmatrix} 1 \\ \xi(\lambda) \end{pmatrix} \begin{pmatrix} 1, & \eta(\lambda) \end{pmatrix} \tag{1.2}$$

where $\psi(\lambda)$ *is a* Q_N*-process,* $\eta(\lambda)$ *and* $\xi(\lambda)$ *satisfy the following conditions* (1.3)–(1.7):

$$(1.3)\qquad \eta(\lambda)\in \mathbf{H}_\psi \equiv \{\eta(\lambda);\ \mathbf{0}\le \eta(\lambda)\in l_N,\quad \eta(\mu_1)-\eta(\mu_2)=(\mu_2-\mu_1)\eta(\mu_1)\psi(\mu_2),\ \forall\ \mu_1,\mu_2>0\}$$

$$(1.4)\qquad \xi(\lambda)\in \mathbf{K}_\psi \equiv \{\xi(\lambda);\ \mathbf{0}\le \xi(\lambda)\le \mathbf{1},\quad \xi(\mu_1)-\xi(\mu_2)=(\mu_2-\mu_1)\psi(\mu_1)\xi(\mu_2),\ \forall\ \mu_1,\mu_2>0\}$$

$$(1.5)\qquad \xi(\lambda)\le \mathbf{1}-\lambda\psi(\lambda)\mathbf{1}$$

$$(1.6)\qquad \lim_{\lambda\to\infty}\lambda\eta(\lambda)=\alpha \quad \textit{and} \quad \lim_{\lambda\to\infty}\lambda\xi(\lambda)=\beta$$

$$(1.7)\qquad \lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\mathbf{1}-\xi\rangle<\infty$$

where $\xi=\lim_{\lambda\to 0}\xi(\lambda)$ *and* c *is a finite constant such that*

$$(1.8)\qquad c\ge \lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\mathbf{1}-\xi\rangle \quad \textit{and} \quad c+\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=q_b.$$

So, if $q_b=\infty$*, it follows that*

$$(1.9)\qquad \lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=\infty \quad \textit{or equivalently} \quad \lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\mathbf{1}\rangle=\infty.$$

$r(\lambda)$ *is determined by*

$$(1.10)\qquad r(\lambda)=\frac{1}{c+\lambda+\lambda\langle\eta(\lambda),\xi\rangle}.$$

If $R(\lambda)$ *is honest, then*

$$(1.11)\qquad \xi(\lambda)=\mathbf{1}-\lambda\psi(\lambda)\mathbf{1},\quad r(\lambda)=\frac{1}{\lambda+\lambda\langle\eta(\lambda),\mathbf{1}\rangle} \quad \textit{and} \quad c\equiv\lambda\langle\eta(\lambda),\mathbf{1}-\xi\rangle$$

In particular, $\lambda\langle\eta(\lambda),\mathbf{1}-\xi\rangle$ *is independent of* λ*.*

Conversely, If we have a Q_N*-process* $\psi(\lambda),\lambda>0$ *and a pair of vectors* $\eta(\lambda)$ *and* $\xi(\lambda)$ *which satisfy* (1.3)–(1.7) *and* $\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle<\infty$*, when* $q_b<\infty$ *or* $\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=\infty$*, when* $q_b=\infty$*, then choose a constant* c *satisfying* (1.8)*, define* $r(\lambda)$ *as* (1.10)*, and finally define* $R(\lambda)$ *as* (1.2)*. The* $R(\lambda)$ *thus constructed then must be a* Q_E*-process. Furthermore, if* $\xi(\lambda)$*,* $r(\lambda)$ *and* c *are chosen as* (1.11)*, then* $R(\lambda)$ *is an honest process.*

Note. (1) Here and elsewhere, $\langle\cdot\,,\,\cdot\rangle$ stands for the product of a row and a column vector, in order to emphasize that the result is scalar. More precisely, for a row vector $x=(x_i)$ and a column vector $y=(y_i)$, $\langle x\,,\,y\rangle\equiv x\,y\equiv\sum_i x_i\,y_i$.

(2) The row vector $\eta(\lambda)$ and the column vector $\xi(\lambda)$ in the above theorem are actually vector families parameterized by $\lambda>0$. However, for simplicity, we do not write '$\lambda>0$' wherever the vector families appear, unless it is required.

We call the process $\psi(\lambda)$ in (1.2) the *projection* (or, the *restriction*) *of* $R(\lambda)$ on $N \times N$, denoting as $\psi(\lambda) = {}^bR(\lambda)$, and call $R(\lambda)$ an expansion process of $\psi(\lambda)$. Note that the expansion process is usually not unique. We shall denote $G_\psi(\lambda) = \{R(\lambda); R(\lambda)$ is an expansion process of $\psi(\lambda)\}$, i.e. $G_\psi(\lambda)$ is the set of all those expansion processes of $\psi(\lambda)$.

The above theorem, especially the converse part, is particularly useful. It allows one to investigate existence/uniqueness conditions and construct processes for complicated cases via simple known results. Note that the recent works of Chen [1]–[3], Tang [29], Liu [19], [20] and Fei [11], [12] are all based on this theorem.

In the next section, we shall concentrate on preparing some lemmas, which will be used in the proofs of the main results. While the main results are given and proved in Section 3.

2. Some lemmas

Hereafter, let the state space $E = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Denote $N_1 = \{\cdots, -n, \cdots, -2, -1\}$, $N_2 = \{1, 2, \cdots, n, \cdots\}$ and $N = N_1 \cup N_2$.

Let $Q_E = (q_{ij}; i, j \in E)$ be a uni-instantaneous bilateral birth-death pre-generator, $\{0\}$ is the instantaneous state, i.e.

$$q_{ij} = \begin{cases} -\infty & \text{if } j = i = 0 \\ q_{0j} & \text{if } j \neq 0, i = 0 \\ a_i & \text{if } j = i-1, i \in N \\ b_i & \text{if } j = i+1, i \in N \\ -(a_i + b_i) & \text{if } j = i, i \in N \\ 0 & \text{if } j \neq i-1, i+1, i \in N \end{cases} \tag{2.1}$$

where $0 \le q_{0j} < \infty, 0 < a_i < \infty, 0 < b_i < \infty$ are all real numbers. Obviously, Q_E is an almost conservative pre-generator. Let Q_N, Q_{N_1} and Q_{N_2} be the restriction of Q_E on $N \times N$, $N_1 \times N_1$ and $N_2 \times N_2$, respectively. Q_N, Q_{N_1} and Q_{N_2} are all totally stable generators.

We call a Q_E-process $R(\lambda) = (r_{ij}(\lambda); i, j \in E)$ an almost B-type process if

$$\lambda r_{ij}(\lambda) = \sum_{k \in E} q_{ik} r_{kj}(\lambda) + \delta_{ij}, \quad \forall\, i \in N, j \in E.$$

Then we have the following

Lemma 2.1. *If Q_E is a generator, then all the Q_E-processes must be almost B-type.*

Proof. Since all the stable states of Q_E are conservative, the conclusion easily follows from Theorem 2.7.3 of [42]. □

Let $\phi^a(\lambda)$ be the Feller minimal Q_{N_a}-process, $a = 1, 2$ and $\phi(\lambda)$ the Feller minimal Q_N-process. Then by a simple argumentation, we can easily get

$$\phi(\lambda) = \begin{pmatrix} \phi^1(\lambda) & \mathbf{0} \\ \mathbf{0} & \phi^2(\lambda) \end{pmatrix}.$$

Since $Q_{N_1} = (q_{ij};\, i,j \in N_1)$ and $Q_{N_2} = (q_{ij};\, i,j \in N_2)$ are two totally stable ordinary birth-death generators, we can define their standard measures, natural scales and boundary points as usual.

Definition 2.2. (1) The standard measure of Q_{N_1} is defined as:

$$\mu_{-1} = 1,\ \mu_{-2} = \frac{a_{-1}}{b_{-2}},\ \cdots,\ \mu_{-n} = \frac{a_{-1}a_{-2}\cdots a_{-n+1}}{b_{-2}b_{-3}\cdots b_{-n}},\ \cdots$$

The natural scale of Q_{N_1} is defined as: $z_{-1} = \frac{1}{b_{-1}},\ z_{-2} = \frac{1}{b_{-1}} + \frac{1}{a_{-1}},\ \cdots,$

$$z_{-n} = \frac{1}{b_{-1}} + \frac{1}{a_{-1}} + \cdots + \frac{b_{-2}b_{-3}\cdots b_{-n+1}}{a_{-1}a_{-2}a_{-3}\cdots a_{-n+1}},\ \cdots$$

And the boundary point of Q_{N_1} is defined as: $z^1 = \lim_{n\to\infty} z_{-n}$.
(2) Similarly, we can define the standard measure of Q_{N_2}:

$$\mu_1 = 1,\ \mu_2 = \frac{b_1}{a_2},\ \cdots,\ \mu_n = \frac{b_1 b_2\cdots b_{n-1}}{a_2 a_3\cdots a_n},\ \cdots$$

The natural scale of Q_{N_2}:

$$z_1 = \frac{1}{a_1},\ z_2 = \frac{1}{a_1} + \frac{1}{b_1},\ \cdots,\ z_n = \frac{1}{a_1} + \frac{1}{b_1} + \cdots + \frac{a_2 a_3\cdots a_{n-1}}{b_1 b_2 b_3\cdots b_{n-1}},\ \cdots$$

And the boundary point of Q_{N_2}: $z^2 = \lim_{n\to\infty} z_n$.
We say z^1 and z^2 are the two boundary points of

$$Q_N = \begin{pmatrix} Q_{N_1} & \mathbf{0} \\ \mathbf{0} & Q_{N_2} \end{pmatrix} \tag{2.2}$$

Definition 2.3. The boundary point $z^a,\ a \in \{1,2\}$, is said to be
(1) regular, if $z^a < \infty$ and $\sum_{n\in N_a} \mu_n < \infty$;
(2) exit, if z^a is not regular and $\sum_{n\in N_a} (z^a - z_n)\mu_n < \infty$;
(3) entrance, if z^a is not regular and $\sum_{n\in N_a} \mu_n < \infty$;
(4) natural, for all other cases.

Now let $u^1(\lambda) = (\cdots, u_{-n}(\lambda), \cdots, u_{-2}(\lambda), u_{-1}(\lambda))$ be the solution of the following equation

$$\begin{cases} (\lambda I - Q_{N_1})u = \mathbf{0} \\ u_{-1}(\lambda) = 1. \end{cases}$$

Then actually, we have

$$(2.3)\qquad \begin{cases} u_{-1}(\lambda)=1 \\ u_{-2}(\lambda)=1+b_{-1}(z_{-2}-z_{-1})+\lambda(z_{-2}-z_{-1})u_{-1}(\lambda)\mu_{-1} \\ \quad\cdots \\ u_{-n}(\lambda)=1+b_{-1}(z_{-2}-z_{-1})+\lambda\displaystyle\sum_{j=1}^{n-1}(z_{-n}-z_{-j})u_{-j}(\lambda)\mu_{-j} \\ \quad\cdots \end{cases}$$

Similarly, let $u^2(\lambda)=(u_1(\lambda),u_2(\lambda),\cdots,u_n(\lambda),\cdots)$ be the solution of the following equation

$$\begin{cases} (\lambda I-Q_{N_2})u=\mathbf{0} \\ u_1(\lambda)=1. \end{cases}$$

We have precisely

$$(2.4)\qquad \begin{cases} u_1(\lambda)=1 \\ u_2(\lambda)=1+a_1(z_2-z_1)+\lambda(z_2-z_1)u_1(\lambda)\mu_1 \\ \quad\cdots \\ u_n(\lambda)=1+a_1(z_2-z_1)+\lambda\displaystyle\sum_{j=1}^{n-1}(z_n-z_j)u_j(\lambda)\mu_j \\ \quad\cdots \end{cases}$$

Let $u(z^1,\lambda)=\lim_{n\to\infty}u_{-n}(\lambda)$, $u(z^2,\lambda)=\lim_{n\to\infty}u_n(\lambda)$. Set $X^a(\lambda)=\dfrac{u^a(\lambda)}{u(z^a,\lambda)}$, $a=1,\,2$. Then $X^a(\lambda)$ is the maximal solution of the following equation

$$(2.5)\qquad \begin{cases} (\lambda I-Q_{N_a})u=\mathbf{0} \\ \mathbf{0}\le u\le \mathbf{1} \end{cases}$$

and satisfies

$$(2.6)\qquad X^a(\mu)=A_{\phi^a}(\lambda,\mu)X^a(\lambda),\quad \forall\ \lambda,\ \mu>0$$

where $A_{\phi^a}(\lambda,\mu)=I+(\lambda-\mu)\phi^a(\mu)$. Moreover, if for some λ, $X^a(\lambda)=\mathbf{0}$, then $X^a(\lambda)\equiv\mathbf{0}$.

Now, define two column vectors on N by

$$\bar X^1(\lambda)=\begin{pmatrix} X^1(\lambda) \\ \mathbf{0}\end{pmatrix},\qquad \bar X^2(\lambda)=\begin{pmatrix} \mathbf{0} \\ X^2(\lambda)\end{pmatrix}.$$

We have the following

Lemma 2.4. *Suppose $u(\lambda)$ is a solution of the following equation*

$$(2.7)\qquad \begin{cases} (\lambda I-Q_N)u=\mathbf{0} \\ \mathbf{0}\le u\in m_N, \end{cases}$$

where m_N is the space of all bounded column vectors on N. Then $u(\lambda)$ can be expressed as a linear combination of $\bar{X}^1(\lambda)$ and $\bar{X}^2(\lambda)$:

$$u(\lambda) = t_1(\lambda)\bar{X}^1(\lambda) + t_2(\lambda)\bar{X}^2(\lambda), \tag{2.8}$$

where $t_a(\lambda), a = 1, 2$ are two non-negative scalar functions of λ.

Moreover, when $u(\lambda)$ further satisfies

$$u(\mu) = A_\phi(\lambda, \mu)u(\lambda), \qquad \forall\ \lambda,\ \mu > 0 \tag{2.9}$$

where $A_\phi(\lambda, \mu) = I + (\lambda - \mu)\phi(\mu)$, then $t_a(\lambda), a = 1, 2$ in the above expression (2.8) can be selected to be independent of λ. More precisely, we have

$$t_a(\lambda) = \begin{cases} 0, & \text{if } \bar{X}^a(\lambda) = \mathbf{0} \\ t_a, & \text{if } \bar{X}^a(\lambda) \neq \mathbf{0}. \end{cases} \tag{2.10}$$

Proof. Let $u(\lambda) = \begin{pmatrix} u_{N_1}(\lambda) \\ u_{N_2}(\lambda) \end{pmatrix}$ is a solution of the equation (2.7), where $u_{N_a}(\lambda)$ is the restriction of $u(\lambda)$ on $N_a, a = 1,\ 2$. Since

$$\lambda I - Q_N = \begin{pmatrix} \lambda I - Q_{N_1} & \mathbf{0} \\ \mathbf{0} & \lambda I - Q_{N_2} \end{pmatrix}, \tag{2.11}$$

$u_{N_a}(\lambda)$ $(a \in \{1, 2\})$ must be a solution of

$$\begin{cases} (\lambda I - Q_{N_a})u = \mathbf{0} \\ \mathbf{0} \leq u \in m_{N_a}, \end{cases}$$

thus there exists some $t_a(\lambda) \geq 0$ such that

$$u_{N_a}(\lambda) = t_a(\lambda)X^a(\lambda). \tag{2.12}$$

Hence $u(\lambda) = \begin{pmatrix} u_{N_1}(\lambda) \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ u_{N_2}(\lambda) \end{pmatrix} = t_1(\lambda)\bar{X}^1(\lambda) + t_2(\lambda)\bar{X}^2(\lambda)$, and we get (2.8).

If further, $u(\lambda)$ satisfies (2.9), since (2.9) is equivalent to

$$u_{N_a}(\mu) = A_{\phi^a}(\lambda, \mu)u_{N_a}(\lambda), \qquad \forall\ \lambda,\ \mu > 0$$

for each $a \in \{1, 2\}$, so together with (2.6) and (2.12), we get

$$u_{N_a}(\mu) = t_a(\lambda)X^a(\mu).$$

Since $u_{N_a}(\mu) = t_a(\mu)X^a(\mu)$, we get $t_a(\lambda) = t_a(\mu)$ when $X^a(\mu) \neq \mathbf{0}$, i.e. $t_a(\lambda)$ is independent of λ. When $X^a(\lambda) = \mathbf{0}$, it is obvious that we can set $t_a(\lambda) = 0$. The proof is complete. ☐

Let

$$\eta^1(\lambda) = (\cdots, \eta_{-n}(\lambda), \cdots, \eta_{-2}(\lambda), \eta_{-1}(\lambda)),$$
$$\eta^2(\lambda) = (\eta_1(\lambda), \eta_2(\lambda), \cdots, \eta_n(\lambda), \cdots)$$

be two row vectors on N_1 and N_2, respectively. Their components are defined as the following

$$\eta_{-n}(\lambda) = \begin{cases} \dfrac{u_{-n}(\lambda)\mu_{-n}}{u(z^1,\lambda)}, & \text{if } z^1 \text{ is regular} \\ \dfrac{b_{-1}u_{-n}(\lambda)\mu_{-n}}{u^+(z^1,\lambda))}, & \text{if } z^1 \text{ is entrance} \\ 0, & \text{if } z^1 \text{ is exit or natural} \end{cases}$$

$$\eta_{n}(\lambda) = \begin{cases} \dfrac{u_{n}(\lambda)\mu_{n}}{u(z^2,\lambda)}, & \text{if } z^2 \text{ is regular} \\ \dfrac{a_{1}u_{n}(\lambda)\mu_{n}}{u^+(z^2,\lambda)}, & \text{if } z^2 \text{ is entrance} \\ 0, & \text{if } z^2 \text{ is exit or natural} \end{cases}$$

where

$$u^+(z^1,\lambda) = \lim_{n\to\infty} u^+_{-n}(\lambda) \qquad u^+_{-n}(\lambda) = \frac{u_{-n-1}(\lambda) - u_{-n}(\lambda)}{z_{-n-1} - z_{-n}} \tag{2.13}$$

$$u^+(z^2,\lambda) = \lim_{n\to\infty} u^+_{n}(\lambda) \qquad u^+_{n}(\lambda) = \frac{u_{n+1}(\lambda) - u_{n}(\lambda)}{z_{n+1} - z_{n}}. \tag{2.14}$$

Each $\eta^a(\lambda)$ $(a = 1, 2)$ satisfies the equation

$$\begin{cases} v(\lambda I - Q_{N_a}) = \mathbf{0} \\ \mathbf{0} \le v \in l_{N_a} \end{cases} \tag{2.15}$$

and has the property

$$\eta^a(\mu) = \eta^a(\lambda)A_{\phi^a}(\lambda,\mu), \qquad \forall\ \lambda,\ \mu > 0\,. \tag{2.16}$$

Moreover, if for some λ, $\eta^a(\lambda) = \mathbf{0}$, then $\eta^a(\lambda) \equiv \mathbf{0}$.

Now define two row vectors on N by

$$\bar\eta^1(\lambda) = (\eta^1(\lambda)\,,\,\mathbf{0}), \quad \bar\eta^2(\lambda) = (\mathbf{0}\,,\,\eta^2(\lambda)).$$

Then we have

Lemma 2.5. *Each solution $v(\lambda)$ of the following equation*

$$\begin{cases} v(\lambda I - Q_{N}) = \mathbf{0} \\ \mathbf{0} \le v \in l_{N} \end{cases}$$

can be expressed as a linear combination of $\bar\eta^1(\lambda)$ and $\bar\eta^2(\lambda)$:

$$v(\lambda) = d_1(\lambda)\bar\eta^1(\lambda) + d_2(\lambda)\bar\eta^2(\lambda), \tag{2.17}$$

where $d_a(\lambda) \ge 0$, $a = 1, 2$ are two scalar functions of λ.

Moreover, when $v(\lambda)$ further satisfies

$$v(\mu) = v(\lambda)A_{\phi}(\lambda,\mu), \qquad \forall\ \lambda,\ \mu > 0$$

then $d_a(\lambda), a = 1, 2$ *in the above expression* (2.17) *can be selected to be independent of* λ. *More precisely, we have*

$$d_a(\lambda) = \begin{cases} 0, & if\ \bar{\eta}^a(\lambda) = \mathbf{0} \\ d_a, & if\ \bar{\eta}^a(\lambda) \neq \mathbf{0}. \end{cases}$$

Proof. Using equation (2.11) and noticing that each $\eta^a(\lambda)$ $(a = 1, 2)$ satisfies equation (2.15), the lemma can be proved in a similar way to Lemma 2.4. □

Remark 2.6. (1) Combining (2.3) with (2.13) and (2.4) with (2.14), it is easy to get (c.f. [42])

$$u^+_{-n}(\lambda) = b_{-1} + \lambda \sum_{j=1}^{n} u_{-j}(\lambda)\mu_{-j}, \qquad u^+_n(\lambda) = a_1 + \lambda \sum_{j=1}^{n} u_j(\lambda)\mu_j,$$

thus the four sequences $\{u^+_{-n}(\lambda)\}$, $\{u^+_n(\lambda)\}$, $\{u_{-n}(\lambda)\}$ and $\{u_n(\lambda)\}$ are all increasing in n and therefore their limits: $u^+(z^1, \lambda)$, $u^+(z^2, \lambda)$, $u(z^1, \lambda)$ and $u(z^2, \lambda)$ exist as n goes to ∞.
(2) $u(z^a, \lambda) < \infty$ $(a = 1, 2)$ iff z^a is regular or exit;
$u^+(z^a, \lambda) < \infty$ $(a = 1, 2)$ iff z^a is regular or entrance. So
$\bar{X}^a(\lambda) \neq \mathbf{0}$, iff z^a is regular or exit;
$\bar{\eta}^a(\lambda) \neq \mathbf{0}$, iff z^a is regular or entrance.

Denote

$$\mathcal{M}^+_\lambda(Q_N) \equiv \{u; \mathbf{0} \leq u \leq \mathbf{1},\ (\lambda I - Q_N)u = \mathbf{0}\} \tag{2.18}$$

$$\mathcal{L}^+_\lambda(Q_N) \equiv \{v;\ \mathbf{0} \leq v \in l_E,\ v(\lambda I - Q_N) = \mathbf{0}\} \tag{2.19}$$

where l_E denotes the space of all absolutely summable vectors on E. $\mathcal{M}^+_\lambda(Q_N)$ and $\mathcal{L}^+_\lambda(Q_N)$ are sometimes simply written as $\mathcal{M}^+_\lambda$ and $\mathcal{L}^+_\lambda$, respectively. It is well-known that both the dimensions of $\mathcal{M}^+_\lambda$ and $\mathcal{L}^+_\lambda$ are independent of λ (see Reuter [22], and also Yang [42]). We shall therefore use $m^+(Q_N)$ (or m^+) and $n^+(Q_N)$ (or n^+) to denote the dimensions of $\mathcal{M}^+_\lambda$ and $\mathcal{L}^+_\lambda$, respectively. When $m^+ = 0$, we say Q_N is *null exit*; When $m^+ = 1$, we say Q_N is *single exit* and when m^+ is finite, we say Q_N is *finite exit*. Similarly, when $n^+ = 0$, 1, we say Q_N is *null entrance* and *single entrance*, respectively, while when n^+ is finite, we say Q_N is *finite entrance*.

Recall the definition of $\mathbf{H}_\psi$ and $\mathbf{K}_\psi$ in (1.3) and (1.4). We have

Lemma 2.7. $\bar{X}^a(\lambda) \in \mathcal{M}^+_\lambda(Q_N) \cap \mathbf{K}_\phi$ *and* $\bar{\eta}^a(\lambda) \in \mathcal{L}^+_\lambda(Q_N) \cap \mathbf{H}_\phi$, *for* $a = 1, 2$.

Proof. Since $X^a(\lambda)$ satisfies (2.5) and (2.6), $\eta^a(\lambda)$ satisfies (2.15) and (2.16), i.e. $X^a(\lambda) \in \mathcal{M}^+_\lambda(Q_{N_a}) \cap \mathbf{K}_{\phi^a}$ and $\eta^a(\lambda) \in \mathcal{L}^+_\lambda(Q_{N_a}) \cap \mathbf{H}_{\phi^a}$, it immediately follows that $\bar{X}^a(\lambda) \in \mathcal{M}^+_\lambda(Q_N) \cap \mathbf{K}_\phi$ and $\bar{\eta}^a(\lambda) \in \mathcal{L}^+_\lambda(Q_N) \cap \mathbf{H}_\phi$. □

Let $\bar{X}^1 = (\bar{X}^1_n;\, n \in N)$, $\bar{X}^2 = (\bar{X}^2_n;\, n \in N)$, $Y^1 = (Y^1_n;\, n \in N)$ and $Y^2 = (Y^2_n;\, n \in N)$ be four column vectors having the following components

$$\bar{X}^1_n = \begin{cases} \dfrac{b_{-1}(z_n - z_{-1}) + 1}{b_{-1}(z^1 - z_{-1}) + 1}, & n \in N_1 \\ 0, & n \in N_2 \end{cases} \qquad \bar{X}^2_n = \begin{cases} 0, & n \in N_1 \\ \dfrac{a_1(z_n - z_1) + 1}{a_1(z^1 - z_1) + 1}, & n \in N_2 \end{cases}$$

$$Y^1_n = \begin{cases} \dfrac{b_{-1}(z^1 - z_n)}{b_{-1}(z^1 - z_{-1}) + 1}, & n \in N_1 \\ 0, & n \in N_2 \end{cases} \qquad Y^2_n = \begin{cases} 0, & n \in N_1 \\ \dfrac{a_1(z^2 - z_n)}{a_1(z^1 - z_1) + 1}, & n \in N_2. \end{cases}$$

Obviously, we have

$$\bar{X}^1 + \bar{X}^2 + Y^1 + Y^2 = \mathbf{1}. \tag{2.20}$$

Next, set $Y^1(\lambda) = \phi_{\cdot -1}(\lambda) b_{-1}$, $Y^2(\lambda) = \phi_{\cdot 1}(\lambda) a_1$, that is

$$Y^1(\lambda) = \begin{pmatrix} \vdots \\ \phi_{-n,-1}(\lambda) b_{-1} \\ \vdots \\ \phi_{-1,-1}(\lambda) b_{-1} \\ \phi_{1,-1}(\lambda) b_{-1} \\ \vdots \\ \phi_{n,-1}(\lambda) b_{-1} \\ \vdots \end{pmatrix}, \qquad Y^2(\lambda) = \begin{pmatrix} \vdots \\ \phi_{-n,1}(\lambda) a_1 \\ \vdots \\ \phi_{-1,1}(\lambda) a_1 \\ \phi_{1,1}(\lambda) a_1 \\ \vdots \\ \phi_{n,1}(\lambda) a_1 \\ \vdots \end{pmatrix}.$$

Denote $\beta_a = (q_{i0};\, i \in N_a)$, $a = 1, 2$, $\beta = (q_{i0};\, i \in N) = \begin{pmatrix} \mathbf{0} \\ b_{-1} \\ a_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.

Then we have

$$Y^1(\lambda) + Y^2(\lambda) = \phi(\lambda)\beta. \tag{2.21}$$

Lemma 2.8. (1) *For each* $a \in \{1, 2\}$,

$$\begin{aligned} &\bar{X}^a(\lambda) \downarrow \mathbf{0} \qquad \lambda \bar{X}^a(\lambda) \to \mathbf{0} \qquad (\lambda \uparrow \infty) \\ &\lambda \phi(\lambda) \bar{X}^a = \bar{X}^a - \bar{X}^a(\lambda). \end{aligned}$$

While when z^a *is regular or exit, or equivalently,* $\bar{X}^a(\lambda) \neq \mathbf{0}$*, we have*

$$\lim_{\lambda \to 0} \bar{X}^a(\lambda) = \bar{X}^a;$$

and when z^a *is entrance or natural, or equivalently,* $\bar{X}^a(\lambda) = \mathbf{0}$*, we have*

$$\lambda \phi(\lambda) \bar{X}^a = \bar{X}^a.$$

(2) *As* $\lambda \uparrow \infty$,

$$\lambda Y_n^1(\lambda) \to \begin{cases} 0, & if\ n \in N\backslash\{-1\} \\ b_{-1}, & if\ n = -1 \end{cases}$$

$$\lambda Y_n^2(\lambda) \to \begin{cases} a_1, & if\ n = 1 \\ 0, & if\ n \in N\backslash\{1\} \end{cases}$$

and for each $a = 1, 2$, $Y^a(\lambda) \downarrow \mathbf{0}$. *Moreover, we have*

$$\lim_{\lambda \to 0} Y^a(\lambda) = Y^a \quad and \quad \lambda\phi(\lambda)Y^a = Y^a - Y^a(\lambda).$$

(3) *The following equality holds*:

$$\lambda\phi(\lambda)\mathbf{1} = \mathbf{1} - \phi(\lambda)\beta - \bar{X}^1(\lambda) - \bar{X}^2(\lambda). \tag{2.22}$$

Proof. (1) Since the restriction of $\bar{X}^a(\lambda)$ on N_a is $X^a(\lambda)$ and on $N\backslash N_a$ is $\mathbf{0}$, and the same assertions hold for $X^a(\lambda)$ by Lemma 6.5.1 and Lemma 6.5.2 of [42], so the conclusions easily follow.
(2) Similar to (1).
(3) Since by Theorem 2.10.5 of [42], $\lambda\phi^a(\lambda)\mathbf{1} = \mathbf{1} - \phi^a(\lambda)\beta_a - X^a(\lambda)$ for each $a \in \{1, 2\}$, (2.22) follows immediately. □

Lemma 2.9. *Suppose* $a, b \in \{1, 2\}$, $a \neq b$, *then*

$$\lambda\bar{\eta}^a(\lambda)\bar{X}^b = 0. \tag{2.23}$$

If z^a *is regular, then*

$$\lim_{\lambda \to \infty} \lambda\bar{\eta}^a(\lambda)\bar{X}^a = \infty. \tag{2.24}$$

Proof. (2.23) is trivial and (2.24) follows from Lemma 6.5.5 of [42]. □

Note that a Q_N-process $\psi(\lambda)$ is called a B-type process if it satisfies the Kolmogorov backward equation

$$(\lambda I - Q_N)\psi(\lambda) = I.$$

We have the following

Lemma 2.10. *If* $R(\lambda)$ *is a uni-instantaneous bilateral birth-death* Q_E-*process and* $\psi(\lambda)$ *is its projection on* $N \times N$, *where* $N = E\backslash\{0\}$, *then* $\psi(\lambda)$ *must be a* B-*type* Q_N-*process and have the following form*

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \bar{X}_i^1(\lambda)F_j^1(\lambda) + \bar{X}_i^2(\lambda)F_j^2(\lambda) \tag{2.25}$$

or in matrix form

$$\psi(\lambda) = \phi(\lambda) + \bar{X}^1(\lambda)F^1(\lambda) + \bar{X}^2(\lambda)F^2(\lambda) \tag{2.26}$$

where $F^a(\lambda) = (F_j^a(\lambda); j \in N) \geq \mathbf{0}$, $a = 1, 2$ *are two row vectors defined on* N *satisfying*

$$\lambda F^a(\lambda)\mathbf{1} \leq 1. \tag{2.27}$$

Proof. By Lemma 2.1, every Q_E-process $R(\lambda)$ must be an almost B-type process, then by Lemma 3.4 of Chen [3], the projection process $\psi(\lambda)$ of $R(\lambda)$ on $N \times N$ is a B-type process. Finally, by Chapter 8 of [42], we know $\psi(\lambda)$ has the stated form of (2.25) or (2.26) and $F^a(\lambda)$ satisfies (2.27). $\square$

Lemma 2.11. *Suppose $\psi(\lambda)$ is a B-type Q_N-process, $\eta(\lambda) \in \mathbf{H}_\psi$. Denote $\Gamma = \lim_{\lambda\to 0} \phi(\lambda)$. Then we have*
(1) $\sup_{\lambda>0} \lambda\eta(\lambda)\Gamma\beta < \infty$, *i.e.* $\sup_{\lambda>0} \lambda\eta(\lambda)(Y^1 + Y^2) < \infty$.
Consequently, for any non-negative row vector α_N such that $\alpha_N\phi(\lambda) \in l_N$, we have

$$\alpha_N(Y^1 + Y^2) \le \liminf_{\lambda\to\infty} \lambda\alpha_N\phi(\lambda)(Y^1 + Y^2) < \infty. \tag{2.28}$$

(2) *If X is a bounded non-negative column vector satisfying $\lambda\phi(\lambda)X = X, \forall\, \lambda > 0$, then $\lim_{\lambda\to\infty} \lambda\eta(\lambda)X < \infty$.*

Proof. (1) Let $\bar{Y}^1(\lambda) = \psi_{\cdot -1}(\lambda)b_{-1}$ and $\bar{Y}^2(\lambda) = \psi_{\cdot 1}(\lambda)a_1$. Denote $\bar{Y}^a = \lim_{\lambda\to 0} \bar{Y}^a(\lambda), a = 1, 2$. Then by doing the same work as in the proof of Lemma 2.11.4 of [42], we can get $\lim_{\lambda\to\infty} \lambda\eta(\lambda)\bar{Y}^a < \infty$, $a = 1, 2$. Since $\lambda\eta(\lambda)\bar{Y}^a$ is increasing in λ and $\bar{Y}^a(\lambda) \ge Y^a(\lambda)$ implies $\bar{Y}^a \ge Y^a$, we get $\sup_{\lambda>0} \lambda\eta(\lambda)Y^a \le \sup_{\lambda>0} \lambda\eta(\lambda)\bar{Y}^a = \lim_{\lambda\to\infty} \lambda\eta(\lambda)\bar{Y}^a < \infty$, which yields

$$\sup_{\lambda>0} \lambda\eta(\lambda)\Gamma\beta = \sup_{\lambda>0} \lambda\eta(\lambda)(Y^1 + Y^2) < \infty.$$

In particular, when $\alpha_N\phi(\lambda) \in l_N$, from that $\phi(\lambda)$ is a B-type Q_N-process and $\alpha_N\phi(\lambda) \in \mathbf{H}_\phi$, we immediately get (2.28).

(2) For $\lambda \ge \mu$,

$$\begin{aligned}
\mu\langle\eta(\mu), X\rangle &= \mu\langle\eta(\lambda) + (\lambda-\mu)\eta(\lambda)\psi(\mu), X\rangle = \mu\langle\eta(\lambda), X + (\lambda-\mu)\psi(\mu)X\rangle \\
&= \mu\langle\eta(\lambda), X\rangle + (\lambda-\mu)\langle\eta(\lambda), \mu\psi(\mu)X\rangle \\
&\ge \mu\langle\eta(\lambda), X\rangle + (\lambda-\mu)\langle\eta(\lambda), \mu\phi(\mu)X\rangle \\
&= \mu\langle\eta(\lambda), X\rangle + (\lambda-\mu)\langle\eta(\lambda), X\rangle = \lambda\langle\eta(\lambda), X\rangle,
\end{aligned}$$

so $\lambda\langle\eta(\lambda), X\rangle$ is decreasing in λ and therefore

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), X\rangle \le \mu\langle\eta(\mu), X\rangle < \infty.$$

$\square$

Lemma 2.12. *Suppose $\psi(\lambda)$ is a B-type Q_N-process of the form* (2.26). *$\eta(\lambda) \in \mathbf{H}_\psi$ satisfying $\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha \equiv (q_{i0};\ i \in N)$. If $F^a(\lambda) \ne \mathbf{0}$ $(a \in \{1, 2\})$, then*

$$\limsup_{\lambda\to\infty} \lambda\eta(\lambda)\bar{X}^a < \infty.$$

Proof. When $\bar{X}^a(\lambda) = \mathbf{0}$, the conclusion easily follows from Lemma 2.8 and Lemma 2.11 (2). Next, assume $\bar{X}^a(\lambda) \ne \mathbf{0}$ and $F^a(\lambda) \ne \mathbf{0}$. By (2.26), we have $\psi(\mu) - \phi(\mu) \geq \bar{X}^a(\mu)F^a(\mu)$. If $\lambda \geq \mu$, then

$$\begin{aligned}(\lambda-\mu)\eta(\lambda)\bar{X}^a(\mu)[1-\mu F^a(\mu)\bar{X}^a] &= (\lambda-\mu)\eta(\lambda)[\bar{X}^a(\mu)-\mu\bar{X}^a(\mu)F^a(\mu)\bar{X}^a]\\ &\geq (\lambda-\mu)\eta(\lambda)[\bar{X}^a(\mu)-\mu(\psi(\mu)-\phi(\mu))\bar{X}^a] = (\lambda-\mu)\eta(\lambda)[\bar{X}^a-\mu\psi(\mu)\bar{X}^a]\\ &= (\lambda-\mu)\eta(\lambda)\bar{X}^a-\mu(\lambda-\mu)\eta(\lambda)\psi(\mu)\bar{X}^a\\ &= (\lambda-\mu)\eta(\lambda)\bar{X}^a-\mu[\eta(\mu)-\eta(\lambda)]\bar{X}^a = \lambda\eta(\lambda)\bar{X}^a-\mu\eta(\mu)\bar{X}^a,\end{aligned}$$

if there exists some $\mu_0 > 0$ such that $\mu_0 F^a(\mu_0)\bar{X}^a = 1$, then from the above inequality we have

$$\lambda\eta(\lambda)\bar{X}^a \leq \mu_0\eta(\mu_0)\bar{X}^a < \infty, \quad \forall\,\lambda \geq \mu_0$$

and so

$$\limsup_{\lambda\to\infty}\lambda\eta(\lambda)\bar{X}^a = \lim_{\lambda\to\infty}\lambda\eta(\lambda)\bar{X}^a < \infty.$$

Otherwise, for all $\mu > 0$, $\mu F^a(\mu)\bar{X}^a < 1$. In this case, set $h(\mu) = \dfrac{F^a(\mu)}{1-\mu F^a(\mu)\bar{X}^a}$, then $h(\mu) \neq 0$ and for any $\lambda \geq \mu$,

$$\begin{aligned}&(\lambda-\mu)\eta(\lambda)\bar{X}^a(\mu)F^a(\mu) \geq [\lambda\eta(\lambda)\bar{X}^a-\mu\eta(\mu)\bar{X}^a]h(\mu),\\ &\eta(\mu) = \eta(\lambda)[I+(\lambda-\mu)\psi(\mu)]\\ &\qquad = \eta(\lambda)[I+(\lambda-\mu)\phi(\mu)+(\lambda-\mu)(\psi(\mu)-\phi(\mu))]\\ &\qquad \geq \eta(\lambda)[I+(\lambda-\mu)\phi(\mu)]+(\lambda-\mu)\eta(\lambda)\bar{X}^a(\mu)F^a(\mu)\\ &\qquad \geq \eta(\lambda)[I+(\lambda-\mu)\phi(\mu)]+\lambda\eta(\lambda)\bar{X}^a h(\mu)-\mu\eta(\mu)\bar{X}^a h(\mu),\end{aligned}$$

so we have $\eta(\lambda)[I+(\lambda-\mu)\phi(\mu)]+\lambda\eta(\lambda)\bar{X}^a h(\mu) \leq \eta(\mu)+\mu\eta(\mu)\bar{X}^a h(\mu)$, thus for any $\lambda \geq \mu$,

$$\begin{aligned}\lambda\eta(\lambda)\bar{X}^a h(\mu) &\leq \eta(\lambda)[I+(\lambda-\mu)\phi(\mu)]+\lambda\eta(\lambda)\bar{X}^a h(\mu)\\ &\leq \eta(\mu)+\mu\eta(\mu)\bar{X}^a h(\mu),\end{aligned}$$

which follows that

$$\limsup_{\lambda\to\infty}\lambda\eta(\lambda)\bar{X}^a h(\mu) \leq \eta(\mu)+\mu\eta(\mu)\bar{X}^a h(\mu) < \infty.$$

Since $h(\mu) \neq 0$, the above inequality implies that $\limsup_{\lambda\to\infty}\lambda\eta(\lambda)\bar{X}^a < \infty$, which completes the proof. □

Lemma 2.13. *Suppose $\psi(\lambda) = \phi(\lambda)+\bar{X}^1(\lambda)F^1(\lambda)+\bar{X}^2(\lambda)F^2(\lambda)$ is a Q_N-process. If both $F^1(\lambda) \neq \mathbf{0}$ and $F^2(\lambda) \neq \mathbf{0}$, then there exist no expansion Q_E-processes of $\psi(\lambda)$, i.e. $G_\psi(\lambda) = \varnothing$.*

Proof. If $G_\psi(\lambda) \neq \varnothing$, then by Chen's resolvent decomposition theorem, there must exist some $\eta(\lambda) \in \mathbf{H}_\psi$ and some $\xi(\lambda) \in \mathbf{K}_\psi$ such that (1.5)–(1.9) hold. However, by (2.20), Lemma 2.11 and Lemma 2.12,

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda\eta(\lambda)\mathbf{1} &= \lim_{\lambda\to\infty}\lambda\eta(\lambda)(\bar{X}^1+\bar{X}^2+Y^1+Y^2)\\ &\le \limsup_{\lambda\to\infty}\lambda\eta(\lambda)(\bar{X}^1+\bar{X}^2)+\sup_{\lambda>0}\lambda\eta(\lambda)\Gamma\beta<\infty,\end{aligned}$$

which contradicts to (1.9). This completes the proof. □

Lemma 2.14. *Suppose $\psi(\lambda)=\phi(\lambda)+\bar{X}^a(\lambda)F^a(\lambda)$, $a\in\{1,2\}$ is a B-type Q_N-process with $\bar{X}^a(\lambda)\ne\mathbf{0}$. Then $F^a(\lambda)$ has the following properties*
(1) $F^a(\lambda)\ge\mathbf{0}$ and $\lambda F^a(\lambda)\mathbf{1}\le 1$.
(2) $F^a(\lambda)$ satisfies the following equation

$$F^a(\lambda)A_\phi(\lambda,\mu)=[1+(\mu-\lambda)\langle F^a(\lambda),\bar{X}^a(\mu)\rangle]F^a(\mu)\qquad \lambda,\,\mu>0 \tag{2.29}$$

or equivalently, $F^a(\lambda)=[1+(\mu-\lambda)\langle F^a(\lambda),\bar{X}^a(\mu)\rangle]F^a(\mu)A_\phi(\mu,\lambda)$, in which $m_{\lambda\mu}\equiv[1+(\mu-\lambda)\langle F^a(\lambda),\bar{X}^a(\mu)\rangle]\ge 1\wedge\dfrac{\mu}{\lambda}>0$ and $F^a(\mu)A_\phi(\mu,\lambda)=m_{\lambda\mu}^{-1}F^a(\lambda)\ge\mathbf{0}$.
(3) Moreover, if $F^a(\lambda)\ne\mathbf{0}$, then it can be expressed as

$$F^a(\lambda)=\frac{{}^a\eta(\lambda)}{c_1+\lambda\langle{}^a\eta(\lambda),\bar{X}^a\rangle} \tag{2.30}$$

for some

$${}^a\eta(\lambda)=\alpha^a\phi(\lambda)+{}^a\bar{\eta}(\lambda)\ne\mathbf{0} \tag{2.31}$$

and some constant

$$c_1\ge\sup_{\lambda>0}\lambda\langle{}^a\eta(\lambda),\mathbf{1}-\bar{X}^a\rangle, \tag{2.32}$$

where ${}^a\bar{\eta}(\lambda)\in\mathcal{L}_\lambda^+\cap\mathbf{H}_\phi$ can be further expressed as

$${}^a\bar{\eta}(\lambda)=d_a\bar{\eta}^a(\lambda)+d_b\bar{\eta}^b(\lambda) \tag{2.33}$$

for some constants d_a, $d_b\ge 0$ ($d_b=0$ when $\lim_{\lambda\to\infty}\lambda\bar{\eta}^b(\lambda)\bar{X}^b=\infty$ or $\bar{\eta}^b(\lambda)=\mathbf{0}$), α^a is a non-negative row vector such that

$$\alpha^a\phi(\lambda)\in l_N\qquad\text{and}\qquad \alpha^a\bar{X}^b<\infty. \tag{2.34}$$

In fact, ${}^a\eta(\lambda)$ and c_1 can be taken as

$${}^a\eta(\lambda)=F^a(\mu_0)A_\phi(\mu_0,\lambda),\qquad c_1=1-\mu_0\langle{}^a\eta(\mu_0),\bar{X}^a\rangle \tag{2.35}$$

for any fixed $\mu_0>0$.

Proof. All the conclusions except (2.33) and $\alpha^a \bar{X}^b < \infty$ in (2.34) come from Theorem 3.2.1 of [42], we only give the proof for (2.33) and $\alpha^a \bar{X}^b < \infty$.

By Lemma 2.5, ${}^a\bar{\eta}(\lambda)$ allows an expression as

$$ {}^a\bar{\eta}(\lambda) = d_a \bar{\eta}^a(\lambda) + d_b \bar{\eta}^b(\lambda), $$

for some constants d_a, $d_b \geq 0$ ($d_b = 0$ when $\bar{\eta}^b(\lambda) = \mathbf{0}$), hence

$$ \begin{aligned} &\lambda\langle {}^a\eta(\lambda), \mathbf{1} - \bar{X}^a\rangle = \lambda\langle {}^a\eta(\lambda), Y^1 + Y^2 + \bar{X}^b\rangle \\ &= \lambda\langle {}^a\eta(\lambda), Y^1 + Y^2\rangle + \lambda\alpha^a\phi(\lambda)\bar{X}^b + \lambda(d_a\bar{\eta}^a(\lambda) + d_b\bar{\eta}^b(\lambda))\bar{X}^b \\ &= \lambda\langle {}^a\eta(\lambda), Y^1 + Y^2\rangle + \lambda\alpha^a\phi(\lambda)\bar{X}^b + d_b\lambda\bar{\eta}^b(\lambda)\bar{X}^b. \end{aligned} $$

While by Lemma 2.11, $\sup_{\lambda>0} \lambda\langle {}^a\eta(\lambda), Y^1 + Y^2\rangle < \infty$, so the above equality and (2.32) imply

$$ \alpha^a \bar{X}^b \leq \liminf_{\lambda\to\infty} \lambda\alpha^a\phi(\lambda)\bar{X}^b < \infty $$

and $d_b = 0$ when $\lim_{\lambda\to\infty} \lambda\bar{\eta}^b(\lambda)\bar{X}^b = \infty$. □

According to Lemma 2.13 and Lemma 2.14, in the following, we shall always assume

(**A**) $\psi(\lambda) = \phi(\lambda) + \bar{X}^a(\lambda)F^a(\lambda)$, $a \in \{1, 2\}$, where $F^a(\lambda)$ possesses the properties (1)–(3) in Lemma 2.14.

Lemma 2.15. *Assume the assumption* (**A**) *with* $\bar{X}^a(\lambda) \neq \mathbf{0}$ *and* $F^a(\lambda) \neq \mathbf{0}$. *Then there exists an* $\eta(\lambda) \in \mathbf{H}_\psi$ *such that*

$$ \lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha \equiv (q_{i0};\ i \in N) \tag{2.36} $$

if and only if $\alpha\bar{X}^a < \infty$ *and* $\alpha\phi(\lambda) \in l_N$.

When $\alpha\bar{X}^a < \infty$ *and* $\alpha\phi(\lambda) \in l_N$ *are satisfied, take a row vector* ${}^a\eta(\lambda) \in \mathbf{H}_\phi$ *such that* (2.30)–(2.34) *hold, then take a constant* $A \geq 0$, *set* $\alpha' = \alpha + A\alpha^a$, *take a row vector* $\bar{\eta}'(\lambda) \in \mathcal{L}^+_\lambda \cap \mathbf{H}_\phi$ *such that* $\bar{\eta}'(\lambda) = d'_a\bar{\eta}^a(\lambda) + d'_b\bar{\eta}^b(\lambda)$ *for some constants* d'_a, $d'_b \geq 0$ *and such that*

$$ \bar{\eta}'(\lambda) \geq A \cdot {}^a\bar{\eta}(\lambda) \tag{2.37} $$

$$ \lim_{\lambda\to\infty} \lambda(d'_a - Ad_a)\bar{\eta}^a(\lambda)\bar{X}^a < \infty, \tag{2.38} $$

furthermore, take a constant c_2 *satisfying*

$$ \alpha\bar{X}^a + \lim_{\lambda\to\infty} \lambda(d'_a - Ad_a)\bar{\eta}^a(\lambda)\bar{X}^a \leq Ac_1 + c_2 \tag{2.39} $$

($\alpha\bar{X}^a + \lim_{\lambda\to\infty} \lambda(d'_a - Ad_a)\bar{\eta}^a(\lambda)\bar{X}^a = Ac_1 + c_2$ *when* $\alpha^a\mathbf{1} + \lim_{\lambda\to\infty} \lambda d_a\bar{\eta}^a(\lambda)\bar{X}^a < \infty$), *let* $\eta'(\lambda) = \alpha'\phi(\lambda) + \bar{\eta}'(\lambda)$ *and finally let*

$$ \eta(\lambda) = \eta'(\lambda) - \frac{\lambda\langle\eta'(\lambda), \bar{X}^a\rangle - c_2}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle}\, {}^a\eta(\lambda). \tag{2.40} $$

Then this $\eta(\lambda)$ *belongs to* $\mathbf{H}_\psi$ *and satisfies* (2.36). *Moreover, any vector belonging to* $\mathbf{H}_\psi$ *and satisfying* (2.36) *can be obtained in the above way.*

Remark 2.16. When the boundary point z^a is regular, by Lemma 2.9, $\lim_{\lambda\to\infty} \lambda\bar{\eta}^a(\lambda)\bar{X}^a = \infty$, so (2.38) is equivalent to $d'_a = Ad_a$, and (2.39) has a simpler form $\alpha\bar{X}^a \le Ac_1 + c_2$, where the equality $\alpha\bar{X}^a = Ac_1 + c_2$ holds when $\alpha^a\mathbf{1} < \infty$ and $d_a = 0$.

Proof of Lemma 2.15. (I) Necessity. Suppose that there exists an $\eta(\lambda) \in \mathbf{H}_\psi$ satisfying (2.36). By Lemma 2.12, we have $\lim_{\lambda\to\infty} \lambda\eta(\lambda)\bar{X}^a < \infty$, so by Fatou's lemma, we immediately get

$$\alpha\bar{X}^a \le \lim_{\lambda\to\infty} \lambda\eta(\lambda)\bar{X}^a < \infty.$$

Furthermore, $\eta(\lambda) \in \mathbf{H}_\psi$ implies, for any $\lambda \ge \mu > 0$,

$$(\lambda-\mu)\eta(\lambda)\phi(\mu)\mathbf{1} \le (\lambda-\mu)\eta(\lambda)\psi(\mu)\mathbf{1} + \eta(\lambda)\mathbf{1} = \eta(\mu)\mathbf{1},$$

so we have

$$\frac{\lambda-\mu}{\lambda}\lambda\eta(\lambda)\phi(\mu)\mathbf{1} \le \eta(\mu)\mathbf{1},$$

hence Fatou's lemma and (2.36) yield $\alpha\phi(\mu)\mathbf{1} \le \eta(\mu)\mathbf{1} < \infty$, i.e. $\alpha\phi(\mu) \in l_N$.

Next we show $\eta(\lambda)$ possesses the form of (2.40). Let ${}^a\eta(\lambda)$ and c_1 be taken, for some fixed $\mu_0 > 0$, as (2.35), which satisfy (2.30)–(2.34). Since

$$\begin{aligned}\eta(\mu_0) &= \eta(\lambda)[I + (\lambda-\mu_0)\psi(\mu_0)]\\ &= \eta(\lambda)[I + (\lambda-\mu_0)\phi(\mu_0)] + (\lambda-\mu_0)\eta(\lambda)[\psi(\mu_0) - \phi(\mu_0)]\\ &= \eta(\lambda)A_\phi(\lambda,\mu_0) + (\lambda-\mu_0)\eta(\lambda)\bar{X}^a(\mu_0)F^a(\mu_0),\end{aligned}$$

where $A_\phi(\lambda,\mu_0) = I + (\lambda-\mu_0)\phi(\mu_0)$. Multiplying $A_\phi(\mu_0,\lambda)$ in the two sides of the above equality and using the property $A_\phi(\lambda,\mu_0)A_\phi(\mu_0,\lambda) = I$, we get

$$\eta(\mu_0)A_\phi(\mu_0,\lambda) = \eta(\lambda) + (\lambda-\mu_0)\eta(\lambda)\bar{X}^a(\mu_0)F^a(\mu_0)A_\phi(\mu_0,\lambda),$$

thus

$$\eta(\lambda) = \eta(\mu_0)A_\phi(\mu_0,\lambda) - (\lambda-\mu_0)\eta(\lambda)\bar{X}^a(\mu_0)F^a(\mu_0)A_\phi(\mu_0,\lambda),$$

that is

$$\eta(\lambda) = \eta'(\lambda) - (\lambda-\mu_0)\eta(\lambda)\bar{X}^a(\mu_0)\cdot{}^a\eta(\lambda), \tag{2.41}$$

where $\eta'(\lambda) = \eta(\mu_0)A_\phi(\mu_0,\lambda) \in \mathbf{H}_\phi$. Denote $d_\lambda = (\lambda-\mu_0)\eta(\lambda)\bar{X}^a(\mu_0)$, and multiplying by $(\lambda-\mu_0)\bar{X}^a(\mu_0)$ in (2.41), we then have

$$\begin{aligned}d_\lambda &= \frac{(\lambda-\mu_0)\eta'(\lambda)\bar{X}^a(\mu_0)}{1 + (\lambda-\mu_0)\langle{}^a\eta(\lambda), \bar{X}^a(\mu_0)\rangle}\\ &= \frac{\lambda\eta'(\lambda)\bar{X}^a - \mu_0\eta'(\mu_0)\bar{X}^a}{1 - \mu_0\langle{}^a\eta(\mu_0), \bar{X}^a\rangle + \lambda\langle{}^a\eta(\lambda), \bar{X}^a\rangle} = \frac{\lambda\eta'(\lambda)\bar{X}^a - c_2}{c_1 + \lambda\langle{}^a\eta(\lambda), \bar{X}^a\rangle},\end{aligned} \tag{2.42}$$

where $c_2 \equiv \mu_0\eta'(\mu_0)\bar{X}^a \leq \mu\eta'(\mu_0)\mathbf{1} < \infty$ (note the denominator $1 + (\lambda - \mu_0)\langle {}^a\eta(\lambda), \bar{X}^a(\mu_0)\rangle = m_{\mu_0\lambda} > 0$). Combining (2.42) with (2.41) we get (2.40).

Next, we turn to show that A, α', $\bar{\eta}'(\lambda)$, and $\eta'(\lambda)$ satisfy the conditions stated in the Lemma. First, we show $d_\lambda \uparrow$ and has a finite limit as $\lambda \uparrow \infty$. Since for any $\lambda, \mu > 0$,

$$\begin{aligned}
&(\lambda-\mu)\eta(\lambda)\bar{X}^a(\mu)[1-(\mu-\mu_0)F^a(\mu)\bar{X}^a(\mu_0)] \\
&= (\lambda-\mu)\eta(\lambda)[\bar{X}^a(\mu)-(\mu-\mu_0)\bar{X}^a(\mu)F^a(\mu)\bar{X}^a(\mu_0)] \\
&= (\lambda-\mu)\eta(\lambda)[\bar{X}^a(\mu)+(\mu-\mu_0)\phi(\mu)\bar{X}^a(\mu_0)-(\mu-\mu_0)\psi(\mu)\bar{X}^a(\mu_0)] \\
&= (\lambda-\mu)\eta(\lambda)\bar{X}^a(\mu_0)-(\mu-\mu_0)(\lambda-\mu)\eta(\lambda)\psi(\mu)\bar{X}^a(\mu_0) \\
&= (\lambda-\mu)\eta(\lambda)\bar{X}^a(\mu_0)-(\mu-\mu_0)[\eta(\mu)-\eta(\lambda)]\bar{X}^a(\mu_0) \\
&= (\lambda-\mu_0)\eta(\lambda)\bar{X}^a(\mu_0)-(\mu-\mu_0)\eta(\mu)\bar{X}^a(\mu_0) \\
&= d_\lambda - d_\mu
\end{aligned}$$

and $(\mu-\mu_0)F^a(\mu)\bar{X}^a(\mu_0) < \mu F^a(\mu)\bar{X}^a(\mu_0) \leq 1$, it is easy to conclude from the above equality that $d_\lambda \uparrow$ as $\lambda \uparrow$. On the other hand, by multiplying $A_\phi(\lambda,\mu)$ in both sides of (2.41) and noticing that $\eta'(\lambda)A_\phi(\lambda,\mu) = \eta'(\mu)$ and ${}^a\eta(\lambda)A_\phi(\lambda,\mu) = {}^a\eta(\mu)$, we have

$$\eta'(\mu) = \eta(\lambda)A_\phi(\lambda,\mu) + d_\lambda \cdot {}^a\eta(\mu).$$

Since $\liminf_{\lambda\to\infty} \eta(\lambda)A_\phi(\lambda,\mu) \geq \alpha\phi(\mu)$, so by letting λ go to ∞ in the right hand of the above equation, we get for any $\mu > 0$

$$\eta'(\mu) \geq \alpha\phi(\mu) + A \cdot {}^a\eta(\mu), \tag{2.43}$$

in particular,

$$\eta'(\mu) \geq A \cdot {}^a\eta(\mu), \tag{2.44}$$

where $A = \lim_{\lambda\to\infty} d_\lambda$. Since ${}^a\eta(\mu) \neq \mathbf{0}$ and $d_\lambda \geq 0$ for $\lambda \geq \mu_0$, we thus get $0 \leq A < \infty$.

Now suppose the Riesz decomposition of $\eta'(\lambda) \in \mathbf{H}_\phi$ is

$$\eta'(\lambda) = \alpha'\phi(\lambda) + \bar{\eta}'(\lambda) \tag{2.45}$$

where α' is a non-negative row vector such that $\alpha'\phi(\lambda) \in l_N$ and, $\bar{\eta}'(\lambda) \in \mathcal{L}^+_\lambda \cap \mathbf{H}_\phi$, according to Lemma 2.5, possesses further an expression as $\bar{\eta}'(\lambda) = d'_a\bar{\eta}^a(\lambda) + d'_b\bar{\eta}^b(\lambda)$ for some constants d'_a, $d'_b \geq 0$.

Recall that ${}^a\eta(\lambda)$ has the similar expression of (2.31) and (2.33) for some constants d_a, $d_b \geq 0$ ($d_b = 0$ when $\lim_{\lambda\to\infty} \lambda\bar{\eta}^b(\lambda)\bar{X}^b = \infty$ or $\bar{\eta}^b(\lambda) = \mathbf{0}$) and α^a satisfying $\alpha^a\phi(\lambda) \in l_N$ and $\alpha^a\bar{X}^b < \infty$.

Since by Lemma 2.11.3 of [42], ${}^a\eta(\lambda)$ and $\eta'(\lambda)$ satisfy

$${}^a\eta(\lambda) \downarrow \mathbf{0}, \quad \lambda \cdot {}^a\eta(\lambda) \to \alpha^a, \quad \lambda \uparrow \infty \tag{2.46}$$

$$\eta'(\lambda) \downarrow \mathbf{0}, \quad \lambda\eta'(\lambda) \to \alpha', \quad \lambda \uparrow \infty \tag{2.47}$$

so by multiplying λ in (2.41), then letting $\lambda \to \infty$ and using the properties (2.46) and (2.47), we obtain

$$\alpha = \alpha' - A\alpha^a \tag{2.48}$$

hence $\alpha' = \alpha + A\alpha^a$. While (2.37) easily follows from (2.43), (2.31), (2.33), (2.45) and the expression of $\bar{\eta}'(\lambda)$.

To verify (2.38) and (2.39), first, noticing that as $\lambda \uparrow \infty$, $d_\lambda \uparrow A$, so we have

$$A - d_\lambda = A - \frac{\lambda\eta'(\lambda)\bar{X}^a - c_2}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \downarrow 0,$$

that is

$$\frac{Ac_1 + c_2 - \lambda[\eta'(\lambda) - A \cdot {}^a\eta(\lambda)]\bar{X}^a}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \downarrow 0.$$

Furthermore, substituting (2.31), (2.45) and (2.48) into the above equation, we have

$$\frac{Ac_1 + c_2 - \lambda\alpha\phi(\lambda)\bar{X}^a - \lambda[\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]\bar{X}^a}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \downarrow 0.$$

Therefore, from the above two equations, we conclude that

$$\begin{aligned} &\lambda[\eta'(\lambda) - A \cdot {}^a\eta(\lambda)]\bar{X}^a \le Ac_1 + c_2, \\ &\lambda\alpha\phi(\lambda)\bar{X}^a + \lambda[\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]\bar{X}^a \le Ac_1 + c_2, \end{aligned} \tag{2.49}$$

and when $\lim\limits_{\lambda\to\infty} \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle < \infty$,

$$\begin{aligned} &\lambda[\eta'(\lambda) - A \cdot {}^a\eta(\lambda)]\bar{X}^a \to Ac_1 + c_2, \\ &\lambda\alpha\phi(\lambda)\bar{X}^a + \lambda[\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]\bar{X}^a \to Ac_1 + c_2. \end{aligned} \tag{2.50}$$

Since

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda\alpha\phi(\lambda)\bar{X}^a &= \alpha\bar{X}^a - \lim_{\lambda\to\infty} \alpha\bar{X}^a(\lambda) = \alpha\bar{X}^a \quad (\because \textit{Lemma } 2.8\ (1)) \\ \lambda[\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]\bar{X}^a &= \lambda[(d_a' - Ad_a)\bar{\eta}^a(\lambda) + (d_b' - Ad_b)\bar{\eta}^b(\lambda)]\bar{X}^a \\ &= \lambda(d_a' - Ad_a)\bar{\eta}^a(\lambda)\bar{X}^a \end{aligned} \tag{2.51}$$

and

$$\begin{aligned} \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle &= \lambda\alpha^a\phi(\lambda)\bar{X}^a + \lambda(d_a\bar{\eta}^a(\lambda) + d_b\bar{\eta}^b(\lambda))\bar{X}^a \\ &= \lambda\alpha^a\phi(\lambda)\bar{X}^a + \lambda d_a\bar{\eta}^a(\lambda)\bar{X}^a, \end{aligned}$$

by taking limit in (2.49) and (2.50), we get

$$\alpha\bar{X}^a + \lim_{\lambda\to\infty} \lambda(d_a' - Ad_a)\bar{\eta}^a(\lambda)\bar{X}^a \le Ac_1 + c_2$$

which holds the equality when

$$\lim_{\lambda\to\infty} \lambda\alpha^a\phi(\lambda)\bar{X}^a + \lim_{\lambda\to\infty} \lambda d_a\bar{\eta}^a(\lambda)\bar{X}^a < \infty. \tag{2.52}$$

Since

$$\lim_{\lambda\to\infty} \lambda\alpha^a\phi(\lambda)\bar{X}^a < \infty \tag{2.53}$$

implies $\alpha^a\bar{X}^a < \infty$ and furthermore, from (2.32) we have $\alpha^a(\mathbf{1} - \bar{X}^a) < \infty$, so (2.53) further implies $\alpha^a\mathbf{1} < \infty$, which in turn, implies (2.53) itself, therefore (2.52) is equivalent to $\alpha^a\mathbf{1} + \lim_{\lambda\to\infty} \lambda d_a\bar{\eta}^a(\lambda)\bar{X}^a < \infty$, which, together with the above argument, shows (2.38) and (2.39). The necessity is proved.

(II) Sufficiency. Suppose $\alpha\bar{X}^a < \infty$ and $\alpha\phi(\lambda) \in l_N$. We show the $\eta(\lambda)$ constructed as (2.40) satisfies (2.36) and $\eta(\lambda) \in \mathbf{H}_\psi$.

Denote $d_\lambda = \dfrac{\lambda\eta'(\lambda)\bar{X}^a - c_2}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle}$, then (2.40) can be simply rewritten as

$$\eta(\lambda) = \eta'(\lambda) - d_\lambda \cdot {}^a\eta(\lambda). \tag{2.54}$$

Since

$$\begin{aligned} A - d_\lambda &= \frac{Ac_1 + c_2 - \lambda[\eta'(\lambda) - A\cdot {}^a\eta(\lambda)]\bar{X}^a}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \\ &= \frac{Ac_1 + c_2 - \lambda\alpha\phi(\lambda)\bar{X}^a - \lambda[\bar{\eta}'(\lambda) - A\cdot {}^a\bar{\eta}(\lambda)]\bar{X}^a}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle}, \end{aligned}$$

from $\eta'(\lambda)$, ${}^a\eta(\lambda) \in \mathbf{H}_\phi$ and (2.31), (2.34), (2.37) and the equality $\alpha' = \alpha + A\alpha^a$, $\bar{\eta}'(\lambda) = d'_a\bar{\eta}^a(\lambda) + d'_b\bar{\eta}^b(\lambda)$, we conclude that

$$\eta'(\lambda) \geq A\cdot {}^a\eta(\lambda), \tag{2.55}$$

and furthermore, by Lemma 2.11.3 and Lemma 2.11.4 of [42], as $\lambda \uparrow \infty$, we have

$$\lambda\bar{\eta}'(\lambda)\bar{X}^a \uparrow \qquad \lambda\eta'(\lambda)\bar{X}^a \uparrow \qquad \lambda\eta'(\lambda) \to \alpha' \tag{2.56}$$

$$\lambda\cdot {}^a\bar{\eta}(\lambda)\bar{X}^a \uparrow \qquad \lambda\cdot {}^a\eta(\lambda)\bar{X}^a \uparrow \qquad \lambda\cdot {}^a\eta(\lambda) \to \alpha^a \tag{2.57}$$

$$\lambda[\eta'(\lambda) - A\cdot {}^a\eta(\lambda)]\bar{X}^a \uparrow \qquad \lambda[\bar{\eta}(\lambda) - A\cdot {}^a\bar{\eta}(\lambda)]\bar{X}^a \uparrow \tag{2.58}$$

$$\lambda\alpha\phi(\lambda)\bar{X}^a \uparrow \alpha\bar{X}^a. \tag{2.59}$$

So from (2.39), (2.51), (2.58) and (2.59), we get $0 \leq A - d_\lambda \to 0$, i.e.

$$d_\lambda \leq A \quad \text{and} \quad d_\lambda \to A \quad \text{as} \quad \lambda \uparrow \infty. \tag{2.60}$$

Furthermore, from (2.54), (2.56), (2.57), (2.60), it follows

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha' - A\alpha^a = \alpha$$

which shows (2.36).

To show $\eta(\lambda) \in \mathbf{H}_\psi$, we need to verify the following two assertions:

$$\eta(\mu) - \eta(\lambda) = (\lambda - \mu)\eta(\lambda)\psi(\mu), \quad \lambda, \mu > 0 \tag{2.61}$$

$$0 \le \eta(\lambda) \in l_N. \tag{2.62}$$

Since $\eta(\lambda)$ can be also written as $\eta(\lambda) = \eta'(\lambda) - (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)$, and $\eta'(\lambda) \in \mathbf{H}_\phi$, i.e. $\eta'(\mu) - \eta'(\lambda) = (\lambda - \mu)\eta'(\lambda)\phi(\mu), \ \forall \lambda, \mu > 0$, we have

$$\begin{aligned}
&\eta(\mu) - \eta(\lambda)\\
&\quad = \eta'(\mu) - \eta'(\lambda) - (\mu\eta'(\mu)\bar{X}^a - c_2)F^a(\mu) + (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)\\
&\quad = (\lambda - \mu)\eta'(\lambda)\phi(\mu) - (\lambda - \mu)(\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)\phi(\mu) +\\
&\quad\quad + (\lambda - \mu)(\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)\phi(\mu) - (\mu\eta'(\mu)\bar{X}^a - c_2)F^a(\mu) +\\
&\quad\quad + (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)\\
&\quad = (\lambda - \mu)[\eta'(\lambda) - (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)]\phi(\mu) +\\
&\quad\quad + (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)[I + (\lambda - \mu)\phi(\mu)] - (\mu\eta'(\mu)\bar{X}^a - c_2)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\lambda)A_\phi(\lambda, \mu) -\\
&\quad\quad - (\mu\eta'(\mu)\bar{X}^a - c_2)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda\eta'(\lambda)\bar{X}^a - c_2)[1 + (\mu - \lambda)F^a(\lambda)\bar{X}^a(\mu)]F^a(\mu) -\\
&\quad\quad - (\mu\eta'(\mu)\bar{X}^a - c_2)F^a(\mu) \quad (\because (2.29))\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda\eta'(\lambda)\bar{X}^a - c_2)F^a(\mu) -\\
&\quad\quad - (\lambda\eta'(\lambda)\bar{X}^a - c_2)(\mu - \lambda)F^a(\lambda)\bar{X}^a(\mu)F^a(\mu) - (\mu\eta'(\mu)\bar{X}^a - c_2)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda\eta'(\lambda)\bar{X}^a - \mu\eta'(\mu)\bar{X}^a)F^a(\mu) -\\
&\quad\quad - (\lambda\eta'(\lambda)\bar{X}^a - c_2)(\mu - \lambda)F^a(\lambda)\bar{X}^a(\mu)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda - \mu)\eta'(\lambda)\bar{X}^a(\mu)F^a(\mu) -\\
&\quad\quad - (\lambda\eta'(\lambda)\bar{X}^a - c_2)(\mu - \lambda)F^a(\lambda)\bar{X}^a(\mu)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda - \mu)[\eta'(\lambda) - (\lambda\eta'(\lambda)\bar{X}^a - c_2)]\bar{X}^a(\mu)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)\phi(\mu) + (\lambda - \mu)\eta(\lambda)\bar{X}^a(\mu)F^a(\mu)\\
&\quad = (\lambda - \mu)\eta(\lambda)[\phi(\mu) + \bar{X}^a(\mu)F^a(\mu)]\\
&\quad = (\lambda - \mu)\eta(\lambda)\psi(\mu),
\end{aligned}$$

which shows (2.61).

Finally, by (2.55), we have

$$\eta(\lambda) = \eta'(\lambda) - d_\lambda \cdot {}^a\eta(\lambda) \ge \eta'(\lambda) - A \cdot {}^a\eta(\lambda) \ge \mathbf{0}.$$

On the other hand, $\eta'(\lambda), {}^a\eta(\lambda) \in \mathbf{H}_\phi$ implies $\eta'(\lambda) \in l_N$ and ${}^a\eta(\lambda) \in l_N$, thus

$$\eta(\lambda) = \eta'(\lambda) - d_\lambda \cdot {}^a\eta(\lambda) \le \max\{|d_\lambda|, 1\}(\eta'(\lambda) + {}^a\eta(\lambda)) \in l_N.$$

This leads to the conclusion (2.62). The proof is complete. □

Lemma 2.17. *Assume assumption* (**A**), $\alpha\bar{X}^a < \infty$ *and* $\alpha\phi(\lambda) \in l_N$. *Let* $\eta(\lambda)$ *be taken as in Lemma* 2.15. *If* $\alpha\mathbf{1} < \infty$, *then as* $\lambda \to \infty$, *we have*
(1) *For any* $b \in \{1,2\}$, $\lambda\eta(\lambda)\bar{X}^b \to \infty$ *if and only if* $\lambda[\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]\bar{X}^b \to \infty$;
(2) $\lambda\eta(\lambda)\mathbf{1} \to \infty$ *if and only if* $\lambda[\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]\mathbf{1} \to \infty$.

Proof. Since

$$\begin{aligned} A - d_\lambda &= \frac{Ac_1 + c_2 - \lambda[\eta'(\lambda) - A \cdot {}^a\eta(\lambda)]\bar{X}^a}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \\ &\le \frac{Ac_1 + c_2}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle}, \\ \mathbf{0} \le (A - d_\lambda) \cdot {}^a\eta(\lambda) &\le \frac{Ac_1 + c_2}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} {}^a\eta(\lambda) = (Ac_1 + c_2)F^a(\lambda), \end{aligned}$$

we have

$$0 \le \lambda(A - d_\lambda) \cdot {}^a\eta(\lambda)\mathbf{1} \le (Ac_1 + c_2)\lambda F^a(\lambda)\mathbf{1} \le Ac_1 + c_2 < \infty. \tag{2.63}$$

On the other hand, by (2.31) and (2.45), we have

$$\begin{aligned} \eta(\lambda) &= \eta'(\lambda) - d_\lambda \cdot {}^a\eta(\lambda) = \alpha'\phi(\lambda) + \bar{\eta}'(\lambda) - d_\lambda[\alpha^a\phi(\lambda) + {}^a\bar{\eta}(\lambda)] \\ &= \alpha\phi(\lambda) + (A - d_\lambda)\alpha^a\phi(\lambda) + \\ &\quad + (A - d_\lambda) \cdot {}^a\bar{\eta}(\lambda) + [\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)] \\ &= \alpha\phi(\lambda) + (A - d_\lambda) \cdot {}^a\eta(\lambda) + [\bar{\eta}'(\lambda) - A \cdot {}^a\bar{\eta}(\lambda)]. \end{aligned} \tag{2.64}$$

Thus the assertions of the lemma easily follow from (2.63), (2.64) and the assumptions. ☐

Lemma 2.18. *Assume assumption* (**A**) *with* $\bar{X}^a(\lambda) \ne \mathbf{0}$ *and* $F^a(\lambda) \ne \mathbf{0}$. $\eta(\lambda) \in \mathbf{H}_\psi$ *satisfying* $\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha$ *and* $\lim_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} = \infty$, *then if and only if* $\bar{X}^b(\lambda) \ne \mathbf{0}$ ($b \ne a$, $b \in \{1,2\}$), *there exists a column vector* $\xi(\lambda)$ *satisfying the following conditions* (2.65)–(2.69):

$$\mathbf{0} \le \xi(\lambda) \le \mathbf{1} \tag{2.65}$$

$$\xi(\lambda) + \lambda\psi(\lambda)\mathbf{1} \le \mathbf{1} \tag{2.66}$$

$$\xi(\lambda) - \xi(\mu) = (\mu - \lambda)\psi(\lambda)\xi(\mu) \tag{2.67}$$

$$\lim_{\lambda\to\infty} \lambda\xi(\lambda) = \beta \tag{2.68}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)(\mathbf{1} - \xi) < \infty \tag{2.69}$$

where $\xi = \lim_{\lambda\to 0} \xi(\lambda)$.

When $\bar{X}^b(\lambda) \ne \mathbf{0}$, $\xi(\lambda)$ *can be expressed as*

$$\xi(\lambda) = \phi(\lambda)\beta + X(\lambda) - \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi, \tag{2.70}$$

where $X(\lambda) \in \mathcal{M}_\lambda^+ \cap \mathbf{K}_\phi$ *has the following expression*

$$X(\lambda) = t_a \bar{X}^a(\lambda) + \bar{X}^b(\lambda) \tag{2.71}$$

for some non-negative constant t_a *satisfying*

$$\lim_{\lambda\to\infty} \frac{1}{c_1} \lambda\langle {}^a\eta(\lambda), \mathbf{1} - \bar{X}^a\rangle \le t_a \le 1 \qquad \textit{and} \tag{2.72}$$

$$\xi = Y^1 + Y^2 + t_a \bar{X}^a + \bar{X}^b. \tag{2.73}$$

In particular, the equality in (2.66) *holds iff* $t_a = 1$.

Proof. (I) Necessity. The resolvent equation of $\psi(\lambda)$ implies that $F^a(\lambda)$ satisfies the following equation $F^a(\lambda) = F^a(\mu) + (\mu - \lambda)F^a(\mu)\psi(\lambda)$, so by Lemma 3.1.3 of [18],

$$(\mu - \lambda)F^a(\lambda)\xi(\mu) = \mu F^a(\mu)\xi - \lambda F^a(\lambda)\xi \tag{2.74}$$

where $\xi = \lim_{\lambda\to 0} \xi(\lambda)$. Furthermore, by (2.67) and noticing that $\bar{X}^a(\lambda) = A_\phi(\mu,\lambda)\bar{X}^a(\mu)$, we have

$$\begin{aligned} \xi(\lambda) &= A_\phi(\mu,\lambda)\xi(\mu) + (\mu - \lambda)\bar{X}^a(\lambda)F^a(\lambda)\xi(\mu) \\ &= A_\phi(\mu,\lambda)\xi(\mu) + \bar{X}^a(\lambda)[\mu F^a(\mu)\xi - \lambda F^a(\lambda)\xi] \\ &= A_\phi(\mu,\lambda)[\xi(\mu) + \mu\bar{X}^a(\mu)F^a(\mu)\xi] - \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi, \end{aligned}$$

hence

$$\xi(\lambda) + \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi = A_\phi(\mu,\lambda)[\xi(\mu) + \mu\bar{X}^a(\mu)F^a(\mu)\xi] \tag{2.75}$$

that is

$$T(\lambda) \equiv \xi(\lambda) + \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi \in \mathbf{K}_\phi. \tag{2.76}$$

By (2.68) and Lemma 2.8, we get

$$\lim_{\lambda\to\infty} \lambda T(\lambda) = \beta \tag{2.77}$$

thus the Riesz decomposition of $T(\lambda)$ must have the following form

$$T(\lambda) = \phi(\lambda)\beta + X(\lambda) \tag{2.78}$$

where $X(\lambda) \in \mathcal{M}_\lambda^+ \cap \mathbf{K}_\phi$ and therefore there exist some non-negative constants

t_a and t_b such that

$$X(\lambda) = t_a \bar{X}^a(\lambda) + t_b \bar{X}^b(\lambda) \tag{2.79}$$

where $t_b = 0$ when $\bar{X}^b(\lambda) = \mathbf{0}$. So $\xi(\lambda)$ is expressed as

$$\begin{aligned} \xi(\lambda) &= T(\lambda) - \lambda \bar{X}^a(\lambda) F^a(\lambda)\xi \\ &= \phi(\lambda)\beta + X(\lambda) - \lambda \bar{X}^a(\lambda) F^a(\lambda)\xi \\ &= \phi(\lambda)\beta + t_a \bar{X}^a(\lambda) + t_b \bar{X}^b(\lambda) - \lambda \bar{X}^a(\lambda) F^a(\lambda)\xi. \end{aligned} \tag{2.80}$$

Now, by (2.67), we have $\xi = \xi(\lambda) + \lambda\psi(\lambda)\xi$, which yields $\lim_{\lambda\to 0} \lambda\psi(\lambda)\xi = \mathbf{0}$, or equivalently

$$\lim_{\lambda\to 0} \lambda\phi(\lambda)\xi = \mathbf{0} \quad \text{and} \quad \lim_{\lambda\to 0} \lambda \bar{X}^a(\lambda) F^a(\lambda)\xi = \mathbf{0}$$

so together with (2.21) and Lemma 2.8, by letting $\lambda \to 0$ in (2.80), we get

$$\xi = \begin{cases} Y^a + Y^b + t_a \bar{X}^a + t_b \bar{X}^b, & \text{when } \bar{X}^b(\lambda) \neq \mathbf{0} \\ Y^a + Y^b + t_a \bar{X}^a, & \text{when } \bar{X}^b(\lambda) = \mathbf{0} \end{cases} \tag{2.81}$$

$$\text{hence} \quad \mathbf{1} - \xi = \begin{cases} (1-t_a)\bar{X}^a + (1-t_b)\bar{X}^b, & \text{when } \bar{X}^b(\lambda) \neq \mathbf{0} \\ (1-t_a)\bar{X}^a + \bar{X}^b, & \text{when } \bar{X}^b(\lambda) = \mathbf{0}. \end{cases}$$

Since $\lambda\eta(\lambda)\mathbf{1} = \lambda\eta(\lambda)(Y^a + Y^b + \bar{X}^a + \bar{X}^b)$, by Lemma 2.11 and Lemma 2.12, we see that $\lim_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} = \infty$ implies $\lim_{\lambda\to\infty} \lambda\eta(\lambda)\bar{X}^b = \infty$, therefore from

$$\lambda\eta(\lambda)(\mathbf{1} - \xi) = \begin{cases} (1-t_a)\lambda\eta(\lambda)\bar{X}^a + (1-t_b)\lambda\eta(\lambda)\bar{X}^b, & \text{when } \bar{X}^b(\lambda) \neq \mathbf{0} \\ (1-t_a)\lambda\eta(\lambda)\bar{X}^a + \lambda\eta(\lambda)\bar{X}^b, & \text{when } \bar{X}^b(\lambda) = \mathbf{0} \end{cases}$$

we see that when $\bar{X}^b(\lambda) = \mathbf{0}$, (2.69) can not be satisfied. So when there exists a $\xi(\lambda)$ satisfying (2.65)–(2.69), the condition $\bar{X}^b(\lambda) \neq \mathbf{0}$ is necessarily required. When $\bar{X}^b(\lambda) \neq \mathbf{0}$ is satisfied, from the above argument and the above equation, it follows that $t_b = 1$. Substituting this into (2.79)–(2.81), we immediately get (2.71), (2.70) and (2.73).

Next, by (2.66), (2.80), (2.20) and (2.22), we have

$$
\begin{aligned}
\mathbf{0} &\le \mathbf{1} - \xi(\lambda) - \lambda\psi(\lambda)\mathbf{1} \\
&= \mathbf{1} - \phi(\lambda)\beta - t_a\bar{X}^a(\lambda) - \bar{X}^b(\lambda) - \lambda\psi(\lambda)\mathbf{1} + \\
&\quad + \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)(Y^1+Y^2+t_a\bar{X}^a+\bar{X}^b)}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} \\
&= \mathbf{1} - \phi(\lambda)\beta - t_a\bar{X}^a(\lambda) - \bar{X}^b(\lambda) - \lambda\psi(\lambda)\mathbf{1} + \\
&\quad + \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)[(Y^1+Y^2+\bar{X}^a+\bar{X}^b)-(1-t_a)\bar{X}^a]}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} \\
&= \mathbf{1} - \phi(\lambda)\beta - t_a\bar{X}^a(\lambda) - \bar{X}^b(\lambda) - \lambda\psi(\lambda)\mathbf{1} + \\
&\quad + \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)\mathbf{1}}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} - \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)(1-t_a)\bar{X}^a}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} \\
&= \mathbf{1} - \phi(\lambda)\beta - t_a\bar{X}^a(\lambda) - \bar{X}^b(\lambda) - \lambda\phi(\lambda)\mathbf{1} - \\
&\quad - \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)(1-t_a)\bar{X}^a}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} \\
&= \bar{X}^a(\lambda) + \bar{X}^b(\lambda) - t_a\bar{X}^a(\lambda) - \bar{X}^b(\lambda) - \\
&\quad - \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)(1-t_a)\bar{X}^a}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} \\
&= (1-t_a)\bar{X}^a(\lambda) - \bar{X}^a(\lambda)\frac{\lambda\cdot {}^a\eta(\lambda)(1-t_a)\bar{X}^a}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle} \\
&= \bar{X}^a(\lambda)\frac{(1-t_a)c_1}{c_1+\lambda\langle {}^a\eta(\lambda),\bar{X}^a\rangle},
\end{aligned}
\tag{2.82}
$$

so $(1-t_a)c_1 \ge 0$, i.e.

$$
t_a \le 1. \tag{2.83}
$$

Apparently, (2.66) holds the equality iff $t_a = 1$.

On the other hand, by(2.68) and Fatou's Lemma, we have

$$
\liminf_{\mu\to\infty} A_\phi(\mu,\lambda)\xi(\mu) \ge \phi(\lambda)\beta
$$

and by (2.74) we know $\mu F^a(\mu)\xi$ is increasing in μ and thus has a limit $B \le 1$ as $\mu \uparrow \infty$, so by letting $\mu \to \infty$ in (2.75), we get $T(\lambda) \ge \phi(\lambda)\beta + B\bar{X}^a(\lambda)$. i.e.

$$
\phi(\lambda)\beta + t_a\bar{X}^a(\lambda) + \bar{X}^b(\lambda) \ge \phi(\lambda)\beta + B\bar{X}^a(\lambda)
$$

therefore $(t_a - B)\bar{X}^a(\lambda) + \bar{X}^b(\lambda) \geq \mathbf{0}$, which implies $t_a \geq B$. Since

$$\begin{aligned}
0 \leq B - \lambda F^a(\lambda)\xi &= B - \frac{\lambda \cdot {}^a\eta(\lambda)(Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b)}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \\
&= \frac{Bc_1 + \lambda \cdot {}^a\eta(\lambda)[(B - t_a)\bar{X}^a - Y^1 - Y^2 - \bar{X}^b]}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \\
&= \frac{Bc_1 + (t_a - B)c_1 - \lambda \cdot {}^a\eta(\lambda)(Y^1 + Y^2 + \bar{X}^b)}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} + B - t_a \\
&\leq \frac{t_a c_1 - \lambda \cdot {}^a\eta(\lambda)(Y^1 + Y^2 + \bar{X}^b)}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle},
\end{aligned}$$

we have

$$t_a \geq \frac{1}{c_1}\lambda \cdot {}^a\eta(\lambda)(Y^1 + Y^2 + \bar{X}^b) = \frac{1}{c_1}\lambda\langle {}^a\eta(\lambda), \mathbf{1} - \bar{X}^a\rangle$$

for all $\lambda \geq 0$. Combining this with (2.83), we get (2.72).

(II) Sufficiency. If $\bar{X}^b(\lambda) \neq \mathbf{0}$ and $\xi(\lambda)$ is taken as in (2.70)–(2.73), we show $\xi(\lambda)$ satisfies (2.65)–(2.69).

First, by (2.72) and (2.82), we immediately get (2.66). And, by (2.70), (2.71), (2.73), Lemma 2.8 (1) and Lemma 2.14 (1), we can also easily get (2.68). Next, from (2.73) we have

$$\lambda\eta(\lambda)(\mathbf{1} - \xi) = (1 - t_a)\lambda\eta(\lambda)\bar{X}^a, \tag{2.84}$$

hence (2.69) is guaranteed by Lemma 2.12. Moreover, by (2.82)

$$\xi(\lambda) = \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} - \bar{X}^a(\lambda)\frac{(1 - t_a)c_1}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle}$$

so by (2.72), we get $(1 - t_a)c_1 \leq c_1 - \lambda\langle {}^a\eta(\lambda), \mathbf{1} - \bar{X}^a\rangle$, hence

$$\begin{aligned}
\xi(\lambda) &\geq \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} - \bar{X}^a(\lambda)\frac{c_1 - \lambda\langle {}^a\eta(\lambda), \mathbf{1} - \bar{X}^a\rangle}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \\
&= \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} - \bar{X}^a(\lambda) + \bar{X}^a(\lambda)\frac{\lambda \cdot {}^a\eta(\lambda)\mathbf{1}}{c_1 + \lambda\langle {}^a\eta(\lambda), \bar{X}^a\rangle} \\
&= \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} - \bar{X}^a(\lambda) + \lambda\bar{X}^a(\lambda)F^a(\lambda)\mathbf{1} \\
&= \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} - \bar{X}^a(\lambda) = \phi(\lambda)\beta + \bar{X}^b(\lambda) \quad (\because \text{ Lemma 2.8 (3)}) \\
&\geq \mathbf{0},
\end{aligned}$$

which, together with (2.66), implies (2.65).

Next, by (2.70) and Lemma 2.8, we have

$$
\begin{aligned}
\xi(\lambda) &= \phi(\lambda)\beta + X(\lambda) - \lambda(\psi(\lambda) - \phi(\lambda))\xi \\
&= Y^1(\lambda) + Y^2(\lambda) + X(\lambda) + \lambda\phi(\lambda)\xi - \lambda\psi(\lambda)\xi \\
&= Y^1(\lambda) + Y^2(\lambda) + t_a\bar{X}^a(\lambda) + \bar{X}^b(\lambda) + \lambda\phi(\lambda)(Y^1 + \\
&\quad + Y^2 + t_a\bar{X}^a + \bar{X}^b) - \lambda\psi(\lambda)(Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b) \\
&= Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b - \lambda\psi(\lambda)(Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b) \\
&= \xi - \lambda\psi(\lambda)\xi,
\end{aligned}
\tag{2.85}
$$

therefore by the resolvent equation of $\psi(\lambda)$, we have

$$
\begin{aligned}
\xi(\lambda) - \xi(\mu) &= \lambda\psi(\mu)\xi - \lambda\psi(\lambda)\xi = (\mu - \lambda)\psi(\lambda)\xi - \mu[\psi(\lambda) - \psi(\mu)]\xi \\
&= (\mu - \lambda)\psi(\lambda)\xi - \mu(\mu - \lambda)\psi(\lambda)\psi(\mu)\xi \\
&= (\mu - \lambda)\psi(\lambda)(\xi - \mu\psi(\mu)\xi) = (\mu - \lambda)\psi(\lambda)\xi(\mu),
\end{aligned}
$$

which shows (2.67).

Finally, we show

$$\xi \equiv \lim_{\lambda\to 0}\xi(\lambda) = Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b.$$

By letting $\mu \downarrow 0$ in (2.67), we obtain

$$\xi(\lambda) = \xi - \lambda\psi(\lambda)\xi, \tag{2.86}$$

hence $\lim_{\lambda\to 0}\lambda\psi(\lambda)\xi = \mathbf{0}$, and thus

$$\lim_{\lambda\to 0}\lambda\phi(\lambda)\xi = \mathbf{0}. \tag{2.87}$$

Denote $\xi' = Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b$. By Lemma 2.8,

$$\lambda\phi(\lambda)\xi' = \xi' - Y^1(\lambda) - Y^2(\lambda) - X(\lambda), \tag{2.88}$$

letting $\lambda \downarrow 0$, we get $\lim_{\lambda\to 0}\lambda\phi(\lambda)\xi' = \mathbf{0}$. Let $X^0 = \xi - \xi'$. Then by (2.87) we have

$$\lim_{\lambda\to 0}\lambda\phi(\lambda)X^0 = \mathbf{0}. \tag{2.89}$$

Noticing that the ξ in (2.85) is actually $Y^1 + Y^2 + t_a\bar{X}^a + \bar{X}^b$, which is denoted here by ξ', we then get from (2.85) and (2.86) that $\lambda\psi(\lambda)X^0 = X^0$. Furthermore,

$$
\begin{aligned}
\xi(\lambda) &= \xi' - \lambda\psi(\lambda)\xi' = \xi' - X^0 - \lambda\psi(\lambda)(\xi' - X^0) \\
&= \xi' - X^0 - \lambda\phi(\lambda)(\xi' - X^0) - \lambda\bar{X}^a(\lambda)F^a(\lambda)(\xi' - X^0) \\
&= \xi' - X^0 - \lambda\phi(\lambda)\xi' + \lambda\phi(\lambda)X^0 - \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi \\
&= Y^1(\lambda) + Y^2(\lambda) + X(\lambda) - X^0 + \lambda\phi(\lambda)X^0 - \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi \quad (\because\ (2.88)) \\
&= \phi(\lambda)\beta + X(\lambda) - X^0 + \lambda\phi(\lambda)X^0 - \lambda\bar{X}^a(\lambda)F^a(\lambda)\xi,
\end{aligned}
$$

so by (2.70) we have $X^0 - \lambda\phi(\lambda)X^0 = \mathbf{0}$, therefore by (2.89)

$$X^0 = \lim_{\lambda\to 0} \lambda\phi(\lambda)X^0 = \mathbf{0}$$

as is required. The proof is complete. □

Remark 2.19. We shall denote $F^a(\lambda) = F^a(\lambda;\, c_1, \alpha^a, d_a, d_b)$ to emphasize that the $F^a(\lambda)$ in Lemma 2.14 (3) satisfying (2.30)–(2.34) is actually decided by c_1, α^a, d_a and d_b. Similarly, the $\eta(\lambda)$ as (2.40) in Lemma 2.15 will be denoted by $\eta(\lambda) = \eta(\lambda;\, c_1, c_2, A, \alpha, \alpha^a, d_a, d_b, d'_a, d'_b)$ and the $\xi(\lambda)$ in Lemma 2.18 satisfying (2.70)–(2.73) will be denoted by $\xi(\lambda) = \xi(\lambda;\, \beta, t_a, F^a(\lambda))$.

3. Existence and construction theorems

Suppose Q is a uni-instantaneous bilateral birth and death pre-generator defined as (2.1), with $\{0\}$ being the instantaneous state. Since Q is written as

$$Q = \begin{pmatrix} Q_{N_1} & \beta_1 & \mathbf{0} \\ \alpha_1 & -\infty & \alpha_2 \\ \mathbf{0} & \beta_2 & Q_{N_2} \end{pmatrix},$$

where $\alpha_a = (q_{i0};\ i \in N_a)$, $\beta_a = (q_{0j};\ j \in N_a)$, $a = 1, 2$, to apply Chen's resolvent decomposition theorem 1.1 in our case, the decomposition (1.2) should be modified into the following form

$$R(\lambda) = \begin{pmatrix} \psi^{11}(\lambda) & \mathbf{0} & \psi^{12}(\lambda) \\ \mathbf{0} & 0 & \mathbf{0} \\ \psi^{21}(\lambda) & \mathbf{0} & \psi^{22}(\lambda) \end{pmatrix} + r(\lambda)\begin{pmatrix} \xi_{N_1}(\lambda) \\ 1 \\ \xi_{N_2}(\lambda) \end{pmatrix} \begin{pmatrix} \eta_{N_1}(\lambda), & 1, & \eta_{N_2}(\lambda) \end{pmatrix} \tag{3.1}$$

where $\psi^{ab}(\lambda)$ is the restriction of the Q_N-process $\psi(\lambda)$ on $N_a \times N_b$, $a, b \in \{1, 2\}$, $\xi_{N_a}(\lambda)$ and $\eta_{N_a}(\lambda)$ are the restrictions of $\xi(\lambda)$ and $\eta(\lambda)$ on N_a ($a = 1, 2$), respectively, with $\xi(\lambda)$ and $\eta(\lambda)$ satisfying (1.3)–(1.11). α, β in (1.6) are defined here by $\alpha = (q_{i0};\ i \in N) = (\alpha_1,\ \alpha_2)$ and $\beta = (q_{0j};\ j \in N) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.

Now we have the following

Theorem 3.1. *If both z^1 and z^2 are entrance or natural, or equivalently, Q_N is null exit, then Q_E-process does not exist.*

Proof. Suppose both z^1 and z^2 are entrance or natural, then Q_N is null exit, i.e. $\bar{X}^a(\lambda) = \mathbf{0}$, $a = 1, 2$, therefore $\psi(\lambda) = \phi(\lambda)$, the Feller minimal Q_N-process, is the unique B-type Q_N-process. If there exists a Q_E-process $R(\lambda)$, then by Chen's theorem, there exist a row vector $\eta(\lambda) \in \mathbf{H}_\phi$ and a column vector $\xi(\lambda) \in \mathbf{K}_\phi$ such that (1.5)–(1.9) hold. Since $\mathcal{M}^+_\lambda = \{0\}$, by taking Riesz decomposition for $\xi(\lambda)$, we have

$$\xi(\lambda) = \phi(\lambda)\beta = Y^1(\lambda) + Y^2(\lambda).$$

Thus $\xi = \lim_{\lambda\to 0} \xi(\lambda) = Y^1 + Y^2$. By Lemma 2.11,

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \xi\rangle = \lim_{\lambda\to\infty} \lambda\eta(\lambda)(Y^1 + Y^2) < \infty$$

which is in contradiction to (1.9). □

Theorem 3.2. *If Q_N is uni-exit, then all the Q_E-processes, if exist, are necessarily expansions of the Feller minimal Q_N-process $\phi(\lambda)$.*

Proof. Suppose Q_E-processes exist. Let $\bar{X}^a(\lambda)$ be the non-zero exit solution of Q_N, thus $\bar{X}^b(\lambda) = \mathbf{0}$, $b \neq a$, $a, b \in \{1, 2\}$. Let $R(\lambda)$ be a Q_E-process and $\psi(\lambda)$ its projection on $N \times N$. Then by Lemma 2.10, $\psi(\lambda)$ has the form $\psi(\lambda) = \phi(\lambda) + \bar{X}^a(\lambda)F^a(\lambda)$. If $F^a(\lambda) \neq \mathbf{0}$, for any $\eta(\lambda) \in \mathbf{H}_\psi$ satisfying (1.6)–(1.9), by Lemma 2.8, Lemma 2.11 and Lemma 2.12, we have

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} = \lim_{\lambda\to\infty} \lambda\eta(\lambda)(X^a + X^b + Y^1 + Y^2) < \infty,$$

which is in contradiction to (1.9). So it must hold $F^a(\lambda) = \mathbf{0}$ and thus $\psi(\lambda) = \phi(\lambda)$. □

Theorem 3.3. *If one of the boundary points, say z^a, is regular or exit, and the other one z^b is entrance or natural ($b \neq a$, $a, b \in \{1, 2\}$), or equivalently, Q_N is uni-exit, then*
(1) when z^a is regular, Q_E-process exists if and only if $\alpha\phi(\lambda) \in l_N$;
(2) when z^a is exit, Q_E-process exists if and only if $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l_N$.

When the existence conditions are satisfied, each Q_E-process $R(\lambda)$ is an expansion of the Feller minimal Q_N-process $\phi(\lambda)$ and can be obtained in the following way: Take

$$\eta(\lambda) = \alpha\phi(\lambda) + d_a\bar{\eta}^a(\lambda) + d_b\bar{\eta}^b(\lambda) \tag{3.2}$$

with $d_a \geq 0$, $d_b \geq 0$ ($d_a > 0$ if $\alpha\mathbf{1} < \infty$, $d_a = 0$ if z^a is exit, $d_b = 0$ if z^b is natural), and

$$\xi(\lambda) = \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} \tag{3.3}$$

select a constant c such that

$$c \geq \lambda\eta(\lambda)\bar{X}^b \equiv \sigma^b = constant \tag{3.4}$$

finally, define $r(\lambda)$ as (1.10) and $R(\lambda)$ as (3.1).

$R(\lambda)$ is honest iff $c = \sigma^b$. Moreover, the honest process is unique when one of the boundary points is exit and the other one is natural, otherwise, there exist infinitely many honest processes.

Proof. (1) Suppose the boundary point z^a is regular, and z^b is entrance or natural ($b \neq a$, $a, b \in \{1, 2\}$), then Q_N is uni-exit and thus, by Theorem 3.2, if the Q_E-process exists, then for any Q_E-process $R(\lambda)$, its projection $\psi(\lambda)$ on

$N \times N$ is just the Feller minimal Q_N-process $\phi(\lambda)$. And furthermore, by Chen's resolvent decomposition theorem, there exist uniquely a row vector $\eta(\lambda) \in \mathbf{H}_\phi$ and a column vector $\xi(\lambda) \in \mathbf{K}_\phi$ such that (1.5)–(1.9) hold. Since every vector $\eta(\lambda) \in \mathbf{H}_\phi$ allows a Riesz decomposition

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda) \tag{3.5}$$

where $\bar{\eta}(\lambda) \in \mathcal{L}_\lambda^+ \cap \mathbf{H}_\phi$. So $\alpha\phi(\lambda) \le \eta(\lambda) \in l_N$ implies $\alpha\phi(\lambda) \in l_N$. The necessity is proved.

Conversely, if $\alpha\phi(\lambda) \in l_N$, then take a row vector $\eta(\lambda)$ as (3.2), a column vector $\xi(\lambda)$ as (3.3) and a constant c as (3.4), finally, define $r(\lambda)$ as (1.10) and $R(\lambda)$ as (3.1). Now we show that the $R(\lambda)$ thus constructed is a Q_E-process.

By Chen's theorem, we only need to show $\eta(\lambda) \in \mathbf{H}_\phi$, $\xi(\lambda) \in \mathbf{K}_\phi$, (1.5)–(1.9).

First, from Lemma 2.7, Lemma 2.8, and noticing that

$$\xi(\lambda) = \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} = \phi(\lambda)\beta + \bar{X}^a(\lambda),$$

it easily follows that $\eta(\lambda) \in \mathbf{H}_\phi$, $\xi(\lambda) \in \mathbf{K}_\phi$, and (1.5)–(1.6) hold. Next, when $\alpha\mathbf{1} = \infty$, we have $\liminf_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} \ge \alpha\mathbf{1} = \infty$, and when $\alpha\mathbf{1} < \infty$, by Lemma 2.9, we have

$$\begin{aligned}\liminf_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} &\ge \liminf_{\lambda\to\infty} \lambda\eta(\lambda)\bar{X}^a \\ &\ge \liminf_{\lambda\to\infty} d_a\lambda\bar{\eta}^a(\lambda)\bar{X}^a = \infty,\end{aligned}$$

therefore (1.9) holds. Finally, by (2.20), (2.21) and Lemma 2.8 (2),

$$\begin{aligned}\xi &= \lim_{\lambda\to 0} \xi(\lambda) = \lim_{\lambda\to 0}(\phi(\lambda)\beta + \bar{X}^a(\lambda)) \\ &= \lim_{\lambda\to 0}(Y^1(\lambda) + Y^2(\lambda) + \bar{X}^a(\lambda)) = Y^1 + Y^2 + \bar{X}^a = \mathbf{1} - \bar{X}^b,\end{aligned}$$

and by Lemma 2.8 (1), for any λ, $\mu > 0$,

$$\begin{aligned}\lambda\eta(\lambda)\bar{X}^b &= \lambda\eta(\mu)[I + (\mu-\lambda)\phi(\lambda)]\bar{X}^b = \lambda\eta(\mu)\bar{X}^b + \lambda\eta(\mu)(\mu-\lambda)\phi(\lambda)\bar{X}^b \\ &= \lambda\eta(\mu)\bar{X}^b + (\mu-\lambda)\eta(\mu)\bar{X}^b = \mu\eta(\mu)\bar{X}^b,\end{aligned}$$

we get

$$\lambda\langle\eta(\lambda), \mathbf{1} - \xi\rangle = \lambda\eta(\lambda)\bar{X}^b \equiv \sigma^b = \text{constant},$$

which shows that (3.4) implies (1.7)–(1.8). The proof of the sufficiency is completed.

Now we show every Q_E-process $R(\lambda)$ can be obtained in the above way. By Chen's theorem, we only need to show each vector in $\mathbf{H}_\phi$ and $\mathbf{K}_\phi$ must have the form of (3.2) and (3.3) respectively. By Lemma 2.11.3 of [42] and (1.6), we have

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda)$$

where $\bar{\eta}(\lambda) \in \mathcal{L}_\lambda^+ \cap \mathbf{H}_\phi$. By Lemma 2.5, $\bar{\eta}(\lambda)$ can be further expressed as

$$\bar{\eta}(\lambda) = d_a \bar{\eta}^a(\lambda) + d_b \bar{\eta}^b(\lambda)$$

with $d_a \geq 0$, $d_b \geq 0$ and $d_b = 0$ when z^b is natural ($\because \bar{\eta}^b(\lambda) = \mathbf{0}$). While when $\alpha\mathbf{1} < \infty$, (1.9) forces $d_a > 0$. Thus (3.2) is shown. Similarly, by Lemma 2.11.3 of [42] and (1.6), we have

$$\xi(\lambda) = \phi(\lambda)\beta + \bar{\xi}(\lambda)$$

where $\bar{\xi}(\lambda) \in \mathcal{M}_\lambda^+ \cap \mathbf{K}_\phi$. By Lemma 2.4, $\bar{\xi}(\lambda)$ can be further expressed as

$$\bar{\xi}(\lambda) = t_a \bar{X}^a(\lambda) \quad (\because \bar{X}^b(\lambda) = \mathbf{0}).$$

So

$$\xi(\lambda) = \phi(\lambda)\beta + t_a \bar{X}^a(\lambda), \qquad \xi = \lim_{\lambda \to 0} \xi(\lambda) = \Gamma\beta + t_a \bar{X}^a.$$

By Lemma 2.11, $\lim_{\lambda \to \infty} \lambda\eta(\lambda)(\Gamma\beta + \bar{X}^b) < \infty$, so (1.9) implies

$$\lim_{\lambda \to \infty} \lambda\eta(\lambda)\bar{X}^a = \infty.$$

On the other hand,

$$\mathbf{1} - \xi = \Gamma\beta + \bar{X}^a + \bar{X}^b - \xi = (1 - t_a)\bar{X}^a + \bar{X}^b,$$

so the above two equalities and (1.8) forces $t_a = 1$ and thus

$$\xi(\lambda) = \phi(\lambda)\beta + \bar{X}^a(\lambda) = \mathbf{1} - \lambda\phi(\lambda)\mathbf{1},$$

as is desired.

Finally, by Chen's theorem, it is obvious that the process $R(\lambda)$ constructed as above is honest iff $c \equiv \lambda\langle\eta(\lambda), \mathbf{1} - \xi\rangle = \sigma^b$. Since the choices of $\eta(\lambda)$ are infinite, there exist infinitely many honest processes as well.

(2) Suppose z^a is exit, and z^b is entrance or natural ($b \neq a$, $a, b \in \{1, 2\}$). So Q_N is uni-exit and thus $\bar{X}^b(\lambda) = \mathbf{0}$, $\bar{\eta}^a(\lambda) = \mathbf{0}$. We only show the necessary condition $\alpha\mathbf{1} = \infty$ and argue the uniqueness for the honest processes. The proof for the other conclusions are similar to the proof of (1).

Suppose that the Q_E-process exists, then for any Q_E-process $R(\lambda)$, its projection $\psi(\lambda)$ on $N \times N$ is the Feller minimal Q_N-process $\phi(\lambda)$. By Chen's theorem, there exist uniquely a row vector $\eta(\lambda) \in \mathbf{H}_\phi$ and a column vector $\xi(\lambda) \in \mathbf{K}_\phi$ such that (1.5)–(1.9) hold. By Lemma 2.11.3 of [42] and Lemma 2.5, we have $\eta(\lambda) = \alpha\phi(\lambda) + d\bar{\eta}^b(\lambda)$ where $d \geq 0$. So if $\alpha\mathbf{1} < \infty$, then by Lemma 2.9,

$$\begin{aligned}\limsup_{\lambda \to \infty} \lambda\eta(\lambda)\bar{X}^a &= \limsup_{\lambda \to \infty} (\lambda\alpha\phi(\lambda)\bar{X}^a + \lambda d_b \bar{\eta}^b(\lambda)\bar{X}^a) \\ &= \limsup_{\lambda \to \infty} \lambda\alpha\phi(\lambda)\bar{X}^a \leq \alpha\mathbf{1} < \infty.\end{aligned}$$

In addition, by Lemma 2.11, we have

$$\sup_{\lambda>0} \lambda\eta(\lambda)(Y^a + Y^b) < \infty, \quad \lim_{\lambda\to\infty} \lambda\eta(\lambda)\bar{X}^b < \infty.$$

So we get

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} = \lim_{\lambda\to\infty} \lambda\eta(\lambda)(Y^a + Y^b + \bar{X}^a + \bar{X}^b) < \infty,$$

which is in contradiction to (1.9). The necessity is shown.

Finally, when the existence conditions are satisfied, we see that when z^a is exit and z^b is natural, $\eta(\lambda) = \alpha\phi(\lambda)$ is uniquely determined, thus the honest process is unique. While in other cases, the choices of $\eta(\lambda)$ are infinite, therefore we have infinitely many honest processes. The proof is complete. □

Theorem 3.4. *If both z^1 and z^2 are regular, then the Q_E-process exists if and only if $\alpha\phi(\lambda) \in l_N$.*

When $\alpha\phi(\lambda) \in l_N$, each Q_E-processes $R(\lambda)$ is either an expansion of the minimal Q_N-process $\phi(\lambda)$ or an expansion of a non-minimal Q_N-process $\psi(\lambda)$.

(1) *The expansion Q_E-processes of the minimal Q_N-process $\phi(\lambda)$ can be obtained in the following way: Take $\psi(\lambda) = \phi(\lambda)$,*

$$\eta(\lambda) = \alpha\phi(\lambda) + d_1\bar{\eta}^1(\lambda) + d_2\bar{\eta}^2(\lambda) \tag{3.6}$$

$$\xi(\lambda) = \phi(\lambda)\beta + t_1\bar{X}^1(\lambda) + t_2\bar{X}^2(\lambda) \tag{3.7}$$

where $d_a \geq 0$, $0 \leq t_a \leq 1, a \in \{1,2\}$ ($d_1 + d_2 > 0$ when $\alpha\mathbf{1} < \infty$; $t_a = 1$ when $d_a > 0$ or $\alpha\bar{X}^a = \infty$), then take a constant c such that (1.8) *holds, and finally, define $r(\lambda)$ as* (1.10) *and $R(\lambda)$ as* (3.1).

$R(\lambda)$ is honest iff $t_1 = t_2 = 1$ and $c = 0$. Moreover, there exist infinitely many honest expansion Q_E-processes of $\phi(\lambda)$.

(2) *The expansion Q_E-processes of the non-minimal Q_N-processes can be obtained in the following way (This case forces $\alpha\bar{X}^1 \wedge \alpha\bar{X}^2 < \infty$):*

When $\alpha\bar{X}^a < \infty$ ($a = 1$ or 2), first take a non-minimal Q_N-process

$$\psi(\lambda) = \phi(\lambda) + \bar{X}^a(\lambda)F^a(\lambda) \tag{3.8}$$

where $F^a(\lambda) = F^a(\lambda; c_1, \alpha^a, d_a, 0)$ is taken as in Lemma 2.14; *then pick up a constant $A \geq 0$, take a row vector*

$$\eta(\lambda) = \eta(\lambda;\, c_1, c_2, A, \alpha, \alpha^a, d_a, 0, d'_a, d'_b) \tag{3.9}$$

as in Lemma 2.15, *where $d'_a = Ad_a$, $d'_b \geq 0$ ($d'_b > 0$ if $\alpha\mathbf{1} < \infty$), c_2 satisfies*

$$Ac_1 + c_2 \geq \alpha\bar{X}^a \quad (Ac_1 + c_2 = \alpha\bar{X}^a \text{ if } \alpha^a\mathbf{1} < \infty \text{ and } d_a = 0); \tag{3.10}$$

then take a column vector

$$\xi(\lambda) = \xi(\lambda;\, \beta, t_a, F^a(\lambda)) \tag{3.11}$$

as in Lemma 2.18, *furthermore, select a constant c satisfying* (1.8) *and finally, define $r(\lambda)$ as* (1.10) *and $R(\lambda)$ as* (3.1).

$R(\lambda)$ is honest iff $t_a = 1$ and $c = 0$. Moreover, there exist infinitely many honest expansion Q_E-processes of the non-minimal Q_N-processes.

Proof. Suppose both the two boundary points z^1 and z^2 are regular. Then by Remark 2.6(2), $\bar{X}^a(\lambda) \neq \mathbf{0}$, $\bar{\eta}^a(\lambda) \neq \mathbf{0}$, for $a = 1, 2$. If Q_E-process exists, by Lemma 2.10, each Q_E-process $R(\lambda)$ is an expansion of a B-type Q_N-process $\psi(\lambda)$, which takes the following form

$$\psi(\lambda) = \phi(\lambda) + \bar{X}^1(\lambda)F^1(\lambda) + \bar{X}^2(\lambda)F^2(\lambda)$$

where $F^a(\lambda) \geq \mathbf{0}$, $a = 1, 2$ are two row vectors defined on N satisfying $\lambda F^a(\lambda)\mathbf{1} \leq 1$. Furthermore, by Lemma 2.13, there necessarily exists at least one $a \in \{1, 2\}$ such that $F^a(\lambda) = \mathbf{0}$.

(1) When $F^1(\lambda) = F^2(\lambda) = \mathbf{0}$, the Q_E-process $R(\lambda)$ is an expansion of the minimal Q_N-process $\psi(\lambda) = \phi(\lambda)$. By Chen's decomposition theorem, there exist uniquely a row vector $\eta(\lambda) \in \mathbf{H}_\phi$, a column vector $\xi(\lambda) \in \mathbf{K}_\phi$ and a constant c such that (1.5)-(1.9) hold. It is easy to show $\alpha\phi(\lambda) \in l_N$ by using the same argument as in Theorem 3.3. Moreover, by Lemma 2.11.3 of [42] and (1.6), we have

$$\eta(\lambda) = \alpha\phi(\lambda) + \bar{\eta}(\lambda), \qquad \xi(\lambda) = \phi(\lambda)\beta + \bar{\xi}(\lambda)$$

where $\bar{\eta}(\lambda) \in \mathcal{L}_\lambda^+ \cap \mathbf{H}_\phi$ and $\bar{\xi}(\lambda) \in \mathcal{M}_\lambda^+ \cap \mathbf{K}_\phi$. By Lemma 2.5 and Lemma 2.4, $\bar{\eta}(\lambda)$, $\bar{\xi}(\lambda)$ can be further expressed as

$$\bar{\eta}(\lambda) = d_1\bar{\eta}^1(\lambda) + d_2\bar{\eta}^2(\lambda), \qquad \bar{\xi}(\lambda) = t_1\bar{X}^1(\lambda) + t_2\bar{X}^2(\lambda)$$

with $d_a \geq 0$, $t_a \geq 0$. So

$$\begin{aligned} \eta(\lambda) &= \alpha\phi(\lambda) + d_1\bar{\eta}^1(\lambda) + d_2\bar{\eta}^2(\lambda) \\ \xi(\lambda) &= \phi(\lambda)\beta + t_1\bar{X}^1(\lambda) + t_2\bar{X}^2(\lambda), \end{aligned} \tag{3.12}$$

therefore

$$\begin{aligned} \xi &= \lim_{\lambda\to 0}\xi(\lambda) = \Gamma\beta + t_1\bar{X}^1 + t_2\bar{X}^2 = Y^1 + Y^2 + t_1\bar{X}^1 + t_2\bar{X}^2 \\ \mathbf{1} - \xi &= Y^1 + Y^2 + \bar{X}^1 + \bar{X}^2 - \xi = (1-t_1)\bar{X}^1 + (1-t_2)\bar{X}^2 \end{aligned}$$

and

$$\begin{aligned} \mathbf{1} - \xi(\lambda) &= \mathbf{1} - \phi(\lambda)\beta - t_1\bar{X}^1(\lambda) - t_2\bar{X}^2(\lambda) \\ &= \lambda\phi(\lambda)\mathbf{1} + \bar{X}^1(\lambda) + \bar{X}^2(\lambda) - t_1\bar{X}^1(\lambda) - t_2\bar{X}^2(\lambda) \quad (\because (2.22)) \\ &= \lambda\phi(\lambda)\mathbf{1} + (1-t_1)\bar{X}^1(\lambda) + (1-t_2)\bar{X}^2(\lambda). \end{aligned}$$

By (1.5) and noticing $\psi(\lambda) = \phi(\lambda)$, we have

$$\begin{aligned} \mathbf{0} &\leq \mathbf{1} - \lambda\phi(\lambda)\mathbf{1} - \xi(\lambda) \\ &= (1-t_1)\bar{X}^1(\lambda) + (1-t_2)\bar{X}^2(\lambda), \end{aligned}$$

which implies $t_1 \leq 1$ and $t_2 \leq 1$. It is easy to see $t_1 = t_2 = 1$ iff (1.5) holds the equality.

Furthermore, we have

$$\begin{aligned}\lambda\eta(\lambda)\mathbf{1} &= \lambda\eta(\lambda)(Y^1+Y^2+\bar{X}^1+\bar{X}^2)\\ &= \lambda\eta(\lambda)(Y^1+Y^2)+\lambda\alpha\phi(\lambda)(\bar{X}^1+\bar{X}^2)+\\ &\quad +(d_1\lambda\bar{\eta}^1(\lambda)+d_2\lambda\bar{\eta}^2(\lambda))(\bar{X}^1+\bar{X}^2)\\ &= \lambda\eta(\lambda)(Y^1+Y^2)+\lambda\alpha\phi(\lambda)(\bar{X}^1+\bar{X}^2)+\\ &\quad +d_1\lambda\bar{\eta}^1(\lambda)\bar{X}^1+d_2\lambda\bar{\eta}^2(\lambda)\bar{X}^2 \quad (\because (2.23)),\end{aligned}$$

so from Lemma 2.9 and Lemma 2.11, we conclude that (1.9) holds if and only if $d_1+d_2>0$ when $\alpha(\bar{X}^1+\bar{X}^2)<\infty$, or equivalently, $\alpha\mathbf{1}<\infty$. Since

$$\begin{aligned}\lambda\eta(\lambda)(\mathbf{1}-\xi) &= \lambda\eta(\lambda)[(1-t_1)\bar{X}^1+(1-t_2)\bar{X}^2]\\ &= \lambda\alpha\phi(\lambda)[(1-t_1)\bar{X}^1+(1-t_2)\bar{X}^2]+\\ &\quad +d_1(1-t_1)\lambda\bar{\eta}^1(\lambda)\bar{X}^1+d_2(1-t_2)\lambda\bar{\eta}^2(\lambda)\bar{X}^2,\end{aligned}$$

(1.8) implies that $t_a=1$ when $d_a>0$ or $\alpha\bar{X}^a=\infty$, $a\in\{1,2\}$.

Conversely, if $\alpha\phi(\lambda)\in l_N$, $\eta(\lambda)$, $\xi(\lambda)$ and constant c are taken as (3.6)–(3.7) and (1.8), then from the above argument, we can easily conclude that $\eta(\lambda)\in\mathbf{H}_\phi$, $\xi(\lambda)\in\mathbf{K}_\phi$ and (1.5)–(1.9) hold, thus by Chen's theorem, the process $R(\lambda)$ constructed as in (1) of Theorem 3.4 is a Q_E-process.

It is obvious that the process $R(\lambda)$ is honest if and only if $t_1=t_2=1$ and $c=0$. Furthermore, since the choices of d_1, d_2 and therefore $\eta(\lambda)$ in (3.12), are infinite, we have infinitely many honest expansion Q_E-processes of the minimal Q_N-process $\phi(\lambda)$.

(2) When $R(\lambda)$ is an expansion of a non-minimal Q_N-process $\psi(\lambda)=\phi(\lambda)+\bar{X}^a(\lambda)F^a(\lambda)$ with $F^a(\lambda)\neq\mathbf{0}$ for some $a\in\{1,2\}$, then by Lemma 2.14, $F^a(\lambda)$ can be expressed as $F^a(\lambda)=F^a(\lambda;c_1,\alpha^a,d_a,d_b)$ with c_1,α^a,d_a,d_b satisfying (2.30)–(2.34). Since z^b is regular, by Lemma 2.9, $\lim_{\lambda\to\infty}\lambda\bar{\eta}^b(\lambda)\bar{X}^b=\infty$, so the constant d_b in (2.33) should be taken as zero. Therefore

$$F^a(\lambda)=F^a(\lambda;c_1,\alpha^a,d_a,0).$$

In addition, by Chen's decomposition theorem, there exist uniquely an $\eta(\lambda)\in\mathbf{H}_\psi$, a $\xi(\lambda)\in\mathbf{K}_\psi$ and a constant c such that (1.5)–(1.9) hold. By Lemma 2.15, we have $\alpha\bar{X}^a<\infty$, hence $\alpha\bar{X}^1\wedge\alpha\bar{X}^2<\infty$ and $\eta(\lambda)$ can be expressed as (3.9). By Lemma 2.9 and Remark 2.16, we get c_2 satisfies (3.10). Moreover, by Lemma 2.11(1) and Lemma 2.12,

$$\limsup_{\lambda\to\infty}\lambda\eta(\lambda)(Y^1+Y^2+\bar{X}^a)<\infty,$$

so (1.9) is equivalent to $\lim_{\lambda\to\infty}\lambda\eta(\lambda)\bar{X}^b=\infty$. Since

$$\lambda(\bar{\eta}'(\lambda)-{}^a\bar{\eta}(\lambda))\bar{X}^b=d_b'\lambda\bar{\eta}^b(\lambda)\bar{X}^b,$$

so when $\alpha\mathbf{1}<\infty$, by Lemma 2.9 and Lemma 2.17, we see that $\lim_{\lambda\to\infty}\lambda\eta(\lambda)\bar{X}^b=\infty$, and therefore (1.9), is equivalent to $d_b'>0$. Combining this with (2.38), it follows that d_a' and d_b' satisfy the desired conditions.

On the other hand, by Lemma 2.18, any vector $\xi(\lambda) \in \mathbf{K}_\psi$ satisfying (1.5)–(1.7) can be expressed as (3.11). Furthermore, the other conclusions are guaranteed by Chen's theorem. The necessity is showed.

As for the sufficiency, suppose $\alpha\phi(\lambda) \in l_N$, $\alpha\bar{X}^a < \infty$ and, $\psi(\lambda)$, $\eta(\lambda)$, and $\xi(\lambda)$ are taken as (3.8)–(3.11), and the constant c is taken as (1.8), then it is easy to see $\psi(\lambda)$ is a non-minimal Q_N-process, c satisfies (1.7)–(1.8) and, by Lemma 2.15 and Lemma 2.18, $\eta(\lambda) \in \mathbf{H}_\psi$, $\xi(\lambda) \in \mathbf{K}_\psi$ and they satisfy (1.5)–(1.6). Moreover, the choosing of the constants d_a, d'_a and d'_b result in (1.9). Therefore by Chen's theorem, the process $R(\lambda)$ defined as (1.10) and (3.1) is an expansion Q_E-process of $\psi(\lambda)$.

Finally, the process $R(\lambda)$ is honest iff (1.5) and (1.8) hold the equality which, by Lemma 2.15 and (2.84), are equivalent to $t_a = 1$ and $c = 0$. Furthermore, since the choices of d_a and therefore $\eta(\lambda)$ in (3.9), are infinite, we have infinitely many honest expansion Q_E-processes of the non-minimal Q_N-processes. The proof is complete. □

Theorem 3.5. *If both z^1 and z^2 are exit, then Q_E-process exists if and only if $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l_N$.*

When $\alpha\mathbf{1} = \infty$ and $\alpha\phi(\lambda) \in l_N$, each Q_E-process $R(\lambda)$ is either an expansion of the minimal Q_N-process $\phi(\lambda)$ or an expansion of a non-minimal Q_N-process $\psi(\lambda)$. More precisely, each $R(\lambda)$ can be obtained in the following way.

(1) *When $R(\lambda)$ is an expansion of $\phi(\lambda)$, put $\psi(\lambda) = \phi(\lambda)$, $\eta(\lambda) = \alpha\phi(\lambda)$, $\xi(\lambda) = \phi(\lambda)\beta + t_1\bar{X}^1(\lambda) + t_2\bar{X}^2(\lambda)$ where t_a $(a = 1, 2)$ are two constants satisfying $0 \le t_a \le 1$ and $t_a = 1$ if $\alpha\bar{X}^a = \infty$, then take a constant c satisfying* (1.8), *and finally define $r(\lambda)$ as* (1.10) *and $R(\lambda)$ as* (3.1).

$R(\lambda)$ is honest iff $t_1 = t_2 = 1$ and $c = 0$. Moreover, the honest expansion Q_E-process of the minimal Q_N-process is unique.

(2) *When $R(\lambda)$ is an expansion of a process $\psi(\lambda) \neq \phi(\lambda)$* (*this case forces $\alpha\bar{X}^1 \wedge \alpha\bar{X}^2 < \infty$ and $\alpha\bar{X}^1 \vee \alpha\bar{X}^2 = \infty$*), *in the case $\alpha\bar{X}^a < \infty$ and $\alpha\bar{X}^b = \infty$, $b \neq a$, $a, b \in \{1, 2\}$, first take $\psi(\lambda) = \phi(\lambda) + \bar{X}^a(\lambda)F^a(\lambda)$, where $F^a(\lambda) = F^a(\lambda; c_1, \alpha^a, 0, 0)$ is taken as in Lemma* 2.14, *then pick up a constant $A \ge 0$, take a row vector $\eta(\lambda) = \eta(\lambda; c_1, c_2, A, \alpha, \alpha^a, 0, 0, 0, 0)$ as in Lemma* 2.15, *where c_2 satisfies $Ac_1 + c_2 \ge \alpha\bar{X}^a$* ($Ac_1 + c_2 = \alpha\bar{X}^a$ *if $\alpha^a\mathbf{1} < \infty$*), *then take a column vector $\xi(\lambda) = \xi(\lambda; \beta, t_a, F^a(\lambda))$ as in Lemma* 2.18, *select a constant c satisfying* (1.8), *and finally, define $r(\lambda)$ as* (1.10) *and $R(\lambda)$ as* (3.1).

The process $R(\lambda)$ is honest iff $t_a = 1$ and $c = 0$. Moreover, there exist infinitely many honest expansion Q_E-processes of the non-minimal Q_N-processes.

Proof. Suppose both the two boundary points z^1 and z^2 are exit, then we have $\bar{X}^a(\lambda) \neq \mathbf{0}$ and $\bar{\eta}^a(\lambda) = \mathbf{0}$ for $a = 1, 2$. If the Q_E-process exists, then by Chen's theorem, for each Q_E-process $R(\lambda)$, there exist a Q_N-process $\psi(\lambda)$, two vectors $\eta(\lambda) \in \mathbf{H}_\psi$ and $\xi(\lambda) \in \mathbf{K}_\psi$ and a constant c such that (3.1) and (1.5)–(1.9) hold. Since by Lemma 2.10 and Lemma 2.13, $\psi(\lambda)$ must possesses the form

$$\psi(\lambda) = \phi(\lambda) + \bar{X}^a(\lambda)F^a(\lambda) \tag{3.13}$$

for some $a \in \{1,2\}$ and $F^a(\lambda)$ having the properties of Lemma 2.14, so corresponding to $F^a(\lambda) = \mathbf{0}$ or $\neq \mathbf{0}$, we have $\psi(\lambda) = \phi(\lambda)$ or $\psi(\lambda) \neq \phi(\lambda)$, which means that the Q_E-process $R(\lambda)$ is either an expansion of the minimal Q_N-process $\phi(\lambda)$ or an expansion of a non-minimal Q_N-process $\psi(\lambda)$. In the former case, $\eta(\lambda) \in \mathbf{H}_\phi$ and $\xi(\lambda) \in \mathbf{K}_\phi$, so $\eta(\lambda)$ takes the simple form

$$\eta(\lambda) = \alpha\phi(\lambda)$$

hence $\eta(\lambda) \in l_N$ is equivalent to $\alpha\phi(\lambda) \in l_N$ and (1.9) equivalent to $\alpha\mathbf{1} = \infty$. This shows the existence conditions for the first situation (1) of the theorem.

As for the second case, i.e. $F^a(\lambda) \neq \mathbf{0}$ in (3.13), it immediately follows from Lemma 2.15 that $\alpha\bar{X}^a < \infty$ and $\alpha\phi(\lambda) \in l_N$. In addition, by Lemma 2.11 (1) and the equality $\lambda\eta(\lambda)\mathbf{1} = \lambda\eta(\lambda)(Y^1 + Y^2) + \lambda\eta(\lambda)(\bar{X}^a + \bar{X}^b)$, we see that (1.9) is equivalent to

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)(\bar{X}^a + \bar{X}^b) = \infty. \tag{3.14}$$

Moreover, since $\bar{\eta}^a(\lambda) = \mathbf{0}$ and $\bar{\eta}^b(\lambda) = \mathbf{0}$ $(b \neq a)$ result in all the constants d_a, d_b in (2.33) and d'_a, d'_b in the expression of $\bar{\eta}'(\lambda)$ in Lemma 2.15 equal zero, by Lemma 2.14 and Lemma 2.15, we have

$$\begin{aligned} &\lambda\eta(\lambda)(\bar{X}^a + \bar{X}^b) = \lambda(\eta'(\lambda) - d_\lambda \cdot {}^a\eta(\lambda))(\bar{X}^a + \bar{X}^b) \\ &= \lambda(\eta'(\lambda) - A \cdot {}^a\eta(\lambda))(\bar{X}^a + \bar{X}^b) + \lambda(A - d_\lambda)^a\eta(\lambda)(\bar{X}^a + \bar{X}^b) \\ &= \lambda(\alpha'\phi(\lambda) - A\alpha^a\phi(\lambda))(\bar{X}^a + \bar{X}^b) + \lambda(A - d_\lambda)^a\eta(\lambda)(\bar{X}^a + \bar{X}^b) \\ &= \lambda\alpha\phi(\lambda)(\bar{X}^a + \bar{X}^b) + \lambda(A - d_\lambda)^a\eta(\lambda)(\bar{X}^a + \bar{X}^b) \end{aligned}$$

where $d_\lambda = \dfrac{\lambda\langle\eta'(\lambda), \bar{X}^a\rangle - c_2}{c_1 + \lambda\langle{}^a\eta(\lambda), \bar{X}^a\rangle}$. From the last equality above, and noticing (2.63), we conclude that (3.14) is further equivalent to

$$\lim_{\lambda\to\infty} \lambda\alpha\phi(\lambda)(\bar{X}^a + \bar{X}^b) = \infty. \tag{3.15}$$

While the following inequality

$$\begin{aligned} \alpha(\bar{X}^a + \bar{X}^b) &\leq \lim_{\lambda\to\infty} \lambda\alpha\phi(\lambda)(\bar{X}^a + \bar{X}^b) \leq \lim_{\lambda\to\infty} \lambda\alpha\phi(\lambda)\mathbf{1} \\ &\leq \alpha\mathbf{1} = \alpha(Y^a + Y^b) + \alpha(\bar{X}^a + \bar{X}^b) \end{aligned}$$

implies apparently that (3.15) is equivalent to $\alpha(\bar{X}^a + \bar{X}^b) = \infty$ and also $\alpha\mathbf{1} = \infty$. This shows the existence conditions and the by product that $\bar{X}^a$ and $\bar{X}^b$ satisfy $\alpha\bar{X}^a \wedge \alpha\bar{X}^b < \infty$ and $\alpha\bar{X}^a \vee \alpha\bar{X}^b = \infty$.

We note that in situation (1), the honest Q_E-process is unique since the vectors $\eta(\lambda)$, $\xi(\lambda)$ and the constant c there are all uniquely determined. The proof of the rest conclusions of situation (1) and (2) is very similar to Theorem 3.4, and therefore is omitted. □

Theorem 3.6. *If one of the two boundary points, say z^a, is regular, and the other one z^b is exit ($b \neq a$, $a, b \in \{1, 2\}$), then Q_E-process exists if and only if $\alpha\phi(\lambda) \in l_N$.*

When the condition is satisfied, each Q_E-process $R(\lambda)$ can be obtained in the following way.

(1) When the $R(\lambda)$ is an expansion of the minimal Q_N-process $\phi(\lambda)$, put $\psi(\lambda) = \phi(\lambda)$, let $\eta(\lambda) = \alpha\phi(\lambda) + d_a\bar{\eta}^a(\lambda)$, $\xi(\lambda) = \phi(\lambda)\beta + t_a\bar{X}^a(\lambda) + t_b\bar{X}^b(\lambda)$, where d_a, t_a, t_b ($b \neq a$) are non-negative constants and $d_a > 0$ when $\alpha\mathbf{1} < \infty$; $t_a = 1$ when $\alpha\bar{X}^a = \infty$ or $d_a > 0$; $t_b = 1$ when $\alpha\bar{X}^b = \infty$. Then take a constant c satisfying (1.8), and define $r(\lambda)$ as (1.10) and $R(\lambda)$ as (3.1).

The process $R(\lambda)$ is honest iff $t_a = t_b = 1$ and $c = 0$. Moreover, there exist infinitely many honest expansion Q_E-processes of the minimal Q_N-process $\phi(\lambda)$.

(2) When $R(\lambda)$ is an expansion of a non-minimal Q_N-process $\psi(\lambda)$ (this case forces $\alpha\bar{X}^k \wedge \alpha\bar{X}^{\bar{k}} < \infty$, where $k \neq \bar{k}$, $k, \bar{k} \in \{1, 2\}$), when $\alpha\bar{X}^k < \infty$, first take $\psi(\lambda) = \phi(\lambda) + \bar{X}^k(\lambda)F^k(\lambda)$, where $F^k(\lambda) = \begin{cases} F^a(\lambda;\, c_1, \alpha^a, d_a, 0), & \text{if } k{=}a \\ F^b(\lambda;\, c_1, \alpha^b, d_a, 0), & \text{if } k{=}b \end{cases}$ is taken as in Lemma 2.14, with α^k satisfying $\alpha^k\phi(\lambda) \in l_N$ and $\alpha^k\bar{X}^{\bar{k}} < \infty$; next pick up a constant $A \geq 0$, take a row vector $\eta(\lambda) = \eta(\lambda;\, c_1, c_2, A, \alpha, \alpha^k, d_a, 0, d'_a, 0)$ as in Lemma 2.15, where $\begin{cases} d'_a = Ad_a, & \text{if } k = a \\ d'_a \geq Ad_a, & \text{if } k \neq a \end{cases}$ and c_2 satisfies $Ac_1 + c_2 \geq \alpha\bar{X}^k$ ($Ac_1 + c_2 = \alpha\bar{X}^k$ if $\alpha^k\mathbf{1} < \infty$ and $d_a 1_{\{a\}}(k) = 0$), then take a column vector $\xi(\lambda) = \xi(\lambda;\, \beta, t_k, F^k(\lambda))$ as in Lemma 2.18, more precisely,

$$\xi(\lambda) = \phi(\lambda)\beta + t_k\bar{X}^k(\lambda) + \bar{X}^{\bar{k}}(\lambda) - \lambda\bar{X}^k(\lambda)F^k(\lambda)(Y^1 + Y^2 + t_k\bar{X}^k + \bar{X}^{\bar{k}})$$

where t_k is a constant satisfying $\lim_{\lambda\to\infty} \frac{1}{c_1}\lambda\langle {}^k\eta(\lambda), \mathbf{1} - \bar{X}^k\rangle \leq t_k \leq 1$, then select a constant c satisfying (1.8), and finally, define $r(\lambda)$ as (1.10) and $R(\lambda)$ as (3.1).

The process $R(\lambda)$ constructed above is honest iff $t_k = 1$ and $c = 0$. Moreover, there exist infinitely many honest expansion Q_E-processes of the non-minimal Q_N-processes.

Proof. Suppose the boundary point z^a is regular, and the other one z^b is exit ($b \neq a$, $a, b \in \{1, 2\}$). Then we have $\bar{X}^a(\lambda) \neq \mathbf{0}$, $\bar{X}^b(\lambda) \neq \mathbf{0}$, $\bar{\eta}^a(\lambda) \neq \mathbf{0}$ and $\bar{\eta}^b(\lambda) = \mathbf{0}$. If the Q_E-process exists, then by the same argument to Theorem 3.5, each Q_E-process $R(\lambda)$ is an expansion of a B-type Q_N-process $\psi(\lambda)$, which takes the following form

$$\psi(\lambda) = \phi(\lambda) + \bar{X}^k(\lambda)F^k(\lambda),$$

where $F^k(\lambda) \geq \mathbf{0}$, $k = a$ or b, is a row vector defined on N satisfying $\lambda F^k(\lambda)\mathbf{1} \leq 1$.

(1) When $R(\lambda)$ is an expansion by the minimal Q_N-process $\phi(\lambda)$, i.e. $F^k(\lambda) = \mathbf{0}$, then by Chen's decomposition theorem, there exist uniquely a row vector $\eta(\lambda) \in \mathbf{H}_\phi$, a column vector $\xi(\lambda) \in \mathbf{K}_\phi$ and a constant c such that

(3.1) and (1.5)–(1.9) hold. By Lemma 2.11.3 of [42] and (1.6), we have

$$\eta(\lambda) = \alpha\phi(\lambda) + d_a\bar{\eta}^a(\lambda), \qquad \xi(\lambda) = \phi(\lambda)\beta + t_a\bar{X}^a(\lambda) + t_b\bar{X}^b(\lambda)$$

for some non-negative constants d_a, t_a and t_b. So by following the same lines of the corresponding proof in Theorem 3.4, we can easily show the existence condition $\alpha\phi(\lambda) \in l_N$ and the first part (1).

(2) When $R(\lambda)$ is an expansion of a non-minimal Q_N-process $\psi(\lambda)$, i.e. $F^k(\lambda) \neq \mathbf{0}$, we can also follow the same way to Theorem 3.4 to show the existence condition $\alpha\phi(\lambda) \in l_N$ and the conclusions of the second part (2).

The proof is completed. □

Finally, we give the equivalent conditions for $\alpha\phi(\lambda) \in l_E$, which enable us to check by using the elements of the pre-generator matrix.

Proposition 3.1. *Let* $\alpha_1 = (q_{0j}; j \in N_1)$, $\alpha_2 = (q_{0j}; j \in N_2)$, $\alpha = (\alpha_1, \alpha_2)$. *Let* $W_1 = (\cdots, w^1_{-j}, \cdots, w^1_{-2}, w^1_{-1})^\tau$, $W_2 = (w^2_1, w^2_2, \cdots, w^2_j, \cdots)^\tau$ *where* $w^1_{-j} = (z^1 - z_{-j})\sum_{n=1}^{j}\mu_{-n} + \sum_{n=j+1}^{\infty}(z^1 - z_{-n})\mu_{-n}$ *and* $w^2_j = (z^2 - z_j)\sum_{n=1}^{j}\mu_n + \sum_{n=j+1}^{\infty}(z^2 - z_n)\mu_n$. *Then*

(1) $\alpha\phi(\lambda) \in l_E$ *is equivalent to* $\alpha_a\phi^a(\lambda) \in l_{N_a}$, $a = 1, 2$.

(2) *If* z^a *is regular or exit, then* $\alpha_a\phi^a(\lambda) \in l_{N_a}$ *is equivalent to* $\alpha_a W_a < \infty$, *or more precisely,* $\sum_{j\in N_a} q_{0j}w^a_j < \infty$.

(3) *If* z^a *is entrance or natural, then* $\alpha_a\phi^a(\lambda) \in l_{N_a}$ *is equivalent to* $\alpha_a\mathbf{1} < \infty$, *or more precisely,* $\sum_{j\in N_a} q_{0j} < \infty$.

Proof. The conclusions easily follow from Theorem 6.9.3, Theorem 6.9.4 and Theorem 6.9.5 of [42]. □

Acknowledgement. The author is deeply grateful to Professor I. Shigekawa for his constant encouragement, valuable suggestions and helpful comments on the work of this paper.

Nippon Institute of Technology
Miyashiro, Saitama, 345-8501 Japan
e-mail: lqpktu@hotmail.com

References

[1] A. Y. Chen, *Construction problems in Q-processes with instantaneous states* (in Chinese), Chinese Ann. Math. **8A**(1) (1987), 52–60.

[2] A. Y. Chen and E. Renshaw, *Markov branching processes with instantaneous immigration*, Probab. Theory Related Fields **87** (1990), 209–240.

[3] ———, *Existence and uniqueness criteria for conservative uni-instantaneous denumerable Markov processes*, Probab. Theory Related Fields **94** (1993), 427–456.

[4] A. Y. Chen, H. J. Zhang and Z. T. Hou, *Feller transition functions, resolvent decomposition theorems and their application in unstable denumerable Markov processes*, in Markov Processes and Controlled Markov Chains, Kluwer Acad. Pub., Netherlands, 15–39, 2002.

[5] A. Y. Chen and K. Liu, *Birth-death processes with an instantaneous reflection barrier*, J. Appl. Probab. **40** (2003), 163–179.

[6] M. F. Chen, *On three classical problems for Markov chains with continuous time parameters*, J. Appl. Probab. **28** (1991), 305–320.

[7] ———, From Markov Chains to Non-equilibrium Particle Systems, World Scientific, Singapore, 1992.

[8] Chung, K. L., Markov chains with stationary transition probabilities, Springer-Verlag, Berlin Gottingen Heidelberg, 1960.

[9] ———, Lectures on boundary theory for Markov chains, Ann. of Math. Stud. **65**, Princeton University Press, 1970.

[10] Doob, J. L., *Markov chains - Denumerable case*, Trans. Amer. Math. Soc. **58** (1945), 455–473.

[11] Fei, Z. L., *A criterion for uniqueness of honest Q-processes with only one instantaneous state* (in Chinese), Chinese J. Appl. Probab. Stat. **7**-3 (1991), 225–230.

[12] ———, *Critreia for the uniqueness of honest Q-processes with finitely many instantaneous states* (in Chinese), Chinese J. Appl. Probab. Stat. **9**-1 (1993), 51–57.

[13] Feller, W., *On the integro-differential equations of purely discontinuous Markov processes*, Trans. Amer. Math. Soc. **48** (1940), 488–515.

[14] ———, *On boundaries and lateral conditions for the Kolmogorov differential equations*, Ann. of Math. **65** (1957), 527–570.

[15] ———, *The birth and death processes as diffusion processes*, J. Math. Pures Appl. **9** (1959), 301–345.

[16] Hou, C., *The uniqueness criterion for Q-processes*, Sci. Sinica **17** (1974), 141–159.

[17] Hou, Z. T. and Guo, Q. F., Homogeneous denumerable Markov chains, Springer, Berlin, 1988.

[18] Hou, Z. T., Zou, J. Z., Zhang, H. J., Liu, Z. M., Xiao, G. N., Chen, A. Y. and Fei, Z. L., The Q-matrix problem for Markov chains (in Chinese), Hunan Science and Technology Publishing House, Changsha, China, 1994.

[19] Liu, Q. P., *The existence for birth and death Q-processes with both infinitely many instantaneous and stable states* (in Chinese), Hunan Ann. Math. **12**-1–2 (1992), 154–161.

[20] ______, *Existence for summable Q-processes with both infinitely many instantaneous and stable states* (in Chinese), J. Cent.-South Inst. Min. Metall. **23**-2 (1992), 20–25.

[21] Neveu, J., *Entrance, exit and fictitious states for Markov chains*, Proc. Aarhus Colloq. Combin. Prob. (1962), 64–68.

[22] Reuter, G. E. H., *Denumerable Markov processes and the associated semigroup on l*, Acta. Math. **97** (1957), 1–46.

[23] ______, *Denumerable Markov processes* (II), J. London Math. Soc. **34** (1959), 81–91.

[24] ______, *Denumerable Markov processes* (III), J. London Math. Soc. **37** (1962), 63–73.

[25] ______, *Remarks on a Markov chain example of Kolmogorov*, Z. Wahrs. View. Geb. **13** (1969), 315–320.

[26] Reuter, G. E. H. and Riley, P. W., *The Feller property for Markov semigroups on a countable state space*, J. London Math. Soc. **5**(2) (1972), 267–275.

[27] Reuter, G. E. H., *Denumerable Markov processes* (IV), *On C.T. Hou's uniqueness theorem for Q-semigroups*, Z. Wahrsch. View. Geb. **33** (1976), 309–315.

[28] L. C. G. Rogers and D. Williams, Diffusions, Markov Processes, and Martingales, Vol. 2, John Wiley & Sons, 1987.

[29] Tang, L. Q., *The construction of uni-instantaneous birth and death processes* (in Chinese), Chinese Ann. Math. **8A**(5) (1987), 565–570.

[30] Wang, Z. K., *The construction thoery of birth-death processes* (in Chinese), Chinese Adv. Math. **5** (1962), 137–187.

[31] Wang, Z. K. and Yang, X. Q., *The construction of stopping birth-death processes* (in Chinese), Acta Math. Sinica **21** (1978), 66–71.

[32] ______, *The probability-analysis method in construction thoery of stopping birth-death processes* (in Chinese), J. Nankai Univ. (Natural Sci.) **3** (1979), 1–32.

[33] ———, Birth and Death Processes and Markov Chains, Springer-Verlag, 1992.

[34] Williams, D., *On the construction problem for Markov chains*, Z. Wahrsch. View. Geb. **3** (1964), 227–246.

[35] ———, *A note on the Q-matrices of Markov chains*, Z. Wahrsch. View. Geb. **7** (1967), 116–121.

[36] ———, *The Q-matrix problem*, In Meyer, P. A. (ed.) Séminaire de Probabilities X (Lecture Notes in Math. **511**, 216–234), Berlin Heidelberg New York, Springer, 1976.

[37] ———, Diffusions, Markov Processes and Martingales, Vol.1: Foundations, John Wiley, Chichester, New York, 1979.

[38] William, J. Anderson, Continuous-time Markov chains: an applications-oriented approach, Springer-Verlag, New York, 1991.

[39] Yang, X. Q., *The bilateral birth and death processes* (in Chinese), J. Nankai Univ. (Natural Sci.), **5**(5) (1964), 9–40.

[40] ———, *A class of birth and death processes* (in Chinese), Acta Math. Sinica **15** (1965), 9–31.

[41] ———, *The lateral conditions of Kolmogorov's backward equations* (in Chinese), Acta Math. Sinica **16** (1966), 429–452.

[42] ———, The Construction Theory of Denumerable Markov processes, John Wiley, England, and Hunan Science and Technology Publishing House, China, 1990.