J. Math. Kyoto Univ. (JMKYAZ)
45-2 (2005), 381–390

On the plurisubharmonicity of the leafwise Poincaré metric on projective manifolds

By

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Introduction

The aim of this paper is to give a new proof of the theorem asserting that the leafwise Poincaré metric has a plurisubharmonic variation. This theorem has been proved in [Br2], building on a method introduced in [Br1], and has later been generalised in [Br3] to the "relative" case. Below we shall recall the precise statement. The techniques used in these papers are sufficiently flexible to work on any compact Kähler manifold. Here, on the contrary, we shall restrict ourselves to complex projective manifolds, and the proof will follow a different strategy.

Let us resume the differences of the present paper with our previous ones.

Given a foliation by curves on a complex projective manifold X, we shall introduce below the *covering tube* U_S, which is roughly speaking a sort of global flow-box obtained by gluing together [Ily] the universal coverings of the leaves through a k-dimensional embedded disc $S \subset X$. It is a complex manifold fibered by curves over S and equipped with a meromorphic map into X. The matter is to prove that the fiberwise Poincaré metric on U_S has a plurisubharmonic variation. This is done in [Br1], [Br2] and [Br3] by showing that U_S has a suitable "holomorphic convexity" property, reducing in this way the problem to a clever result of [Yam].

Here, using the projectivity of X, we shall firstly remove from U_S an hypersurface, in such a way that the remaining part U_S'' appears as a Riemann domain over an Euclidean space. Then, using the Stein machinery, we shall prove that U_S'' has a special "pseudoconvex" exhaustion. This will allow to apply the main result of [M-Y] (more delicate than the corresponding particular case of [Yam] that we used before) to get that the fiberwise Bergman metric on U_S'' has a plurisubharmonic variation. Finally, by an L^2 removal–of–singularities argument, we shall pass from the Bergman metric on U_S'' to the Poincaré metric on U_S.

Received January 31, 2005

1. From Poincaré to Bergman

Let X be a smooth connected complex projective manifold of dimension n. Let $\mathcal{F}$ be a holomorphic foliation by curves on X, with singular set $Sing(\mathcal{F})$, analytic of codimension at least 2. Let $T \subset X^0 = X \setminus Sing(\mathcal{F})$ be an embedded $(n-1)$-disc transverse to the foliation, and let $S \subset T$ be an embedded k-disc, for some k between 1 and $n-1$. For reader's convenience, let us briefly recall some definitions and constructions from [Br3, §2].

For $s \in S$, let L_s^0 be the leaf of $\mathcal{F}^0 = \mathcal{F}|_{X^0}$ through s, and let $\widehat{L_{s,S}^0}$ be its **S-holonomy covering**, i.e. the covering associated to the subgroup $G_{s,S} \subset \pi_1(L_s^0, s)$ composed by those elements $\gamma \in \pi_1(L_s^0, s)$ whose holonomy $hol_\gamma : (T, s) \to (T, s)$ is the identity on $(S, s) \subset (T, s)$. Then the union $V_S^0 = \bigcup_{s\in S} \widehat{L_{s,S}^0}$ has a natural structure of complex manifold of dimension $k+1$, fibered over S and equipped with a holomorphic immersion $\pi_S^0 : V_S^0 \to X^0$ sending fibres to leaves.

Consider a parabolic end K of some fibre $\widehat{L_{s,S}^0}$, i.e., a closed subset K of $\widehat{L_{s,S}^0}$ isomorphic to the closed punctured disc $\overline{\mathbb{D}}^*$. For $r \in (0,1)$, set $A_r = \{r < |w| \le 1\}$ and $\partial A_r = \{|w| = 1\}$. Suppose that, for some $r \in (0,1)$, there exists an embedding $f : \mathbb{D}^k \times A_r \to V_S^0$ such that:

(i) f sends fibres of $\mathbb{D}^k \times A_r \to \mathbb{D}^k$ to fibres of $V_S^0 \to S$;

(ii) f sends $\{0\} \times \partial A_r$ to ∂K, respecting orientations;

(iii) the composition $\pi_S^0 \circ f : \mathbb{D}^k \times A_r \to X$ extends to a meromorphic family of discs (this means [Br3, §1] that there exist a complex manifold W, fibered over $\mathbb{D}^k$ with fibers $\overline{\mathbb{D}}$, a fibered embedding $j : \mathbb{D}^k \times A_r \to W$ sending $\mathbb{D}^k \times \partial A_r$ to ∂W, a meromorphic map $g : W - - \to X$, such that $\pi_S^0 \circ f = g \circ j$).

If this occurs, then we say that K is a **S-vanishing end**.

By compactifying all the S-vanishing ends of $\widehat{L_{s,S}^0}$ we obtain the **completed S-holonomy covering** $\widehat{L_{s,S}}$. Then

$$V_S = \bigcup_{s\in S} \widehat{L_{s,S}}$$

is still a complex manifold in a natural way, equipped with a submersion $Q_S : V_S \to S$, a section $q_S : S \to V_S$ giving the basepoints, and a meromorphic map $\pi_S : V_S - - \to X$. It is called **holonomy tube** (associated to S).

Finally, the **covering tube** (associated to S) U_S is the fiberwise universal covering of V_S:

$$U_S = \bigcup_{s\in S} \widetilde{L_{s,S}}$$

where $\widetilde{L_{s,S}}$ is the universal covering of $\widehat{L_{s,S}}$ with basepoint $q_S(s)$. Once a time, it is a complex manifold (but now this is a nontrivial fact, whose proof requires the unparametrised Levi continuity principle [Br3, Lemma 0]), equipped with a submersion $P_S : U_S \to S$, a section $p_S : S \to U_S$ giving the basepoints, and a meromorphic map $\Pi_S : U_S - - \to X$.

We also have a natural map

$$F_S : U_S \to V_S$$

obtained by gluing together the coverings $(\widetilde{L_{s,S}}, p_S(s)) \to (\widehat{L_{s,S}}, q_S(s))$. It is a local diffeomorphism but, in general, it is *not* a covering map (we shall see below an example); this is a source of difficulties. Note that $\Pi_S = \pi_S \circ F_S$.

On each fibre of U_S (isomorphic to $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$) we put its Poincaré metric (which is identically zero when the fibre is $\mathbb{C}$ or $\mathbb{P}$). We are interested in the case in which at least one fibre is hyperbolic, otherwise this fiberwise Poincaré metric is identically zero. Referring to [Yam], [Kiz], [M-Y] and [Br2] for the meaning of "plurisubharmonic variation", we shall prove:

Theorem 1.1. *Suppose that at least one fibre of U_S is hyperbolic. Then the fiberwise Poincaré metric on U_S has a plurisubarmonic variation.*

Let us reformulate this theorem in a different way, by using the projectivity of the ambient manifold.

Because X is projective, we may find an ample hypersurface (an hyperplane section) $D \subset X$ such that:

(i) $Sing(\mathcal{F}) \subset D$;

(ii) no leaf of $\mathcal{F}$ is entirely contained in D, i.e. every leaf cuts D along a discrete subset.

Define $X' = X \setminus D$ (a Stein manifold) and $\mathcal{F}' = \mathcal{F}|_{X'}$ (a nonsingular foliation). Let V'_S be the part of V_S which is over X', that is $V'_S = V_S \setminus \Gamma$ where Γ is an hypersurface, the strict transform of D under $\pi_S : V_S - - \to X$. Similarly, let $U'_S = U_S \setminus \Sigma$ be the part of U_S over X'. Thanks to condition (i) above, the maps $\pi'_S = \pi_S|_{V'_S}$ and $\Pi'_S = \Pi_S|_{U'_S}$ are *holomorphic*, more precisely holomorphic immersions into X'. The fibres of V'_S are in fact equal to the S-holonomy coverings of the leaves of $\mathcal{F}'$. The fibres of U'_S are coverings of those of V'_S, but *not* universal coverings.

Because X' is Stein, we may find a holomorphic map

$$H : X' \to \mathbb{C}^{k+1}$$

such that the composition $H \circ \pi'_S : V'_S \to \mathbb{C}^{k+1}$ has maximal rank outside a hypersurface $\Gamma' \subset V'_S$ cutting each fibre along a discrete subset. Hence, setting $V''_S = V'_S \setminus \Gamma'$ and $\pi''_S = \pi_S|_{V''_S}$, we obtain a so-called Riemann domain over $\mathbb{C}^{k+1}$, that is a local diffeomorphism

$$h = H \circ \pi''_S : V''_S \to \mathbb{C}^{k+1}.$$

Similarly for $U''_S = U'_S \setminus \Sigma'$, where Σ' is the preimage in U'_S of Γ' under $F'_S = F_S|_{V'_S}$.

Let us resume in a diagram:

$$\begin{array}{ccccccc}
U_S & \stackrel{F_S}{\longrightarrow} & V_S & \stackrel{\pi_S}{- - \rightarrow} & X & & \\
\cup & & \cup & & \cup & & \\
U'_S & \stackrel{F'_S}{\longrightarrow} & V'_S & \stackrel{\pi'_S}{\longrightarrow} & X' & = X \setminus D \\
\cup & & \cup & & \downarrow H & & \\
U''_S & \stackrel{F''_S}{\longrightarrow} & V''_S & \stackrel{h}{\longrightarrow} & \mathbb{C}^{k+1} & &
\end{array}$$

Recall the definition of *Bergman metric* on a Riemann surface R: if $p \in R$ and $v \in T_pR$ then $\|v\| = \sup |\omega_p(v)|$, where the supremum is taken over all the holomorphic 1-forms ω with L^2-norm equal to 1 (i.e. $\int_R \omega \wedge \overline{\omega} = 1$). Because

$$\mathcal{O}(\overline{\mathbb{D}}^*) \cap L^2(\mathbb{D}) = \mathcal{O}(\overline{\mathbb{D}})$$

we see that the Bergman metric on $R \setminus \{\text{discrete subset}\}$ is equal to (the restriction of) the Bergman metric on R. Moreover, when R is simply connected then the Bergman metric coincides with the Poincaré metric.

Now, each fibre of U''_S is obtained from the corresponding simply connected fibre of U_S by deleting a discrete subset. Therefore, the fiberwise Bergman metric on U''_S coincides with (the restriction of) the fiberwise Poincaré metric on U_S. As a consequence, Theorem 1.1 above can be restated as follows:

Theorem 1.2. *Suppose that at least one fibre of U_S is hyperbolic. Then the fiberwise Bergman metric on U''_S has a plurisubharmonic variation.*

This Theorem would be a consequence of [M-Y] if we were able to prove that U''_S is a Stein manifold. Unfortunately, we are unable to do so; for instance, we are unable to prove that U''_S is holomorphically separable (compare with [Ily] for a related problem). However, we shall be able to construct on U''_S a special pseudoconvex exhaustion, which finally will still reduce our problem to [M-Y], even without asserting the Steinness of U''_S.

It is worth noting that the proof above gives the plurisubharmonicity of the fiberwise Bergman metric also on V''_S, and therefore on the full holonomy tube V_S. In fact, this is even simpler, for V''_S has stronger pseudoconvexity properties than U''_S, as we will see. Thus, if the fiberwise covering $U_S \to V_S$ is not trivial, we obtain on U_S a *second* naturally defined fibrewise metric with a plurisubharmonic variation.

2. Holomorphic completion

Lemma 2.1. *The Riemann domain V''_S is holomorphically separable.*

Proof. Let us recall a standard fact about Riemann domains [G-R]: given such a domain (over an Euclidean space) we may take its quotient by the equivalence relation "$x \sim y$ if x and y are not separated by holomorphic functions", and then this quotient is still a Riemann domain (over the same Euclidean space). In other words, the previous equivalence relation is *open*: if $x \sim y$ then there are neighbourhoods B_x of x and B_y of y and a biholomorphism $b : B_x \to B_y$ (induced by the projection to the Euclidean space) such that $x' \sim y' = b(x')$ for every $x' \in B_x$.

Take now our Riemann domain $h : V''_S \to \mathbb{C}^{k+1}$. Suppose, by contradiction, that there are two points $p_1, p_2 \in V''_S$ which are not holomorphically separated. Of course, p_1 and p_2 belong to the same fibre of V''_S, and they have the same image in X' under the map π''_S. Let $\widehat{L'_{q,S}}$ be the fibre of V'_S which contains p_1 and p_2: it is the S-holonomy covering of the leaf L'_q of $\mathcal{F}'$ through some point $q \in S$. Take two paths γ_1 and γ_2 in $\widehat{L'_{q,S}}$ joining the basepoint q to p_1 and p_2. Then the composition of γ_1 and γ_2 gives, in L'_q, a closed path γ_q based at q and passing through $\pi''_S(p_1) = \pi''_S(p_2)$. By definition of S-holonomy covering, the S-holonomy of $\mathcal{F}'$ along this loop is *nontrivial*.

On the other hand, for every p'_1 close to p_1 there is a p'_2 close to p_2 such that p'_1 and p'_2 are not holomorphically separated. By the previous construction, we obtain a continuous family of loops $\gamma_{q'} \subset L'_{q'}$, $q' \in S$ close to q, which lift γ_q. This contradicts the nontriviality of the S-holonomy. □

As a consequence of this, we may embed V''_S into its *envelope of holomorphy* [G-R]: a Riemann domain

$$\overline{h} : \overline{V''_S} \longrightarrow \mathbb{C}^{k+1}$$

which is the maximal holomorphic extension of $h : V''_S \to \mathbb{C}^{k+1}$ among holomorphically separable Riemann domains. We shall identify V''_S with the corresponding open subset of $\overline{V''_S}$. A fundamental theorem of Cartan–Thullen and Oka asserts that $\overline{V''_S}$ is a Stein manifold.

The holomorphic immersion $\pi''_S : V''_S \to X'$ extends to a holomorphic map

$$\overline{\pi''_S} : \overline{V''_S} \to X',$$

because X' is Stein and $\mathcal{O}(\overline{V''_S}) = \mathcal{O}(V''_S)$. Of course, from $h = H \circ \pi''_S$ it follows that $\overline{h} = H \circ \overline{\pi''_S}$. Because $\overline{h}$ has maximal rank, the same holds also for $\overline{\pi''_S}$, which therefore is a holomorphic immersion into X', transverse to the fibres of H.

The submersion $Q''_S : V''_S \to S$ also extends to a map

$$\overline{Q''_S} : \overline{V''_S} \to S,$$

which however could fail to be a submersion. A priori, it could even happen that $\overline{Q''_S}$ has some higher dimensional fibre, or some disconnected fibre.

For every $s \in S$ consider the fibre $(Q''_S)^{-1}(s)$ as a subset of the fibre $(\overline{Q''_S})^{-1}(s)$, via the inclusion $V''_S \subset \overline{V''_S}$. The next lemma is similar to the Closure Lemma of [Ily], see also [Suz, §3] for a related result. In our case everything is simpler because the foliation $\mathcal{F}'$ is nonsingular, but also we need some care due to the additional presence of the projection H.

Lemma 2.2. *For every $s \in S$, $(Q''_S)^{-1}(s)$ is a connected component of $(\overline{Q''_S})^{-1}(s)$ (more precisely, the one which contains the basepoint $q_S(s)$).*

Proof. Obviously $(Q''_S)^{-1}(s) = (\overline{Q''_S})^{-1}(s) \cap V''_S$ is open in $(\overline{Q''_S})^{-1}(s)$. We need to prove that it is also closed.

It is sufficient to prove the following: if $\gamma : [0,1] \to (\overline{Q''_S})^{-1}(s)$ is a path with $\gamma([0,1)) \subset (Q''_S)^{-1}(s)$, then also the endpoint $\gamma(1)$ belongs to $(Q''_S)^{-1}(s)$.

Set $q = \overline{\pi''_S}(\gamma(1)) \in X'$, and let L'_q be the leaf of $\mathcal{F}'$ through q. Then $\eta = \overline{\pi''_S} \circ \gamma$ is a path in L'_q from $\overline{\pi''_S}(\gamma(0))$ to q. It can be lifted to $(Q'_S)^{-1}(s)$, the S-holonomy covering of L'_q, as a path ζ from $\gamma(0)$ (which belongs to $(Q''_S)^{-1}(s)$ and hence also to $(Q'_S)^{-1}(s)$) to some point $\zeta(1)$ over q. Of course, $\zeta([0,1))$ is contained in $(Q''_S)^{-1}(s)$ and ζ on $[0,1)$ coincides with γ on $[0,1)$, after identification of $(Q''_S)^{-1}(s)$ as a subset of $(\overline{Q''_S})^{-1}(s)$ (where γ lives) and $(Q''_S)^{-1}(s)$ as a subset of $(Q'_S)^{-1}(s)$ (where ζ lives). Now, recall that $\overline{\pi''_S}$ is an immersion transverse to H, even at the point $\gamma(1)$. It follows from this that $\zeta(1)$ also belongs to $(Q''_S)^{-1}(s)$, and therefore $\gamma(1)$ belongs to $(Q''_S)^{-1}(s)$ too. Whence the closedness of $(Q''_S)^{-1}(s)$. □

Thus, $\overline{V''_S} \setminus V''_S$ is a closed union of analytic subsets of $\overline{V''_S}$, connected components of fibres of $\overline{Q''_S}$. It is likely that V''_S is Stein (so that, finally, $\overline{V''_S}$ would be equal to V''_S). For instance, this is the case when $\dim S = 1$ (compare with [Suz, §3]), because in that case the above analytic subsets are necessarily hypersurfaces: a Stein surface minus a closed union of curves is still Stein, by the solution of the Levi problem [G-R]. When $\dim S > 1$ the problem is more subtle: a partial answer can be done using the Fibers Connection Lemma of [Ily], whose proof however strongly requires the simply-connectedness of the fibres of V''_S, which is almost never satisfied in our case. For the same reason (absence of simply-connectedness) we cannot apply the methods of [Ily] to prove the (presumable) holomorphic separability of U''_S.

Anyway, the possible nonempty difference between $\overline{V''_S}$ and V''_S will not cause serious troubles in the following.

We may further remark that the construction of the holomorphic completion can be done also starting from the possibly not holomorphically separable Riemann domain U''_S [G-R]. But firstly we need to quotient U''_S by the nonseparated equivalence relation. We thus obtain a Stein $\overline{U''_S}$ and a *locally* injective map

$$U''_S \xrightarrow{j} \overline{U''_S}$$

with $j(U''_S)$ equal to the holomorphically separated quotient of U''_S. As before, $\overline{U''_S}$ has a fibration over S and $\overline{U''_S} \setminus j(U''_S)$ is filled by connected components of fibres.

Now, by [M-Y] the fiberwise Bergman metric on $\overline{V''_S}$ or $\overline{U''_S}$ has a plurisubharmonic variation, and hence also on V''_S or $j(U''_S)$. However, this last fact does *not* give the plurisubharmonic variation of the Bergman metric on U''_S: the Bergman metric is not functorial with respect to coverings, and so the pullback of the fiberwise Bergman metric under $U''_S \to V''_S$ or $U''_S \to j(U''_S)$ is not the fiberwise Bergman metric on U''_S.

3. Plurisubharmonicity

Fix a smooth strictly plurisubharmonic exhaustive function $\psi : \overline{V''_S} \to \mathbb{R}$, and set $\Omega_k = \{\psi < a_k\} \subset\subset \overline{V''_S}$ for some increasing sequence of regular values $\{a_k\}_{k\in\mathbb{N}}$ diverging to $+\infty$ as $k \to +\infty$. Set

$$\widetilde{\Omega}_k = (F''_S)^{-1}(\Omega_k \cap V''_S) \subset U''_S$$

where $F''_S : U''_S \to V''_S$ is the fiberwise covering of Section 1.

Even if the open subsets $\widetilde{\Omega}_k$ are not relatively compact in U''_S, they still form an increasing sequence which exhausts U''_S. By the monotonicity of the Bergman metric with respect to inclusion, in order to prove that the fiberwise Bergman metric has a psh variation on $U''_S \to S$, it is sufficient to prove the same property on $\widetilde{\Omega}_k \to S$, for every $k \in \mathbb{N}$.

We shall use an idea of [Kiz], which consists in finding a "local" embedding of $\widetilde{\Omega}_k$ into a Stein manifold, so that we will be reduced to [M-Y].

As a preliminary fact, we need a key lemma concerning the fiberwise covering $F''_S : U''_S \to V''_S$. Fix $s_0 \in S$ and let $E \subset (Q''_S)^{-1}(s_0)$ be a relatively compact connected open subset. Let $\widetilde{E} \subset (P''_S)^{-1}(s_0)$ be a connected component of $(F''_S)^{-1}(E)$. The next lemma says that the covering $F''_S : \widetilde{E} \to E$ can be extend on "uniform" (over S) neighbourhoods of $\widetilde{E}$ in U''_S and of E in V''_S; this is perhaps nonevident, because $\widetilde{E}$ is possibly noncompact.

Lemma 3.1. *There exist a neighbourhood $U \subset S$ of s_0, a neighbourhood $W \subset V''_S$ of E, a neighbourhood $\widetilde{W} \subset U''_S$ of $\widetilde{E}$, such that:*

(i) *$W \cap (Q''_S)^{-1}(s_0) = E$, $Q''_S(W) = U$, and $Q''_S : W \to U$ is a holomorphically trivial fibration, i.e. $W \simeq U \times E$;*

(ii) *$\widetilde{W} \cap (P''_S)^{-1}(s_0) = \widetilde{E}$, $P''_S(\widetilde{W}) = U$, and $P''_S : \widetilde{W} \to U$ is a holomorphically trivial fibration, i.e. $\widetilde{W} \simeq U \times \widetilde{E}$;*

(iii) *F''_S sends $\widetilde{W}$ to W, as a covering respecting the above trivialisations.*

Proof. By Nishino's trick [Kiz] [Suz, §2], the relatively compact $E \subset (Q''_S)^{-1}(s_0)$ can be holomorphically deformed into nearby fibres, producing W and U as in (i). We claim that, up to reducing the size of U, this holomorphic deformation can be lifted to U''_S.

Indeed, fix $e \in \widetilde{E}$, a path $\gamma : [0, 1] \to (P''_S)^{-1}(s_0)$ from the basepoint $p_S(s_0)$ to e, and for every $x \in \widetilde{E}$ a path $\gamma_x : [1, 2] \to \widetilde{E}$ from e to x. Let

$$\gamma^x : [0, 2] \to (P''_S)^{-1}(s_0)$$

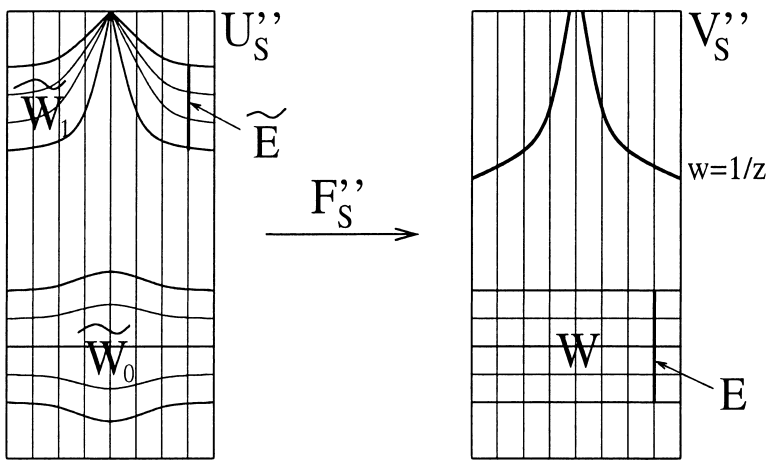
be the composition of γ and γ_x, a path from $p_S(s_0)$ to x. Then

$$F''_S \circ \gamma^x : [0, 2] \to (Q''_S)^{-1}(s_0)$$

is a path from the basepoint $q_S(s_0)$ to $F''_S(x)$. Moreover, all these paths $\{F''_S \circ \gamma^x\}_{x\in\widetilde{E}}$ are contained in the *compact* set $clos(E) \cup F''_S(\gamma([0,1]))$. Still by Nishino's trick, $E \cup F''_S(\gamma([0,1]))$ can be holomorphically deformed into nearby fibres, and therefore also the totality of these paths can be holomorphically deformed, as paths from the basepoint of the fibre to the holomorphically varying endpoint. This gives the desired holomorphic deformation of $\widetilde{E}$ into nearby fibres. □

Remark 3.1. In general, a product neighbourhood $W = U \times E$ cannot be lifted to a product neighbourhood $\widetilde{W} = U \times \widetilde{E}$ unless we reduce U. The problem comes from the deformation of $F''_S \circ \gamma$, which perhaps cannot be done over the full U.

To see an example (see the figure below), suppose $V''_S = (\mathbb{D}\times\mathbb{C})\setminus\{w = 1/z\}$, $U''_S = \mathbb{D} \times \mathbb{C}$, so that the fiberwise covering is $F''_S(z,w) = (z, \frac{1}{z}(e^{zw} + 1))$. Consider $W = \mathbb{D} \times \mathbb{D}(\frac{1}{2}) \subset V''_S$, seen as a thickening of $E = \{\frac{1}{2}\} \times \mathbb{D}(\frac{1}{2})$: then $(F''_S)^{-1}(W)$ has infinitely many connected components $\{\widetilde{W}_j\}$, but only one of them (say, $\widetilde{W}_0$) has a product structure over the full $\mathbb{D}$, the others do not intersect the fibre of U''_S over 0. Here $F''_S \circ \gamma$ is a path in the fibre $\mathbb{C} \setminus \{2\}$ of V''_S over $\frac{1}{2}$, starting at the basepoint 0 and ending at some point of E, after some revolutions around 2. If the number of these revolutions is not 0 (i.e., if we choose $\widetilde{E}$ not in $\widetilde{W}_0$) then, clearly, $F''_S \circ \gamma$ cannot be deformed until the fibre of V''_S over 0, as path from the basepoint 0 to $\{0\} \times \mathbb{D}(\frac{1}{2})$. This explains why $\widetilde{W}_j$, $j \neq 0$, escapes to infinity when approaching the fibre of U''_S over 0.



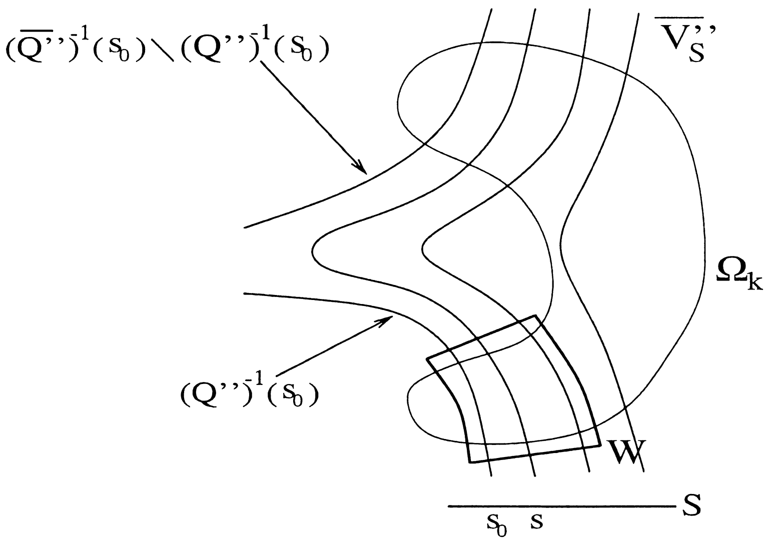
For a similar reason, it is essential, for the lemma above, that $\widetilde{E}$ is *connected*: in general, we cannot find a uniform product structure around the full $(F''_S)^{-1}(E)$ (we have such a structure $\widetilde{W}_j = U_j \times \widetilde{E}_j$ for every connected component $\widetilde{E}_j$ of $(F''_S)^{-1}(E)$, but it may happen that the size of U_j is not uniformly positive).

We can now return to the study of $\widetilde{\Omega}_k \to S$ and its fiberwise Bergman metric.

Fix $s_0 \in S$ and $x_0 \in \widetilde{\Omega}_k \cap (P''_S)^{-1}(s_0)$.

Consider $clos(\Omega_k) \cap (Q''_S)^{-1}(s_0)$: by Lemma 2.2 it is a compact subset of the fibre $(Q''_S)^{-1}(s_0)$. Hence we can find an open, connected, relatively compact subset $E \subset (Q''_S)^{-1}(s_0)$ which contains it. Let $\widetilde{E} \subset (P''_S)^{-1}(s_0)$ be the connected component of $(F''_S)^{-1}(E)$ containing x_0. Take U, $W \simeq U \times E$, $\widetilde{W} \simeq U \times \widetilde{E}$ as in Lemma 3.1.

Remark that it is well possible that, even for s close to s_0, the compact set $clos(\Omega_k) \cap (Q''_S)^{-1}(s)$ is *not* contained in W: there could be a part of $clos(\Omega_k) \cap (Q''_S)^{-1}(s)$ which, as $s \to s_0$, diverges on V''_S but converge on $\overline{V''_S}$ to $clos(\Omega_k) \cap [(\overline{Q''_S})^{-1}(s_0) \setminus (Q''_S)^{-1}(s_0)]$ (see figure below). However, and because $\partial E \subset (Q''_S)^{-1}(s_0)$ is disjoint from $clos(\Omega_k)$, we see that up to reducing U the horizontal boundary $\partial_{hor} W = U \times \partial E$ is also disjoint from $clos(\Omega_k)$, and therefore $\Omega_k \cap W$ is an open subset of W whose closure does not touch $\partial_{hor} W$.



Take now the preimage of $\Omega_k \cap W$ in $\widetilde{W}$ under $F''_S : \widetilde{W} \to W$, that is $\widetilde{\Omega}_k \cap \widetilde{W}$. For every $x \in \widetilde{\Omega}_k \cap \widetilde{W}$ the connected component of the fibre of $\widetilde{\Omega}_k$ which contains x is entirely contained in $\widetilde{W}$. Hence, in order to compute the Bergman metric on the fibres of $\widetilde{\Omega}_k$ at points of $\widetilde{\Omega}_k \cap \widetilde{W}$ (e.g., around x_0), we may replace $\widetilde{\Omega}_k$ with $\widetilde{\Omega}_k \cap \widetilde{W}$ without affecting the result.

Now, $\widetilde{W}$ is a Stein manifold, and $\widetilde{\Omega}_k$ is Levi-pseudoconvex in $\widetilde{W}$, because Ω_k is Levi-pseudoconvex in W. It follows [G-R] that $\widetilde{\Omega}_k \cap \widetilde{W}$ is also a Stein manifold. We can now apply the result of [M-Y] to obtain the plurisubharmonicity of the fiberwise Bergman metric on $\widetilde{\Omega}_k \cap \widetilde{W}$ and hence on $\widetilde{\Omega}_k$.

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