J. Math. Kyoto Univ. (JMKYAZ)
46-1 (2006), 147–165

Stable homotopy groups of spheres and higher singularities

By

Yoshifumi ANDO

Abstract

We will construct an isomorphism of the group of all cobordism classes of fold-maps of degree 0 of n-dimensional closed oriented manifolds to the n-sphere to the n-th stable homotopy group π_n^s of spheres. As an application we will show that elements of π_n^s are detected by higher singularities of certain maps in dimensions $n < 8$.

1. Introduction

Let N and P be smooth (C^∞) manifolds of dimension n. Let $k \gg n$ (k maybe ∞). Let $J^k(N,P)$ denote the k-jet bundle of manifolds N and P with projection $\pi_N^k \times \pi_P^k$ onto $N \times P$, whose canonical fiber is the space $J^k(n,n)$ of all k-jets of map germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. Here, π_N^k and π_P^k map a k-jet to its source and target respectively. Let $I = (i_1, i_2, \ldots, i_r)$ be a Thom-Boardman symbol (simply symbol) where $i_1, i_2, \ldots, i_r$ are a finite number of integers with $i_1 \geq i_2 \geq \cdots \geq i_r \geq 0$. In [11] there have been defined what is called the Boardman manifold $\Sigma^I(N,P)$ in $J^k(N,P)$. A smooth map germ $f : (N,x) \to (P,y)$ has x as a singularity of the symbol I if and only if $j_x^k f \in \Sigma^I(N,P)$. Let $\Omega^I(N,P)$ denote the open subset of $J^k(N,P)$ which consists of all Boardman manifolds $\Sigma^{I'}(N,P)$ with symbols I' of length r and $I' \leq I$ in the lexicographic order. It is known that $\Omega^I(N,P)$ is an open subbundle of $J^k(N,P)$ over $N \times P$, whose canonical fiber in $J^k(n,n)$ is denoted by $\Omega^I(n,n)$. A smooth map $f : N \to P$ is called an Ω^I-*regular map* if and only if $j^k f(N) \subset \Omega^I(N,P)$. When $I = (1,0)$, an $\Omega^{(1,0)}$-regular map is called a *fold-map*.

Let I be a Thom-Boardman symbol with $I \geq (1,0)$, namely either $i_1 > 1$ or $i_1 = 1$ and $i_2 \geq 0$.

Let P be a closed connected oriented smooth manifold of dimension n. We define the notion of oriented Ω^I-cobordism classes of fold-maps. Let $f_i : N_i \to P$ ($i = 0,1$) be two fold-maps of degree d, where N_i are closed oriented smooth n-dimensional manifolds. We say that they are *oriented* Ω^I-*cobordant* when there exists an Ω^I-regular map, say Ω^I-*cobordism* $E : (W, \partial W) \to (P \times$

2000 *Mathematics Subject Classification*(*s*). Primary 57R45; Secondary 55Q45
Received September 13, 2005

$[0,1], P\times 0\cup P\times 1)$ of degree d such that, for a sufficiently small positive number ϵ,

(i) W is an oriented smooth manifold of dimension $n+1$ with $\partial W = N_0\cup(-N_1)$ and the collar of ∂W is identified with $N_0\times[0,\epsilon]\cup N_1\times[1-\epsilon,1]$,

(ii) $E|N_0\times[0,\epsilon]=f_0\times id_{[0,\epsilon]}$ and $E|N_1\times[1-\epsilon,1]=f_1\times id_{[1-\epsilon,1]}$.

Let $\Omega^I_{fold}(P)$ (resp. $\Omega^I_{fold,d}(P)$) denote the set of all oriented Ω^I-cobordism classes of fold-maps to P (resp. of degree d). When $I=(1,0)$, we simply write $\Omega_{fold}(P)$ and $\Omega_{fold,d}(P)$ for $\Omega^I_{fold}(P)$ and $\Omega^I_{fold,d}(P)$ respectively. We provide $\Omega_{fold}(P)$ and $\Omega_{fold,0}(P)$ the structures of modules in the usual way.

Let F_k (resp. F^d_k) denote the space of all base point preserving maps (resp. of degree d) of S^{k-1} with compact-open topology. The suspension induces the inclusions $F_k\to F_{k+1}$ and $F^d_k\to F^d_{k+1}$. Let F and F^d denote the space $\lim_{k\to\infty}F_k$ and $\lim_{k\to\infty}F^d_k$ respectively. Then we have the following theorem.

Theorem 1.1. *Let $n\geq 2$ and P be a closed connected oriented n-dimensional manifold. Then there exists the isomorphism $\omega:\Omega_{fold}(P)\to[P,F]$, which induces the bijection $\omega_d:\Omega_{fold,d}(P)\to[P,F^d]$.*

We have proved that ω is an epimorphism in [7, Corollary 2], while ω turns out to be an isomorphism. This fact has also been proved in [10] from a different point of view. Therefore, F^d is the classifying space of the cobordism set $\Omega_{fold,d}(P)$. We will first construct the isomorphism of $\Omega_{fold}(P)$ to $\pi_{n+k}(T(\nu^k_P))$, where $T(\nu^k_P)$ is the Thom space of the stable k-dimensional normal bundle ν^k_P of P by using the results in [6]. By using S-dual spaces and duality maps in the suspension category in [24] and [28], we can prove that $\pi_{n+k}(T(\nu^k_P))$ is isomorphic to the set of homotopy classes $[P,F]$ even if we take the degree d into consideration.

Let $\pi^s_n=\lim_{k\to\infty}\pi_{n+k}(S^k)$ denote the n-th stable homotopy group of spheres. It follows from [2] that $[S^n,F^0]$ is canonically isomorphic to π^s_n. So identifying $[S^n,F^0]$ with π^s_n, we have the following corollary.

Corollary 1.1. *The map $\omega_0:\Omega_{fold,0}(S^n)\to\pi^s_n$ is an isomorphism for $n\geq 1$.*

For two symbols I and J of any lengths, we write $I\leq J$ when $\Omega^I(m,m)\subset\Omega^J(m,m)$ for any number m and write $I<J$ when $I\leq J$ and $\Omega^I(m,m)\subsetneq\Omega^J(m,m)$ for some number m in this paper. Let $\mathrm{j}^I:\Omega_{fold,0}(S^n)\to\Omega^I_{fold,0}(S^n)$ denote the homomorphism which maps an $\Omega^{(1,0)}$-cobordism class $[f]$ to the Ω^I-cobordism class of $f:N\to S^n$. If $\mathrm{j}^I([f])=0$, then there exists an Ω^I-cobordism $E^f:(V,N)\to(S^n\times I,S^n\times 0)$ with $\partial V=N$ and $E^f|N=f$. We call E^f an *extension* of f. Let $I(f)$ denote the smallest symbol I such that $\mathrm{j}^I([f])$ is a null element. It is obvious that $I(f)$ depends only on the cobordism class $[f]$ in $\Omega_{fold,0}(S^n)$. We denote a generic $\Omega^{I(f)}$-regular extension E^f by $E^{I(f)}$ in this paper. In dimensions $n<8$ we will calculate $I(f)$ and show that if V is parallelizable in addition, then the singularities of certain type with symbol $I(f)$ of an extension $E^{I(f)}$ detect the stable homotopy class $\omega_0([f])\in\pi^s_n$.

Let us explain the result. Recall that $\pi_1^s \approx \pi_2^s \approx \mathbb{Z}/(2)$, $\pi_3^s \approx \mathbb{Z}/(24)$, $\pi_6^s \approx \mathbb{Z}/(2)$, $\pi_7^s \approx \mathbb{Z}/(240)$ and $\pi_n^s \approx \{0\}$ for $n = 4, 5$. In the dimension $n = 7$, we have to review an elaborate work in [15] to state the result. Let $IV_4 = (x^2 + y^2, x^4)$ and $(x^2 + y^3, xy^2)$ stand for the orbit of the k-jets of the C^∞-stable germs $(\mathbb{R}^8, 0) \to (\mathbb{R}^8, 0)$ of the symbols $(2, 0)$ and $(2, 1, 0)$, which are characterized by the local algebras $\mathbb{R}[[x, y]]/(x^2 + y^2, x^4)$ and $\mathbb{R}[[x, y]]/(x^2 + y^3, xy^2)$, by the group action of $\mathrm{Diff}(\mathbb{R}^8, 0)\times\mathrm{Diff}(\mathbb{R}^8, 0)$ respectively. These classes of the singularities have been defined in [22]. It has been proved in [15, Theorem 2.7] that there have been defined the cycle $\langle(x^2 + y^3, xy^2) - 2IV_4\rangle$ under the integer coefficients of the Vassilyev complex and the integer Thom polynomial of $\langle(x^2 + y^3, xy^2) - 2IV_4\rangle$. We apply this result to a fold-map $f : N \to S^n$ of degree 0 and an extension $E^{I(f)}$, and denote the algebraic numbers of the singular points of types $(x^2 + y^3, xy^2)$ and IV_4 of $E^{I(f)}$ by A and B respectively. Then it will turn out that $A - 2B$ is divisible by $6 \cdot 9 = 54$.

Theorem 1.2. *Let $[f] \in \Omega_{fold,0}(S^n)$ and $E^{I(f)}$ be an extension of f as above. Suppose that $\omega_0([f]) \neq 0$ in π_n^s. Then we have the following.*

If $n = 1$, then $E^{I(f)}$ must have the odd number of singularities of the symbol $(1, 1, 0)$.

If $n = 2$, then $E^{I(f)}$ must have the 1-dimensional singularities of the symbol $(1, 1, 0)$.

If $n = 3$, then we identify $\omega_0([f]) \in \pi_3^s \approx \mathbb{Z}/(24)$ with the corresponding number modulo 24. Then the algebraic number of singular points of the symbol $(2, 0)$ of $E^{I(f)}$ is equal to $2\omega_0([f])$ modulo 48.

If $n = 6$, then $E^{I(f)}$ must have the 3-dimensional singularities of the symbol $(2, 0)$.

If $n = 7$, then we have that $I(f) = (2, 0)$ or $(2, 1, 0)$. If we take V to be parallelizable and denote the algebraic numbers of the singular points of types (x^2+y^3, xy^2) and IV_4 of $E^{I(f)}$ by A and B respectively, then $A-2B$ is divisible by $6 \cdot 9 = 54$ and the integer $(A - 2B)/54$ modulo 240 corresponds to the stable homotopy class $\omega_0([f]) \in \pi_7^s \approx \mathbb{Z}/(240)$.

In general it will be a hard problem to detect a non-zero element $\omega_0([f]) \in \pi_n^s$ by higher singularities of $E^{I(f)}$ in dimensions $n \geq 8$. This range lies outside the Mather's nice range in [22] and there are many difficulties for the study of singularities such as integer Thom polynomials.

In Section 2 we will explain notations used in this paper. In Section 3 we will review the results which are necessary for the definition of ω_d and will prove Theorem 1.1. In Section 4 we will prove that an Ω^1-regular map is homotopic relative to a fold-map to $\Omega^{(1,1,0)}$-regular map. In Section 5 we will study the obstructions for finding simpler extensions E^f of fold-maps f in order to determine $I(f)$. In Section 6 we will construct a special fold-map f such that $\omega_0([f])$ generates π_6^s and an extension E^f to determine $I(f)$. In Section 7 we will prove Theorem 1.2.

2. Preliminaries

Throughout the paper all manifolds are smooth of class C^∞. Maps are continuous, but may be smooth (of class C^∞) if necessary. Given a fiber bundle $\pi^G : G \to X$ and a subset C in X, we denote $\pi^{-1}(C)$ by $G|_C$. Let $\pi^H : H \to Y$ be another fiber bundle. A map $\tilde{b} : G \to H$ is called a fiber map over a map $b : X \to Y$ if $\pi^H \circ \tilde{b} = b \circ \pi^G$ holds. The restriction $\tilde{b}|(G|_C) : G|_C \to H$ (or $H|_{b(C)}$) is denoted by $\tilde{b}|_C$. In particular, for a point $x \in X$, $G|_x$ and $\tilde{b}|_x$ are simply denoted by G_x and $\tilde{b}_x : G_x \to H_{b(x)}$ respectively. The trivial bundle $X \times \mathbb{R}^k$ is denoted by ε^k_X.

Let $G \to X$ and $H \to Y$ be n-dimensional vector bundles. Define the vector bundle $J^k(G, H)$ over $X \times Y$ by

$$J^k(G,H) = \bigoplus_{i=1}^{k} \mathrm{Hom}(S^i(\pi_X^*(G)), \pi_Y^*(H)) \tag{2.1}$$

with the canonical projections $\pi^k_X : J^k(G,H) \to X$ and $\pi^k_Y : J^k(G,H) \to Y$. Here, $S^i(G)$ is the vector bundle $\cup_{x\in X} S^i(G_x)$ over X, where $S^i(G_x)$ denotes the i-fold symmetric product of G_x. The fiber $\bigoplus_{i=1}^k \mathrm{Hom}(S^i(\mathbb{R}^n), \mathbb{R}^n)$ is canonically identified with $J^k(n,n)$. The origin of $\mathbb{R}^n$ is simply denoted by 0. Let $GL^+(n)$, $O(n)$ and $SO(n)$ denote the group of orientation preserving linear isomorphisms of $\mathbb{R}^n$, the orthogonal group and the rotation group of degree n respectively. Let $L^k(n)$ denote the group of all k-jets of local diffeomorphisms of $(\mathbb{R}^n, 0)$. Let $h_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ $(i = 1, 2)$ be local diffeomorphisms. We define the action of $L^k(n) \times L^k(n)$ on $J^k(n,n)$ by $(j^k_0 h_1, j^k_0 h_2) \cdot j^k_0 f = j^k_0(h_1 \circ f \circ h_2^{-1})$. In particular, $O(n) \times O(n)$ acts on $J^k(n,n)$. Then $\Omega^I(n,n)$ is an open subset of $J^k(n,n)$ which is invariant with respect to the action of $L^k(n) \times L^k(n)$ ([12]). Let $\Omega^I(G,H)$ be an open subbundle of $J^k(G,H)$ associated to $\Omega^I(n,n)$.

If we provide N and P with Riemannian metrics, then the Levi-Civita connections induce the exponential maps $\exp_{N,x} : T_xN \to N$ and $\exp_{P,y} : T_yP \to P$. In dealing with the exponential maps we always consider the convex neighborhoods ([20]). We define the smooth bundle map

$$J^k(N,P) \to J^k(TN,TP) \quad \text{over } N \times P \tag{2.2}$$

by sending $z = j^k_x f \in (\pi^k_N \times \pi^k_P)^{-1}(x,y)$ to the k-jet of $(\exp_{P,y})^{-1} \circ f \circ \exp_{N,x}$ at $\mathbf{0} \in T_xN$, which is regarded as an element of $J^k(T_xN, T_yP)(= J^k_{x,y}(TN,TP))$ (see [20, Proposition 8.1] for the smoothness of exponential maps). More strictly, (2.2) gives a smooth equivalence of the fiber bundles under the structure group $L^k(n) \times L^k(n)$. Namely, it gives a smooth reduction of the structure group $L^k(n) \times L^k(n)$ of $J^k(N,P)$ to $O(n) \times O(n)$, which is the structure group of $J^k(TN,TP)$. Let us recall Boardman submanifolds $\Sigma^I(N,P)$ in $J^k(N,P)$ and $\Sigma^I(n,n)$ in $J^k(n,n)$ (see [11] and [21]). Let $\Sigma^I(TN,TP)$ and $\Omega^I(TN,TP)$ denote the subbundles $J^k(TN,TP)$ associated to $\Sigma^I(n,n)$ and $\Omega^I(n,n)$, which are identified with $\Sigma^I(N,P)$ and $\Omega^I(N,P)$ under (2.2).

3. ω_d is bijective

We first review the results of [5], [6] and [7] necessary for the definition of the map $\omega_d : \Omega_{fold,d}(P) \to [P, F^d]$. Let (O_1, O_2) be an element of $SO(n) \times SO(n)$ and M be an element of $SO(n+1)$. Then define the actions of $SO(n) \times SO(n)$ on $SO(n+1)$ and on $J^2(n,n)$ by

$$(O_1, O_2) \cdot M = (O_1 \dot{+} (1)) M ({}^tO_2 \dot{+} (1)),$$
$$(O_1, O_2) \cdot j_0^2 f = j_0^2 (O_1 \circ f \circ {}^tO_2),$$

where O_1 and O_2 are identified with the corresponding linear maps of $\mathbb{R}^n$ and $\dot{+}$ denotes the direct sum of matrices. Then we have the following theorem ([5, Theorem (ii)] and [6, Proposition 2.4]).

Theorem 3.1 ([5], [6])**.** *There exists a topological embedding $i_n : SO(n+1) \to \Omega^{(1,0)}(n,n)$ such that i_n is equivariant with respect to the above actions and that the image of i_n is a deformation retract of $\Omega^{(1,0)}(n,n)$.*

Let N and P be oriented manifolds of dimension n. If we choose an orthonormal basis of $\mathbb{R}^n$, then there are canonical inclusions of $GL^+(n)$ into $L^2(n)$ and of $SO(n)$ into $GL^+(n)$. Providing N and P with Riemannian metrics, we reduce the structure group $L^2(n) \times L^2(n)$ of the fibre bundle $\Omega^{(1,0)}(N,P)$ over $N \times P$ to $SO(n) \times SO(n)$. Let $GL^+_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$ and $SO_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$ be the subbundle of $\mathrm{Hom}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$ associated with $GL^+(n+1)$ and $SO(n+1)$ respectively. Then we have the inclusion $i_{SO_{n+1}} : SO_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P) \to GL^+_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$, which is a homotopy equivalence of fibre bundles covering $id_{N\times P}$.

Considering the fiber homotopy equivalence

$$i(N,P) : SO_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P) \longrightarrow \Omega^{(1,0)}(N,P) \tag{3.1}$$

associated with i_n and its homotopy inverse $(i(N,P))^{-1} : \Omega^{(1,0)}(N,P) \to SO_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$, we obtain the fiber homotopy equivalence

$$\begin{aligned} i_{SO_{n+1}} \circ (i(N,P))^{-1} : \Omega^{(1,0)}(N,P) &\longrightarrow SO_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P) \\ &\longrightarrow GL^+_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P). \end{aligned} \tag{3.2}$$

It has been shown in [6, Proposition 3.1] that the homotopy class of the fibre map $i_{SO_{n+1}} \circ (i(N,n))^{-1}$ over $id_{N\times P}$ does not depend on the choice of Riemannian metrics of N and P.

The set of all continuous sections of $GL^+_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$ over N corresponds bijectively to the set of all orientation preserving bundle maps of $TN \oplus \varepsilon^1_N$ to $TP \oplus \varepsilon^1_P$. Thus we have the following theorem.

Theorem 3.2 ([6, Corollary 2])**.** *Given a fold-map $f : N \to P$, the section $j^2 f$ determines the homotopy class of the section $i_{SO_{n+1}} \circ (i(N,P))^{-1} \circ j^2 f$ of $GL^+_{n+1}(TN \oplus \varepsilon^1_N, TP \oplus \varepsilon^1_P)$. It induces a bundle map $\mathcal{T}(f) : TN \oplus \varepsilon^1_N \to TP \oplus \varepsilon^1_P$ determined up to homotopy* (*this is denoted by $\bar{f}$ in* [6]).

Let N and P be embedded in $\mathbb{R}^{n+k}$ with the stable normal bundles ν_N and ν_P respectively. Then we have the trivializations $t_N : \tau_N \oplus \nu_N \to \varepsilon_N^{2k}$ and $t_P : \tau_P \oplus \nu_P \to \varepsilon_P^{2k}$ respectively. Let $\tau(f)$ denote the bundle map $\mathcal{T}(f) \oplus (f \times id_{\mathbf{R}^{k-n-1}})$. Then we have the following proposition.

Proposition 3.1 ([6, Proposition 3.2])**.** *Let $k \gg n$. Let N and P be oriented manifolds of dimension n embedded in $\mathbb{R}^{n+k}$ with the above trivializations t_N and t_P respectively. Then a fold-map $f : N \to P$ determines the homotopy class of a bundle map $\nu(f) : \nu_N \to \nu_P$ over f such that $t_P \circ (\tau(f) \oplus \nu(f)) \circ t_N^{-1}$ is homotopic to $f \times id_{\mathbb{R}^{2k}}$.*

According to [28], let $\{X, Y\}$ denote the set of S-homotopy classes of S-maps $S^i \wedge X \to S^i \wedge Y$ $(i \geq 0)$. Let us define the bijection $c_F : \{S^k P^0, S^k\} \to [P, F]$ for $k \gg n$. Let $\{\beta\} \in \{S^k P^0, S^k\}$ be represented by $\beta : S^k P^0 \to S^k$. For a point $x \in P$ we define $\beta(x) : S^k = S^0 \wedge S^k \to S^k$ by $(\beta|\{*_P \cup x\} \wedge S^k) \circ (\iota_x \wedge id_{S^k})$, where $\iota_x : S^0 \to \{*_P \cup x\}$ is the canonical identification. Then we set $c_F(\{\beta\})(x) = \{\beta(x)\}$. Let $\{S^k P^0, S^k\}_d$ be the subset of $\{S^k P^0, S^k\}$ which consists of all $\{\beta\}$ such that $\beta(x)$ is of degree d, namely $c_F(\{\beta\})(x) \in F^d$ for any $x \in P$. It is not difficult to see that c_F induces the bijection $c_{F^d} : \{S^k P^0, S^k\}_d \to [P, F^d]$.

An element of $\{S^{n+k}, T(\nu_P^k)\}$ represented by a map $\alpha : S^i \wedge S^{n+k} \to S^i \wedge T(\nu_P^k)$ is written as $\{\alpha\}$. Since $k \gg n$, $\{S^{n+k}, T(\nu_P^k)\}$ is isomorphic to $\pi_{n+k}\left(T(\nu_P^k)\right)$. It has been proved in [24, Lemma 2] that $T(\nu_P^k)$ is the S-dual space of $P^0 = P \cup *_P$, where $*_P$ is the base point. Namely, we have the isomorphism $\mathcal{D} : \{S^{n+k}, T(\nu_P^k)\} \to \{S^k P^0, S^k\}$. Let $\{S^{n+k}, T(\nu_P^k)\}_d$ denote the subset of $\{S^{n+k}, T(\nu_P^k)\}$ which consists of all S-maps of degree d. It has been proved in [7, Lemma 2.4] that $\mathcal{D}$ induces the bijection $\{S^{n+k}, T(\nu_P^k)\}_d \to \{S^k P^0, S^k\}_d$.

Now we are ready to define the map $\omega_d : \Omega_{fold,d}(P) \to [P, F^d]$. Let $\alpha_N : S^{n+k} \to T(\nu_N)$ denote the Pontrjagin-Thom construction for an embedding $N \to \mathbb{R}^{n+k}$. Given a fold-map $f : N \to P$ of degree d, there is a bundle map $\tau(f) : \tau_N \to \tau_P$ and a bundle map $\nu(f) : \nu_N \to \nu_P$ determined up to homotopy by Theorem 3.2 and Proposition 3.1 respectively. Let $T(\nu(f)) : T(\nu_N) \to T(\nu_P)$ be the Thom map associated with $\nu(f)$. Then we set $\omega_d(f) = c_{F^d}(\mathcal{D}(\{T(\nu(f)) \circ \alpha_N\}))$. Since $T(\nu(f))$ is of degree d, $\mathcal{D}(\{T(\nu(f)) \circ \alpha_N\})$ is of degree d. It has been proved in [7, Lemma 3.4] that $\omega_d(f) = c_{F^d}(\mathcal{D}(\{T(\nu(f)) \circ \alpha_N\}))$ does not depend on the choice of embeddings of N and P to $\mathbb{R}^{n+k}$ and does not depend on the choice of a representative f of the $\Omega^{(1,0)}$-cobordism class $[f] \in \Omega_{fold,d}(P)$.

The author has missed in [7] the fact that ω_d is injective. Here we give its proof.

Proof of Theorem 1.1. We have proved in [7, Theoem1] that ω_d is surjective. The rest is to prove that ω_d is injective. Take two fold-maps $f_i : N_i \to P$ $(i = 0, 1)$ of degree d such that $\omega_d([f_0]) = \omega_d([f_1])$. Recall that $\omega_d([f_i]) = c_{F^d}(\mathcal{D}\{T(\nu(f_i)) \circ \alpha_{N_i}\})$. Since c_{F^d} and $\mathcal{D}$ are bijections, it follows that there is a homotopy $H : S^{n+k} \times [0,1] \to T(\nu_P) \times [0,1]$ satisfying the following properties. Set $I(0, \epsilon) = [0, \epsilon]$ and $I(1, \epsilon) = [1 - \epsilon, 1]$ for a sufficiently small $\epsilon > 0$.

Set $P^{[0,1]} = P \times [0,1]$. Then we have, for $i = 0, 1$,

(i) $H(x,t) = (T(\nu(f_i)) \circ \alpha_{N_i}(x), t)$ for $x \in S^{n+k}$ and $t \in I(i,\epsilon)$,

(ii) H is smooth around $H^{-1}(P^{[0,1]})$ and is transverse to $P^{[0,1]}$.

We set $W = H^{-1}(P^{[0,1]})$, where the zero-section of ν_P is identified with P. Then we have

(iii) $W \cap S^{n+k} \times I(i,\epsilon) = N_i \times I(i,\epsilon)$,

(iv) $H(x,t) = (f_i(x), t)$ for $x \in N_i$ and $t \in I(i,\epsilon)$ under (iii),

(v) $TW|_{N_i \times I(i,\epsilon)} = T(N_i \times I(i,\epsilon)) = (TN_i \oplus \varepsilon^1_{N_i}) \times I(i,\epsilon)$ under (iii),

(vi) $\nu_W|_{N_i \times I(i,\epsilon)} = \nu_{N_i} \times I(i,\epsilon)$, where ν_W is the normal bundle of W in $S^{n+k} \times [0,1]$.

By (ii) we have the bundle map $B_{\nu_W} : \nu_W \to \nu_P \times [0,1]$ covering $H|W$ such that

(vii) $B_{\nu_W}(\mathbf{v}_x, t) = (\nu(f_i)(\mathbf{v}_x), t)$ for $x \in N_i$, $\mathbf{v}_x \in \nu_{N_i}$ and $t \in I(i,\epsilon)$ under (vi).

It follows from Proposition 3.1 that there exists a bundle map

$$B_{\tau_W} : \tau_W \to \tau_{P^{[0,1]}}$$

covering $H|W : W \to P^{[0,1]}$ such that $t_{P^{[0,1]}} \circ (B_{\tau_W} \oplus B_{\nu_W}) \circ t_W^{-1}$ is homotopic to $(H|W) \times id_{\mathbb{R}^{n+2k+2}}$. We may assume by (iii), (iv) and (v) that

(viii) $B_{\tau_W}((\mathbf{v}_x, w) \oplus a\partial/\partial t, \mathbf{w}, t) = (\mathcal{T}(f_i)(\mathbf{v}_x, w) \oplus a\partial/\partial t, \mathbf{w}, t)$ for $\mathbf{v}_x \in TN_i$, $w \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^k$ and $t \in I(i,\epsilon)$.

Let us consider

$$\begin{aligned} i_{SO_{n+1}} \circ i(W, P^{[0,1]})^{-1} : \Omega^{(1,0)}(W, P^{[0,1]}) \to & \\ & GL^+_{n+2}(TW \oplus \varepsilon^1_W, T(P^{[0,1]}) \oplus \varepsilon^1_{P^{[0,1]}}), \\ \mathbf{j}_{GL} : GL^+_{n+2}(TW \oplus \varepsilon^1_W, T(P^{[0,1]}) \oplus \varepsilon^1_{P^{[0,1]}}) \longrightarrow & \ GL^+_{n+2+k}(\tau_W, \tau_{P^{[0,1]}}), \end{aligned}$$

where $\mathbf{j}_{GL}$ is the fiber map over $W \times P^{[0,1]}$ associated to the inclusion $GL^+_{n+2} \to GL^+_{n+2+k}$. We consider the obstructions for finding a bundle map

$$b_{TW} : TW \oplus \varepsilon^1_W \to T(P^{[0,1]}) \oplus \varepsilon^1_{P^{[0,1]}}$$

covering $H|W$ such that

(ix) $b_{TW}|_{N_i \times I(i,\epsilon)} = \mathcal{T}(f_i \times id_{I(i,\epsilon)}) = \mathcal{T}(f_i) \times id_{I(i,\epsilon)}$,

(x) $\mathbf{j}_{GL}(b_{TW})$ is homotopic relative to $N_0 \times [0,\epsilon] \cup N_1 \times [1-\epsilon, 1]$ to B_{τ_W} which is regarded as a section of $GL^+_{n+2+k}(\tau_W, \tau_{P^{[0,1]}})$ over W.

Since $H^i(W,\ N_0 \cup N_1; \pi_i(SO(n+2+k)/SO(n+2))) = \{0\}$, all of these obstructions vanish and hence, there exists such a bundle map b_{TW}. By the fiber homotopy equivalence $i_{SO_{n+1}} \circ i(W, P^{[0,1]})^{-1}$, we obtain a section $s_W : W \to \Omega^{(1,0)}(W, P^{[0,1]})$ such that

$$s_W | N_i \times I(i,\epsilon) = j^2(f_i \times id_{I(i,\epsilon)}) | N_i \times I(i,\epsilon)$$

and that $i_{SO_{n+1}} \circ i(W, P^{[0,1]})^{-1}_{SO_{n+2}} \circ s_W$ is homotopic relative to $N_0 \times [0,\epsilon] \cup N_1 \times [1-\epsilon, 1]$ to b_{TW} as a section over W.

By the relative homotopy principle on the existence level for fold-maps in [7, Theorem 4.1] (see also [8, Theorem 0.5]), there exists a fold-map $E : W \to$

$P^{[0,1]}$ of degree d such that $E(x,t) = (f_0(x),t)$ for $0 \le t \le \epsilon/2$, $E(x,t) = (f_1(x),t)$ for $1-\epsilon/2 \le t \le 1$ under (iii) and that j^2E is homotopic to s_W relative to $N_0 \times [0,\epsilon/2] \cup N_1 \times [1-\epsilon/2,1]$. This implies that the fold-maps f_0 and f_1 are $\Omega^{(1,0)}$-cobordant. This proves that ω_d is injective. This proves Theorem 1.1. □

4. Ω^1-regular maps

Let us briefly review the fundamental properties of Boardman manifolds introduced in [11]. Let V and Y be smooth manifolds of dimension $n+1$. Let $\mathbf{D}$ and $\mathbf{P}$ denote the total tangent bundle defined on $J^\infty(V,Y)$ and $(\pi_Y^\infty)^*(TY)$ respectively. Let $f:(V,x)\to(Y,y)$ be a map defined on a neighborhood U_x of x with coordinates $(x_1,\dots,x_{n+1})$ and $\mathcal{F}$ be a smooth function in the sense of [11, Definition 1.4] defined on a neighborhood of $z=j_x^\infty f$. We have the local vector fields D_i defined around z with the property

$$D_i\mathcal{F}\circ j^\infty f = \frac{\partial}{\partial x_i}(\mathcal{F}\circ j^\infty f) \quad (1\le i\le n+1), \tag{4.1}$$

which span $\mathbf{D}$. Hence, we have that $d(j^\infty f)(\partial/\partial x_i)(\mathcal{F}) = D_i\mathcal{F}(j^\infty f)$, where $d(j^\infty f): TV \to T(J^\infty(V,Y))$ around x. This implies $d(j^\infty f)(\partial/\partial x_i) = D_i$. Hence, we have $\mathbf{D}\cong(\pi_V^\infty)^*(TV)$. There have been defined the homomorphism $\mathbf{d}_1:\mathbf{D}\to\mathbf{P}$ over $J^\infty(V,Y)$ such that $\mathbf{d}_{1,z}(D_i) = (z, d_xf(\partial/\partial x_i))$. The submanifold $\Sigma^1(V,Y)$ is defined to be the subset of $J^\infty(V,Y)$ which consists of all jets z such that the kernel rank of $\mathbf{d}_{1,z}$ is 1. The open subbundle $\Omega^1(V,Y)$ of $J^\infty(V,Y)$ consists of all regular jets and $\Sigma^1(V,Y)$. Since $\mathbf{d}_1|_{\Sigma^1(V,Y)}$ is of constant rank n, we set $\mathbf{K}_1 = \mathrm{Ker}(\mathbf{d}_1)$ and $\mathbf{P}_1 = \mathrm{Cok}(\mathbf{d}_1)$, which are vector bundles over $\Sigma^1(V,Y)$. Let $\mathbf{1}_r$ denote $(\overbrace{1,\dots,1}^{r})$ in this paper. The Boardman manifold $\Sigma^{\mathbf{1}_r}(V,Y)$ $(r\ge 1)$ has the following properties.

(4-i) There exists the $(r+1)$-th intrinsic derivative

$$\mathbf{d}_{r+1}: T(\Sigma^{\mathbf{1}_{r-1}}(V,Y))|_{\Sigma^{\mathbf{1}_r}(V,Y)} \longrightarrow \mathrm{Hom}(S^r\mathbf{K}_1,\mathbf{P}_1)|_{\Sigma^{\mathbf{1}_r}(V,Y)} \longrightarrow \mathbf{0},$$

so that $\mathrm{Ker}(\mathbf{d}_{r+1}) = T(\Sigma^{\mathbf{1}_r}(V,Y))$. Namely, $\mathbf{d}_{r+1}$ induces the isomorphism of the normal bundle $(T(\Sigma^{\mathbf{1}_{r-1}}(V,Y))|_{\Sigma^{\mathbf{1}_r}(V,Y)})/T(\Sigma^{\mathbf{1}_r}(V,Y))$ of $\Sigma^{\mathbf{1}_r}(V,Y)$ in $\Sigma^{\mathbf{1}_{r-1}}(V,Y)$ onto $\mathrm{Hom}(S^r\mathbf{K}_1,\mathbf{P}_1)|_{\Sigma^{\mathbf{1}_r}(V,Y)}$.

(4-ii) $\Sigma^{\mathbf{1}_{r+1}}(V,Y)$ is defined to be the submanifold of $\Sigma^{\mathbf{1}_r}(V,Y)$ which consists of all jets z such that $\mathbf{d}_{r+1,z}|\mathbf{K}_{1,z}$ vanishes.

(4-iii) The $(r+2)$-th intrinsic derivative $\mathbf{d}_{r+2}$ is defined to be the intrinsic derivative $d(\mathbf{d}_{r+1}|(\mathbf{K}_1|_{\Sigma^{\mathbf{1}_r}(V,Y)}))$:

$$T(\Sigma^{\mathbf{1}_r}(V,Y))|_{\Sigma^{\mathbf{1}_{r+1}}(V,Y)} \to \mathrm{Hom}(S^{r+1}\mathbf{K}_1,\mathbf{P}_1)|_{\Sigma^{\mathbf{1}_{r+1}}(V,Y)}.$$

(4-iv) The submanifold $\Sigma^{\mathbf{1}_r}(V,Y)$ is actually defined so that it coincides with the inverse image of a submanifold $\widetilde{\Sigma}^{\mathbf{1}_r}(V,Y)$ in $J^r(V,Y)$ by π_r^∞. The codimension of $\Sigma^{\mathbf{1}_r}(V,Y)$ in $J^\infty(V,Y)$ is r.

In the proof of the following theorem the homotopy principle on the existence level in [4] and [9] plays an important role. This homotopy principle has been proved by using [13], [14] and [16].

Theorem 4.1. *Let V be an oriented $(n+1)$-manifold with ∂V, which may be empty, Y be an oriented $(n+1)$-manifold and let C be a closed subset of V. Let s be a section of $\Omega^1(V,Y)$ over V which has a fold-map g defined on a neighborhood of C into Y, where $j^\infty g = s$. Then there exists an $\Omega^{(1,1,0)}$-regular map $E : V \to Y$ and a homotopy s_λ of sections of $\Omega^1(V,Y)$ over V relative to a neighborhood $U(C)$ of C such that $s_0 = s$ and $s_1 = j^\infty E$. In particular, we have $E|U(C) = g|U(C)$.*

Proof. In the proof we use the notation introduced in [11]. By (2.2) we always identify $J^r(V,Y)$ with $J^r(TV,TY)$. We may assume that s is transverse to $\Sigma^{\mathbf{1}_r}(V,Y)$ and we set $S^{\mathbf{1}_r}(s) = s^{-1}(\Sigma^{\mathbf{1}_r}(V,Y))$. It follows that $(\pi_r^\infty \circ s)(V \setminus (S^{\mathbf{1}_r}(s))) \subset \Omega^{\mathbf{1}_{r-1},0}(V \setminus S^{\mathbf{1}_r}(s), Y)$.

Let us construct a section $\mathfrak{s}$ of $\Omega^{(1,1,0)}(V,Y)$ such that $\pi_2^\infty \circ \mathfrak{s} = \pi_2^\infty \circ s$. We set $(s|S^1(s))^*\mathbf{K}_1 = K_1$ and $(s|S^1(s))^*\mathbf{P}_1 = P_1$. Since V and Y are oriented and since K_1 and P_1 are line bundles, we have that K_1 and P_1 are isomorphic. In particular, we have the isomorphism $K_1|_{S^{\mathbf{1}_2}(s)} \to P_1|_{S^{\mathbf{1}_2}(s)}$. Consider the homomorphism

$$\mathbf{r}^3 : \mathrm{Hom}(S^3(TV), TY)|_{S^{\mathbf{1}_2}(s)} \longrightarrow \mathrm{Hom}(S^3K_1, P_1)|_{S^{\mathbf{1}_2}(s)}$$

which is induced from the inclusion $S^3K_1|_{S^{\mathbf{1}_2}(s)} \to S^3(TV)|_{S^{\mathbf{1}_2}(s)}$ and the projection $TY|_{S^{\mathbf{1}_2}(s)} \to P_1|_{S^{\mathbf{1}_2}(s)}$. Since $S^3K_1 \approx K_1$, there exists the isomorphism $\iota^3 : S^3K_1|_{S^{\mathbf{1}_2}(s)} \to P_1|_{S^{\mathbf{1}_2}(s)}$, which induces the isomorphism $K_1|_{S^{\mathbf{1}_2}(s)} \to \mathrm{Hom}(S^2K_1, P_1)|_{S^{\mathbf{1}_2}(s)}$. Since $S^{\mathbf{1}_2}(s)$ is a closed submanifold of V such that $S^{\mathbf{1}_2}(s) \cap C = \emptyset$ in V, there exists a homomorphism $\mathbf{h}^3 : S^3(TV)|_{S^{\mathbf{1}_2}(s)} \to (\pi_Y^\infty \circ s)^*(TY)|_{S^{\mathbf{1}_2}(s)}$ such that $\mathbf{r}^3\circ\, \mathbf{h}^3 = \iota^3$. We extend $\mathbf{h}^3$ to a homomorphism $\mathbf{H}^3 : S^3(TV) \to (\pi_Y^\infty \circ s)^*(TY)$. If $(\pi_Y^\infty \circ s)_{TY} : (\pi_Y^\infty \circ s)^*(TY) \to TY$ denote the canonical bundle map covering $\pi_Y^\infty \circ s$, then we define the section $\mathfrak{s} : V \to J^\infty(TV,TY)$ to be the composite of

$$\pi_3^\infty \circ \mathfrak{s}(x) = \pi_2^\infty \circ s(x) \oplus (\pi_Y^\infty \circ s)_{TY} \circ \mathbf{H}^3|_x$$

and the canonical inclusion $J^3(TV,TY) \to J^\infty(TV,TY)$.

We now show that $\mathfrak{s}(V) \subset \Omega^{(1,1,0)}(V,Y)$. In fact, it is obvious that $\mathfrak{s}(V) \subset \Omega^{\mathbf{1}_2}(V,Y)$. It remains to prove that if $x \in S^{\mathbf{1}_2}(s)$, then

$$\text{(4.2)} \qquad \mathbf{d}_{3,\mathfrak{s}(x)} : \mathbf{K}_{1,\mathfrak{s}(x)} \longrightarrow \mathrm{Hom}(S^2\mathbf{K}_{1,\mathfrak{s}(x)}, \mathbf{P}_{1,\mathfrak{s}(x)})$$

is an isomorphism. In other words the homomorphism $S^3\mathbf{K}_{1,\mathfrak{s}(x)} \to \mathbf{P}_{1,\mathfrak{s}(x)}$ induced from $\mathbf{d}_{3,\mathfrak{s}(x)}$ is an isomorphism. For any point $x \in S^{\mathbf{1}_2}(s)$, let $y = \pi_Y^\infty \circ \mathfrak{s}(x)$, U_x and V_y be convex neighborhoods of x and y respectively. Let t and u be the coordinates of $\exp_{V,x}(K_{1,x})$ and $\exp_{Y,y}((\pi_Y^\infty \circ \mathfrak{s})_{TY}(P_{1,y}))$ respectively, where P_1 is regarded as a line subbundle of $(\pi_Y^\infty \circ s)^*(TY)|_{S^{\mathbf{1}_2}(s)}$ by virtue of the Riemannian metric of Y. It follows from (4.1), (4.2), the definition of $\mathbf{d}^3$ in (4-i), (2.2) and the definition of ι^3 that

$$(\bigcirc^3 D_t)u|_{\mathfrak{s}(x)} = \partial^3 u/\partial t^3(x) \neq 0 \quad \text{for } x \in S^{\mathbf{1}_2}(s).$$

Hence, we have that $\mathfrak{s}(S^{\mathbf{1}_2}(s)) \subset \Sigma^{(1,1,0)}(V, Y)$.

By the homotopy principle on the existence level for $\Omega^{(1,1,0)}$-regular maps in [9] there exists an $\Omega^{(1,1,0)}$-regular map $E : V \to Y$ such that $j^\infty E$ and $\mathfrak{s}$ are homotopic relative to a neighborhood of C as sections of $\Omega^1(V, Y)$ over V. $\square$

5. Obstructions

In order to determine $I(f)$ for a fold-map f and singularities of an extension $E^{I(f)}$ we have to prepare some machinery, although the dimensions are low.

Let V be an $(n+1)$-manifold with $\partial V = N$, and let τ_V be the stable k-dimensional tangent bundle of V. Given a fold-map $f : N \to S^n$ of degree 0, we have the bundle maps $\mathcal{T}(f) : TN \oplus \varepsilon^1_N \to \varepsilon^{n+1}_{S^n}$ in Theorem 3.2 and $\tau(f)$. Let us consider the obstruction for $\tau(f)$ to be extended to the trivialization of τ_V, in particular, the primary obstruction $\mathfrak{o}(\tau_V, \tau(f))$ defined in $H^{i+1}(V, N; \pi_i(SO(k)))$ for some i ([29]). Let $\widehat{V} = V \cup_N CN$, which is obtained by pasting V and the cone CN of N. Let $\tau(\widehat{V}, \tau(f))$ be the k-dimensional vector bundle, which is obtained by pasting τ_V and ε^k_{CN} by using $\tau(f)$. We have the primary obstruction $\mathfrak{o}(\tau(\widehat{V}, \tau(f))) \in H^{i+1}(\widehat{V}; \pi_i(SO(k))) \approx H^{i+1}(V, N; \pi_i(SO(k)))$ for $\tau(\widehat{V}, \tau(f))$ to be trivial. It is not difficult to see that $\mathfrak{o}(\tau_V, \tau(f)) = \mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ under the isomorphism.

Remark 1. In this case we may take $k = n+2$ and consider the subbundle $SO_{n+2}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}})$ of $\mathrm{Hom}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}})$ associated to $SO(n+2)$. Since $i^{SO}_{n+2} : SO(n+2) \to \Omega^{(1,0)}(n+1, n+1)$ is a homotopy equivalence, we have the fiber homotopy equivalence

$$SO_{n+2}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}}) \longrightarrow \Omega^{(1,0)}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}}).$$

Therefore, $\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ coincides with the obstruction to find a section of $\Omega^{(1,0)}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}})$, which is equal to the Thom polynomial of the closure $Cl(\Sigma^{(1,1)}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}}))$ in $H^2(\widehat{V}; \pi_1(SO(n+2)))$ (see, for example, [3, Proposition 3.1]). This Thom polynomial is equal to the second Stiefel-Whitney class $w_2(\tau(\widehat{V}, \tau(f)))$ by [25].

If $n+1 = 4m$ and $\omega_0([f])$ lies in what is called the J-image of

$$J : \pi_{4m-1}(SO(k)) \longrightarrow \pi^s_{4m-1}$$

of order j_m in [1], then we can choose an fold-map f such that $N = S^n$ by [7, Proposition 5.1]. This is also true for the case $n = 1$. Furthermore, we can take V to be a parallelizable manifold. Hence, $\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ lies in the $(n+1)$-th cohomology group.

We have the following lemma due to [23, Lemma 2]. Let a_m denote 2 for m odd and 1 for m even.

Lemma 5.1 ([23]). *Let $n+1 = 4m$. Let V be a parallelizable manifold. Then $\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ is related to the m-th Pontrjagin class $P_m(\tau(\widehat{V}, \tau(f)))$ by the identity $P_m(\tau(\widehat{V}, \tau(f))) = \pm a_m(2m-1)!\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$.*

We next see how $\mathfrak{o}(\tau_V, \tau(f))$ varies depending on the choice of V and f (the following argument is available for the case $n = 1$). Let two fold-maps $f_i : S^n \to S^n$ of degree 0 ($i = 0, 1$) be $\Omega^{(1,0)}$-cobordant by a cobordism $E : (W, \partial W) \to (S^n \times [0,1], S^n \times 0 \cup S^n \times 1)$ of degree 0 as in Introduction. Assume that there exists a parallelizable $(n+1)$-manifold V_i with $\partial V_i = S^n \times i$. Then we have the bundle maps $\mathcal{T}(f_i) : T(S^n \times i) \oplus \varepsilon^1_{S^n \times i} \to \varepsilon^{n+1}_{S^n}$ and $\mathcal{T}(E) : TW \oplus \varepsilon^1_W \to \varepsilon^{n+2}_{S^n \times [0,1]}$ by Theorem 3.2 such that $TW|_{S^n \times i} = T(S^n \times i) \oplus \varepsilon^1_{S^n \times i}$ and the stabilizations of $\mathcal{T}(E)|_{N_i}$ and $\mathcal{T}(f_i)$ are equal. Consider the almost parallelizable manifold $\widehat{W} = V_0 \cup_{S^n \times 0} W \cup_{S^n \times 1} (-V_1)$, which is obtained by pasting V_0, W and V_1 with orientation reversed. Let

$$\mathfrak{o}(\tau(\widehat{W})) \in H^{n+1}(\widehat{W}; \pi_n(SO(k))) \approx \pi_n(SO(k))$$

be the unique primary obstruction for $\tau(\widehat{W})$ to be trivial. This is equal to the primary obstruction for extending $\tau(E) : \tau(\widehat{W})|_W \to \varepsilon^k_W$ to a bundle map to $\varepsilon^k_{\widehat{W}}$ over the whole space $\widehat{W}$. Therefore, it is not difficult to see that

$$\mathfrak{o}(\tau_{V_0}, \tau(f_0)) - \mathfrak{o}(\tau_{V_1}, \tau(f_1)) = \pm\mathfrak{o}(\tau(\widehat{W})) \quad \text{in } \pi_n(SO(k)).$$

Define the integer $\mathfrak{m}(n)$ for $n > 1$ to be the minimal nonnegative number such that there exists an $(n+1)$-dimensional almost parallelizable closed manifold $\widehat{W}'$ such that $\mathfrak{o}(\tau(\widehat{W}')) = \mathfrak{m}(n)$. We will see that it is reasonable to set $\mathfrak{m}(1) = 2$ later. We have the following theorem due to [23, Theorems 1 and 2].

Theorem 5.1 ([23]). *Let $n+1 = 4m$. Then we have*

(*i*) *The Pontrjagin class $P_m(\widehat{W}')$ of an almost parallelizable closed manifold $\widehat{W}'$ is divisible by $\pm j_m a_m(2m-1)!$.*

(*ii*) *There exists an almost parallelizable closed manifold $\widehat{W_0}$ such that $P_m(\widehat{W_0}) = \pm j_m a_m(2m-1)!$.*

Consequently, we have $\mathfrak{m}(n) = j_m$.

Lemma 5.2. *Let $n+1 = 4m$. Let $f : N \to S^n$ be a fold-map of degree 0 and $E^{I(f)} : V \to S^n$ be an extension. If V is parallelizable, then $\mathfrak{o}(\widehat{V}, \tau(f_1))$ is well-defined in $\mathbb{Z}/(j_m)$.*

In the rest of the paper we are only concerned with the case $n < 8$. By the definition of $\tau(\widehat{V}, \tau(f))$ in the case $\partial V = S^n$, $\tau(f)$ yields the section of $\Omega^{(1,0)}(\tau(\widehat{V}, \tau(f)), \varepsilon^{n+2}_{\widehat{V}})|_{CS^n}$, which we denote by $s_{\tau(f)}$, for $k = n+2$.

Lemma 5.3. *Let $n < 8$. Let $f : N \to S^n$ be a fold-map and $E^f : V \to S^n \times [0,1]$ be an extension. Then we have the following.*

(i) If $P_1(\tau(\widehat{V}, \tau(f))$ does not vanish, then any extension E^f has the singularities of codimension 4 *and of symbol* (2).

(ii) Let V be parallelizable in addition and $P_1(\tau(\widehat{V}, \tau(f))$ vanish. Then $s_{\tau(f)}$ is extendable to a section of $\Omega^{(1,0)}(\tau(\widehat{V}, \tau(f)), \varepsilon_{\widehat{V}}^{n+2})$ over $\widehat{V}$ if and only if $P_2(\tau(\widehat{V}, \tau(f))$ vanishes.

Proof. (i) follows from the fact that $P_1(\tau(\widehat{V}, \tau(f))$ is the integer Thom polynomial of the topological closure of $\Sigma^2(\tau(\widehat{V}, \tau(f)), \varepsilon_{\widehat{V}}^{n+2})$ ([26]).

(ii) is clear. □

6. π_6^s

Let us recall a map $S^{k+6} \to S^k$ which generates π_6^s. Let $H : S^7 \to S^4$ denote the Hopf map and $\nu_k : S^{k+3} \to S^k$ denote the $(k-4)$-fold suspension of H. Then the composite $\nu_k \circ \nu_{k+3}$ generates π_6^s ([30, Proposition 5.11]). Since $(\nu_k)^{-1}(a\ point)$ is diffeomorphic to S^3, the inverse image of a regular value of $\nu_k \circ \nu_{k+3}$ is diffeomorphic to $S^3 \times S^3$. Therefore, it follows from Corollary 1.2 that there exists a fold-map $f : S^3 \times S^3 \to S^6$ of degree 0 such that $\omega_0([f])$ is the non-zero element of π_6^s. Let us construct a precise example of f such that $I(f) = (2,0)$. Let $\mathbb{Q}$ denote the field of quarternion numbers which is canonically identified with $\mathbb{R}^4$. The product of elements x, $y \in \mathbb{Q}$ is denoted by $x \cdot y$. Let $\mathbb{S}$ denote the set of $x \in \mathbb{Q}$ which is denoted by $x = x_1e_0 + x_2e_1 + x_3e_2 + x_4e_3$ $(= (x_1, x_2, x_3, x_4))$ with $||x|| = 1$. Let $t_{T\mathbb{S}} : T\mathbb{S} \to T_{\mathbf{e}_1}\mathbb{R}^3 = \mathbb{R}^3$ denote the trivialization given by $t_{T\mathbb{S}}(x, \mathbf{v}) = x^{-1} \cdot \mathbf{v}$, where $x \in \mathbb{S}$, $\mathbf{v} \in T_x\mathbb{S}$ is identified with a vector in $\mathbb{R}^4$ and $\mathbb{R}^3$ is spanned by $\mathbf{e}_i$ for $i = 2,3,4$. Let $i_{\mathbb{S}} : \mathbb{S} \to \mathbb{R}^4 \times \mathbb{R}^3 = \mathbb{R}^7$ denote the composite of the inclusions $\mathbb{S} \to \mathbb{R}^4 = \mathbb{R}^4 \times 0$ and $\mathbb{R}^4 \times 0 \to \mathbb{R}^7$. Let $\mathfrak{n}_{\mathbb{S}}$ denote the normal bundle of $i_{\mathbb{S}}(\mathbb{S})$. Let $t_{\mathfrak{n}_{\mathbb{S}}} : \mathfrak{n}_{\mathbb{S}} \to \mathbb{R}^4$ denote the trivialization defined by

$$t_{\mathfrak{n}_{\mathbb{S}}}(x, tx + a_5\mathbf{e}_5 + a_6\mathbf{e}_6 + a_7\mathbf{e}_7) = t\mathbf{e}_1 + a_5\mathbf{e}_5 + a_6\mathbf{e}_6 + a_7\mathbf{e}_7,$$

where $\mathbb{R}^4$ is spanned by $\mathbf{e}_i$ for $i = 1,5,6,7$. They yields the trivialization

$$t_{\mathbb{S}} = t_{T\mathbb{S}} \oplus t_{\mathfrak{n}_{\mathbb{S}}} : T\mathbb{S} \oplus \mathfrak{n}_{\mathbb{S}} \longrightarrow \mathbb{R}^7.$$

Let $\tau_{\mathbb{S}}$ and $\nu_{\mathbb{S}}$ denote the stable tangent and normal bundles of $T\mathbb{S}$ and $\mathfrak{n}_{\mathbb{S}}$ without specifying the dimensions respectively. The trivialization of $\tau_{\mathbb{S}} \oplus \nu_{\mathbb{S}}$ induced from $t_{\mathbb{S}}$ is also denoted by the same letter $t_{\mathbb{S}}$. Let $\beta : S^3 \to SO(4)$ denote the map defined by $\beta(x)\mathbf{v} = x \cdot \mathbf{v}$ where $x \in \mathbb{S}$, $\mathbf{v} \in \mathbb{R}^4$. It is known that the composite of β and $SO(4) \to SO(k)$, $k \gg 4$ generates $\pi_3(SO(k)) \approx \mathbb{Z}$. This composite is also denoted by the same letter β. Let $\beta_k : \mathbb{S} \times \mathbb{R}^k \to \mathbb{R}^k$ be the bundle map defined by $\beta_k(x, \mathbf{v}_1 \oplus \mathbf{v}_2) = x \cdot \mathbf{v}_1 \oplus \mathbf{v}_2$ for $x \in \mathbb{S}$, $\mathbf{v}_1 \in \mathbb{R}^4$ and $\mathbf{v}_2 \in \mathbb{R}^{k-4}$. Let $\beta_k^\tau : T\mathbb{S} \times \mathbb{R} \times \mathbb{R}^{k-4} \to \mathbb{R}^k$ be the bundle map defined by $\beta_k^\tau(x, \mathbf{w}_1, w_2, \mathbf{w}_3) = (x^{-1} \cdot (w_2 \oplus (x^{-1} \cdot \mathbf{w}_1)), \mathbf{w}_3)$ for $x \in \mathbb{S}$, $\mathbf{w}_1 \in T_x\mathbb{S}$, $w_2 \in \mathbb{R}$

and $\mathbf{w}_3 \in \mathbb{R}^{k-4}$. Then we can prove by an analogous argument in the proof of [6, Proposition 3.3] that $\beta_\ell^\tau \oplus \beta_k$ is homotopic to $t_{\mathbb{S}}$ for sufficiently large numbers ℓ and k.

It follows from Theorem 3.2 that there exists a fold-map $f_{\mathbb{S}} : \mathbb{S} \to \mathbb{R}^3$ such that $\tau(f_{\mathbb{S}})$ is homotopic to β_ℓ^τ for some large number ℓ. Let $\overline{\beta_4^\tau} : T\mathbb{S} \times \mathbb{R} \to T\mathbb{R}^4$ be the bundle map defined by $\overline{\beta_4^\tau}(x, \mathbf{w}) = (x, \beta_4^\tau(\mathbf{w}))$. By the Smale-Hirsch Immersion Theorem ([17] and [27]) there exists an immersion $j_{\mathbb{S}} : \mathbb{S} \to \mathbb{R}^4$ such that $d(j_{\mathbb{S}})$ and $\overline{\beta_4^\tau}|T\mathbb{S}$ are homotopic as monomorphisms. Let us try to express $d(f_{\mathbb{S}})_x : T_x\mathbb{S} \approx T_{\mathbf{e}_1}\mathbb{S} \to T_{f_{\mathbb{S}}(x)}\mathbb{R}^3 \approx \mathbb{R}^3$ by a 3×3 matrix and $d(j_{\mathbb{S}})_x : T_x\mathbb{S} \approx T_{\mathbf{e}_1}\mathbb{S} \to T_{f_{\mathbb{S}}(x)}\mathbb{R}^4 \approx \mathbb{R}^4$ by a 4×3 matrix under the trivialization $t_{\mathbb{S}}$. Then we may choose $f_{\mathbb{S}}$ and $j_{\mathbb{S}}$ to satisfy the following property (P).

(P) *$d(f_{\mathbb{S}})_x$ is equal to $2E_3$ and $d(j_{\mathbb{S}})_x$ is equal to the 4×3 matrix $2(\delta_{1+i,j})$ for $0 \le i \le 3$, where E_3 is the unit-matrix of degree 3, $\delta_{1+i,j} = 1$ for $i = j$ and $\delta_{1+i,j} = 0$ for $i \ne j$ if x lies in a very small disk neighborhood $D(\mathbf{e}_1)$ of $\mathbf{e}_1$ in $\mathbb{S}$.*

In fact, we first deform $j_{\mathbb{S}}$ and $f_{\mathbb{S}}$ so that $j_{\mathbb{S}}(\mathbf{e}_1) = \mathbf{e}_1$, $f_{\mathbb{S}}(\mathbf{e}_1) = 0 \in \mathbb{R}^3$, $d(j_{\mathbb{S}})_{\mathbf{e}_1} = 2(\delta_{1+i,j})$ and $d(f_{\mathbb{S}})_{\mathbf{e}_1} = 2E_3$. By a standard argument in differential topology we next deform $j_{\mathbb{S}}$ and $f_{\mathbb{S}}$ so that

$$j_{\mathbb{S}}(x_1, x_2, x_3, x_4) = (1, 2x_2, 2x_3, 2x_4),$$
$$f_{\mathbb{S}}(x_1, x_2, x_3, x_4) = (2x_2, 2x_3, 2x_4)$$

on a very small disk neighborhood $D(\mathbf{e}_1)$ of $\mathbf{e}_1$ in $\mathbb{S}$. This assures the property (P).

Let $\mathbf{e}_{j_{\mathbb{S}}(x)}$ denote the vector of length 1 in $\mathbb{R}^4$ such that

(i) $\mathbf{e}_{j_{\mathbb{S}}(x)}$ is orthogonal to $j_{\mathbb{S}}(\mathbb{S})$ at $j_{\mathbb{S}}(x)$,

(ii) the orientation determined by $(\mathbf{e}_{j_{\mathbb{S}}(x)}, dj_{\mathbb{S}}(x \cdot \mathbf{e}_2), dj_{\mathbb{S}}(x \cdot \mathbf{e}_3), dj_{\mathbb{S}}(x \cdot \mathbf{e}_4))$ coincides with the canonical orientation of $\mathbb{R}^4$.

Let $\mathfrak{n}_{j_{\mathbb{S}}} = \mathbb{S} \times \mathbb{R}$ be the orthogonal normal bundle to the immersion $j_{\mathbb{S}}$. We define the map $\exp_{j_{\mathbb{S}}} : \mathfrak{n}_{j_{\mathbb{S}}} \to \mathbb{R}^4$ by

$$\exp_{j_{\mathbb{S}}}(x, t) = j_{\mathbb{S}}(x) + t\mathbf{e}_{j_{\mathbb{S}}(x)}.$$

Then for a sufficiently small positive real number ϵ and the disk bundle $D_{2\epsilon}(\mathfrak{n}_{j_{\mathbb{S}}})$ with radius 2ϵ, we find a small neighborhood O_x for each $x \in \mathbb{S}$ such that $\exp_{j_{\mathbb{S}}}|(D_{2\epsilon}(\mathfrak{n}_{j_{\mathbb{S}}})|_{O_x})$ is an embedding.

Using $\exp_{j_{\mathbb{S}}}$ we define a fold-map $f_{\mathbb{S}\times\mathbb{S}} : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2$ by

$$f_{\mathbb{S}\times\mathbb{S}}(x, y) = (j_{\mathbb{S}}(x) + \epsilon f_{\mathbb{S}}^1(y)\mathbf{e}_{j_{\mathbb{S}}(x)}, \epsilon f_{\mathbb{S}}^2(y), \epsilon f_{\mathbb{S}}^3(y))$$

where $f_{\mathbb{S}}(y) = (f_{\mathbb{S}}^1(y), f_{\mathbb{S}}^2(y), f_{\mathbb{S}}^3(y))$. Then we have the following proposition.

Proposition 6.1. *$\omega_0([f_{\mathbb{S}\times\mathbb{S}}])$ is the generator of π_6^s.*

Let us prepare the following lemma for the proof of the proposition. There exists the trivialization of $T(\mathbb{S} \times \mathbb{S})$ which is induced from $t_{\mathbb{S}}$. Let

$$\beta_{\mathbb{S}\times\mathbb{S}}^\tau : T(\mathbb{S} \times \mathbb{S}) \times \mathbb{R}^{k-6} = \mathbb{S} \times \mathbb{S} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k,$$
$$\beta_{\mathbb{S}\times\mathbb{S}} : \mathbb{S} \times \mathbb{S} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

denote the bundle maps defined by

$$\beta^{\tau}_{\mathbb{S}\times\mathbb{S}}(x,y)(\mathbf{v}_1,\mathbf{v}_2,v_3,v_4,\mathbf{v}_5) = (\beta^{\tau}_4(x,\mathbf{v}_1,v_3),\beta^{\tau}_4(y,\mathbf{v}_2,v_4),\mathbf{v}_5),$$
$$\beta_{\mathbb{S}\times\mathbb{S}}(x,y)\mathbf{v} = \beta(x)\beta(y)\mathbf{v},$$

where x, $y\in\mathbb{S}$, $\mathbf{v}_1,\mathbf{v}_2\in T(\mathbb{S})$, $v_3,v_4\in\mathbb{R}$, $\mathbf{v}_5\in\mathbb{R}^{k-8}$ and $\mathbf{v}\in\mathbb{R}^k$.

Lemma 6.1. *Under the above trivialization of $T(\mathbb{S}\times\mathbb{S})$ and the canonical trivialization of $T\mathbb{R}^6$, $\tau(f_{\mathbb{S}\times\mathbb{S}})$ is homotopic to the bundle map $\beta^{\tau}_{\mathbb{S}\times\mathbb{S}}$, and so $\nu(f_{\mathbb{S}\times\mathbb{S}})$ is homotopic to the bundle map $\beta_{\mathbb{S}\times\mathbb{S}}$.*

Proof. We will show that $\tau(f_{\mathbb{S}\times\mathbb{S}})|\mathbb{S}\times\{\mathbf{e}_1\}$ and $\tau(f_{\mathbb{S}\times\mathbb{S}})|\{\mathbf{e}_1\}\times\mathbb{S}$ are homotopic to $\beta^{\tau}_k : T\mathbb{S}\oplus\mathbb{R}^{k-3}\to\mathbb{R}^k$ under the identification $\mathbb{S}\times\{\mathbf{e}_1\}=\mathbb{S}=\{\mathbf{e}_1\}\times\mathbb{S}$. Let $D(\mathbf{e}_1)$ be the above very small disk neighborhood of $\mathbf{e}_1$ in $\mathbb{S}$. There exists a deformation retraction of $\mathbb{S}\times\mathbb{S}\backslash\mathrm{Int}\{D(\mathbf{e}_1)\times D(\mathbf{e}_1)\}$ to $\mathbb{S}\times\{\mathbf{e}_1\}\cup\{\mathbf{e}_1\}\times\mathbb{S}$. Hence, it follows from $\pi_6(SO(6))\approx\{0\}$ that $\tau(f_{\mathbb{S}\times\mathbb{S}})$ is homotopic to $\beta^{\tau}_{\mathbb{S}\times\mathbb{S}}$.

By the property (P) the differential $(df_{\mathbb{S}\times\mathbb{S}})_{x,y}$ is equal to $2E_3\dot{+}(df_{\mathbb{S}})_y$ for $x\in D(\mathbf{e}_1)$ and $y\in\mathbb{S}$. Since $f_{\mathbb{S}}$ is a fold-map, it follows from the definitions of $\tau(f_{\mathbb{S}})$ and $\tau(f_{\mathbb{S}\times\mathbb{S}})$ that $\tau(f_{\mathbb{S}\times\mathbb{S}})|\{\mathbf{e}_1\}\times\mathbb{S}$ are homotopic to β^{τ}_k. Next recall that $j_{\mathbb{S}}:\mathbb{S}\to\mathbb{R}^4$ is defined by the bundle map $\overline{\beta^{\tau}_4}$, the differential $df_{\mathbb{S}\times\mathbb{S}}|\mathbb{S}\times\{\mathbf{e}_1\}$ is homotopic to $\overline{\beta^{\tau}_4}\oplus id_{\mathbb{R}^2}$. This proves the lemma. □

Let $\pi_{\mathfrak{n}_{\mathbb{S}}}:\mathfrak{n}_{\mathbb{S}}\to\mathbb{S}$ be the projection and $\alpha_{\mathbb{S}}$ be the Pontrjagin-Thom construction for $i_{\mathbb{S}}$. It is not difficult to see that the composite

$$T(\beta_4)\circ T(\pi_{\mathfrak{n}_{\mathbb{S}}}\times t_{\nu_{\mathbb{S}}})\circ\alpha_{\mathbb{S}} : S^7\to T(\mathfrak{n}_{\mathbb{S}})\to T(\mathbb{S}\times\mathbb{R}^4)\to S^4$$

is homotopic to the Hopf map H.

Proof of Proposition 6.1. Consider the embedding $i_{\mathbb{S}}\times i_{\mathbb{S}}:\mathbb{S}\times\mathbb{S}\to\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^{k-2}$ with the normal bundle $\nu^k_{\mathbb{S}\times\mathbb{S}}$ which consists of all points $((x,y),tx\oplus t'y\oplus\mathbf{v})$ for x, $y\in\mathbb{S}$, t, $t'\in\mathbb{R}$ and $\mathbf{v}\in\mathbb{R}^{k-2}$. Let $t_{\nu^k_{\mathbb{S}\times\mathbb{S}}}:\nu^k_{\mathbb{S}\times\mathbb{S}}\to\mathbb{S}\times\mathbb{S}\times\mathbb{R}^k$ be the bundle map defined by $t_{\nu^k_{\mathbb{S}\times\mathbb{S}}}((x,y),tx\oplus t'y\oplus\mathbf{v})=((x,y),t\mathbf{e}_1+t'\mathbf{e}_2+\mathbf{v})$. Let $B:\mathbb{S}\times\mathbb{S}\times\mathbb{R}^k\to\mathbb{R}^k$ be the bundle map defined by $B(x,y,\mathbf{v})=\beta(x)\beta(y)\mathbf{v}$. Let $\alpha_{\mathbb{S}\times\mathbb{S}}$ denote the Pontrjagin-Thom construction for the embedding $i_{\mathbb{S}}\times i_{\mathbb{S}}$. Then it is not difficult to see by [30, Proposition 5.11] that $\{T(B)\circ T(t_{\nu^k_{\mathbb{S}\times\mathbb{S}}})\circ\alpha_{\mathbb{S}\times\mathbb{S}}\}$ and its dual map $\mathcal{D}(\{T(B)\circ T(t_{\nu^k_{\mathbb{S}\times\mathbb{S}}})\circ\alpha_{\mathbb{S}\times\mathbb{S}}\})$ are homotopic to the generator of π^s_6. If we recall the isomorphism $\{S^{6+k},S^k\}\approx\{(s^6)^0\wedge S^k,S^k\}$ for $k\gg 6$ (see also [2, Proof of Lemma 1.3]), then we have

$$\omega_0([f_{\mathbb{S}\times\mathbb{S}}])=c_{F^0}(\mathcal{D}(\{T(B)\circ T(t_{\nu^k_{\mathbb{S}\times\mathbb{S}}})\circ\alpha_{\mathbb{S}\times\mathbb{S}}\})).$$

This proves the proposition. □

Recall that $f_{\mathbb{S}}$ has a smooth extension $E^{f_{\mathbb{S}}}:D^4\to\mathbb{R}^4$ with $E^{f_{\mathbb{S}}}|\mathbb{S}=(f_{\mathbb{S}},0)$ and $E^{f_{\mathbb{S}}}|\mathrm{Int}D^4$ being contained in $\mathbb{R}^3\times(0,\infty)$ such that $E^{f_{\mathbb{S}}}$ has non-empty isolated singularities of type $I_{2,2}$ or $II_{2,2}$ of symbol $(2,0)$ in the terminology in [22]. Define the smooth extension $E^{f_{\mathbb{S}\times\mathbb{S}}} : \mathbb{S}\times D^4 \to \mathbb{R}^6\times(0,\infty)$ of $f_{\mathbb{S}\times\mathbb{S}}$ by

$$E^{f_{\mathbb{S}\times\mathbb{S}}}(x,y) = ((j_{\mathbb{S}}(x) + \epsilon E^1_{\mathbb{S}}(y)\mathbf{e}_{j_{\mathbb{S}}(x)}), \epsilon E^2_{\mathbb{S}}(y), \epsilon E^3_{\mathbb{S}}(y), \epsilon E^4_{\mathbb{S}}(y)),$$

where $E^{f_{\mathbb{S}}}(y) = (E^1_{\mathbb{S}}(y), E^2_{\mathbb{S}}(y), E^3_{\mathbb{S}}(y), E^4_{\mathbb{S}}(y))$, where ϵ is a small positive real number as before. It is obvious that $E^{f_{\mathbb{S}\times\mathbb{S}}}$ has singularities of types A_i $(1\leq i\leq 4)$ $I_{2,2}$ or $II_{2,2}$. This shows $I(f_{\mathbb{S}\times\mathbb{S}})\leq(2,0)$.

Proposition 6.2. *Let $[f]\in\Omega_{fold,0}(S^6)$. If $\omega_0([f])\neq 0$ in π^s_6, then $I(f)=(2,0)$.*

Proof. Let $f : N\to S^6$. Since $I(f_{\mathbb{S}\times\mathbb{S}})\leq(2,0)$, we have $I(f)\leq(2,0)$. Suppose that $I(f)<(2,0)$. Then there exists an extension $E^f : (V',N)\to(S^6\times[0,1], S^6\times 0)$ such that E^f is an Ω^1-regular cobordism. By Theorem 4.1 we may assume that E^f is an $\Omega^{(1,1,0)}$-regular cobordism of f and $I(f)\leq(1,1,0)$. Namely, we may assume that the singularities of E^f are of symbols $(1,0)$ or $(1,1,0)$. Since $[f]=[f_{\mathbb{S}\times\mathbb{S}}]$, there exists an $\Omega^{(1,0)}$-regular cobordism $E : (W,\partial W)\to(S^6\times[0,1], S^6\times 0\cup S^6\times 1)$ such that $\partial W = N\cup(-\mathbb{S}\times\mathbb{S})$, $E|N=f$ and $E|\mathbb{S}\times\mathbb{S} = f_{\mathbb{S}\times\mathbb{S}}$. Let $G_{\mathbb{S}\times\mathbb{S}} : V = V'\cup_N W\to S^6\times[0,1]$ be an extension of $f_{\mathbb{S}\times\mathbb{S}}$ which is obtained by pasting E^f and E on N. Let $S^{(1,1,0)}(G_{\mathbb{S}\times\mathbb{S}})$ denote the set of singularities of symbol $(1,1,0)$ of $G_{\mathbb{S}\times\mathbb{S}}$. Then $G_{\mathbb{S}\times\mathbb{S}}|(V\backslash S^{(1,1,0)}(G_{\mathbb{S}\times\mathbb{S}}))$ is a fold-map, and $V\backslash S^{(1,1,0)}(G_{\mathbb{S}\times\mathbb{S}})$ is stably parallelizable by Theorem 3.2.

By applying the surgery technique introduced in [19] we may assume that $V\backslash S^{(1,1,0)}(G_{\mathbb{S}\times\mathbb{S}})$ is 2-connected and that the inclusion $\{\mathbf{e}_1\}\times\mathbb{S}\to\mathbb{S}\times\mathbb{S}$ is null-homotopic as a map to $V\backslash S^{(1,1,0)}(G_{\mathbb{S}\times\mathbb{S}})$. The last assertion follows from the fact that we can deform this inclusion to an embedding into $\mathrm{Int}(V\backslash S^{(1,1,0)}(G_{\mathbb{S}\times\mathbb{S}}))$ and its normal bundle is trivial by $\pi_2(SO(4))=\{0\}$. Therefore, we can apply the surgery to this embedding. Let $\tau(\widehat{V},\tau(f_{\mathbb{S}\times\mathbb{S}}))$ over $\widehat{V}$ (resp. $\xi=\tau(S^4,\tau(f_{\mathbb{S}}))$ over S^4) be the vector bundle which is constructed by pasting $\tau(V)$ (resp. $\varepsilon^k_{D^4}$) and $C(\mathbb{S}\times\mathbb{S})\times\mathbb{R}^k$ (resp. $\varepsilon^k_{D^4}$) by the bundle map $\tau(f_{\mathbb{S}\times\mathbb{S}})$ (resp. $\tau(f_{\mathbb{S}})$). Let $j:\mathbb{S}\to\{\mathbf{e}_1\}\times\mathbb{S}\subset\mathbb{S}\times\mathbb{S}$ be the inclusion. Then it is extended to a map $\mathbf{j} : D^4\to V\backslash S(G_{\mathbb{S}\times\mathbb{S}})\subset V$. Since $\tau(f_{\mathbb{S}\times\mathbb{S}})|\{\mathbf{e}_1\}\times\mathbb{S}$ is homotopic to $\tau(f_{\mathbb{S}})$ under the identification $\mathbb{S}=\{\mathbf{e}_1\}\times\mathbb{S}$, it follows that there exists a bundle map $\xi\to\tau(\widehat{V},\tau(f_{\mathbb{S}\times\mathbb{S}}))$ covering $\mathbf{j}$. This implies that $\mathbf{j}^*(P_1(\tau(\widehat{V},\tau(f_{\mathbb{S}\times\mathbb{S}})))) = P_1(\xi)$. We have proved that $P_1(\xi)\neq 0$ in Lemma 5.3 (i), and hence $P_1(\tau(\widehat{V},\tau(f_{\mathbb{S}\times\mathbb{S}})))\neq 0$. Consequently, it follows that $G_{\mathbb{S}\times\mathbb{S}}$ must have the singularities of symbol (2). This is a contradiction. Hence, we have $I(f)=(2,0)$. □

As is observed, the worst singularities of $E^{I(f_{\mathbb{S}\times\mathbb{S}})}$ are of type $I_{2,2}$ or $II_{2,2}$. In the equidimension 7 we have the singularities of many types of symbol $(2,0)$ other than $I_{2,2}$ and $II_{2,2}$ as described in [22, Section 7]. This fact suggests that Boardman symbols are not sufficient, and in order to detect elements of π^s_n in higher dimensions we need some nice classification of higher singularities. This view will turn out to be more evident in the case π^s_7.

7. Proof of Theorem 1.2

Recall the homomorphism $\mathfrak{j}^I : \Omega_{fold,0}(S^n) \to \Omega^I_{fold,0}(S^n)$ and the symbol $I(f)$ for $[f] \in \Omega_{fold,0}(S^n)$ defined in Introduction. Then there exists an extension $E^{I(f)} : (V, \partial V) \to (S^n \times [0,1], S^n \times 0)$ such that $\partial V = N$, the collar of ∂V is identified with $N \times [0,\epsilon]$, and $E^{I(f)}|N \times [0,\epsilon] = f \times id_{[0,\epsilon]}$. In this section we show that the singularities of certain type with symbol $I(f)$ of $E^{I(f)}$ detect the non-zero stable homotopy class $\omega_0([f]) \in \pi^s_n$. Note that in dimensions $n = 1, 2$, stable tangent bundles τ_V is trivial (an orientable 3-manifold is parallelizable).

Proof of Theorem 1.2. (Case: $n = 1$) We may take $N = S^1$. We have by [25] that $\mathfrak{o}(\tau_V, \tau(f)) = \mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ is equal to the second Stiefel-Whitney class $w_2(\widehat{V})$ by Remark 1. This is, as an invariant in $\mathbb{Z}/(2)$, coincides with the number of the singularities of the symbol $(1,1,0)$ of $E^{(1,1,0)}$ modulo 2, since the Thom polynomial is the dual class of $S^{(1,1,0)}(E^{(1,1,0)})$. Hence, we have $\mathfrak{m}(1) = 2$.

(Case: $n = 2$) It follows from Theorem 4.1 that we can choose an extension $E^{(1,1,0)}$ for a fold-map f. Hence, $\mathfrak{o}(\tau_V, \tau(f))$, namely $\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ lies in $H^2(\widehat{V}; \pi_1(SO(k)))$ and coincides with $w_2(\tau(\widehat{V}, \tau(f)))$ by Remark 1. Suppose that $w_2(\tau(\widehat{V}, \tau(f)))$ vanishes. Since $\pi_2(SO(3)) \approx \{0\}$, the second obstruction in $H^3(\widehat{V}; \pi_2(SO(3)))$, for $\tau(\widehat{V}, \tau(f))$ to be trivial, always vanishes. This implies that $\omega_0([f]) \neq 0$ if and only if $w_2(\tau(\widehat{V}, \tau(f)))$ does not vanish for any choice of V and $E^{(1,1,0)}$. Hence, $E^{(1,1,0)}$ must have 1-dimensional singularities of the symbol $(1,1,0)$.

(Case: $n = 3$) By Lemma 5.1 and Theorem 5.1 we have that $\mathfrak{m}(3) = j_1 = 24$ and $a_1 = 2$. By Lemma 5.1 we have that $\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ is equal to $\pm P_1(\tau(\widehat{V}, \tau(f)))/2$. The Thom polynomial of $\Sigma^2(\tau(\widehat{V}, \tau(f)), \varepsilon^4_{\widehat{V}})$ is equal to $P_1(\tau(\widehat{V}, \tau(f))$ by [26] (see also [3]). Consequently, for an element $\omega_0([f]) \in \pi^s_3 \approx \mathbb{Z}/(24)$, the algebraic number of the singular points of the symbol $(2,0)$ of an extension $E^{I(f)}$ is equal to $2\omega_0([f])$ modulo 48. We note that $\mathrm{codim}\Sigma^{2,1}(n,n) = 7$.

(Case: $n = 6$) By Proposition 6.2 we have $I(f) = (2,0)$ and $E^{(2,0)}$ has 3-dimensional singularities of type $I_{2,2}$ or $II_{2,2}$ with the symbol $(2,0)$.

(Case : $n = 7$) By [19, Section 7, Discussions and commputations], an element of $\pi^s_7 \approx \mathbb{Z}/(240)$ is detected by $P_2(\tau(\widehat{V}, \tau(f))/6$ modulo 240. We note $\mathrm{codim}\Sigma^3(n+1, n+1) = 9$ and $\mathrm{codim}\Sigma^{2,1,1}(n,n) = 10$. By Lemma 5.1 and Theorem 5.1 we have that $\mathfrak{m}(7) = j_2 = 240$ and $a_1 = 1$. Let $f : S^7 \to S^7$ be a fold-map with $\omega_0([f]) \neq 0$. If $P_1(\tau(\widehat{V}, \tau(f))$ does not vanish, then we have that $I(f) = (2,0)$ or $(2,1,0)$ by Lemma 5.3 (i). If V is parallelizable in addition, then $P_2(\tau(\widehat{V}, \tau(f))$$= \pm 6\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ do not vanish for any extension $E^{I(f)} : (V, S^7) \to (S^7 \times [0,1], S^7 \times 0)$ by Lemmas 5.3 (ii).

Let us recall the orbits $IV_4 = (x^2+y^2, x^4)$ and (x^2+y^3, xy^2) of the k-jets of the C^∞-stable germs $(\mathbb{R}^8, 0) \to (\mathbb{R}^8, 0)$ of the symbols $(2,0)$ and $(2,1,0)$ which are characterized by the local algebras $\mathbb{R}[[x,y]/(x^2+y^2, x^4)$ and $\mathbb{R}[[x,y]/(x^2+y^3, xy^2)$ respectively. They have been defined in [22]. If we apply an elaborate work in [15] to the jet bundle $J^k(\tau(\widehat{V}, \tau(f)), \varepsilon^8_{\widehat{V}})$, then we obtain the cycle $\langle (x^2+y^3, xy^2) - 2IV_4 \rangle$ under the integer coefficients of the Vassilyev complex ([15, Theorem 2.7]) and the Thom polynomial of $\langle (x^2+y^3, xy^2) - 2IV_4 \rangle$ is equal to $9P_2(\tau(\widehat{V}, \tau(f)))$ ([15, Section 3]). We denote the algebraic numbers of the singular points of types (x^2+y^3, xy^2) and IV_4 by A and B respectively. Then $A - 2B = 9 \cdot 6\mathfrak{o}(\tau(\widehat{V}, \tau(f)))$ is divisible by $6 \cdot 9 = 54$ and $(A-2B)/54$ modulo 240 corresponds to the stable homotopy class $\omega_0([f])$. Hence, we have $I(f) = (2, 0)$ or $(2, 1, 0)$. $\square$

We remark that the numbers of the singular points in the case $n = 3, 7$ correspond the e invariants introduced in [1] and [30]. The author would like to propose a problem: To what extent do higher singularities of extensions detect the stable homotopy groups of spheres?. We will find two difficulties in the study of this problem. First we may not find parallelizable manifolds V for extensions in general. This makes harder to determine types of singularities as in the case $n = 3, 7$. Next, as far as the author knows, we do not yet have a nice classification of singularities outside the Mather's nice range for this purpose.

Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
753-8512, Japan
e-mail: andoy@yamaguchi-u.ac.jp

References

[1] J. F. Adams, *On the groups $J(X)$ IV*, Topology **5** (1966), 21–71.

[2] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. **11** (1961), 291–310.

[3] Y. Ando, *Elimination of certain Thom-Boardman singularities of order two*, J. Math. Soc. Japan **34** (1982), 241–267.

[4] ———, *On the elimination of Morin singularities*, J. Math. Soc. Japan **37** (1985), 471–487.

[5] ———, *The homotopy type of the space consisting of regular jets and folding jets in $J^2(n, n)$*, Japan. J. Math. **24** (1998), 169–181.

[6] ———, *Folding maps and the surgery theory on manifolds*, J. Math. Soc. Japan **53** (2001), 357–382.

[7] ———, *Fold-maps and the space of base point preserving maps of spheres*, J. Math. Kyoto Univ. **41** (2002), 691–735.

[8] ———, *Existence theorems of fold-maps*, Japan. J. Math. **30** (2004), 29–73.

[9] ———, *The homotopy principle in the existence level for maps with only singularities of types A, D and E*, http://front.math.ucdavis.edu/math.GT/0411399.

[10] ———, *A homotopy principle for maps with prescribed Thom-Boardman singularities*, http://front.math.ucdavis.edu/math.GT/0309204.

[11] J. M. Boardman, *Singularities of differentiable maps*, Inst. Hautes Études Sci. Publ. Math. **33** (1967), 21–57.

[12] A. du Plessis, *Maps without certain singularities*, Comment. Math. Helv. **50** (1975), 363–382.

[13] J. M. Èliašberg, *On singularities of folding type*, Math. USSR. Izv. **4** (1970), 1119–1134.

[14] ———, *Surgery of singularities of smooth mappings*, Math. USSR. Izv. **6** (1972), 1302–1326.

[15] L. Fehér and R. Rimányi, *Thom polynomials with integer coefficients*, Illinois J. Math. **46** (2002), 1145–1158.

[16] M. Gromov, *Stable mappings of foliations into manifolds*, Math. USSR. Izv. **3** (1969), 671–694.

[17] M. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276.

[18] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer-Verlag, Berlin Heidelberg, 1956.

[19] M. Kervaire and J. Milnor, *Groups of homotopy spheres*: I, Ann. Math. **77** (1963), 504–537.

[20] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1, Interscience Publishers, New York, 1963.

[21] H. I. Levine, *Singularities of differentiable maps*, Proc. Liverpool Singularities Symposium, Springer Lecture Notes **192** (1971), 1–85.

[22] J. N. Mather, *Stability of C^∞ mappings*, VI: *The nice dimensions*, Proc. Liverpool Singularities Symp. I, Springer Lecture Notes **192** (1971), 207–253.

[23] J. Milnor and M. Kervaire, *Bernoulli numbers, homotopy groups, and a theorem of Rohlin*, Proc. Int. Congress of Math. Edinburgh, 1958, pp. 454–458.

[24] J. Milnor and E. Spanier, *Two remarks on fiber homotopy type*, Pacific J. Math. **10** (1960), 585–590.

[25] I. Porteous, *Simple singularities of maps*, Proc. Liverpool Singularities Symp. I, Springer Lecture Notes **192** (1971), 286–307.

[26] F. Ronga, *Le calcul de la classe de cohomologie entière dual a* Σ^k, Proc. Liverpool Singularities Symp. I, Springer Lecture Notes **192** (1971), 313–315.

[27] S. Smale, *The classification of immersions of spheres in Euclidian Spaces*, Ann. Math. **69** (1959), 327–344.

[28] E. Spanier, *Function spaces and duality*, Ann. Math. **70** (1959), 338–378.

[29] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, Princeton, 1951.

[30] H. Toda, Composition Methods in Homotopy Groups of Spheres, Ann. Math. Study **49**, Princeton, 1962.